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CONDITIONS FOR THE NONOSCILLATION OF THIRD ORDER DIFFERENTIAL EQUATIONS WITH NONNEGATIVE COEFFICIENTS*

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Abstract. This article is concerned with the oscillation of solutions of third order, linear differential equations with nonnegative coefficients. The objective of the paper is to obtain necessary conditions and sufficient conditions for the nonoscillation of such equations. The techniques which are developed and the results which are obtained depend heavily on the fundamental papers of M. Hanan and A. C. Lazer on third order differential equations. Extensive use is also made of the survey article on oscillation theory by J. H. Barrett.

1. Introduction. This paper is concerned with the development of necessary conditions and sufficient conditions for the nonoscillation of third order linear differential equations of the form

$$L[y] = y''' + p(x)y' + q(x)y = 0, \tag{E}$$

where $p(x)$ and $q(x)$ are continuous, nonnegative functions on an interval $[a, \infty)$. A nontrivial solution $y(x)$ of (E) is said to be *oscillatory* if the set of zeros of $y(x)$ on $[a, \infty)$ is not bounded above, or, equivalently, $y(x)$ is oscillatory if $y(x)$ has infinitely many zeros on $[a, \infty)$. A nontrivial solution which is not oscillatory is called *nonoscillatory*. The equation (E) is *oscillatory* if it has at least one nontrivial oscillatory solution, otherwise (E) is said to be *nonoscillatory*. Finally, (E) is said to be *disconjugate* on the subinterval $[b, c)$, $a \leqq b < c \leqq \infty$, if no nontrivial solution of (E) has more than two zeros, counting multiplicities, on $[b, c)$. It is easily seen that if (E) is disconjugate on the subinterval $[b, \infty)$ of $[a, \infty)$, then (E) is nonoscillatory. In contrast to the situation for linear second order equations, however, the nonoscillation of (E) does not imply the existence of a subinterval $[c, \infty)$ of $[a, \infty)$ on which (E) is disconjugate. See, for example, the discussion by J. H. Barrett [2, p. 213]. The relationship between disconjugacy and nonoscillation of (E) has been investigated by the authors [5].

The results in this paper are derived primarily from the fundamental work of M. Hanan [6] and we shall be referring to his results throughout. We also have relied heavily on the survey article by J. H. Barrett [3] and the work of A. C. Lazer [8].

2. Nonoscillation and disconjugacy. As indicated above, the relationship between nonoscillation and disconjugacy for (E) is more complicated than in the case of second order equations. In particular, as a consequence of the Sturm separation theorem, the second order equation

$$(f(x)y')' + g(x)y = 0, \tag{E_2}$$

where $f(x)$ and $g(x)$ are continuous functions on $[a, \infty)$ with $f(x) > 0$ on this

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interval, is nonoscillatory if and only if there exists a number $b \geqq a$ such that (E_2) is disconjugate on $[b, \infty]$. On the other hand, there are examples of third order equations which are nonoscillatory on an interval $[a, \infty)$ and yet fail to be disconjugate on every subinterval $[b, \infty)$ of $[a, \infty)$. Barrett [2, p. 213] cites the existence of several examples and J. M. Dolan [4, p. 385] provides an example.

The following results are either well known, or are the combination of known results. They establish a connection between the notions of disconjugacy, nonoscillation and oscillation of (E), and they will be of considerable use in the development of our necessary conditions and sufficient conditions for the nonoscillation of (E).

THEOREM A (N. Azbelev and Z. Caljuk [1, Lemma 1, Thm. 2]). *The differential equation* (E) *is disconjugate on* $[b, \infty)$, $a \leqq b < \infty$, *if and only if the adjoint*

$$L^*(y) = [y'' + p(x)y]' - q(x)y = 0 \tag{E*}$$

of (E) *is disconjugate on* $[b, \infty)$. *Moreover, if* $u(x, b)$ *and* $u^*(x, b)$ *are the solutions of* (E) *and* (E*), *respectively, satisfying the initial conditions*

$$y(b) = y'(b) = 0, \qquad y''(b) = 1, \qquad b \geqq a, \tag{1}$$

then a necessary and sufficient condition for each of (E) *and* (E*) *to be disconjugate on* $[b, \infty)$ *is* $u(x, b) > 0$ *and* $u^*(x, b) > 0$ *on* (b, ∞).

THEOREM B ([3, Thm. 2.15] and [4, Thm. 1]). *If* (E) *is nonoscillatory on* $[a, \infty)$, *then either all solutions of* (E*) *are oscillatory, or all solutions of* (E*) *are nonoscillatory. Moreover, if each of* (E) *and* (E*) *is nonoscillatory, then each equation is disconjugate on some subinterval* $[b, \infty)$ *of* $[a, \infty)$.

THEOREM C ([5, Thm. 5]). *If* (E) *is not disconjugate on any subinterval* $[b, \infty)$ *of* $[a, \infty)$, *then each of the following is a sufficient condition for* (E) *to be oscillatory*:

(a) $y'' + p(x)y = 0$ *is nonoscillatory on* $[a, \infty)$,

(b) $p(x) \in C'[a, \infty)$ *with* $2q(x) - p'(x)$ *of one sign on* $[a, \infty)$ *and not identically zero on any subinterval.*

In the work which follows, we shall assume that the coefficients $p(x)$ and $q(x)$ of (E) satisfy the following hypothesis:

(H) $\quad p(x) \in C'[a, \infty), \quad p(x) \geqq 0, \quad q(x) \geqq 0, \quad q(x) - p'(x) \geqq 0 \quad$ on $[a, \infty)$
with these functions not identically zero on any subinterval.

Assuming that the coefficients satisfy (H), we have $2q(x) - p'(x) \geqq 0$ on $[a, \infty)$. Thus Theorem C applies and we have the following.

THEOREM D. *Let the coefficients of* (E) *satisfy* (H). *Then* (E) *is nonoscillatory if and only if* (E) *is disconjugate on* $[b, \infty)$ *for some* $b \geqq a$.

3. Necessary conditions. In this section we establish necessary conditions for the nonoscillation of (E). In view of Theorems A, B and D, our necessary conditions for the nonoscillation of (E) are, at the same time, necessary conditions for the nonoscillation of (E*), and, in fact, are actually necessary conditions for the eventual disconjugacy of both (E) and (E*).

THEOREM 1. *Let coefficients of* (E) *satisfy* (H). *If* (E) *is nonoscillatory, then each of the following holds*:

(i) *there exists a number* $b \geqq a$ *such that the second order equation*

(2) $$y'' + [p(x) + (1/2)(x - c)q(x)]y = 0$$

is disconjugate on $[b, \infty)$ *for all* $c \geqq b$;

(ii) *the second order equation*

(3) $$y'' + \left[p(x) + \int_x^\infty q(t)\, dt\right]y = 0$$

is nonoscillatory;

(iii) $\int_a^\infty x[q(x) - p'(x)]\, dx < \infty$.

By Theorem D, if (E) is nonoscillatory, then (E) is disconjugate on $[b, \infty)$ for some $b \geqq a$. In addition, by Theorem A, (E*) is disconjugate on $[b, \infty)$, and $u(x, b) > 0$, $u^*(x, b) > 0$ on (b, ∞), where u and u^* are the solutions of (E) and (E*) satisfying (1). We remark that since $u(x, b)$ and $u^*(x, b)$ are each positive on (b, ∞), and since $q(x)$ and $q(x) - p'(x)$ are not identically zero on any subinterval, neither $u'(x, b)$ nor $u^{*\prime}(x, b)$ is identically zero on any subinterval of $[b, \infty)$.

We now establish a sequence of lemmas which establish the behavior of the solutions $u(x, b)$ and $u^*(x, b)$.

LEMMA 1.1. *Let the coefficients of* (E) *satisfy* (H) *and let* $u(x, b)$ *and* $u^*(x, b)$ *be the solutions of* (E) *and* (E*), *respectively, satisfying* (1). *If* (E) *is disconjugate on* $[b, \infty)$, *then* $D_2u(x, b) = u''(x, b) + p(x)u(x, b) > 0$ *on* $[b, \infty)$.

Proof. The disconjugacy of (E) on $[b, \infty)$ implies $u(x, b) > 0$ on (b, ∞). Substituting $u(x, b)$ into (E) and integrating from b to x, yields the equation

$$u''(x, b) - 1 + p(x)u(x, b) - \int_b^x [p'(t) - q(t)]u(t, b)\, dt = 0.$$

Thus,

$$D_2u(x, b) = 1 - \int_b^x [q(t) - p'(t)]u(t, b)\, dt,$$

and we conclude $D_2u(x, b) \leqq 1$ on $[b, \infty)$. Now $[D_2u(x, b)]' = -[q(x) - p'(x)]u(x, b) \leqq 0$ and, consequently, $D_2u(x, b)$ is nonincreasing on $[b, \infty)$. If $D_2u(x, b) \leqq 0$ on $[c, \infty)$ for some $c \geqq b$, then $u''(x, b) \leqq -p(x)u(x, b) \leqq 0$ on $[c, \infty)$, which implies $u'(x, b)$ is nonincreasing on this interval. If $u'(x, b) \geqq 0$ on $[c, \infty)$, then $u'''(x, b) = -p(x)u'(x, b) - q(x)u(x, b) \leqq 0$ on $[c, \infty)$. But $u'''(x, b) \leqq 0$ and $u''(x, b) \leqq 0$ on $[c, \infty)$, together with the fact that the coefficients are not identically zero and $u(x, b)$ is nontrivial, implies $u'(x, b) \to -\infty$ as $x \to \infty$, contradicting our assumption $u'(x, b) \geqq 0$ on $[c, \infty)$. Thus there exists a number d, $d \geqq c$, such that $u'(x, b) < 0$ on (d, ∞). However, $u''(x, b) \leqq 0$ and $u'(x, b) < 0$ on (d, ∞) implies $u(x, b) \to -\infty$ as $x \to \infty$, and this is impossible. We conclude, therefore, that $D_2u(x, b) = u''(x, b) + p(x)u(x, b) > 0$ on $[b, \infty)$.

LEMMA 1.2. *Let the hypothesis of Lemma* 1.1 *hold. Then* $u^{*\prime}(x, b) \geqq 0$ *on* (b, ∞).

Proof. Suppose $u^{*\prime}(x, b)$ changes sign on (b, ∞) and let $x = c$ be the first point at which $u^{*\prime}(x, b)$ has a sign change. Then we have $u^{*\prime}(x, b) \geqq 0$ on $[b, c)$, $u^{*\prime}(c, b) = 0$ and $u^{*\prime}(x, b) < 0$ on some interval to the right of c. Assume that $u^{*\prime}(x, b)$ has a zero on (c, ∞) and let the first such zero be at $x = d$. Let $v^*(x, b)$

be the solution of (E*) satisfying the initial conditions

$$y(b) = 0, \quad y'(b) = 1, \quad y''(b) = 0, \tag{4}$$

and put $\lambda^*(x) = v^{*\prime}(x, b)/u^{*\prime}(x, b)$ on (c, d). Using the fact that the Wronskian of two solutions of (E*) is a solution of (E) (and vice versa), it is readily verified that

$$u(x, b) \equiv v^*(x, b)u^{*\prime}(x, b) - u^*(x, b)v^{*\prime}(x, b). \tag{5}$$

We can now conclude that $v^{*\prime}(d, b) < 0$. Hence $\lambda^*(x) \to +\infty$ as $x \uparrow d$. Calculating the derivative of $\lambda^*(x)$ and using (5), we find $\lambda^{*\prime}(x) = -D_2u(x, b)/[u^{*\prime}(x, b)]^2$. Thus $\lambda^{*\prime}(x) < 0$ on (c, d), contradicting the fact that $\lambda^*(x) \to +\infty$ as $x \uparrow d$. Therefore $u^{*\prime}(x, b) < 0$ on (c, ∞).

We now consider $u^{*\prime\prime}(x, b)$. Since

$$u^{*\prime\prime\prime}(x, b) = -p(x)u^{*\prime}(x, b) - [p'(x) - q(x)]u^*(x, b) \geqq 0$$

on $[c, \infty)$, we have $u^{*\prime\prime}(x, b)$ nondecreasing on this interval. If $u^{*\prime\prime}(x, b) \leqq 0$ on $[c, \infty)$, then we could conclude that $u^*(x, b) \to -\infty$ and this is impossible. Thus $u^{*\prime\prime}(x, b)$ is eventually positive, i.e., $u^{*\prime\prime}(x, b) > 0$ on some subinterval (e, ∞), $e \geqq c$. But $u^{*\prime\prime\prime}(x, b) \geqq 0$ and $u^{*\prime\prime}(x, b) > 0$ on (e, ∞) imply $u^{*\prime}(x, b) \to +\infty$, which contradicts $u^{*\prime}(x, b) < 0$ on (c, ∞). We conclude, therefore, that $u^{*\prime}(x, b) > 0$ on $[b, \infty)$.

LEMMA 1.3. *Let the hypothesis of Lemma* 1.1 *hold. Then* $D_2u^*(x, b) \equiv u^{*\prime\prime}(x, b) + p(x)u^*(x, b) > 0$ *on* $[b, \infty)$ *and* $\int_b^\infty D_2u^*(x, b)\,dx = \infty$.

Proof. Substituting $u^*(x, b)$ into (E^*) and integrating from b to x yields

$$D_2u^*(x, b) = 1 + \int_b^x q(t)u^*(t, b)\,dt.$$

Since $u^*(x, b) > 0$ and $q(x) \geqq 0$ on (b, ∞), the result follows.

LEMMA 1.4. *Let the hypothesis of Lemma* 1.1 *hold. Then* $u'(x, b) > 0$ *and* $u''(x, b) > 0$ *on* (b, ∞).

Proof. If we assume that $u'(x, b)$ changes sign on (b, ∞), say at the point $x = c$, then using the same argument as in the proof of Lemma 1.2, we find $u'(x, b) < 0$ on (c, ∞).

The linear operators L and L^* are related by the Lagrange identity

$$zL(y) + yL^*(z) = \{y; z\}', \tag{6}$$

where

$$\{y; z\} = zy'' - z'y' + [z'' + p(x)z]y.$$

Using the initial values of $u(x, b)$ and $u^*(x, b)$, we obtain $\{u(x, b); u^*(x, b)\} = 0$. Thus $u(x, b)$ is a solution of the second order equation

$$u^*(x, b)y'' - u^{*\prime}(x, b)y' + D_2u^*(x, b)y = 0 \tag{7}$$

on (b, ∞). Of course, $u(x, b)$ is a solution of (E) so that upon eliminating the y-term from (E) and (7), we find that $u(x, b)$ is a solution of third order equation

$$D_2u^*(x, b)y''' - q(x)u^*(x, b)y'' + [p(x)D_2u(x, b) + q(x)u^{*\prime}(x, b)]y' = 0. \tag{8}$$

Letting $z = y'$ and dividing (8) by $D_2u^*(x,b)$, we obtain

$$z'' - \frac{q(x)u^*(x,b)}{D_2u^*(x,b)}z' + \frac{p(x)D_2u^*(x,b) + q(x)u^{*\prime}(x,b)}{D_2u^*(x,b)}z = 0,$$

and this equation can be rewritten

$$\left[\frac{1}{D_2u^*(x,b)}z'\right]' + \frac{p(x)D_2u^*(x,b) + q(x)u^{*\prime}(x,b)}{[D_2u^*(x,b)]^2}z = 0. \tag{9}$$

Now, (9) is disconjugate on $[c,\infty)$ since the function $z(x) \equiv u'(x,b)$ is a solution such that $z(c) = 0$ and $z(x) < 0$ on (c,∞). From Lemmas 1.2 and 1.3, $p(x)D_2u^*(x,b) + q(x)u^{*\prime}(x,b) \geqq 0$ on (b,∞) and $\int_c^\infty D_2u^*(x,b)\,dx = \infty$. Therefore, by a theorem of Hille [7], $z(x)\cdot z'(x) > 0$ on (c,∞) and we conclude $z'(x) < 0$ on (c,∞). But $z(x) \equiv u'(x,b) < 0$ and $z'(x) \equiv u''(x,b) < 0$ on (c,∞) implies $u(x,b) \to -\infty$ as $x \to \infty$, contradicting the fact that $u(x,b) > 0$ on (b,∞).

We can now conclude that $u'(x,b) \geqq 0$ on $[b,\infty)$. Finally, since $u(x,b) > 0$ and $u'(x,b) \geqq 0$, $p(x) \geqq 0$, $q(x) \geqq 0$ on (b,∞) with none of these functions being identically zero on any subinterval, it follows that $u'''(x,b) \leqq 0$ on (b,∞) and is not identically zero on any subinterval. Therefore, $u'(x,b) > 0$ and $u''(x,b) > 0$ on (b,∞).

Proof of Theorem 1. Assuming that (E) is nonoscillatory, we have (E) disconjugate on some subinterval $[b,\infty)$ of $[a,\infty)$, and the solutions $u(x,b)$ and $u^*(x,b)$ of (E) and (E*) satisfying (1) have the properties:

$$u(x,b) > 0, \quad u'(x,b) > 0, \quad u''(x,b) > 0, \quad D_2u(x,b) > 0,$$
$$u^*(x,b) > 0, \quad u^{*\prime}(x,b) \geqq 0, \quad D_2u^*(x,b) > 0 \quad \text{on } (b,\infty).$$

Now, $u(x,b) > 0$, $u'(x,b) > 0$ and $u''(x,b) > 0$ on (b,∞) implies $u'''(x,b) \leqq 0$ on $[b,\infty)$, and not identically zero on any subinterval. Thus the second order equation

$$y'' + \left[p(x) + \frac{u(x,b)}{u'(x,b)}q(x)\right]y = 0$$

is satisfied by the positive solution $u'(x,b)$ on (b,∞). Using a result of Lazer [8, Lemma 3.2], it follows that $u(x,b)/u'(x,b) \geqq (x-b)/2$. Therefore, by the Sturm comparison theorem, the second order equation

$$y'' + [p(x) + (1/2)(x-b)q(x)]y = 0 \tag{10}$$

is disconjugate on (b,∞) and the first necessary condition holds.

The argument provided by Barrett [2, Lemma 5.2] establishes the second necessary condition.

To establish the third condition, fix a number $c > b$. Then $u''(x,b) > 0$ and $u(x,b)/u'(x,b) > (x-b)/2$ implies $u(x,b) > A(x-b)$ on $[c,\infty)$, where

$$A = u'(c,b)/2.$$

Now substituting $u(x,b)$ into (E) and integrating from c to x, yields

$$u''(x,b) - u''(c,b) + p(x)u(x,b) - p(c)u(c,b) + \int_c^x [q(t) - p'(t)]u(t,b)\,dt = 0,$$

which can be written

$$D_2u(c,b) = D_2u(x,b) + \int_c^x [q(t) - p'(t)]u(t,b)\,dt.$$

Thus

$$D_2u(c,b) > \int_c^x [q(t) - p'(t)]\cdot A(t-b)\,dt,$$

and we conclude $\int_a^\infty x[q(x) - p'(x)]\,dx < \infty$. This completes the proof of the theorem.

In view of our remarks concerning the relationship between disconjugacy and nonoscillation, we have the following equivalent of Theorem 1 providing oscillation criteria for (E).

COROLLARY. *Let the coefficients of* (E) *satisfy* (H). *Each of the following is a sufficient condition for the oscillation of* (E):

(i) $$y'' + [p(x) + (1/2)(x-b)q(x)]y = 0$$

is oscillatory for each $b \in [a, \infty)$;

(ii) $$y'' + \left[p(x) + \int_x^\infty q(t)\,dt\right]y = 0$$

is oscillatory;

(iii) $$\int_a^\infty x[q(x) - p'(x)]\,dx = \infty.$$

Remark. Hanan obtained the oscillation criterion (iii) under the added hypothesis: $y'' + p(x)y = 0$ is nonoscillatory [6, Thm. 5.12]. P. Waltman [9] obtained an oscillation criterion for nonlinear third order equations and the linear version of his criterion is the same as our condition (iii).

4. Sufficient conditions. In this section we present two conditions which are sufficient for the nonoscillation and eventual disconjugacy of (E).

THEOREM 2. *Let the coefficients of* (E) *satisfy* (H). *If* $\int_a^\infty xp(x)\,dx < \infty$ *and* $\int_a^\infty x^2q(x)\,dx < \infty$, *then* (E) *is nonoscillatory. In fact* (E) *is disconjugate on some subinterval* $[b, \infty)$ *of* $[a, \infty)$.

Proof. Assume that (E) is not nonoscillatory, i.e., assume (E) is oscillatory. The condition $q(x) - p'(x) \geqq 0$ on $[a, \infty)$ implies that (E) belongs to Hanan's class $C_I[a, \infty)$ [6, Thm. 2.2] and, consequently, every solution which vanishes once is oscillatory [6, Thm. 3.4]. Thus for each $b \geqq a$, the solution $u(x, b)$ of (E) determined by (1) is oscillatory. Fixing b, $b \geqq a$, substituting $u(x, b)$ into (E) and integrating three times, we obtain

$$u(x,b) - \frac{1}{2}(x-b)^2 + \int_b^x (x-t)p(t)u(t,b)\,dt + \frac{1}{2}\int_b^x (x-t)^2[q(t) - p'(t)]u(t,b)\,dt = 0. \tag{11}$$

Let c be the first zero of $u(x, b)$ to the right of b. Then, from (11), we have $u(x, b) \leqq \frac{1}{2}(x-b)^2$ on $[b, c]$.

Replacing x by c in (11), we get

$$\frac{1}{2}(c-b)^2 = \int_b^c (c-t)p(t)u(t,b)\,dt + \frac{1}{2}\int_b^c (c-t)^2[q(t)-p'(t)]u(t,b)\,dt,$$

which implies

$$\frac{1}{2}(c-b)^2 \leqq \frac{(c-b)}{2}\int_b^c (t-b)^2 p(t)\,dt + \frac{(c-b)^2}{2}\int_b^c \frac{(t-b)^2}{2}[q(t)-p'(t)]\,dt.$$

Therefore

$$(12)\qquad \begin{aligned} 1 &\leqq \int_b^c (t-b)p(t)\,dt + \frac{1}{2}\int_b^c (t-b)^2[q(t)-p'(t)]\,dt \\ &\leqq \int_b^c tp(t)\,dt + \frac{1}{2}\int_b^c t^2 q(t)\,dt - \frac{1}{2}\int_b^c t^2 p'(t)\,dt. \end{aligned}$$

Integrating $\int_b^c tp(t)\,dt$ by parts yields

$$(13)\qquad \int_b^c tp(t)\,dt = \frac{1}{2}t^2 p(t)|_b^c - \frac{1}{2}\int_b^c t^2 p'(t)\,dt.$$

Substituting (13) into (12), we obtain the inequality

$$(14)\qquad 1 \leqq 2\int_b^c tp(t)\,dt + \frac{1}{2}\int_b^c t^2 q(t)\,dt + \frac{1}{2}b^2 p(b).$$

Now since $\int_a^\infty tp(t)\,dt < \infty$ and $\int_a^\infty t^2 q(t)\,dt < \infty$, we may assume that b was chosen large enough such that $2\int_b^\infty tp(t)\,dt < 1/3$ and $(1/2)\int_b^\infty t^2 q(t)\,dt < 1/3$. Also, $\int_a^\infty tp(t)\,dt < \infty$ implies that $\liminf_{x\to\infty} x^2 p(x) = 0$. Thus we may assume that $(1/2)b^2 p(b) < 1/3$. Hence, from (14), we have

$$1 \leqq 2\int_b^\infty tp(t)\,dt + \frac{1}{2}\int_b^\infty t^2 q(t)\,dt + \frac{1}{2}b^2 p(b) < \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.$$

Therefore (E) is nonoscillatory and is disconjugate on some subinterval $[b, \infty)$ of $[a, \infty)$.

As in the previous section, we have, as a corollary, the equivalent of Theorem 2 giving a necessary condition for the oscillation of (E).

COROLLARY. *Let the coefficients of* (E) *satisfy* (H). *If* (E) *is oscillatory, then* $\int_a^\infty xp(x)\,dp = \infty$ *or* $\int_a^\infty x^2 q(x)\,dx = \infty$.

Our final result has been stated by Hanan [6, Thm. 5.13] but there is a mistake in sign in case (ii), p. 943, lines 15 and 16, of his proof invalidating his result. We present an alternative proof.

THEOREM 3. *Let the coefficients of* (E) *satisfy* (H) *and assume that the second order equation* $y'' + p(x)y = 0$ *is nonoscillatory. If* $\int_a^\infty x^2[q(x) - p'(x)]\,dx < \infty$, *then* (E) *is nonoscillatory.*

Proof. By a result of Hanan [6, Thm. 2.3], the nonoscillation of $y'' + p(x)y = 0$ together with the fact that $q(x) \geqq 0$, implies that (E) belongs to class C_I, i.e., if $y(x)$ is a solution of (E) such that $y(x)$ has a double zero at $x = b$, $b \geqq a$, then $y(x) \neq 0$ on $[a, b)$.

Now, by assumption,

$$\lim_{b\to\infty}\int_a^b x^2[q(x)-p'(x)]\,dx<\infty. \tag{15}$$

Integrating $\int_a^b xp(x)\,dx$ by parts, we obtain

$$\int_a^b xp(x)\,dx=\frac{1}{2}x^2p(x)\Big|_a^b-\frac{1}{2}\int_a^b x^2p'(x)\,dx. \tag{16}$$

Substituting (16) into (15) yields

$$\lim_{b\to\infty}\left(\int_a^b x^2q(x)\,dx+2\int_a^b xp(x)\,dx-x^2p(x)\Big|_a^b\right)<\infty.$$

Since $y''+p(x)y=0$ is nonoscillatory, we have $\liminf_{x\to\infty}x^2p(x)\leqq 1/4$. Thus there is a sequence $\{b_n\}$ such that $b_n\to\infty$ as $n\to\infty$ and $b_n^2p(b_n)<1$ for all n. Hence

$$\lim_{n\to\infty}\left(\int_a^{b_n} x^2q(x)\,dx+\int_a^{b_n} xp(x)\,dx\right)<\infty,$$

and we can conclude that $\int_a^\infty xp(x)\,dx<\infty$ and $\int_a^\infty x^2q(x)\,dx<\infty$. The proof is now completed by applying Theorem 2.

Again, we have the equivalent result.

COROLLARY. *Let the coefficients of* (E) *satisfy* (H), *and let* $y''+p(x)y=0$ *be nonoscillatory. If* (E) *is oscillatory, then* $\int_a^\infty x^2[q(x)-p'(x)]\,dx=\infty$.

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ON CONVERGENCE OF PADÉ APPROXIMANTS*

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Abstract. Three theorems are given concerning the convergence of sequences of Padé approximants. The first shows that in a neighborhood of the origin uniform boundedness of the approximants is necessary and sufficient for uniform convergence. The other two results give sufficient conditions to insure that uniformly convergent sequences of Padé approximants have as their limit the value of the expanded function. The first two theorems are proved along the lines of analogous results in continued fraction theory. The third theorem is based on a recent result of Pommerenke on convergence in capacity.

1. Introduction. Padé approximants have recently been employed in a variety of problems in theoretical physics, chemistry and engineering [2], [3], [6], [7], [10]. Although, in practice, numerical computations frequently indicate that Padé approximants converge satisfactorily, the general theory of convergence is still incomplete [1], [2], [5], [13], [15], [18]. We provide here some additional results in this area. In Theorem 1 it is shown that uniform boundedness is both necessary and sufficient for uniform convergence of a large class of sequences of Padé approximants (§ 3). In Theorems 2 and 3 we present sufficient conditions to insure that when a sequence of Padé approximants is uniformly convergent, its limit is equal to the value of the expanded function (§ 4). The proofs of Theorems 1 and 2 are similar to those of analogous results in the theory of continued fractions [11], [17]. The proof of Theorem 3 is based on a recent result of Pommerenke [15] on convergence in capacity. Pommerenke's theorem (stated in § 4) does not imply pointwise convergence of the sequence $\{R_\nu(z)\}$ of Padé approximants at any point, but asserts that the error of approximation tends uniformly to zero as $\nu \to \infty$, except on sets of arbitrarily small capacity. Wallin [18] has given an example of a sequence of Padé approximants which converges in capacity but diverges and even is unbounded at each point in the complex plane. In spite of this, the application made of Pommerenke's theorem to obtain Theorem 3 shows that convergence in capacity is a useful property.

Before stating and proving these theorems, we shall give in § 2 some definitions and terminology that are employed in the sequel.

2. Preliminaries. For a given formal power series

$$P(z) = c_0 + c_1 z + c_2 z^2 + \cdots, \qquad (c_0 \neq 0) \tag{2.1}$$

and for nonnegative integers m and n, the (m, n)-*Padé approximant* $R_{m,n}(z)$ is defined to be the (uniquely determined) rational function

$$R_{m,n}(z) = \frac{A_{m,n}(z)}{B_{m,n}(z)}, \tag{2.2}$$

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satisfying the formal identity

$$P(z)B_{m,n}(z) - A_{m,n}(z) = Kz^{m+n+1} + (\text{terms of higher degree}), \tag{2.3}$$

where $A_{m,n}(z)$ and $B_{m,n}(z)$ are polynomials in the complex variable z of degrees not exceeding n and m, respectively. It is well known [17] that for each $m, n = 0, 1, 2, \cdots$ the (m, n)-Padé approximant exists and is uniquely determined by the conditions stated above. The *Padé table* of (1) is the doubly infinite array of approximants

$$\begin{array}{cccc} R_{0,0} & R_{0,1} & R_{0,2} & \cdots \\ R_{1,0} & R_{1,1} & R_{1,2} & \cdots \\ R_{2,0} & R_{2,1} & R_{2,2} & \cdots \\ \cdots & \cdots & \cdots & \cdots . \end{array} \tag{2.4}$$

A power series (1) is said to be *normal* iff each entry in its Padé table occurs exactly once. In that case the polynomials $A_{m,n}(z)$ and $B_{m,n}(z)$ in (2) have degrees equal to n and m, respectively.

If f is a function and A is a subset of the complex plane, we shall mean by $f(A)$ the set $\{w | w = f(z), z \in A\}$. By the symbols $\bar{A}$, comp A and diam A we shall mean the *closure*, *complement* and *diameter* of the set A, respectively. A *domain* D will mean an open connected subset of complex numbers.

A sequence of meromorphic functions $\{R_n\}$ converges uniformly on a compact set K if and only if

(i) there exists $N(K)$ such that R_n is defined and consequently holomorphic on K for every $n \geqq N(K)$;

(ii) given $\varepsilon > 0$ there exists $N_\varepsilon \geqq N(K)$ such that

$$\sup_{z \in K} |R_m(z) - R_n(z)| < \varepsilon \quad \text{when } n, m \geqq N_\varepsilon .$$

3. Uniform convergence.

THEOREM 1. *Let $\{m_\nu\}$ and $\{n_\nu\}$ be sequences of nonnegative integers such that for some ε with $0 < \varepsilon < 1$,*

$$\sum_{\nu=0}^{\infty} \varepsilon^{n_\nu} < \infty . \tag{3.1}$$

For each $\nu = 0, 1, 2, \cdots$ let $R_\nu(z)$ denote the (m_ν, n_ν)-Padé approximant of a given power series (2.1). Let D be a domain containing the origin. Then a necessary and sufficient condition for $\{R_\nu(z)\}$ to be uniformly convergent on each compact subset of D is that for ν sufficiently large $\{R_\nu(z)\}$ is uniformly bounded on each compact subset of D.

Proof. It suffices to prove the sufficiency of the condition in the theorem, since the proof of its necessity is immediate. Let K denote an arbitrary compact subset of D. Then clearly there exists an open connected set K_0 containing the origin such that the closure $\bar{K}_0$ is compact and such that $K \subset K_0 \subset \bar{K}_0 \subset D$. By hypothesis there exist positive numbers M and N such that

$$|R_\nu(z)| \leqq M \quad \text{for } \nu \geqq N \text{ and } z \in \bar{K}_0 . \tag{3.2}$$

Thus for $\nu \geqq N$, $R_\nu(z)$ is holomorphic on $\bar{K}_0$. Therefore there exists $r > 0$ such that for all z with $|z| \leqq r$, $z \in K_0$ and the Maclaurin expansion

$$R_\nu(z) = \sum_{k=0}^{\infty} \gamma_k^{(\nu)} z^k, \qquad \nu \geqq N, \tag{3.3}$$

converges. Cauchy's inequality implies that

$$|\gamma_k^{(\nu)}| \leqq \frac{M}{r^k}, \qquad \nu \geqq N, \quad k = 0, 1, 2, \cdots. \tag{3.4}$$

From (2.3) it can be seen that for $\nu \geqq 0$,

$$\gamma_k^{(\nu)} = c_k, \qquad k = 0, 1, \cdots, n_\nu. \tag{3.5}$$

Therefore, letting

$$N_\nu = \min \{n_\nu, n_{\nu+1}\}, \qquad \nu \geqq 0, \tag{3.6}$$

we obtain, for each $\nu \geqq N + 1$ and z with $|z| < r$,

$$\begin{aligned} |R_{\nu+1}(z) - R_\nu(z)| &= \left| \sum_{k=N_\nu+1}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)}) z^k \right| \\ &\leqq 2M \sum_{k=N_\nu+1}^{\infty} \left| \frac{z}{r} \right|^k = \frac{2M|z/r|^{N_\nu+1}}{1 - |z/r|}. \end{aligned} \tag{3.7}$$

It follows that

$$R_N(z) + \sum_{\nu=N+1}^{\infty} [R_{\nu+1}(z) - R_\nu(z)] \tag{3.8}$$

is uniformly convergent on $\{z \,||z| \leqq \varepsilon r\}$ provided that

$$\sum_{\nu=0}^{\infty} \varepsilon^{N_\nu} < \infty. \tag{3.9}$$

But (3.9) is implied by (3.1) as can be seen from the inequalities

$$\varepsilon^{N_\nu} \leqq \varepsilon^{n_\nu} + \varepsilon^{n_{\nu+1}}.$$

We have shown that $\{R_\nu(z)\}$ is uniformly convergent on $\{z\,||z| \leqq \varepsilon r\}$. To complete the proof it suffices to apply Stieltjes' theorem [9, p. 251]: A uniformly bounded sequence of functions holomorphic in a domain K_0 converges uniformly on K_0 provided the sequence converges uniformly on some subdomain of K_0.

4. The limit of Padé approximants. When the power series (2.1) is normal, for each $k \geqq 0$ there exists a unique continued fraction of the form

$$\sum_{j=0}^{k-1} c_j z^j + \frac{a_0^{(k)} z^k}{1+} \frac{a_1^{(k)} z}{1+} \frac{a_2^{(k)} z}{1+} \cdots \qquad (a_j^{(k)} \neq 0) \tag{4.1}$$

whose approximants occur in the Padé table as the stairlike sequence $R_{0,k-1}$, $R_{0,k}$, $R_{1,k}$, $R_{1,k+1}$, $R_{2,k+1}$, $R_{2,k+2}$, $\cdots$ [17, Thm. 9.6.1]. For particular values of z it is possible to have the continued fraction (4.1) and corresponding power series

(2.1) both converge, or both diverge, or either one converge while the other diverges [14, pp. 145–146]. However, an early result of VanVleck and Pringsheim [14, p. 148] shows that when the continued fraction (4.1) converges uniformly for $|z| < r$, the corresponding power series (2.1) also converges to the same limit for $|z| < r$. The following theorem is an extension of the VanVleck–Pringsheim result to more general sequences of Padé approximants. It is noteworthy that the condition that the power series be normal is not needed. Our proof, though similar to the earlier ones, is included here for completeness.

THEOREM 2. *Let $\{m_\nu\}$ and $\{n_\nu\}$ be sequences of nonnegative integers such that $\{n_\nu\}$ tends to infinity. For each $\nu = 0, 1, 2, \cdots$ let $R_\nu(z)$ denote the (m_ν, n_ν)-Padé approximant corresponding to a given power series* (2.1). *Let D be a domain containing the origin. If $\{R_\nu(z)\}$ converges uniformly on compact subsets of D to a function $f(z)$, then $f(z)$ is holomorphic in D and the power series* (2.1) *converges to $f(z)$ for all z in the largest circular disk with center at the origin lying entirely within D.*

Proof. Let K denote an arbitrary compact subset of D. Since $\{R_\nu(z)\}$ converges uniformly on K, there exists an index N such that, for all $\nu \geqq N$, $R_\nu(z)$ is holomorphic in K. It follows that $f(z)$ is holomorphic at each point of D.

Now let $r > 0$ be chosen such that the disk $K_r = \{z \mid |z| \leqq r\}$ is contained in D. Let $N(r)$ be chosen such that $R_\nu(z)$ is holomorphic in K_r for all $\nu \geqq N(r)$. If we define $\{f_\nu(z)\}$ by

$$f_{N(r)}(z) = R_{N(r)}(z), \qquad f_{\nu+1}(z) = R_{\nu+1}(z) - R_\nu(z), \qquad \nu \geqq N(r), \tag{4.2}$$

then, for all $z \in K_r$, $f(z)$ has the uniformly convergent expansion

$$f(z) = \sum_{\nu=N(r)}^{\infty} f_\nu(z), \tag{4.3}$$

each term of which is a holomorphic rational function for $z \in K_r$. Using the notation (3.3), we obtain the Maclaurin expansions

$$\begin{gathered} f_{N(r)}(z) = \sum_{k=0}^{\infty} \gamma_k^{(N(r))} z^k, \\ f_{\nu+1}(z) = \sum_{k=0}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)}) z^k, \qquad \nu \geqq N(r), \end{gathered} \tag{4.4}$$

each of which is uniformly convergent for $|z| \leqq \rho r$ for each ρ with $0 < \rho < 1$. It follows from Weierstrass's double series theorem [8, Thm. 8.1.5], that, for $|z| < r$, $f(z)$ has the convergent Maclaurin expansion

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{4.5}$$

where, for each $k = 0, 1, 2, \cdots$, a_k is given by the convergent series

$$a_k = \gamma_k^{(N(r))} + \sum_{\nu=N(r)}^{\infty} (\gamma_k^{(\nu+1)} - \gamma_k^{(\nu)}). \tag{4.6}$$

Since the $\gamma_k^{(\nu)}$ satisfy (3.5) and by hypothesis $n_\nu \to \infty$, it follows from (4.6) that

$$a_k = \lim_{\nu \to \infty} \gamma_k^{(\nu)} = c_k, \qquad k = 0, 1, 2, \cdots. \tag{4.7}$$

Thus

$$f(z) = \sum_{k=0}^{\infty} c_k z^k \quad \text{for } z \in K_r. \tag{4.8}$$

Let Γ denote the largest circular disk with center at the origin lying entirely within D. If z is an arbitrary point in Γ, we have shown that there exists a closed disk K_r such that $z \in K_r \subset D$ and (4.8) holds. This completes the proof.

Before stating Theorem 3, we shall summarize a few basic facts about logarithmic capacity that we shall use (see [4, Chap. VII], [9, Chap. 16] and [16, Chap III] for more details). If E is a compact subset of the complex plane, the *capacity* of E, denoted by cap E, is defined by

$$\operatorname{cap} E = \lim_{n \to \infty} \delta_n(E), \tag{4.9}$$

where, for each $n = 2, 3, \cdots$, $\delta_n(E)$ is defined by

$$[\delta_n(E)]^{(n/2)(n-1)} = \max_{z_j \in E} \prod_{1 \leqq j < k \leqq n} |z_j - z_k|. \tag{4.10}$$

The sequence $\{\delta_n(E)\}$ decreases to its limit and $\delta_2(E) = \operatorname{diam} E$. If $m(E)$ denotes the two-dimensional Lebesgue measure of E, then

$$m(E) \leqq \pi(\operatorname{cap} E)^2. \tag{4.11}$$

A countable set has zero capacity, but an infinite connected set has positive capacity. In particular, the capacity of a circle (or disk) is equal to its radius and the capacity of a line segment is one fourth its length. The following lemmas will be used.

LEMMA 1. *Let E be a compact subset of the complex plane such that $0 \in E$. Then*

$$\operatorname{cap} \frac{1}{E} \leqq \left(\max_{w \in 1/E} |w|^2 \right) \operatorname{cap} E. \tag{4.12}$$

Thus cap $1/E = 0$ *if and only if* cap $E = 0$.

Proof. The set $1/E$ is compact, since $0 \notin E$. For each $n = 2, 3, \cdots$, choose $w_{1,n}, \cdots, w_{n,n}$ contained in $1/E$ such that

$$\left[\delta_n \frac{1}{E}\right]^{(n/2)(n-1)} = \prod_{1 \leqq j < k \leqq n} |w_{j,n} - w_{k,n}|, \tag{4.13}$$

and define $z_j^{(n)} = 1/w_{j,n}$, $j = 1, 2, \cdots, n$. It follows from (4.10) and (4.13) that

$$\begin{aligned} \delta_n\left(\frac{1}{E}\right) &= \left[\prod_{1 \leqq j < k \leqq n} |w_{j,n} w_{k,n}| \cdot |z_j^{(n)} - z_k^{(n)}|\right]^{1/((n/2)(n-1))} \\ &\leqq \left[\max_{w \in 1/E} |w|^2\right] \delta_n(E). \end{aligned} \tag{4.14}$$

Our assertion follows from (4.9) and (4.14).

In order to deal with unbounded sets, we make the following definition.

DEFINITION. Let E be a closed subset of the extended complex plane and, for each $n = 1, 2, \cdots$, let E_n be the compact subset of E given by

$$E_n = E \cap \{z \mid |z| \leqq n\}. \tag{4.15}$$

We then define

$$\operatorname{cap}^* E = \lim_{n\to\infty} \operatorname{cap} E_n, \tag{4.16}$$

provided the limit exists.

LEMMA 2. *Let E be closed and bounded. Then*

$$\begin{aligned} &\text{(A)} && \operatorname{cap} E = \operatorname{cap}^* E, \\ &\text{(B)} && \operatorname{cap} E = 0 \quad \textit{if and only if } \operatorname{cap}^* 1/E = 0. \end{aligned} \tag{4.17}$$

Proof. (A) follows immediately from (4.16) and the fact that $E_n = E$ for n sufficiently large. To prove (B), first suppose that $\operatorname{cap} E = 0$. Let

$$F_n = \frac{1}{E} \cap \{w \mid |w| \leqq n\}.$$

Then, for each $n = 1, 2, 3, \cdots$, $1/F_n = E \cap \{z \mid |z| \geqq 1/n\}$ is a compact subset of E. Since the capacity of a set is never less than the capacity of a subset, it follows that $\operatorname{cap}(1/F_n) \leqq \operatorname{cap} E = 0$ and hence $\operatorname{cap}(1/F_n) = 0$, $n = 1, 2, \cdots$. Since $0 \notin 1/F_n$, F_n is compact and Lemma 1 implies that $\operatorname{cap} F_n = 0$, $n = 1, 2, \cdots$. Therefore

$$\operatorname{cap}^* 1/E = \lim_{n\to\infty} \operatorname{cap} F_n = 0 \tag{4.18}$$

as asserted in (B).

Conversely, suppose that $\operatorname{cap}^* 1/E = 0$ and let F_n be defined as above. Since

$$0 = \operatorname{cap}^* 1/E = \lim_{n\to\infty} \operatorname{cap} F_n, \tag{4.19}$$

$\{\operatorname{cap} F_n\}$ is a nondecreasing sequence of nonnegative numbers whose limit is zero; hence $\operatorname{cap} F_n = 0$, $n = 1, 2, \cdots$. Since $0 \notin F_n$ and F_n is compact, Lemma 1 implies that $\operatorname{cap}(1/F_n) = 0$, $n = 1, 2, \cdots$. But

$$E = \frac{1}{F_n} \cup H_n, \tag{4.20}$$

where $H_n = E \cap \{z \mid |z| \leqq 1/n\}$ is compact and $\operatorname{cap} H_n \leqq 1/n$. Also it is well known [12, p. 127] that if $\operatorname{cap} A_m \leqq \alpha$, $m = 1, 2, \cdots, M$, then

$$\operatorname{cap}(A_1 \cup \cdots \cup A_M) \leqq \alpha^{1/M}[\operatorname{diam}(A_1 \cup \cdots \cup A_M)]^{1-1/M}. \tag{4.21}$$

It follows that $\operatorname{cap} E \leqq (1/n)^{1/2}[\operatorname{diam} E]^{1/2}$, $n = 1, 2, \cdots$. Hence $\operatorname{cap} E = 0$ as asserted. This completes the proof.

THEOREM 3. *Let $\{m_\nu\}$ and $\{n_\nu\}$ be sequences of nonnegative integers such that for some fixed λ with $\lambda > 1$,*

$$\frac{1}{\lambda} \leqq \frac{n_\nu}{m_\nu} \leqq \lambda \quad \textit{and} \quad m_\nu \to \infty \textit{ as } \nu \to \infty. \tag{4.22}$$

Let $f(z)$ be a function holomorphic on an open connected subset G of the extended complex plane such that $0 \in G$ and

$$\text{cap}^* (\text{comp } G) = 0. \tag{4.23}$$

For each $v = 0, 1, 2, \cdots$ let $R_v(z)$ denote the (m_v, n_v)-Padé approximant corresponding to the (convergent) Maclaurin expansion of $f(z)$. If $\{R_v(z)\}$ converges uniformly on compact subsets of an open subset D of G, then

$$\lim_{v \to \infty} R_v(z) = f(z) \quad \textit{for all } z \in D. \tag{4.24}$$

Remarks. (a) The conclusion in Theorem 3 is similar to that of Theorem 2. However, in the present case, it is not necessary to assume that the sequence $\{R_v(z)\}$ converges uniformly in a neighborhood of the origin. (b) If in Theorem 3, the set G also contains infinity, then comp G is compact and hence, by Lemma 2, (4.23) can be replaced by the condition cap (comp G) = 0. (c) Our proof of Theorem 3 is based on the following theorem due to Pommerenke, which is stated here in a more convenient but equivalent form.

POMMERENKE'S THEOREM [15]. *Let $\{m_v\}$ and $\{n_v\}$ be sequences of nonnegative integers such that for some fixed λ with $\lambda > 1$ (4.22) holds. Let $f(z)$ be a function holomorphic on an open connected subset G of the extended complex plane such that $0 \in G$ and*

$$\text{cap} \left(\frac{1}{\text{comp } G} \right) = 0. \tag{4.25}$$

For each $v = 0, 1, 2, \cdots$, let $R_v(z)$ denote the (m_v, n_v)-Padé approximant corresponding to the (convergent) *Maclaurin expansion of $f(z)$. Let $\varepsilon > 0$ and $r > 1$ be given. Then*

$$\lim_{v \to \infty} \text{cap} \left\{ w | w = \frac{1}{z}, \frac{1}{r} \leqq |z| \leqq r, |R_v(z) - f(z)| \geqq \varepsilon^{m_v} \right\} = 0. \tag{4.26}$$

Remarks. In view of Lemma 2, conditions (4.23) and (4.25) are equivalent. Thus Pommerenke's theorem may be applied to the function $f(z)$ of Theorem 3.

Proof of Theorem 3. For all sufficiently large v, $R_v(z)$ must be bounded (hence holomorphic) on D, since by hypothesis the convergence is uniform on compact subsets. It follows that $R(z) = \lim R_v(z)$ is holomorphic on D.

Assume that $R(z_0) \neq f(z_0)$ for some $z_0 \in D$. Then, by the continuity of $f(z)$ and $R(z)$, $f(z) \neq R(z)$ for all z on a closed disk Δ contained in D. Moreover, the disk may be chosen so that $0 \notin \Delta$. Again by continuity,

$$M = \min_{z \in \Delta} |f(z) - R(z)| > 0. \tag{4.27}$$

Then there exists N such that

$$|R_v(z) - R(z)| < M/2, \qquad v \geqq N, \quad z \in \Delta. \tag{4.28}$$

Hence, for $v \geqq N$ and $z \in \Delta$,

$$|f(z) - R_v(z)| \geqq |f(z) - R(z)| - |R(z) - R_v(z)| > M - \frac{M}{2} = \frac{M}{2}. \tag{4.29}$$

Now choose ε such that $0 < \varepsilon < \min\{1, M/2\}$, choose r such that

$$\Delta \subset \{z \mid r^{-1} \leqq |z| \leqq r\}$$

and let

$$E_\nu = \left\{ w \mid w = \frac{1}{z}, \frac{1}{r} \leqq |z| \leqq r, |f(z) - R_\nu(z)| \geqq \varepsilon \right\}, \qquad \nu \geqq 0. \tag{4.30}$$

Then by Pommerenke's theorem,

$$\lim_{\nu \to \infty} \operatorname{cap}(E_\nu) = 0. \tag{4.31}$$

But

$$\frac{1}{\Delta} \subset E_\nu \quad \text{for } \nu \geqq N, \tag{4.32}$$

and $\operatorname{cap}(1/\Delta) > 0$, since $1/\Delta$ is a disk. Since the capacity of a set is not exceeded by the capacity of any subset, (4.31) and (4.32) lead to a contradiction. This completes the proof.

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A MIXED PROBLEM FOR THE EULER–POISSON–DARBOUX EQUATION IN TWO SPACE VARIABLES*

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Abstract. An explicit solution of a mixed problem for the Euler–Poisson–Darboux equation $u_{tt} + (k/t)u_t - u_{xx} - u_{yy} = f(x, y, t)$ is obtained in the quarter space $x > 0$, $-\infty < y < \infty$, $t > 0$ for $k > 0$, in which case uniqueness of solution holds. The uniqueness property is lost when $k < 0$ as a nontrivial solution of the corresponding homogeneous mixed problem can be easily found. The method used is similar to that employed by Davis [6] and Young [20], which is based on analytic continuation of a generalized Riemann–Liouville integral developed by Riesz. Although only the case of two-space variables is treated, the method is also applicable to the general case of n-space variables provided $k > n - 1$ ($k > n - 2$ when n is even, $n \geqq 4$).

1. Introduction. This paper is concerned with an explicit solution of the mixed problem

$$Lu \equiv u_{tt} + \frac{k}{t}u_t - u_{xx} - u_{yy} = f(x, y, t), \tag{1}$$

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0, \tag{2}$$

$$u(0, y, t) = g(y, t), \tag{3}$$

in the quarter space $x > 0$, $-\infty < y < \infty$, $t > 0$, where $k > 0$ is a real parameter and f and g are twice continuously differentiable functions such that $g(y, 0) = g_t(y, 0) = 0$.

Equation (1) is the well-known Euler–Poisson–Darboux (abbreviated EPD) equation in two-space variables. While the equation for different numbers of space variables and special values of k has occurred in many classical problems for over two centuries, the general case of n-space variables and arbitrary values of k was fully treated only twenty years ago by Weinstein [16], [17], Diaz and Weinberger [8] and Blum [1]. Since then the EPD equation has been the object of much investigation, see, for example, Diaz and Ludford [7], Walter [15], Davis [6], Lions [13], Carroll [3] and just recently, Bresters [2]. A concise survey of more recent work on the EPD equation can be found in Gilbert [10]. All of these works, however, have been concerned with the Cauchy problem for the EPD equation. Indeed, as far as the author knows, the only other mixed problems considered for the EPD equation were done in one space variable by Weinstein [18], Lieberstein [12] and Fusaro [9]. Copson [4] and Copson and Erdélyi [5] solved a mixed problem for a generalized EPD equation but only in the case of one-space variable.

When $k = 0$, equation (1) reduces to the wave equation in which case the problem (1), (2), (3) is then classical (see Hadamard [11, pp. 247–253]). We consider here only the case $k > 0$ because, then, our solution is uniquely determined by the use of Green's formula. There is no uniqueness when $k < 0$ as evidenced

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by the existence of a nontrivial solution, $u = xt^{1-k}$, of the corresponding homogeneous mixed problem $Lu = 0$, $u(x, y, 0) = 0$, $u_t(x, y, 0) = 0$, $u(0, y, t) = 0$.

As is well known, for points (x, y, t) such that $x \geqq t > 0$, the boundary condition (3) plays no role in the determination of a solution and, hence, it may be ignored. In such a case, the unique solution of the problem (1), (2) was obtained by Diaz and Ludford [7] who showed that

$$u(x, y, t) = \frac{2^{k-1}}{\pi} \int_D \frac{\tau^k f(\xi, \eta, \tau)}{R^{1/2} R'^{k/2}} F\left(\frac{k}{2}, \frac{k-1}{2}; \frac{1}{2}; \frac{R}{R'}\right) d\xi \, d\eta \, d\tau, \tag{4}$$

where D is the domain bounded by the plane $\tau = 0$ and the retrograde characteristic cone

$$R \equiv (t - \tau)^2 - (x - \xi)^2 - (y - \eta)^2 = 0, \qquad (t - \tau > 0) \tag{5}$$

with vertex at (x, y, t), and where

$$R' \equiv (t + \tau)^2 - (x - \xi)^2 - (y - \eta)^2. \tag{6}$$

The solution (4) is no longer valid in the region $0 < x < t$. Therefore, the basic problem in (1), (2), (3) is the determination of a solution in the region $0 < x < t$. We propose to consider this problem by using a method due to Riesz as was adopted by Davis [6] and Young [19], [20]. We remark that although we treat here only the case of two-space variables, our method and procedure can be applied to the case of $n \geqq 3$ space variables as well, provided $k > n - 1$ ($k > n - 2$ when n is even, $n \geqq 4$).

2. The solution formula. The Riesz method consists essentially in finding a kernel function $V^\alpha(x, y, t; \xi, \eta, \tau)$, depending on two points (x, y, t) and (ξ, η, τ), and a parameter α, such that V^α vanishes together with its first derivatives on the retrograde cone (5) and satisfies the relation

$$LV^{\alpha+2}(x, y, t; \xi, \eta, \tau) = V^\alpha(x, y, t; \xi, \eta, \tau). \tag{7}$$

Moreover, with respect to the point (ξ, η, τ), it satisfies an analogous relation

$$MV^{\alpha+2}(x, y, t; \xi, \eta, \tau) = V^\alpha(x, y, t; \xi, \eta, \tau), \tag{8}$$

where M is the adjoint of L defined by

$$Mv \equiv v_{\tau\tau} - k\left(\frac{v}{\tau}\right)_\tau - v_{\xi\xi} - v_{\eta\eta}.$$

The kernel function for the operator L was determined by Davis [6] (also by Young [19]) who showed that

$$\begin{aligned} V^\alpha(x, y, t; \xi, \eta, \tau) = {} & \frac{2^k \tau^k}{2^{\alpha-1}\pi^{1/2}\Gamma(\alpha/2)\Gamma[(\alpha-1)/2]} \\ & \cdot \frac{R^{(\alpha-3)/2}}{R'^{k/2}} F\left(\frac{k}{2}, \frac{\alpha+k-3}{2}; \frac{\alpha-1}{2}; \frac{R}{R'}\right), \end{aligned} \tag{9}$$

where R and R' are defined in (5) and (6), respectively. For a fixed point (x, y, t) in the quarter space $\xi > 0$, $\tau > 0$, we note that the hypergeometric function F in

(9) converges absolutely and uniformly with respect to (ξ, η, τ) for $\tau \geqq \varepsilon > 0$, because in that space $|R/R'| < 1$. Moreover, for $\alpha > 3$, V^α remains bounded at $\tau = 0$.

Now let us seek a solution u of the problem (1), (2), (3) at the point (x, y, t) where $0 < x < t$. Let D denote the domain bounded by the retrograde cone $R = 0$ with vertex at (x, y, t) and the planes $\xi = 0$ and $\tau = 0$. For $\alpha > 3$, we have, by Green's theorem,

$$\begin{aligned} &\int_D (V^{\alpha+2}Lu - uMV^{\alpha+2})\, d\xi\, d\eta\, d\tau \\ &= \int_{\partial D} \left(V^{\alpha+2}\frac{\partial u}{\partial n} - u\frac{\partial V^{\alpha+2}}{\partial n} + k\frac{uV^{\alpha+2}}{\tau}\nu_\tau \right) dS, \end{aligned} \tag{10}$$

where $\partial u/\partial n$ is the conormal derivative

$$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial \tau}\nu_\tau - \frac{\partial u}{\partial \xi}\nu_\xi - \frac{\partial u}{\partial \eta}\nu_\eta$$

with ν_ξ, ν_η, ν_τ being the components of the outward unit normal vector on ∂D, the boundary of D. Since $V^{\alpha+2}$ and its first derivatives vanish on $R = 0$, and in view of (8), formula (10) yields

$$I^\alpha u(x, y, t) = I^{\alpha+2}f(x, y, t) - \int_T \left(V^{\alpha+2}\frac{\partial u}{\partial \xi} - \frac{\partial V^{\alpha+2}}{\partial \xi}g \right) d\eta\, d\tau \tag{11}$$

in which we have introduced the notation

$$I^\alpha u(x, y, t) = \int_D u(\xi, \eta, \tau)V^\alpha(x, y, t; \xi, \eta, \tau)\, d\xi\, d\eta\, d\tau \tag{12}$$

and substituted the data given in (1), (2), (3). The domain of integration T of the last integral in (11) is that intercepted off the plane $\xi = 0$ by the cone $R = 0$ $(\tau > 0)$.

In order to eliminate the term $\partial u/\partial \xi$ from (11), we consider the kernel function $V_*^{\alpha+2}(-x, y, t; \xi, \eta, \tau)$ corresponding to the point $(-x, y, t)$ which is symmetric to (x, y, t) with respect to the plane $\xi = 0$. Denote by D^* the domain lying in the quarter space $\xi > 0$, $\tau > 0$ and bounded by the retrograde cone

$$R_* \equiv (t - \tau)^2 - (x + \xi)^2 - (y - \eta)^2 = 0$$

and vertex at $(-x, y, t)$. Notice that D^* is in the interior of D. For $\alpha > 3$, it is clear that $V_*^{\alpha+2}$ vanishes together with its first derivatives on $R_* = 0$, and satisfies both the relations (7) and (8). Hence for $V_*^{\alpha+2}$ and the domain D^*, we obtain by applying formula (10),

$$I_*^\alpha u(-x, y, t) = I_*^{\alpha+2}f(-x, y, t) - \int_T \left(V_*^{\alpha+2}\frac{\partial u}{\partial \xi} - \frac{\partial V_*^{\alpha+2}}{\partial \xi}g \right) d\eta\, d\tau. \tag{13}$$

Here $I_*^\alpha u$ denotes the integral (12) over D^* with the kernel function V_*^α. Now, since $V^\alpha = V_*^\alpha$ on T, it follows by subtracting (13) from (11) that

$$\begin{aligned} I^\alpha u(x, y, t) - I_*^\alpha u(-x, y, t) &= I^{\alpha+2}f(x, y, t) - I_*^{\alpha+2}f(-x, y, t) \\ &\quad + J^{\alpha+2}g(y, t), \end{aligned} \tag{14}$$

where

$$J^\alpha g(y,t) = \int_T \left(\frac{\partial V^\alpha}{\partial \xi} - \frac{\partial V_*^\alpha}{\partial \xi}\right) g(\eta,\tau)\, d\eta\, d\tau. \tag{15}$$

For $\alpha > 3$, we observe that each of the integrals in (14) converges and defines an analytic function of α. Moreover, if we extend the function u as an odd function of x for $x < 0$, that is, $u(-x,y,t) = -u(x,y,t)$, then the integral $I_*^\alpha u(-x,y,t)$ over D^* becomes the negative of the integral $I^\alpha u(x,y,t)$ whose domain of integration is the reflection of D^* with respect to $\xi = 0$. Hence the term $I^\alpha u(x,y,t) - I_*^\alpha u(-x,y,t)$ (and also $I^{\alpha+2}f - I_*^{\alpha+2}f$) can be expressed as the volume integral over the domain extending into the quarter space $\xi < 0$, $\tau > 0$, and bounded simply by the retrograde cone $R = 0$ and the plane $\tau = 0$. Then, since $V^{\alpha+2}$ and its first derivatives vanish on $R = 0$, the differentiation implied in the operator L may be applied under the integral sign of each of the integrals in (14). In view of relation (7), we then have

$$\begin{aligned} LI^\alpha u(x,y,t) - LI_*^\alpha u(-x,y,t) &= LI^{\alpha+2}f(x,y,t) - LI_*^{\alpha+2}f(-x,y,t) \\ &\quad + LJ^{\alpha+2}g(y,t) \\ &= I^\alpha f(x,y,t) - I_*^\alpha f(-x,y,t) + J^\alpha g(y,t). \end{aligned} \tag{16}$$

As will be shown in the next section, when u and g are continuously differentiable, it is possible to continue analytically each of the integrals in (16) with respect to α into $-1 < \alpha \leqq 3$, and that

$$I^0 u(x,y,t) = u(x,y,t), \tag{17a}$$

$$I_*^0 u(-x,y,t) = 0, \tag{17b}$$

$$J^0 g(y,t) = 0. \tag{17c}$$

Then, by the principle of analytic continuation, (16) will yield the result

$$Lu(x,y,t) = f(x,y,t),$$

thus verifying that $I^0 u(x,y,t) = u(x,y,t)$ satisfies equation (1). Further, by performing the analytic continuation of (14) to $\alpha = 0$, we will obtain the explicit solution

$$u(x,y,t) = I^{0+2}f(x,y,t) - I_*^{0+2}f(-x,y,t) + J^{0+2}g(y,t), \tag{18}$$

which will be shown to satisfy conditions (2), (3). It is in this sense that formula (14) provides a solution of the problem (1), (2), (3).

3. The analytic continuation. We first establish the identity (17a). Let D_1 denote the conical domain bounded by $R = 0$ $(t - \tau > 0)$ and $\tau = t - x \geqq 0$, and set $D_2 = D - D_1$. Then, as was proved by Davis [6], the part of the integral $I^\alpha u$ over D_1 yields the desired identity when it is continued analytically to $\alpha = 0$. On the other hand, by the same procedure the other part of $I^\alpha u$ over D_2 and the integral $I_*^\alpha u$ both vanish when they are continued analytically to $\alpha = 0$, since the domains of integration D_2 and D^* do not contain the points (x, y, t) and $(-x, y, t)$, respectively. We shall demonstrate here the technique for carrying out this analytic continuation in connection with the integral

$$K^\alpha g(y, t) = \int_T V^\alpha(x, y, t; 0, \eta, \tau) g(\eta, \tau)\, d\eta\, d\tau. \tag{19}$$

The result of this will be used to establish the identity (17c).

Let $z = R_0/R_0'$, where R_0 and R_0' denote the values of R and R' on T, that is, when $\xi = 0$. Since $|z| < 1$ for $\tau > 0$, we may expand the hypergeometric function in V^α in infinite series and write

$$\begin{aligned} F\left(\frac{k}{2}, \frac{\alpha + k - 3}{2}; \frac{\alpha - 1}{2}; z\right) &= 1 + \frac{\Gamma[(\alpha - 1)/2]}{\Gamma(k/2)\Gamma[(\alpha + k - 3)/2]} \\ &\quad \cdot z \sum_{r=0}^{\infty} \frac{\Gamma(k/2 + 1 + r)\Gamma[(\alpha + k - 1)/2 + r]}{(r + 1)!\Gamma[(\alpha + 1)/2 + r]} z^r. \end{aligned} \tag{20}$$

The infinite series on the right-hand side of (20) converges uniformly for $|z| < 1$; in fact, it may be written as

$$C(\alpha, k)\, {}_3F_2\left(\frac{k + 2}{2}, \frac{\alpha + k - 1}{2}, 1; 2, \frac{\alpha + 1}{2}; z\right), \tag{21}$$

where C is some factor depending on α and k, and ${}_3F_2$ is a generalized hypergeometric function (14, p. 73). Thus the integral (19) may be written as the sum of two integrals, the first being given by

$$\begin{aligned} U_1(\alpha) &= A(\alpha, k) \int_T \frac{\tau^k g(\eta, \tau) R_0^{(\alpha - 3)/2}}{R_0'^{k/2}}\, d\eta\, d\tau \\ &= A(\alpha, k) \int_{y - (t^2 - x^2)^{1/2}}^{y + (t^2 - x^2)^{1/2}} d\eta \int_0^{t - [x^2 + (y - \eta)^2]^{1/2}} \frac{\tau^k g(\eta, \tau) R_0^{(\alpha - 3)/2}}{R_0'^{k/2}}\, d\tau, \end{aligned} \tag{22}$$

where

$$A(\alpha, k) = \frac{2^{k - \alpha + 1}}{\pi^{1/2}\Gamma(\alpha/2)\Gamma[(\alpha - 1)/2]}. \tag{23}$$

The integral (22) defines an analytic function of α for $\alpha > 1$. In order to continue it analytically into $0 \leqq \alpha \leqq 1$, we introduce in the inner integral the new variable s, where $\tau = s\{t - [x^2 + (y - \eta)^2]^{1/2}\}$. Then (22) becomes

$$\begin{aligned} U_1(\alpha) &= A(\alpha, k) \int_{y - (t^2 - x^2)^{1/2}}^{y + (t^2 - x^2)^{1/2}} \{t - [x^2 + (y - \eta)^2]^{1/2}\}^{(k + \alpha - 1)/2}\, d\eta \\ &\quad \cdot \int_0^1 G(x, y, t, s) s^k (1 - s)^{(\alpha - 3)/2}\, ds, \end{aligned} \tag{24}$$

where G represents all the other factors in the integrand of (22). We consider the inner integral and integrate it by parts. We obtain

$$
\begin{aligned}
&\int_0^1 G(x, y, t, s)s^k(1 - s)^{(\alpha-3)/2}\, ds \\
(25)\qquad &= -\frac{2}{\alpha - 1} G(x, y, t, s)s^k(1 - s)^{(\alpha-1)/2}\Big|_0^1 \\
&\quad + \frac{2}{\alpha - 1}\int_0^1 \frac{\partial}{\partial s}(G(x, y, t, s)s^k)(1 - s)^{(\alpha-1)/2}\, ds \\
&= \frac{2}{\alpha - 1}\int_0^1 \frac{\partial}{\partial s}(G(x, y, t, s)s^k)(1 - s)^{(\alpha-1)/2}\, ds,
\end{aligned}
$$

since the boundary terms vanish for $\alpha > 1$. But now the last integral in (25) converges for $-1 < \alpha \leqq 1$ and defines an analytic function of α there. In view of the factor $1/(\Gamma(\alpha/2))$ in (23), it follows that (22) vanishes when it is continued analytically to $\alpha = 0$.

The second integral of (19) is given by

$$
(26)\qquad U_2(\alpha) = \frac{2^{k-\alpha+1}C(\alpha, k)}{\pi^{1/2}\Gamma(\alpha/2)\Gamma[(\alpha - 1)/2]}\int_T \frac{\tau^k g(\eta, \tau)R_0^{(\alpha-1)/2}}{R_0'^{1+k/2}}\, {}_3F_2(z)\, d\eta\, d\tau
$$

involving the function (21). This integral converges for $\alpha > -1$, and, again, because of the factor $1/(\Gamma(\alpha/2))$, it vanishes at $\alpha = 0$. Thus we have shown that (19) can be continued analytically with respect to α into $-1 < \alpha \leqq 1$, and that $K^0 g(y, t) = 0$. We shall use this result to verify that $J^0 g(y, t) = 0$.

We observe that on T,

$$
\frac{\partial V^\alpha}{\partial \xi} - \frac{\partial V_*^\alpha}{\partial \xi} = -\frac{\partial V^\alpha}{\partial x} - \frac{\partial V_*^\alpha}{\partial x} = -2\frac{\partial V^\alpha}{\partial x}(x, y, t; 0, \eta, \tau).
$$

Since V^α vanishes on the intersection of ∂T with $R = 0$ for large α, we may write

$$
J^\alpha g(y, t) = -2\frac{\partial}{\partial x}\int_T V^\alpha(x, y, t; 0, \eta, \tau)g(\eta, \tau)\, d\eta\, d\tau.
$$

By the principle of analytic continuation and from the previous results, we see that $J^\alpha g$ vanishes as α is continued analytically to $\alpha = 0$.

4. The explicit solution. We notice that all the integrals on the right-hand side of (14) converge for $\alpha > -1$. Thus by analytic continuation to $\alpha = 0$, and by (17), we obtain our explicit solution

$$
\begin{aligned}
u(x, y, t) &= \frac{2^{k-1}}{\pi}\int_D \frac{\tau^k f(\xi, \eta, \tau)}{R^{1/2}R'^{k/2}} F\left(\frac{k}{2}, \frac{k-1}{2}; \frac{1}{2}; \frac{R}{R'}\right) d\xi\, d\eta\, d\tau \\
(27)\qquad &\quad - \frac{2^{k-1}}{\pi}\int_{D^*} \frac{\tau^k f(\xi, \eta, \tau)}{R_*^{1/2}R_*'^{k/2}} F\left(\frac{k}{2}, \frac{k-1}{2}; \frac{1}{2}; \frac{R_*}{R_*'}\right) d\xi\, d\eta\, d\tau \\
&\quad - \frac{2^k}{\pi}\frac{\partial}{\partial x}\int_T \frac{\tau^k g(\eta, \tau)}{R_0^{1/2}R_0'^{k/2}} F\left(\frac{k}{2}, \frac{k-1}{2}; \frac{1}{2}; \frac{R_0}{R_0'}\right) d\eta\, d\tau.
\end{aligned}
$$

Now, for $0 < t \leqq x$, the domains of integration D^* and T in (27) are both empty and, hence, (27) reduces to the formula (4) which solves the problem (1), (2). On the other hand, as x tends to zero, the integrals over D and D^* cancel out since D^* approaches D, and R and R' coincide with R_* and R'_*, respectively. Thus we need only show that

$$\lim_{x\to 0} J^{0+2}g(y,t) = g(y,t). \tag{28}$$

To this end, we write

$$J^{0+2}g(y,t) = -\frac{2^k}{\pi}\frac{\partial}{\partial x}\int_0^{t-x}\int_{y-[(t-\tau)^2-x^2]^{1/2}}^{y+[(t-\tau)^2-x^2]^{1/2}} \frac{\tau^k g(\eta,\tau)}{R^{1/2}R_0'^{k/2}}F(R_0/R_0')\,d\eta\,d\tau,$$

where, for convenience, we have omitted writing the parameters of the hypergeometric function. If we introduce the new variable $s = (y-\eta)/\sqrt{(t-\tau)^2 - x^2}$ in the inner integral and perform the indicated differentiation, we obtain

$$\begin{aligned} J^{0+2}g(y,t) &= \frac{2^k}{\pi}(t-x)^k\int_{-1}^{1}\frac{g(y,t-x)}{[(2t-x)^2-x^2]^{k/2}}(1-s^2)^{-1/2}\,ds \\ &\quad - \frac{2^k}{\pi}\int_0^{t-x}\tau^k\int_{-1}^{1}\frac{\partial}{\partial x}\left[\frac{g(\eta,\tau)}{R_0'^{k/2}}F(R_0/R_0')\right](1-s^2)^{-1/2}\,ds\,d\tau. \end{aligned} \tag{29}$$

Now the differentiation with respect to x in the second integral on the right-hand side of (29) gives rise to a factor x, and so, the integral vanishes as x tends to zero. On the other hand, the first integral yields

$$\lim_{x\to 0}\frac{2^k}{\pi}(t-x)^k\int_{-1}^{1}\frac{g(y,t-x)}{[(2t-x)^2-x^2]^{k/2}}(1-s^2)^{-1/2}\,ds = g(y,t),$$

thus establishing (28). This completes the verification that (27) indeed satisfies the conditions (2), (3).

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NONLINEAR DEGENERATE EVOLUTION EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS OF MIXED TYPE*

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Abstract. The Cauchy problem for the evolution equation $Mu'(t) + N(t, u(t)) = 0$ is studied, where M and $N(t, \cdot)$ are, respectively, possibly degenerate and nonlinear monotone operators from a vector space to its dual. Sufficient conditions for existence and for uniqueness of solutions are obtained by reducing the problem to an equivalent one in which M is the identity but each $N(t, \cdot)$ is multivalued and accretive in a Hilbert space. Applications include weak global solutions of boundary value problems with quasilinear partial differential equations of mixed Sobolev-parabolic-elliptic type, boundary conditions with mixed space-time derivatives, and those of the fourth or fifth type. Similar existence and uniqueness results are given for the semilinear and degenerate wave equation $Bu''(t) + F(t, u'(t)) + Au(t) = 0$, where each nonlinear $F(t, \cdot)$ is monotone and the nonnegative B and positive A are self-adjoint operators from a reflexive Banach space to its dual.

1. Introduction. Suppose we are given a nonnegative and symmetric linear operator $\mathscr{M}$ from a vector space E into its (algebraic) dual E^*. This is equivalent to specifying the nonnegative and symmetric bilinear form $m(x, y) = \langle \mathscr{M}x, y \rangle$ on E, where the brackets denote $E^* - E$ duality. Since m is a semiscalar-product on E, we have a (possibly non-Hausdorff) topological vector space (E, m) with seminorm "$x \to m(x, x)^{1/2}$", and its dual $(E, m)' = E'$ is a Hilbert space which contains the range of $\mathscr{M}$. We let $\mathscr{N}(t, \cdot)$ be a family of (possibly) nonlinear functions from E into E^*, $0 \leqq t \leqq T$, and consider the evolution equation

$$\frac{d}{dt}(\mathscr{M}u(t)) + \mathscr{N}(t, u(t)) = 0, \qquad 0 \leqq t \leqq T. \tag{1.1}$$

By a *solution* of (1.1) we mean a function $u:[0, T] \to E$ such that $\mathscr{M}u:[0, T] \to E'$ is absolutely continuous (hence, differentiable almost everywhere), with $\mathscr{N}(t, u(t)) \in E'$ for all t, and (1.1) is satisfied at almost every $t \in [0, T]$. The *Cauchy problem* is to find a solution u of (1.1) for which $\mathscr{M}u(0)$ is specified in E'.

The plan of the paper is as follows. In § 2 we use elementary linear algebra to show that (1.1) is equivalent to an evolution problem essentially of the form

$$-u'(t) \in \mathscr{M}^{-1} \circ \mathscr{N}(t, u(t)), \tag{1.2}$$

where $\mathscr{M}^{-1}$ denotes the (possibly) multivalued operator or relation that is inverse to $\mathscr{M}$. Our main results on the existence and uniqueness of solutions of (1.1) (or (1.2)) are stated and proved in § 3, and provide a natural application of nonlinear evolution problems with multivalued operators. Section 4 gives some applications of our results to various nonlinear boundary value problems which may contain derivatives in time of at most first order. Each such problem is reduced to (1.1) in an appropriate space. The examples include boundary value problems for

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equations of the form

$$\frac{\partial}{\partial t}\left(m_0(x)u(x,t) - \frac{\partial}{\partial x}\left(m(x)\frac{\partial u}{\partial x}\right)\right) - \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) = 0,$$

where $m_0(x) \geqq 0$, $m(x) \geqq 0$, and $p \geqq 2$. A first order time derivative may also appear in boundary conditions such as in boundary value problems of the fourth and fifth type. In § 5 we study the abstract wave equation

$$Bu''(t) + F(t, u'(t)) + Au(t) = 0, \tag{1.3}$$

where A and B are self-adjoint with A strictly positive and B nonnegative and each $F(t, \cdot)$ is monotone. When the operators in (1.3) are realizations of partial differential equations, we obtain results on the solvability of (e.g.)

$$\frac{\partial^2}{\partial t^2}\left(m_0(x)u(x,t) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}m_j(x)\frac{\partial u}{\partial x_j}\right) + \left|\frac{\partial u}{\partial t}\right|^{p-1}\frac{\partial u}{\partial t} - \Delta u = f,$$

wherein each m_j is nonnegative and bounded, and $p \geqq 2$, and boundary conditions may contain second order time derivatives.

Abstract equations of the form (1.1) have been considered by C. Bardos, H. Brezis, O. Grange and F. Mignot, H. Levine, J.-L. Lions, M. Visik and this writer. Our results of § 3 are closest to those in § 5 of [3] and those of [14], while in [4] it is assumed the leading operator in (1.1) is bounded from a Hilbert space into itself. The writers in [21, pp. 69–73] and [29] consider linear equations with time-dependent operators uniformly bounded from below by a positive quantity, hence, nondegenerate, and this last assumption was removed in [28]. Each of the preceding works has been directed toward the solution of boundary value problems, many of which have been studied by more direct methods. We refer the reader to the extensive bibliographies of [26], [27] for the theory and application of nondegenerate equations with mixed space and time derivatives (i.e., of Sobolev type) and to [7], [28] for additional references to (degenerate) mixed elliptic-parabolic type. See [24] for a treatment of (1.3) when B is the identity.

2. Two Cauchy problems. Let m denote the nonnegative and symmetric bilinear form given on the vector space E. Let K be the kernel of m, i.e., the subspace of those $x \in E$ with $m(x, x) = 0$, and denote the corresponding quotient space by E/K. Then the quotient map $q: E \to E/K$ given by

$$q(x) = \{y \in E : m(x - y, x - y) = 0\}$$

is a linear surjection, and it determines a scalar product $\mathbf{m}$ on E/K by

$$\mathbf{m}(q(x), q(y)) = m(x, y), \qquad x, y \in E. \tag{2.1}$$

The completion of $(E/K, \mathbf{m})$ is a Hilbert space W whose scalar product is the extension by continuity of $\mathbf{m}$, and we denote this extension also by $\mathbf{m}$.

Let E' denote the strong dual of the seminormed topological vector space (E, m). E' is a Hilbert space which is important in the discussion below, so we consider it briefly. Letting $(E/K)'$ and W' denote the duals of the indicated scalar product space and Hilbert space, respectively, and noting that E/K is dense in W, we have each $f \in W'$ uniquely determined by its restriction to E/K. This restriction gives a bijection of W' onto $(E/K)'$ and we hereafter identify these spaces. Regard q as a map from E into W. Its dual is the linear map $q^*: W' \to E'$ defined by

$$\langle q^*(f), x\rangle = \langle f, q(x)\rangle, \qquad f \in W', \quad x \in E. \tag{2.2}$$

Since $q(E) = E/K$ is dense in W, q^* is injective. Furthermore, each $g \in E'$ necessarily vanishes on K, so there is a unique element $f \in (E/K)'$ with $f \circ q = g$. That is, $g = q^*(f)$, so q^* is a bijection of W' onto E'. It follows from (2.1) and (2.2) that q^* is norm-preserving.

We easily relate the linear map $\mathcal{M}: E \to E'$ given to us by

$$\langle \mathcal{M}x, y\rangle = m(x, y), \qquad x, y \in E,$$

to the Hilbert space isomorphism $\mathcal{M}_0: W \to W'$ of F. Riesz defined by

$$\langle \mathcal{M}_0 x, y\rangle = \mathbf{m}(x, y), \qquad x, y \in W.$$

For any pair $x, y \in E$ we have $\langle q^*\mathcal{M}_0 qx, y\rangle = \langle \mathcal{M}_0 qx, qy\rangle = \mathbf{m}(qx, qy) = m(x, y) = \langle \mathcal{M}x, y\rangle$, and, hence,

$$\mathcal{M} = q^*\mathcal{M}_0 q. \tag{2.3}$$

The notion of a relation $\mathcal{R}$ on a Cartesian product $X \times Y$ of linear spaces will be essential. A *relation* $\mathcal{R}$ on $X \times Y$ is a subset of $X \times Y$. For each $x \in X$, the *image* of x by $\mathcal{R}$ is the set $\mathcal{R}(x) = \{y \in Y : [x, y] \in \mathcal{R}\}$, and the *domain* of $\mathcal{R}$ is the set of $x \in X$ for which $\mathcal{R}(x)$ is nonempty. The *range* of $\mathcal{R}$ is $\bigcup\{\mathcal{R}(x) : x \in X\}$. The graph of every function from a subset of X into Y is a relation on $X \times Y$, and we so identify functions as relations. The *inverse* of $\mathcal{R}$ is the relation $\mathcal{R}^{-1} = \{[y, x] : [x, y] \in \mathcal{R}\}$ on $Y \times X$. If a is a real number, we define

$$a\mathcal{R} = \{[x, ay] : [x, y] \in \mathcal{R}\}.$$

If $\mathcal{S}$ is a second relation on $X \times Y$, then

$$\mathcal{R} + \mathcal{S} = \{[x, y + z] : [x, y] \in \mathcal{R} \text{ and } [x, z] \in \mathcal{S}\}.$$

If $\mathcal{T}$ is a relation on $Y \times Z$, then the *composition* of $\mathcal{R}$ and $\mathcal{T}$ is

$$\mathcal{T} \circ \mathcal{R} = \{[x, z] : [x, y] \in \mathcal{R} \text{ and } [y, z] \in \mathcal{T} \text{ for some } y \in Y\}.$$

If $\mathcal{P}$ is a relation on $W \times X$, then composition is associative, i.e.,

$$(\mathcal{T} \circ \mathcal{R}) \circ \mathcal{P} = \mathcal{T} \circ (\mathcal{R} \circ \mathcal{P}).$$

Also, we identify the identity function I_Y on Y with its graph $\{[y, y] : y \in Y\}$, and easily obtain the inclusion $\mathcal{R} \circ \mathcal{R}^{-1} \supseteq I_Y$. These sets are equal if (and only if) $\mathcal{R}$ is a function, i.e., each $\mathcal{R}(x)$ is a singleton. Finally we note that

$$(\mathcal{T} \circ \mathcal{R})^{-1} = \mathcal{R}^{-1} \circ \mathcal{T}^{-1}.$$

Suppose that for each $t \in [0, T]$ we are given a (not necessarily linear) function $\mathcal{N}(t): E \to E^*$. Define a corresponding relation $\mathcal{N}_0(t)$ on $W \times W'$ as follows: $[w, f] \in \mathcal{N}_0(t)$ if and only if there is an $x \in E$ such that $q(x) = w$ and $\mathcal{N}(t, x) = q^*(f)$. Since q^* is onto E', it follows that the domain of $\mathcal{N}_0(t)$ is precisely the image $q(D(t))$, where we define $D(t) = \{x \in E : \mathcal{N}(t, x) \in E'\}$. Also, for each $t \in [0, T]$ and

$x \in D(t)$, there is exactly one $f \in W'$ with $\mathcal{N}(t, x) = q^*(f)$, so we have

$$\mathcal{N}(t, x) = q^* \circ \mathcal{N}_0(t) \circ q(x), \qquad 0 \leqq t \leqq T, \quad x \in D(t). \tag{2.4}$$

Finally, we define a family of composite relations on $W \times W$ by $\mathscr{A}(t) = \mathcal{M}_0^{-1} \circ \mathcal{N}_0(t)$, $t \in [0, T]$. That is, $[x, z] \in \mathscr{A}(t)$ if and only if there is a $y \in W'$ for which $[x, y] \in \mathcal{N}_0(t)$ and $y = \mathcal{M}_0 z$. Since $\mathcal{M}_0$ is a bijection, $\mathscr{A}(t)$ and $\mathcal{N}_0(t)$ have the same domain, $q(D(t))$.

Remark 1. Note that $\mathcal{N}_0(t)$ is a function (as is $\mathscr{A}(t)$) if and only if $\mathcal{N}(t, x) = \mathcal{N}(t, y)$ for every pair $x, y \in E$ such that $\mathcal{N}(t, x)$ and $\mathcal{N}(t, y)$ belong to E' and $\mathcal{M}x = \mathcal{M}y$. This is frequently (but not always) the case in applications, even where $\mathcal{M}$ is not injective.

We shall relate solutions of the evolution equation (1.1) to those of an evolution problem determined by the relations $\mathscr{A}(t)$, $0 \leqq t \leqq T$. A function $v: [0, T] \to W$ is called a *solution* of the evolution problem

$$v'(t) + \mathscr{A}(t, v(t)) \ni 0, \qquad 0 \leqq t \leqq T, \tag{2.5}$$

if it is (strongly) absolutely continuous (hence, differentiable a.e.), $v(t) \in q(D(t))$ for every t, and (2.5) is satisfied at a.e. t. Since the domain of each $\mathscr{A}(t)$ is contained in E/K and since the maps $\mathcal{M}_0: W \to W'$ and $q^*: W' \to E'$ are linear isometries, it follows that v is a solution of (2.5) if and only if $v: [0, T] \to E/K$, is absolutely continuous, (hence, $q^*\mathcal{M}_0 v: [0, T] \to E'$ is differentiable a.e.), $v(t) \in q(D(t))$ for every t, and

$$(q^*\mathcal{M}_0 v(t))' \in -q^*\mathcal{N}_0(t, v(t))$$

at a.e. t. Let v be such a solution, and for each $t \in [0, T]$ choose a representative $u(t) \in D(t)$ from the coset $v(t) \in E/K$. Then $q(u(t)) = v(t)$, $\mathcal{M}u(t) = q^*\mathcal{M}_0 v(t)$ and $\mathcal{N}(t, u(t)) = q^*\mathcal{N}_0(t, v(t))$ for each t, so u is a solution of (1.1). Conversely, if u is any solution of (1.1), then the function $v \equiv q \circ u$ is a solution of (2.5), so we have the following result.

PROPOSITION 1. *If v is a solution of* (2.5) *and for each $t \in [0, T]$, $u(t) \in D(t)$ belongs to the coset $v(t) \in E/K$, then u is a solution of* (1.1). *Conversely, if u is a solution of* (1.1), *then $v \equiv q \circ u$ is a solution of* (2.5).

COROLLARY 1. *Let $u_0 \in D(0)$. Then there exists a solution v of* (2.5) *with $v(0) = q(u_0)$ if and only if there exists a solution u of* (1.1) *with $\mathcal{M}u(0) = \mathcal{M}u_0$.*

COROLLARY 2. *Let $u_0 \in D(0)$. Then there is at most one solution v of* (2.5) *with $v(0) = q(u_0)$ if and only if for every pair of solutions u_1, u_2 of* (1.1) *with $\mathcal{M}u_1(0) = \mathcal{M}u_2(0) = \mathcal{M}u_0$, we have $\mathcal{M}u_1(t) = \mathcal{M}u_2(t)$ for all $t \in [0, T]$, hence*

$$\mathcal{N}(t, u_1(t)) = \mathcal{N}(t, u_2(t)).$$

Remark 2. In the situation of Corollary 2, uniqueness holds for solutions of the Cauchy problem for (1.1) if for each $t \in [0, T]$ and each pair $x, y \in E$, $\mathcal{M}x = \mathcal{M}y$ and $\mathcal{N}(t, x) = \mathcal{N}(t, y) \in E'$ imply that $x = y$.

Example. Take $E = \mathbb{R}^2 = E^*$ with $\langle [x_1, x_2], [y_1, y_2] \rangle = x_1 y_1 + x_2 y_2$. Let $\mathcal{M}([x_1, x_2]) = [x_1, 0]$ and $\mathcal{N}(t)[x_1, x_2] = [x_2, -x_1]$. Then the kernel of $\mathcal{M} + \mathcal{N}(t)$ is null, so uniqueness holds. Note, however, $\mathcal{N}_0(t)$ is *not* a function. This corresponds to the (trivial) evolution equation

$$u_1'(t) = 0, \quad u_2(t) = 0, \qquad t \geqq 0,$$

for $u(t) \equiv [u_1(t), u_2(t)]$.

3. Existence and uniqueness. Evolution problems of the form (2.5) have been considered by many writers, and we refer to the recent work of M. Crandall and A. Pazy [10] and J. Dorroh [12] for references in this direction. In particular, a sufficient condition for uniqueness of solutions of Cauchy problems associated with (2.5) is that each $\mathscr{A}(t)$ be accretive.

DEFINITION 1. A relation $\mathscr{A}$ on $W \times W$ is *accretive* if for every pair $[x_1, w_1]$ and $[x_2, w_2]$ in $\mathscr{A}$ we have $\mathbf{m}(w_1 - w_2, x_1 - x_2) \geqq 0$.

A related notion is that of monotonicity for functions (or relations) mapping a subset of a vector space into its dual. Such a condition holds for many operators associated with the variational formulation of (possibly nonlinear) elliptic boundary value problems [5], [7].

DEFINITION 2. Let D be a subset of the vector space E and denote by E^* the (algebraic) dual of E. A function $\mathscr{N}: D \to E^*$ is *D-monotone* if for each pair x_1, $x_2 \in D$ we have $\langle \mathscr{N}(x_1) - \mathscr{N}(x_2), x_1 - x_2 \rangle \geqq 0$.

Because of our intended applications, it is essential that D-monotonicity of each $\mathscr{N}(t)$ imply accretiveness of the corresponding $\mathscr{A}(t)$, where $\mathscr{N}(t)$ and $\mathscr{A}(t)$ are the functions and relations of § 2. This was our motivation for portions of the construction in § 2, and its success in this direction is reflected in the following central result.

PROPOSITION 2. *For each $t \in [0, T]$ let $\mathscr{N}(t)$ and $\mathscr{A}(t)$ be the respective function and relation of* § 2. *Then $\mathscr{A}(t)$ is accretive if and only if $\mathscr{N}(t)$ is $D(t)$-monotone.*

Proof. Let $[x_1, w_1]$ and $[x_2, w_2]$ belong to $\mathscr{A}(t)$. Then there are u_1, u_2 in $D(t)$ such that $x_j = q(u_j)$ and $\mathscr{N}(t, u_j) = q^*\mathscr{M}_0 w_j$, $j = 1, 2$. Thus we have

$$\begin{aligned}\mathbf{m}(w_1 - w_2, x_1 - x_2) &= \langle \mathscr{M}_0(w_1 - w_2), q(u_1 - u_2)\rangle \\ &= \langle q^*\mathscr{M}_0(w_1 - w_2), u_1 - u_2\rangle \\ &= \langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2\rangle .\end{aligned}$$

Hence, if $\mathscr{N}(t)$ is $D(t)$-monotone, then $\mathscr{A}(t)$ is accretive.

Conversely, if u_1, $u_2 \in D(t)$, there is a unique pair w_j $(j = 1, 2)$ in W with $\mathscr{N}(t, u_j) = q^*\mathscr{M}_0 w_j$. Then $[q(u_j), w_j] \in \mathscr{A}(t)$ and, as above, $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2 \rangle = \mathbf{m}(w_1 - w_2, x_1 - x_2)$, so $\mathscr{A}(t)$ being accretive implies $\mathscr{N}(t)$ is $D(t)$-monotone.

DEFINITION 3. If in the definition of accretive (or D-monotone) the inequality is strict whenever $x_1 \neq x_2$, then we say that $\mathscr{A}$ is *strictly accretive* (respectively, $\mathscr{N}$ is *strictly D-monotone*).

If u_1, $u_2 \in D(t)$, then $\mathscr{N}(t, u_1) - \mathscr{N}(t, u_2) \in E'$, so there is a constant k such that

$$|\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), e\rangle| \leqq k m(e, e)^{1/2}, \qquad e \in E.$$

If $\mathscr{M}u_1 = \mathscr{M}u_2$, then setting $e = u_1 - u_2$ in the above identity shows that $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2 \rangle = 0$. Thus, if $\mathscr{N}(t)$ is strictly $D(t)$-monotone then $u_1 = u_2$, and Remark 1 shows that $\mathscr{A}(t)$ is a function. The first part of the proof of Proposition 2 shows then that $\mathscr{A}(t)$ is strictly accretive. Conversely, if $\mathscr{A}(t)$ is a strictly accretive function, and if $\langle \mathscr{N}(t, u_1) - \mathscr{N}(t, u_2), u_1 - u_2 \rangle = 0$ in the second part of the proof of Lemma 1, then $w_1 = w_2$, hence $\mathscr{N}(t, u_1) = \mathscr{N}(t, u_2)$. These remarks prove the following.

COROLLARY 1. *In the situation of Proposition* 2, $\mathcal{N}(t)$ *is strictly* $D(t)$*-monotone if and only if it is injective and* $\mathscr{A}(t)$ *is a strictly accretive function.*

From Remark 2 and the preceding argument applied to $\mathcal{M} + \mathcal{N}(t)$, we obtain a sufficient condition for uniqueness.

THEOREM 1. *Let* $\mathcal{N}(t)$ *be* $D(t)$*-monotone and let* $\mathcal{M} + \mathcal{N}(t)$ *be strictly* $D(t)$*-monotone for each* $t \in [0, T]$. *Then, for each* $u_0 \in D(0)$, *there is at most one solution* u *of* (1.1) *for which* $\mathcal{M}u(0) = \mathcal{M}u_0$.

We turn now to the considerably more difficult question of existence. Our results in this direction will be obtained from recent results of M. Crandall, T. Liggett and A. Pazy on the existence of solutions of evolution problems like (2.5) in general Banach space [9], [10]. We shall present the special case of their results as they apply to Hilbert space and obtain through Proposition 1 a corresponding set of sufficient conditions for the existence of a solution of (1.1).

To begin, we shall describe the existence results of [10] that are relevant for (2.5). We assume that $\mathscr{A}(t)$ is accretive and that the range of $I + \mathscr{A}(t)$ is all of W for every $t \in [0, T]$ (Each $\mathscr{A}(t)$ is *m-accretive* [16] or *hyper-accretive* [12]). It follows that the range of $I + \lambda\mathscr{A}(t)$ is W for every $\lambda > 0$, so its inverse

$$J_\lambda(t) \equiv (I + \lambda\mathscr{A}(t))^{-1}, \qquad \lambda > 0,$$

is a function defined on all of W. The dependence on t will be restricted in two ways. First, the domain of $\mathscr{A}(t)$ is independent of t, and we shall denote it by $q(D)$. Second, there is a monotone increasing function $L:[0, \infty) \to [0, \infty)$ such that

$$\|J_\lambda(t, x) - J_\lambda(\tau, x)\|_W \leqq \lambda|t - \tau|L(\|x\|_W)(1 + \inf\{\|y\|_W : [x, y] \in \mathscr{A}(\tau)\}),$$
$$t, \tau \geqq 0, \quad x \in W, \quad 0 < \lambda \leqq 1. \tag{3.1}$$

It is shown in [10], under hypotheses somewhat more general than those above, that for each $v_0 \in q(D)$ there exists a (unique) solution v of (2.5) with $v(0) = v_0$.

The preceding result will be used to prove the following.

THEOREM 2. *Let the nonnegative and symmetric linear operator* $\mathcal{M}$ *from the vector space* E *into its dual* E^* *be given. Let* E' *denote the dual of the topological vector space* E *with the seminorm* $\langle \mathcal{M}x, x\rangle^{1/2}$; E' *is a Hilbert space with norm*

$$\|f\|_{E'} = \sup\{|\langle f, x\rangle| : x \in E, \langle \mathcal{M}x, x\rangle \leqq 1\}.$$

For each $t \in [0, T]$ *let* $\mathcal{N}(t) : E \to E^*$ *be a (possibly nonlinear) function, and define* $D(t) = \{x \in E : \mathcal{N}(t, x) \in E'\}$. *Assume the following: for each* t, $\mathcal{N}(t)$ *is* $D(t)$*-monotone and the range of* $\mathcal{M} + \mathcal{N}(t)$ *contains* E'; $\mathcal{M}(D(t))$ *is independent of* t; *and there is a monotone increasing function* $L:[0, \infty) \to [0, \infty)$ *such that*

$$\|\mathcal{N}(t, w) - \mathcal{N}(\tau, w)\|_{E'} \leqq |t - \tau|L(\langle \mathcal{M}w, w\rangle)(1 + \|\mathcal{N}(t, w)\|_{E'}),$$
$$t, \tau \geqq 0, \quad w \in D(t). \tag{3.2}$$

Then, for each $u_0 \in D(0)$, *there exists a solution* u *of* (1.1) *with* $\mathcal{M}u(0) = \mathcal{M}u_0$.

Proof. From Proposition 1 it follows that we need only to verify that the relations $\mathscr{A}(t)$ constructed in § 2 satisfy the conditions listed above. Proposition 2 shows that each $\mathscr{A}(t)$ is accretive, and $\mathcal{M}(D(t))$ being constant implies that the domain $q(D(t))$ of $\mathscr{A}(t)$ is constant. Since $q^*\mathcal{M}_0$ maps W *onto* E', the identity

$$(\mathcal{M} + \lambda\mathcal{N}(t))(x) = q^* \circ \mathcal{M}_0 \circ (I + \lambda\mathscr{A}(t)) \circ q(x), \qquad x \in E, \tag{3.3}$$

shows that $I + \lambda\mathscr{A}(t)$ maps $q(D(t))$ onto W if (and only if) $\mathscr{M} + \lambda\mathscr{N}(t)$ maps $D(t)$ onto E'. Thus, we need only to verify the estimate (3.1).

Before proceeding to the verification of (3.1), we obtain some identities and estimates. First, we recall $q^*\mathscr{M}_0$ is an isomorphism of W onto E', and

$$\|q^*\mathscr{M}_0 w\|_{E'} = \|w\|_W, \qquad w \in W. \tag{3.4}$$

Also, we have from (2.1) and (2.3),

$$\|\mathscr{M}x\|_{E'} = \langle \mathscr{M}x, x\rangle^{1/2}, \qquad x \in E. \tag{3.5}$$

From (2.3), (2.4) and (3.3) follow the identities

$$\mathscr{A}(t)q = (q^*\mathscr{M}_0)^{-1}\mathscr{N}(t), \tag{3.6}$$

$$I + \lambda\mathscr{A}(t) = (q^*\mathscr{M}_0)^{-1}q^*(\mathscr{M}_0 + \lambda\mathscr{N}_0(t)). \tag{3.7}$$

This last result with the properties of relations mentioned in § 2 (e.g., $q \circ q^{-1} = I_{E/K}$) gives us

$$\begin{aligned} J_\lambda(t) &= [q^*(\mathscr{M}_0 + \lambda\mathscr{N}_0(t))]^{-1}q^*\mathscr{M}_0 \\ &= q(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}q^*\mathscr{M}_0 \\ &= (q^*\mathscr{M}_0)^{-1}\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(q^*\mathscr{M}_0). \end{aligned} \tag{3.8}$$

This shows that $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a function. Thus, if $x_1,\ x_2 \in D(t)$ and $(\mathscr{M} + \lambda\mathscr{N}(t))x_1 = (\mathscr{M} + \lambda\mathscr{N}(t))x_2$, then $\mathscr{M}x_1 = \mathscr{M}x_2$. Furthermore, this shows $\mathscr{N}(t)x_1 = \mathscr{N}(t)x_2$, so $\mathscr{N}(t)(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ also is a function on E'. From (3.8) we now obtain

$$\lambda^{-1}(I - J_\lambda(t)) = (q^*\mathscr{M}_0)^{-1}\mathscr{N}(t)(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(q^*\mathscr{M}_0). \tag{3.9}$$

Since $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a function, we have

$$\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}(\mathscr{M} + \lambda\mathscr{N}(t))x = \mathscr{M}x, \qquad x \in D(t). \tag{3.10}$$

Also, $\mathscr{M} + \lambda\mathscr{N}(t)$ is a function and so follows

$$(\mathscr{M} + \lambda\mathscr{N}(t))(\mathscr{M} + \lambda\mathscr{N}(t))^{-1} = I_{E'}. \tag{3.11}$$

We recall that each $J_\lambda(t)$ is a contraction on W, i.e.,

$$\|J_\lambda(t)w_1 - J_\lambda(t)w_2\|_W \leqq \|w_1 - w_2\|_W, \qquad w_1, w_2 \in W,$$

and this implies through (3.4) and (3.8),

$$\begin{gathered} \|\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}x_1 - \mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}x_2\|_{E'} \leqq \|x_1 - x_2\|_{E'}, \\ x_1, x_2 \in E'. \end{gathered} \tag{3.12}$$

That is, $\mathscr{M}(\mathscr{M} + \lambda\mathscr{N}(t))^{-1}$ is a contraction on E'. Also, (3.6) shows that $\mathscr{A}(t)q$ is a function and

$$\|\mathscr{A}(t)q(x)\|_W = \|\mathscr{N}(t)x\|_{E'}, \qquad x \in D(t), \tag{3.13}$$

and the identity [10]

$$\|\lambda^{-1}(I - J_\lambda(t))w\|_W \leqq \inf\{\|y\|_W : [w, y] \in \mathscr{A}(t)\}, \qquad w \in q(D),$$

together with $w = q(x)$ and (3.9) gives

$$\|\mathcal{N}(t)(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'} \leqq \|\mathcal{N}(t)x\|_{E'}, \qquad x \in D(t),$$

so we have the estimate

$$\|\mathcal{N}(t)(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'} \leqq \inf\{\|\mathcal{N}(t)y\|_{E'} : q(x - y) = 0\}, \qquad x \in D(t). \tag{3.14}$$

After this lengthy preparation, we are ready to verify (3.1) and thus complete the proof of Theorem 2. Let t, $\tau \in [0, T]$, $0 < \lambda \leqq 1$, and $x \in E$. From (3.4) and (3.8) we have

$$\|J_\lambda(\tau)q(x) - J_\lambda(t)q(x)\|_W = \|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'}.$$

Using (3.10), we have this quantity given by

$$\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}(\mathcal{M} + \lambda\mathcal{N}(t))(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(t))^{-1}\mathcal{M}x\|_{E'}.$$

Let $w \in (\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}(x)$. Then from (3.12) it follows that the above is bounded by

$$\|(\mathcal{M} + \lambda\mathcal{N}(t))(w) - \mathcal{M}x\|_{E'}.$$

By (3.11), this is equal to

$$\begin{aligned}
&\|(\mathcal{M} + \lambda\mathcal{N}(t))w - (\mathcal{M} + \lambda\mathcal{N}(\tau))(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'} \\
&\quad = \|(\mathcal{M} + \lambda\mathcal{N}(t))w - (\mathcal{M} + \lambda\mathcal{N}(\tau))w\|_{E'} \\
&\quad = \lambda\|\mathcal{N}(t, w) - \mathcal{N}(\tau, w)\|_{E'}.
\end{aligned}$$

From our hypothesis (3.2), the estimate (3.14), and (3.5) we now obtain

$$\begin{aligned}
\|J_\lambda(\tau)q(x) - J_\lambda(t)q(x)\|_W &\leqq \lambda|t - \tau|L(\|\mathcal{M}w\|_{E'}^2) \\
&\qquad \cdot (1 + \|\mathcal{N}(\tau)(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'}) \\
&\leqq \lambda|t - \tau|L(\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'}^2) \\
&\qquad \cdot (1 + \inf\{\|\mathcal{N}(\tau)y\|_{E'} : q(x - y) = 0\})
\end{aligned} \tag{3.15}$$

if $q(x) \in q(D)$. In order to estimate the term involving L in the above, we pick x_0 with $q(x_0) \in q(D)$ and then use (3.10) and (3.12) to obtain

$$\begin{aligned}
&\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}x_0\|_{E'} \\
&\quad = \|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x - \mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}(\mathcal{M} + \lambda\mathcal{N}(\tau))x_0\|_{E'} \\
&\quad \leqq \|\mathcal{M}x - (\mathcal{M} + \lambda\mathcal{N}(\tau))x_0\|_{E'}.
\end{aligned}$$

From (3.2) it follows that

$$K_T \equiv \sup\{\|\mathcal{N}(t, x_0)\|_{E'} : 0 \leqq t \leqq T\} + 2\|\mathcal{M}x_0\|_{E'} < \infty,$$

and so we have

$$\|\mathcal{M}(\mathcal{M} + \lambda\mathcal{N}(\tau))^{-1}\mathcal{M}x\|_{E'} \leqq \|\mathcal{M}x\|_{E'} + K_T, \qquad q(x) \in q(D), \quad 0 < \lambda \leqq 1.$$

This estimate together with (3.13) and (3.5) show that (3.15) implies (3.1) with L replaced by the monotone increasing function

$$L_1(\xi) = L((\xi + K_T)^2), \qquad \xi \geqq 0.$$

Remark 3. If each $\mathscr{A}(t)$ is a function, then Theorem 2 follows from T. Kato's result [16]. This will be the case in many of the applications below.

4. Boundary value problems. We shall describe realizations of the abstract evolution equation (1.1) as initial and boundary value problems for some partial differential equations of mixed elliptic-parabolic-Sobolev type. Our intent is to indicate a variety of such problems to which our results imply existence or uniqueness of solutions, so we do not attempt to attain technically best results in any sense. In particular, we shall limit consideration here to autonomous equations with spatial derivatives of at most second order. After introducing the Banach spaces which we shall use, we construct a quasilinear elliptic partial differential operator following the technique of F. Browder [5], [7]. Then we deduce from the appropriate surjectivity results of [5], [7] the information necessary to apply Theorem 2 of § 3. We illustrate the application of our resulting Theorem 3 to boundary value problems through the methods of J.-L. Lions [20], [21], [23].

Let G be a bounded open set in Euclidean space R^n with G locally on one side of its smooth boundary ∂G. The space of (equivalence classes of) functions on G with Lebesgue summable pth powers is denoted by $L^p(G)$, and $W^p(G)$ is the Sobolev space of those $\phi \in L^p(G)$ for which each of the (distribution) partial derivatives $D_j\phi$ belongs to $L^p(G)$, $1 \leqq j \leqq n$. Letting D_0 be the identity on $L^p(G)$, we can express the norm on $W^p(G)$ by

$$\|\phi\|_{W^p} = \left(\sum_{j=0}^{n} (\|D_j\phi\|_{L^p})^p \right)^{1/p}.$$

There is a continuous and linear *trace* map $\gamma: W^p(G) \to L^p(\partial G)$ with dense range, and it coincides with "restriction to ∂G" on smooth functions on G. (When it is appropriate to mention the variable $s \in \partial G$, we shall suppress the trace map by writing, e.g., $\phi(s) \equiv (\gamma\phi)(s)$ for $\phi \in W^p(G)$.) Since ∂G is smooth, there is a unit (outward) normal $n(s) = [n_1(s), \cdots, n_n(s)]$ at each $s \in \partial G$ for which we have

$$\int_G D_j\phi(x)\,dx = \int_{\partial G} \phi(s)n_j(s)\,ds, \qquad 1 \leqq j \leqq n, \tag{4.1}$$

for functions $\phi \in W^1(G)$.

Let V be a Banach space continuously embedded in $W^p(G)$ and containing the space $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G. Suppose we are given a family of functions $N_j: G \times R^{n+1} \to R$, $0 \leqq j \leqq n$, for which we assume the following:

Each $N_j(x, y)$ is measurable in x for fixed $y \in R^{n+1}$, continuous in y for fixed $x \in G$, and there are a $C > 0$ and $g \in L^q(G)$ with $q = p/(p-1)$ and $p \geqq 2$, such that

$$|N_j(x, y)| \leqq C \sum_{k=0}^{n} |y_k|^{p-1} + g(x), \qquad x \in G, \quad y \in R^{n+1}, \quad 0 \leqq j \leqq n, \tag{4.2}$$

$$\sum_{j=0}^{n} (N_j(x, y) - N_j(x, z))(y_j - z_j) \geqq 0, \qquad y, z \in R^{n+1}, \quad x \in G, \tag{4.3}$$

and there are a $c > 0$ and $h \in L^q(G)$ such that

$$\sum_{j=0}^{n} N_j(x, y)y_j + h(x) \geqq c|y|^p, \qquad y \in R^{n+1}, \quad x \in G. \tag{4.4}$$

Letting $D\phi \equiv \{D_j\phi : 0 \leqq j \leqq n\}$ for $\phi \in W^p(G)$, we find that each $N_j(x, D\phi(x))$ belongs to $L^q(G)$, so we can define $\mathcal{N} : V \to V'$ by

$$\langle \mathcal{N}\phi, \psi \rangle \equiv \sum_{j=0}^{n} \int_G N_j(x, D\phi(x)) D_j\psi(x)\, dx, \qquad \phi, \psi \in V. \tag{4.5}$$

Note that the restriction of $\mathcal{N}\phi$ to $C_0^\infty(G)$ is the distribution on G given by

$$N(\phi) \equiv -\sum_{j=1}^{n} (D_j N_j(\cdot, D\phi)) + N_0(\cdot, D\phi). \tag{4.6}$$

This defines our quasilinear elliptic partial differential operator $N : V \to \mathcal{D}'(G)$, the space of distributions on G. From (4.1) we obtain the (formal) Green's identity

$$\langle \mathcal{N}\phi - N\phi, \psi \rangle = \int_{\partial G} \frac{\partial\phi(s)}{\partial N} \psi(s)\, ds, \qquad \psi \in V, \tag{4.7}$$

whenever $\mathcal{N}(\phi)$, and hence $N(\phi)$, belong to $L^q(G)$, and we have let

$$\frac{\partial\phi(s)}{\partial N} \equiv \sum_{j=1}^{n} N_j(s, D\phi(s)) n_j(s), \qquad s \in \partial G, \tag{4.8}$$

denote the conormal derivative.

(Note that $L^q(G)$ is simultaneously identified by duality with subsets of V' and $\mathcal{D}'(G)$.)

One can use (4.2) and dominated convergence to show that $\mathcal{N}$ is hemicontinuous, i.e., continuous from line segments in V to V' with the weak topology. Also, (4.3) shows that $\mathcal{N}$ is V-monotone while (4.4) implies

$$\langle \mathcal{N}\phi, \phi \rangle \geqq c(\|\phi\|_{W^p})^p - |h|_{L^q}\|\phi\|_{L^p}, \qquad \phi \in V,$$

so $\mathcal{N}$ is coercive: $\langle \mathcal{N}\phi, \phi \rangle \to \infty$ as $\|\phi\|_{W^p} \to \infty$. These three properties are sufficient to make $\mathcal{N}$ surjective [5], [7].

Suppose we are given a continuous, linear and monotone $\mathcal{M} : V \to V'$. Then $\mathcal{M} + \mathcal{N}$ is hemicontinuous, coercive and monotone, hence maps onto V'. Assume also $\mathcal{M}$ is symmetric and let E denote the space V with the seminorm induced by $\mathcal{M}$. Then the injection $V \to E$ is continuous, and hence $E' \subset V'$, so the range of $\mathcal{M} + \mathcal{N}$ includes E'. From Theorems 1 and 2 we obtain the following result.

THEOREM 3. *Let V be a reflexive Banach space and $\mathcal{M} : V \to V'$ a symmetric, continuous, linear and monotone operator. Let $\mathcal{N} : V \to V'$ be hemicontinuous, monotone and coercive, and $u_0 \in V$ with $\mathcal{N}u_0 \in E'$, where E is the space V with the seminorm induced by $\mathcal{M}$. Then there exists an absolutely continuous $u : [0, T) \to E$, such that $\mathcal{N}u(t) \in E'$ for all $t \in [0, T]$,*

$$\frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t) = 0, \qquad a.e.\ t \in [0, t],$$

and $\mathcal{M}(u(0) - u_0) = 0$. The solution is unique if $\mathcal{M} + \mathcal{N}$ is strictly monotone.

Remark 4. By our choice of V, we may obtain *stable* boundary conditions from the inclusions $u(t) \in V$, $t \in [0, T]$, or *variational* boundary conditions from the identity

$$\left\langle \frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t), v \right\rangle = \int_G \left\{ \frac{d}{dt}(\mathcal{M}u(t)) + \mathcal{N}u(t) \right\} v(x)\, dx, \tag{4.9}$$
$$v \in V, \quad t \in [0, T].$$

We shall illustrate the application of Theorem 3 to boundary value problems by four examples. In the first two examples, we choose $V = W_0^p(G)$, the space of those $\phi \in W^p(G)$ for which $\gamma(\phi) = 0$.

Example 1. *Degenerate elliptic-parabolic equations.* Let $m_0 \in L^r(G)$ with $m_0(x) \geqq 0$, a.e. $x \in G$, $r = p/(p-2)$, and define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G m_0(x)\phi(x)\psi(x)\, dx, \qquad \phi, \psi \in V.$$

Let $u_0 \in V = W_0^p(G)$ be given with $N(u_0) = m_0^{1/2} g$ for some $g \in L^2(G)$. Then Theorem 3 asserts the existence of a solution of the equation

$$\frac{\partial}{\partial t}(m_0(x)u(x,t)) + N(u(x,t)) = 0, \qquad x \in G, \quad t > 0, \tag{4.10}$$

with $u(s,t) = 0$ for $s \in \partial G$ and $t \geqq 0$, and $u(x,0) = u_0(x)$ for all $x \in G$ with $m_0(x) > 0$. Such problems arise, e.g., in classical models of heat propagation, and $m_0(x)$ then denotes a variable specific heat capacity of the material.

Example 2. *Degenerate parabolic-Sobolev equations.* Let m_0 be as above, but define

$$\langle \mathcal{M}\phi, \psi \rangle \equiv \int_G \left\{ \phi\psi + m_0 \sum_{j=1}^{n} (D_j\phi)(D_j\psi) \right\}, \qquad \phi, \psi \in V.$$

Since $\langle \mathcal{M}\phi, \phi \rangle \geqq (\|\phi\|_{L^2})^2$, we have $L^2(G) \subset E'$, and so Theorem 3 shows that for each $u_0 \in W_0^p(G)$ with $N(u_0) \in L^2(G)$, there is a unique solution of the equation

$$\frac{\partial}{\partial t}\left\{ u(x,t) - \sum_{j=1}^{n} D_j(m_0(x)D_j u(x,t)) \right\} + N(u(x,t)) = 0, \qquad x \in G,$$

with $u(s,t) = 0$ for $s \in \partial G$, $t \in [0, T]$ and $u(x,0) = u_0(x)$ for all $x \in G$. Such equations have been used to describe diffusion processes wherein m_0 is a material constant with the dimensions of viscosity [11], [26]. Also see [8], [18], [25].

Many variations on the preceding examples are immediate. For example, one can use Sobolev imbedding results to get a smaller choice of r in the first example, and other choices of V could replace the Dirichlet boundary condition (in part) by a condition on the conormal derivative (4.8). Such is the case in our next two examples which consider equation (4.10) with *evolutionary boundary conditions.*

Example 3. *Parabolic boundary conditions.* In order to simplify some computations below, assume that ∂G intersects the hyperplane $R^{n-1} \times \{0\}$ in a set with relative interior S. Let $a \in L^\infty(S)$ be given with $a(s) \geqq 0$, $s \in S$, and define the space

$$V \equiv \{\phi \in W^p(G) : \phi(s) = 0 \text{ if } s \in \partial G \sim S,\ a^{1/2}(s)D_j\phi(s) \in L^2(S) \text{ for } 1 \leqq j \leqq n-1\}$$

with the norm

$$\|\phi\|_V \equiv \|\phi\|_{W^p} + \left(\int_S a(s) \sum_{j=1}^{n-1} (D_j\phi(s))^2\, ds\right)^{1/2}.$$

Let m_0 be given as in Example 1 and $n_0 \in L^r(S)$ with $n_0(s) \geqq 0$, a.e. $s \in S$. Define

$$\langle \mathscr{M}\phi, \psi\rangle \equiv \int_G m_0\phi\psi + \int_S n_0(s)\phi(s)\psi(s)\, ds,$$

and

$$\langle \mathscr{N}\phi, \psi\rangle \equiv (4.5) + \int_S a(s)\left(\sum_{j=1}^{n-1} D_j\phi(s) D_j\psi(s)\right) ds.$$

For u_0 as in Example 1, Theorem 3 asserts the existence of a solution of equation (4.10) which satisfies the initial conditions

$$m_0(x)(u(x,0) - u_0(x)) = 0, \qquad x \in G,$$
$$n_0(s)(u(s,0) - u_0(s)) = 0, \qquad s \in S.$$

Since for $\phi \in V$ we have $\phi(s) = 0$ for $s \in \partial G \sim S$ we obtain from (4.7), (4.9) and (4.1) (applied to S) the variational boundary condition

$$\frac{\partial}{\partial t}(n_0(s)u(s,t)) + \frac{\partial}{\partial N}(u(s,t)) = \sum_{j=1}^{n-1} D_j(a(s)D_j u(s,t)), \qquad s \in S, \quad t \in [0, T].$$

Also, we have the stable condition $u(s,t) = 0$ for $s \in \partial G \sim S$ and $t \in [0, T]$. Boundary value problems of this form describe models of fluid flow wherein S is an approximation of a narrow fracture characterized by a very high permeability. Thus, most of the flow in S occurs in the tangential directions. See [6], [28] for applications and references.

Example 4. *The fourth boundary value problem.* (This terminology is not ours, but comes from [1].) Let $V \equiv \{\phi \in W^p(G) : \gamma(\phi)$ is constant on $\partial G\}$ with the norm of $W^p(G)$, and define $\mathscr{N}$ by (4.5). Let m_0 be given as in Example 1 and define

$$\langle \mathscr{M}\phi, \psi\rangle \equiv \int_G m_0\phi\psi + \gamma(\phi)\cdot\gamma(\psi), \qquad \phi, \psi \in V.$$

Then from Theorem 3 it follows that for each $u_0 \in V$, with $N(u_0) = m_0^{1/2} g$ for some $g \in L^2(G)$, there exists a solution of equation (4.10) which satisfies the boundary conditions of the fourth kind

$$u(s,t) = f(t), \qquad s \in \partial G, \quad t \in [0, T],$$
$$f'(t) + \int_{\partial G} \frac{\partial u(s,t)}{\partial N}\, ds = 0,$$

as well as the initial conditions

$$m_0(x)(u(x,0) - u_0(x)) = 0, \qquad x \in G,$$
$$f(0) = u_0(s), \quad \text{a constant}.$$

Such problems are used to describe, for example, heat conduction in a region G which is submerged in a highly conductive material of finite mass, so the heat flow from G affects the temperature $f(t)$ in the enclosing material. This problem was introduced in [1], together with a problem of the fifth kind (to which our results can be applied).

5. Two degenerate wave equations. We shall give results on existence and uniqueness of two second order evolution equations with (possibly degenerate) operator coefficients on the time derivatives and then indicate some applications. As before, we illustrate the variety of potential applications through the simplest examples.

THEOREM 4. *Let A and B be symmetric and continuous linear operators from a reflexive Banach space V into its dual V', where B is monotone and A is coercive: there is a $k > 0$ such that*

$$\langle A\phi, \phi\rangle \geqq k\|\phi\|_V^2, \qquad \phi \in V.$$

Denote by V_b the space V with the seminorm induced by B and let $F: V \to V'$ be a (possibly nonlinear) monotone and hemicontinuous function. Then, for each pair u_1, $u_2 \in V$ with $Au_1 + F(u_2) \in V'_b$, there exists a unique absolutely continuous $u:[0, T] \to V$ with $Bu':[0, T] \to V'_b$ absolutely continuous, $u(0) = u_1$, $Bu'(0) = Bu_2$,

$$F(u'(t)) + Au(t) \in V'_b, \quad \text{a.e. } t \in [0, T], \tag{5.1}$$

and

$$(Bu'(t))' + F(u'(t)) + Au(t) = 0, \quad \text{a.e. } t \in [0, T]. \tag{5.2}$$

Proof. Define a pair of operators from the product space $E \equiv V \times V$ into $E^* = V^* \times V^*$ by

$$\mathscr{M}([\phi_1, \phi_2]) \equiv [A\phi_1, B\phi_2],$$

$$\mathscr{N}([\phi_1, \phi_2]) \equiv [-A\phi_2, A\phi_1 + F(\phi_2)].$$

The symmetric and nonnegative $\mathscr{M}$ gives a seminorm on E for which the dual is $E' = V' \times V'_b$. The operator $A: V \to V'$ is an isomorphism, so u is a solution of the Cauchy problem for (5.2) if and only if $[u, u']$ is a solution of the Cauchy problem for (1.1) with the operators above. Uniqueness follows from Remark 2 of § 2, and existence will follow from Theorem 2 if we can verify that the range of $\mathscr{M} + \mathscr{N}$ contains E'. Since A is surjective, an easy exercise shows we need only to verify that $A + B + F$ maps onto V'. This follows by [5], [7], since $A + B + F$ is hemicontinuous, monotone and coercive.

The Cauchy problem solved by Theorem 4 appears to ask for too much in two directions. First, our previous results suggest we should specify (essentially) $F(u(0)) = F(u_1)$ instead of $u(0) = u_1$, since, e.g., we may take $B \equiv 0$ in (5.2). The second point to be noticed in the Cauchy problem associated with Theorem 4 is the inclusion (5.1). In applications, (5.1) can actually imply that a differential equation is satisfied, so this Cauchy problem possibly contains a *pair* of differential equations.

In our next and final result, we obtain a considerably weaker solution of a single equation similar to (5.2) subject to initial conditions with data that need not satisfy the compatibility condition, $Au_1 + F(u_2) \in V'_b$.

THEOREM 5. *Let the operators A, B and F and spaces V and V_b be given as in Theorem 4. Then for each pair u_0, $u_1 \in V$, there exists a unique summable function $w:[0, T] \to V$ for which $Bw:[0, T] \to V'_b$ is absolutely continuous,*

$$(Bw)' + F(w):[0, T] \to V'$$

is (equal a.e. to a function which is) absolutely continuous,

$$(Bw)(0) = Bu_0, \qquad ((Bw') + F(w))(0) = Au_1,$$

and

$$((Bw)' + F(w))' + Aw = 0 \tag{5.3}$$

a.e. in $[0, T]$.

Proof. The Cauchy problem above for (5.3) is equivalent to that of Theorem 4 as well as to that of (1.1) with the operators given in the proof of Theorem 4. In short, if u is the solution of (5.2), then $w \equiv u'$ is the solution of (5.3) and $[u, w]$ is the solution of (1.1).

We continue our listing of examples with references to their applications and history.

Example 5. *Degenerate wave-parabolic-Sobolev-elliptic equations.* Take

$$V \equiv W_0^2(G),$$

the indicated Sobolev space introduced in § 4, and define the coercive form

$$\langle A\phi, \psi\rangle \equiv \int_G \sum_{j=1}^n D_j\phi D_j\psi, \qquad \phi, \psi \in V.$$

Let $m_j \in L^\infty(G)$ with $m_j(x) \geqq 0$, a.e. $x \in G$, for $0 \leqq j \leqq n$, and define the operator B by

$$\langle B\phi, \psi\rangle \equiv \int_G \sum_{j=0}^n m_j(x) D_j\phi(x) D_j\psi(x)\, dx, \qquad \phi, \psi \in V.$$

Finally, let $F \equiv \mathcal{N}$ be given by (4.5), where we assume (4.2), (4.3) and $1 < p \leqq 2$. (This last requirement is quite restrictive but is relevant here since it gives the continuous inclusions $L^2(G) \to L^p(G)$ and $L^q(G) \to L^2(G)$. Then, Theorem 5 shows there is a unique generalized solution w of the equation

$$\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial t}(m_0 w(x, t) - \sum_{j=1}^n D_j m_j(x) D_j w(x, t)) + N(w)\right\} - \sum_{j=1}^n D_j^2 w(x, t) = 0, \tag{5.4}$$
$$x \in G, \quad t \in [0, T],$$

where N is given by (4.6), and w satisfies the boundary conditions

$$w(s, t) = 0, \qquad s \in \partial G, \quad t \in [0, T],$$

and the initial conditions

$$B(w(\cdot, 0) - u_0) = 0, \qquad (Bw)' + N(w)|_{t=0} = w_1,$$

where $u_0 \in V$ and $w_1 \in V'$ are given.

Equation (5.4) includes the classical wave equation as well as the equation

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t} - \lambda \Delta w\right) - \Delta w = 0,$$

which arises in classical hydrodynamics and the theory of elasticity [15]. Applications in which B is a homogeneous differential operator of order two include the modeling of infinitesimal waves [22] by the equation

$$\frac{\partial^2}{\partial t^2}\left(\sum_{j=1}^{2} D_j^2 w(x,t)\right) + \sum_{j=1}^{3} D_j^2 w(x,t) = 0$$

and the Sobolev equation

$$\frac{\partial^2}{\partial t^2}\left(\sum_{j=1}^{3} D_j^2 w(x,t)\right) + D_3^2 w(x,t) = 0,$$

which describes the motion of a fluid in a rotating vessel [27], [29]. (An elementary change of variables will bring this last equation to the form of (5.4).)

Example 6. *A gas diffusion equation.* Taking the special case of (5.4) with $B \equiv 0$, we can solve problems for the equation

$$\frac{\partial}{\partial t}N(w) - \sum_{j=1}^{n} D_j^2 w = 0, \tag{5.5}$$

in which N is given by (4.6). Setting $N_j = 0$ for $1 \leqq j \leqq n$ and

$$N_0(x,s) \equiv m_0(x)|s|^{p-1}\operatorname{sgn}(s),$$

where $m_0 \in L^\infty(G), m_0(x) \geqq 0$, and $1 < p \leqq 2$ gives us the degenerate and nonlinear

$$\frac{\partial}{\partial t}(m_0(x)|w|^{p-1}\operatorname{sgn}(w)) - \Delta w = 0.$$

The change of variable $u \equiv |w|^{p-1}\operatorname{sgn}(w)$ puts this in the form [3], [4]

$$\frac{\partial}{\partial t}(m_0(x)u(x,t)) - (q-1)\sum_{j=1}^{n} D_j(|u|^{q-2}D_j u(x,t)) = 0, \tag{5.6}$$

with $q - 2 = (2-p)/(p-1) \geqq 0$.

Note that (5.6) is not of the form suitable for the results of § 3, since the nonlinear part is not monotone, but it can be rewritten as

$$\frac{\partial}{\partial t}(A^{-1}m_0 u) - |u|^{q-2}u(x,t) = 0,$$

where A is given in Example 5. We also note that (5.5) includes one of the Stefan free-boundary problems [4], [17], and the nonlinear term can contain spatial derivatives.

Our final example illustrates an application of both Theorem 4 and Theorem 5 to a situation in which B acts only on the boundary ∂G and F is multiplication by a nonnegative function on G. Other combinations are possible and useful, but the following is typical of higher order evolutionary boundary conditions.

Example 7. Second order boundary conditions. Let $S \subset \partial G$ and define V to be the subspace of $W^2(G)$ consisting of those functions which vanish on $\partial G \sim S$. Let the operator A and the function m_0 be given as in Example 5, and define

$$\langle F\varphi, \psi\rangle = \int_G m_0(x)\varphi(x)\psi(x)\,dx,$$

$$\langle B\varphi, \psi\rangle = \int_S \varphi(s)\psi(s)\,ds, \qquad \varphi, \psi \in V.$$

Let $w_0 \in V$, $w_1 \in L^2(G)$ and $w_2 \in L^2(S)$. Since A is an isomorphism, there is a $u_1 \in V$ with

$$\langle Au_1, v\rangle = \int_G w_1(x)v(x)\,dx + \int_S w_2(s)v(s)\,ds.$$

From Theorem 5 it follows that there is a unique $w:[0, T] \to V$ which is a generalized solution of the partial differential equation

$$\frac{\partial}{\partial t}(m_0(x)w(x,t)) - \Delta w(x,t) = 0, \qquad x \in G, t > 0, \tag{5.7}$$

subject to the boundary conditions

$$w(s,t) = 0, \qquad s \in \partial G \sim S, \quad t > 0,$$

$$\frac{\partial^2 w(s,t)}{\partial t^2} + \frac{\partial w(s,t)}{\partial n} = 0, \qquad s \in S, \qquad t > 0,$$

and the initial conditions

$$w(s,0) = w_0(s), \qquad s \in S,$$

$$\frac{\partial w(s,0)}{\partial t} = w_2(s), \qquad s \in S,$$

$$w(x,0) = w_1(x), \qquad \text{where } m_0(x) > 0.$$

Since $V_b' = L^2(S)$, the pair $u_1, u_2 \in V$ satisfies $Au_1 + Fu_2 \in V_b'$ if and only if

$$-\Delta u_1 + m_0 u_2 = 0 \quad \text{in } G,$$

and

$$\frac{\partial u_1}{\partial n} \in L^2(S).$$

These conditions imply a regularity result for u_1 depending on the smoothness of m_0. If u_1 and u_2 are so given, and if w denotes the solution of the Cauchy problem of Theorem 4, then (5.1) implies that w is regular (depending on m_0) and satisfies (5.7) and the null boundary condition on $\partial G \sim S$. The equation (5.2) implies the second boundary condition above, and the initial conditions in Theorem 4 assert that w satisfies $w(x,0) = u_1(x)$, $x \in \bar{G}$, and $\partial w(s,0)/\partial t = u_2(s)$, $s \in S$. The data in this case are more restricted and the conditions stronger than above, but we obtain a correspondingly stronger solution. Problems of the above type (with $m_0(x) \equiv 0$) originate from the equations of water waves or gravity waves. See [13], [22] for additional results and references.

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AN INVESTIGATION OF STABILITY OF MOTION UNDER CONSTANTLY ACTING DISTURBANCES*

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Abstract. This article discusses the total stability, or stability under constantly acting disturbances, of a system of nonlinear ordinary differential equations. Total stability differs significantly from Lyapunov stability in that the former allows for a disturbance in the equations of motion, as well as a disturbance in the initial condition. The purpose of this study is to extend the mathematical theory for total stability into a form which can be used directly in applications. To do this, a specific Lyapunov function is constructed. Then, using this Lyapunov function in a new total stability theorem we obtain explicit expressions for maximum magnitudes of the initial condition and the disturbances in the equations of motion. These maximum magnitudes are expressed in terms of the prescribed bound of the motion from the equilibrium, and in terms of the parameters of the physical system which the differential equations describe.

1. Introduction. Consider a set of first order ordinary differential equations

$$\dot{x}(t) = f(x,t), \tag{1}$$

where x is an n-vector and the dot denotes the total derivative with respect to time, t. The vector function $f(x,t)$ is zero at the equilibrium $x(t) = 0$, i.e., $f(0,t) = 0$, continuous and in class E, i.e., it assures the existence of a unique solution to (1) for all time. Let $\|\cdot\|$ be the norm of a vector. The equilibrium of (1) is said to be Lyapunov stable [1], [4] if there exists for each $\varepsilon > 0$ a function $\delta_1(\varepsilon, t_0) > 0$ such that $\|x(t_0)\| < \delta_1$ implies $\|x(t)\| < \varepsilon$ for $t \geqq t_0$. If (1) is not the precise mathematical model of the physical system that it is supposed to describe, or if there exist disturbances acting on the physical system, then an additional error term or perturbing force term would be added to (1), namely,

$$\dot{x}(t) = f(x,t) + g(x,t), \tag{2}$$

where the perturbation function $g(x,t)$ is assumed to be continuous and such that the right-hand side of (2) is in class E. In general, it is not required that $g(0,t) = 0$, so the equilibrium of (1) may not be a solution to (2). To investigate the effect of the perturbation term it is necessary to extend the concept of Lyapunov stability to that of stability under constantly acting disturbances (or total stability) as first introduced by Dubošin [2]. The equilibrium of (1) is said to be totally stable if for every $\varepsilon > 0$ there exist two positive numbers $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ such that if $\|x(t_0)\| < \delta_1$ and $\|g(x,t)\| < \delta_2$ in $\{(x,t) | \|x\| \leqq \varepsilon, t \geqq t_0\}$ then $\|x(t)\| < \varepsilon$ for all $t \geqq t_0$, where $x(t)$ is the solution to (2). A major theorem on total stability was given by Malkin [6], which is stated here: If there exists a function $v(x,t)$ such that in some bounded domain including $x = 0$ it is positive definite, has its first partial derivatives with respect to the components of x bounded, and has its

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total time derivative for (1) negative definite, then the equilibrium of (1) is totally stable. The proof of Malkin's theorem may be found in [3] and [6]. Nevertheless, Malkin's theorem cannot be put into use unless the expressions of the upper bounds of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ are found for a given $\varepsilon > 0$. But these bounds of δ_1 and δ_2 cannot be determined unless a Lyapunov function is constructed, which has its total time derivative negative definite along a solution of (1). Recent work by Yoshizawa [7] also establishes the total stability of (1). However, his work does not include the construction of a Lyapunov function for the actual calculations of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$. It is therefore seen that in order to apply the theory of total stability it is necessary to overcome two essential difficulties. The first is to find a Lyapunov function for (1). The second is to incorporate into the actual physical system an energy dissipation mechanism such that the total time derivative of the Lyapunov function following the motion of the physical system, as represented by the solution of (1), is negative definite. Such a damping scheme should be realistic, as the motion of the physical system and its stability behavior depend on the specific damping scheme. This is a key point often overlooked in stability studies.

The Hamiltonian of the physical system, which is sometimes used as a Lyapunov function in stability analysis, cannot be used here as a Lyapunov function in total stability. This is because the total time derivative of the Hamiltonian of a nonconservative system incorporated with damping can be shown to be only nonpositive but not negative definite. Therefore, in the investigation of total stability the class of Lyapunov functions is more restrictive than in stability analysis. Due to these difficulties, little work has been done on obtaining explicit expressions for $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ in terms of the parameters of the physical system described by (1).

In this paper we first construct a Lyapunov function for (1) following a method suggested in [4], and then develop some lemmas and a theorem for the total stability of (1) which explicitly yields the desired bounds. In the work presented here the use of a particular Lyapunov function permits the calculation of upper bounds of $\delta_1(\varepsilon)$ and $\delta_2(\varepsilon)$ in terms of the physical parameters of the system. The class of functions $f(x, t)$ in (1) considered in this paper is such that $f(x, t)$ can be decomposed into linear and nonlinear parts such that

$$\dot{x}(t) = f(x, t) = Ax(t) + h(x, t). \tag{3}$$

Here, the $n \times n$ constant matrix A is diagonalizable and its eigenvalues have negative real parts, i.e., $\operatorname{Re}(\lambda_i) < 0$, $i = 1, 2, \cdots, n$, (due to the damping mechanism provided in the physical system), and $h(x, t) = o(\|x\|)$, or

$$\lim_{\|x\| \to 0} \frac{\|h(x, t)\|}{\|x\|} = 0.$$

The eigenvector of A corresponding to λ_i shall be denoted by ϕ_i, i.e., $A\phi_i = \lambda_i\phi_i$.

2. Definitions. In order for this paper to be self-contained, a few necessary definitions shall be stated here which may be found in some books on stability (as, e.g., [3]).

DEFINITION 2.1. A continuous real function, $w(r)$, *is in class* K if it is defined on $0 \leqq r \leqq R$ for some $R > 0$ such that $w(0) = 0$ and $w(r)$ increases strictly monotonically with r.

DEFINITION 2.2. A continuous real function, $v(x, t)$, is *decrescent* if there exists a function $\Psi(r)$ in class K such that $v(x, t) \leqq \Psi(\|x\|)$ for $\|x\| \leqq R$ and $t \geqq t_0$ for some t_0.

DEFINITION 2.3. A continuous real function, $v(x, t)$, is *positive* (*or negative*) *definite* if there exists a function $\phi(r)$ (or $\chi(r)$) in class K such that $v(x, t) \geqq \phi(\|x\|)$ (or $\leqq -\chi(\|x\|)$) for $\|x\| \leqq R$ and $t \geqq t_0$ for some t_0.

DEFINITION 2.4. A function which is positive definite, decrescent and which has its total time derivative negative definite along any solution of (1) is a Lyapunov function for (1).

Unless otherwise stated it shall be assumed that all n-vectors are written in the basis $\{e_i\}_{i=1}^n$, where each element of the n-vector e_i is zero except for the ith, which is unity.

If A is an $n \times n$ matrix, (Ax, x) shall represent $(Ax)^T\bar{x}$, where the superscript T denotes the transpose and the bar denotes the complex conjugate.

Denote by S the matrix which transforms the basis vector e_i onto the eigenvector ϕ_i, i.e., $S e_i = \phi_i$, $i = 1, 2, \cdots, n$.

3. The construction of a Lyapunov function. In [4] a technique is presented which assures the existence of a Lyapunov function for an equation of the form $\dot{x} = Ax$ if A is a matrix whose eigenvalues have negative real parts. However, it is not possible in general to actually obtain such a Lyapunov function. If A is diagonalizable then the method of [4] does explicitly yield a Lyapunov function, as will be shown later. Such a Lyapunov function will be used in a revised form of Malkin's original theorem on total stability to obtain expressions for bounds of δ_1 and δ_2.

It is known that if the $n \times n$ constant matrix A in (3) is diagonalizable then its eigenvectors form a basis for n-space. In this new basis, D, the diagonal form of A has λ_i as the ith diagonal element. Now, rewrite the linearized part of (3) in the basis $\{\phi_i\}_{i=1}^n$, i.e.,

$$\dot{x}(t) = Dx(t). \tag{4}$$

Consider a diagonal matrix B, written also in the basis $\{\phi_i\}_{i=1}^n$, whose ith diagonal element is $-1/(2 \operatorname{Re} \lambda_i)$.

LEMMA 3.1. *In the basis* $\{\phi_i\}_{i=1}^n$, $v(x, t) = (Bx, x)$ *is positive definite, i.e.,*

$$v(x, t) \geqq \phi(\|x\|), \quad \textit{where } \phi(\|x\|) = \frac{\|x\|^2}{2 \max_i |\operatorname{Re} \lambda_i|}, \tag{5}$$

and is decrescent, i.e.,

$$v(x, t) \leqq \Psi(\|x\|), \quad \textit{where } \Psi(\|x\|) = \frac{\|x\|^2}{2 \min_i |\operatorname{Re} \lambda_i|}. \tag{6}$$

Proof. The proof is trivial due to the definition of the matrix B.

It is known [1] that the existence of the function $v(x, t)$ given by Lemma 3.1 establishes the asymptotic stability of the equilibrium of (4) and hence of (1). If one wishes to calculate the bound for the norm of the initial condition so that the norm of the solution to (1) is less than ε, one could use this Lyapunov function. Here, we are interested in the more realistic problem of total stability, and must do more to obtain bounds for δ_1 and δ_2.

LEMMA 3.2. *The function* $v(x, t)$ *given in Lemma* 3.1 *has its total time derivative negative definite along solutions of* (1) *and* (3), *expressed in the basis* $\{\phi_i\}_{i=1}^n$.

Proof. Choose a constant α between 0 and 1. Select a number δ such that in the basis $\{\phi_i\}_{i=1}^n$,

$$2\|B\|\frac{\|h(x,t)\|}{\|x\|} \leqq \alpha \quad \text{whenever } \|x\| \leqq \delta. \tag{7}$$

Such a $\delta > 0$ exists since $h(x, t) = o(\|x\|)$. The total time derivative of $v(x, t)$ along a solution of (1) or (3) is (here, * denotes adjoint)

$$\begin{aligned} \dot{v}_1(x,t) &= ((D^*B + BD)x, x) + (Bh(x,t), x) + (Bx, h(x,t)) \\ &= -\|x\|^2 + 2\,\mathrm{Re}\,(Bh(x,t), x). \end{aligned}$$

It follows that

$$\dot{v}_1(x,t) \leqq \left[-1 + 2\|B\|\frac{\|h(x,t)\|}{\|x\|}\right]\|x\|^2. \tag{8}$$

Consequently, based on (7), $\dot{v}_1(x, t)$ is negative definite, i.e.,

$$\dot{v}_1(x,t) \leqq -\chi(\|x\|), \quad \text{where } \chi(\|x\|) = (1-\alpha)\|x\|^2 \tag{9}$$

in a bounded domain $\|x\| \leqq \delta$. This completes the proof.

THEOREM 3.1. *Let* A *in* (3) *have the properties specified in* (3). *Then the equilibrium of* (1) *is totally stable, with upper bounds for* $x(t_0)$ *and the perturbation* $g(x, t)$ *given, for any preassigned positive constant* ε, *by*

$$\delta_1 \leqq \varepsilon\left[\frac{\min_i |\mathrm{Re}\,\lambda_i|}{\max_i |\mathrm{Re}\,\lambda_i|}\right]^{1/2}, \tag{10}$$

$$\delta_2 \leqq \frac{\varepsilon(1-\alpha)}{2\|B\|}\left[\frac{\min_i |\mathrm{Re}\,\lambda_i|}{\max_i |\mathrm{Re}\,\lambda_i|}\right]. \tag{11}$$

Proof. Express (1) and (2) in the basis $\{\phi_i\}_{i=1}^n$. With δ determined as in the proof of Lemma 3.2, choose ε such that $0 < \varepsilon \leqq \delta$. Take the Lyapunov function $v(x, t)$ as established in Lemma 3.1. Let $b = \min v(x, t)$ on the surface $\|x\| = \varepsilon$. Since $v(x, t)$ is positive definite as in (5), $b \geqq \phi(\varepsilon) = \varepsilon^2/(2\max_i |\mathrm{Re}\,\lambda_i|)$. Choose a number l between 0 and $\phi(\varepsilon)$. Consider a solution $x(t)$ to (2) such that $v(x, t) = l$ at some time, t^*. Since $v(x, t)$ is descrescent, if follows from (6) that

$$\|x\|^2 \geqq 2l\left(\min_i |\mathrm{Re}\,\lambda_i|\right)$$

at $x(t^*)$. Then, (9) implies that

$$\dot{v}_1(x, t^*) \leqq -2(1-\alpha)l\left(\min_i |\mathrm{Re}\,\lambda_i|\right) < 0. \tag{12}$$

Now, the total time derivative of $v(x, t)$ along a solution of (2) is

$$\dot{v}_2(x,t) = \dot{v}_1(x,t) + 2\,\mathrm{Re}\,(Bx, g(x,t)).$$

So at t^* this means

$$\dot{v}_2(x, t^*) \leqq -2(1-\alpha)l\left(\min_i |\operatorname{Re} \lambda_i|\right) + 2\|B\|\,\|x\|\,\|g(x, t^*)\|.$$

By virtue of (5), $\|x(t^*)\| \leqq [2l \max_i |\operatorname{Re} \lambda_i|]^{1/2}$, so that

$$\dot{v}_2(x, t^*) \leqq -2(1-\alpha)l\left(\min_i |\operatorname{Re} \lambda_i|\right) + 2\|B\|\,\|g(x, t^*)\|\left[2l\left(\max_i |\operatorname{Re} \lambda_i|\right)\right]^{1/2}.$$

If, in the basis $\{\phi_i\}_{i=1}^n$, there exists a positive number δ_2 such that $\|g(x, t)\| < \delta_2$ at $x(t^*)$, then

$$\dot{v}_2(x, t^*) < -2l(1-\alpha)\left(\min_i |\operatorname{Re} \lambda_i|\right) + 2\|B\|\delta_2\left[2l \max_i |\operatorname{Re} \lambda_i|\right]^{1/2}. \tag{13}$$

Since l is an arbitrary number in the interval between 0 and $\phi(\varepsilon)$, it can be taken arbitrarily close to $\phi(\varepsilon)$. Hence, $\dot{v}_2(x, t^*) < 0$ if $\|g(x, t^*)\| < \delta_2$, where δ_2 can easily be shown to have the upper bound given in (11).

Now, pick a positive number δ_1 less than or equal to ε such that in the basis $\{\phi_i\}_{i=1}^n$, $v(x(t_0), t_0) < l$ whenever $\|x(t_0)\| < \delta_1$. By virtue of (6) and the fact that l has been taken arbitrarily close to $\phi(\varepsilon)$, the maximum possible upper bound, δ_1, for $\|x(t_0)\|$ is such that $\Psi(\delta_1) = \phi(\varepsilon)$. That is, $\delta_1 = \Psi^{-1}[\phi(\varepsilon)]$ which is the bound given in (10).

Let $x(t)$ be a solution to (2) under an initial condition $\|x(t_0)\| < \delta_1$. Suppose that $\|x(t)\|$ reaches the value ε at some time $t_1 > t_0$. Then $v(x, t_1) \geqq \phi(\varepsilon) > l$, indicating that $v(x, t)$ has increased and passed through the point $v(x, t) = l$. But this is impossible, since at any point $x(t^*)$ for which $v(x, t^*) = l$ it has been shown that $\dot{v}_2(x, t^*) < 0$. Therefore, if $\|x(t_0)\| < \delta_1$ and $\|g(x, t)\| < \delta_2$ in $\{(x, t) | \|x\| \leqq \varepsilon, t \geqq t_0\}$, then $\|x(t)\| < \varepsilon$ for all $t \geqq t_0$. This proves the total stability of the equilibrium of (1).

COROLLARY 3.1. *For any δ_1 and δ_2 satisfying* (10) *and* (11) *respectively, for a given $\varepsilon > 0$, the total stability defined in the original basis, $\{e_i\}_{i=1}^n$, takes the form*

$$\|x(t_0)\| < \delta_1' = \delta_1/\|S^{-1}\|$$

and

$$\|g(x, t)\| < \delta_2' = \delta_2/\|S^{-1}\| \quad \text{in } \{(x, t) | \|x\| \leqq \varepsilon', t \geqq t_0\}$$

imply

$$\|x(t)\| < \varepsilon' = \varepsilon\|S\| \quad \text{for all } t \geqq t_0.$$

Proof. The proof is trivial, as x in the basis $\{\phi_i\}_{i=1}^n$ is now replaced by $S^{-1}x$ with x in the basis $\{e_i\}_{i=1}^n$.

The corollary together with (10) and (11) indicate that in order to assure $\|x(t)\| < \varepsilon'$ for all $t \geqq t_0$ for some given $\varepsilon' > 0$, it is sufficient to have $\|x(t_0)\| < \delta_1'$ and $\|g(x, t)\| < \delta_2'$ in $\{(x, t) | \|x\| \leqq \varepsilon', t \geqq t_0\}$, where all vectors are written in the original basis, $\{e_i\}_{i=1}^n$, and δ_1' and δ_2' satisfy the inequalities

$$\begin{aligned} \delta_1' &\leqq \frac{\varepsilon'}{\|S\|\,\|S^{-1}\|}\left[\frac{\min_i |\operatorname{Re} \lambda_i|}{\max_i |\operatorname{Re} \lambda_i|}\right]^{1/2}, \\ \delta_2' &\leqq \frac{\varepsilon'(1-\alpha)}{2\|B\|\,\|S\|\,\|S^{-1}\|}\left[\frac{\min_i |\operatorname{Re} \lambda_i|}{\max_i |\operatorname{Re} \lambda_i|}\right]. \end{aligned} \tag{14}$$

In (7) a positive number δ was defined such that whenever $\varepsilon \leqq \delta$, the theorem is established. There is no reason for requiring δ to be greater than ε. Thus (7) becomes

$$2\|B\|\frac{\|h(x,t)\|}{\|x\|} \leqq \alpha \quad \text{whenever } \|x\| \leqq \varepsilon. \tag{15}$$

It is seen from (14) that the number δ_2' may be made large by making α small. Hence, the optimal α is the smallest value allowed by (15).

4. Concluding remarks. The foregoing result on the total stability of a nonlinear differential equation may be applied to many physical systems. While there are theorems in the literature which establish that such a system is totally stable [3], [6], [7], those works cannot be directly used. This is because these theorems neither include expressions for bounds of δ_1' and δ_2' in terms of parameters of the physical system, nor indicate how to construct a Lyapunov function so as to obtain such expressions. In this work, however, we have converted these theorems to a form which can be used in applications and have obtained explicit expressions for bounds of δ_1' and δ_2'.

As stated earlier, the physical system must be incorporated with a damping mechanism such that the eigenvalues of the constant matrix A in (3) have negative real parts. It is not difficult to devise such a damping scheme in practice. For such a given physical system, of which (1) or (3) is the mathematical model, the eigenvalues λ_i of the matrix A, the matrices B and S, and the number α are known functions of the physical parameters. If a bound, ε', for the deviation of the motion from the equilibrium is chosen, then one can compute from (14) the upper bounds for the norms of the initial condition and the constantly acting disturbing force in order for $\|x\|$ not to exceed ε' at any time. On the other hand, if the magnitude of the perturbing force or the estimate of the error term is known, then (14) will allow one either to determine a bound of the motion for a given physical system or to design the physical system for a desired bound of the motion from the equilibrium.

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ON THE LAPLACE TRANSFORM FOR DISTRIBUTIONS*

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Abstract. A new characterization of the Laplace transform for Schwartz distributions is developed, using sequences of linear transformations on the space of distributions. The standard theorems on analyticity, uniqueness and invertibility of the transform are proved, using the new characterization as the definition of the Laplace transform. It is shown that this sequential definition is equivalent to Schwartz's extension of the ordinary Laplace transform to distributions which he obtained from the Fourier transform.

1. Introduction. The Laplace transform has been an important tool of applied mathematicians and engineers for many years. The properties of the ordinary Laplace transform have been well known at least since Widder published his book, *The Laplace Transform* [12], in 1941. L. Schwartz [11] extended the Laplace transform to distributions in 1952, and there have been many other extensions since then; e.g., Zemanian [13], [14], [15] and Ishihara [9]. In this paper we give another characterization of the Laplace transform for distributions and use it to prove the standard theorems on analyticity, uniqueness and invertibility of the transform.

The work which led to this study was motivated by a paper of E. Gesztelyi on linear operator transformations [7]. Two classes of transformations he considers are the dilatations U_n and exponential shifts T^{-p} which are defined for ordinary functions f, complex numbers p and positive integers n by

$$U_n f(t) = nf(nt),$$
$$T^{-p} f(t) = e^{-pt} f(t).$$

Gesztelyi shows that whenever the sequence $U_n f$ converges (in the sense of Mikusiński convergence [10]), the limit is necessarily a complex number. In addition, he proves that if f is a function which has a Laplace transform at p, then the sequence of functions $\{U_n T^{-p} f(t)\}$ converges (in the Mikusiński sense) as $n \to \infty$ to the classical Laplace transform of f at p. He then defines the Laplace transform of a Mikusiński operator x as the limit (whenever it exists in the sense of Mikusiński convergence) of the sequence $\{U_n T^{-p} x\}$, and shows that this definition generalizes the previous formulations of the Laplace transform for Mikusiński operators of G. Doetsch [4] and V. A. Ditkin [2], [3]. Since the dilatations U_n and shifts T^{-p} may be defined on the space of Schwartz distributions, we were led to consider whether there might be results analogous to Gesztelyi's in this different setting.

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Denote by $\mathscr{D}(R^n)$ the space of all infinitely differentiable complex-valued functions of the n-dimensional real variable $t = (t_1, t_2, \cdots, t_n)$ with compact support. If $j = j_1, j_2, \cdots, j_n$ is a multi-index, then by $\phi^{(j)}(t)$ or $\partial^j\phi(t)$ we mean

$$\frac{\partial^{|j|}\phi(t_1, \cdots, t_n)}{\partial t_1^{j_1}\, \partial t_2^{j_2} \cdots \partial t_n^{j_n}},$$

where $|j| = j_1 + j_2 + \cdots + j_n$. If $\{\phi_k\}$ is a sequence in $\mathscr{D}(R^n)$, then we say that $\{\phi_k\}$ converges to zero in $\mathscr{D}(R^n)$ as $k \to \infty$ if there is a fixed compact set K containing the support of every ϕ_k; and for every multi-index j, $\{\phi_k^{(j)}\}$ converges to zero uniformly on K as $k \to \infty$.

Denote by $\mathscr{D}'(R^n)$ the space of all linear transformations f from $\mathscr{D}(R^n)$ to the complex field which are continuous in the sense that if $\{\phi_k\}$ converges to zero in $\mathscr{D}(R^n)$, then the sequence of complex numbers $\{\langle f, \phi_k\rangle\}$ converges to zero as $k \to \infty$. Although there are several different ways to assign topologies to $\mathscr{D}(R^n)$ and arrive at the set $\mathscr{D}'(R^n)$ of continuous linear functionals on $\mathscr{D}(R^n)$, we shall not define a topology for $\mathscr{D}(R^n)$ explicitly, since the notion of sequential continuity is sufficient for our needs. The elements of $\mathscr{D}'(R^n)$ are the distributions defined by L. Schwartz in [11]. In the sequel, when the dimension of the space R^n is understood, we shall write $\mathscr{D}$ and $\mathscr{D}'$ for $\mathscr{D}(R^n)$ and $\mathscr{D}'(R^n)$. Following Schwartz [11] and Zemanian [13], we associate a locally integrable function $f(t)$ with the distribution f which assigns to each test function ϕ in $\mathscr{D}$ the value

$$\langle f, \phi\rangle = \int_{R^n} f(t)\phi(t)\, dt.$$

With this convention we have the result that for any infinitely differentiable function $\psi(t)$,

$$\langle \psi(t)f(t), \phi(t)\rangle = \langle f(t), \psi(t)\phi(t)\rangle.$$

Let $\mathscr{S}(R^n)$ (or $\mathscr{S}$) denote the space of infinitely differentiable complex-valued functions of $t = (t_1, t_2, \cdots, t_n)$ which approach zero faster than any power of $1/|t|$ as $|t| \to \infty$. Give $\mathscr{S}$ the locally convex topology defined by the family $\{q_{k,j}\}$ of seminorms, where

$$q_{k,j}(\phi) = \max\{|(1 + t^2)^k \partial^j\phi(t)| : t \in R^n\}$$

for every positive integer k and multi-index j. The space $\mathscr{S}'$ of weakly continuous linear functionals on $\mathscr{S}$ consists of the tempered distributions, or distributions of slow growth.

Let $\mathscr{E}(R^n)$ (or $\mathscr{E}$) denote the space of all infinitely differentiable complex-valued functions on R^n. For each compact set K and each multi-index j, define the seminorm $q_{K,j}$ by

$$q_{K,j}(\phi) = \max\{|\partial^j\phi(t)| : t \in K\}.$$

Equip $\mathscr{E}$ with the locally convex topology defined by the family $\{q_{K,j}\}$ of seminorms, and let $\mathscr{E}'(R^n)$ (or $\mathscr{E}'$) denote the space of weakly continuous linear functionals on $\mathscr{E}$. Then $\mathscr{E}'$ is the space of distributions of compact support. Standard results in the theory of distributions tell us that $\mathscr{D} \subset \mathscr{S} \subset \mathscr{E}$, that $\mathscr{D}$ is dense in both $\mathscr{S}$ and $\mathscr{E}$ with their respective topologies, and that $\mathscr{E}' \subset \mathscr{S}' \subset \mathscr{D}'$.

In § 2 we introduce the space $\mathscr{B}$ of bounded infinitely differentiable functions and its subset $\mathscr{B}_0$ consisting of those functions in $\mathscr{B}$ which converge to zero along with each derivative as $|t| \to \infty$. We characterize distributions in $\mathscr{B}'_0$, sometimes called integrable distributions, as those which satisfy certain a priori bounds when applied to test functions in $\mathscr{D}$, then show that each distribution in $\mathscr{B}'_0$ may be extended to all of $\mathscr{B}$. This allows us to prove that if f is in $\mathscr{B}'_0$, then $\{U_j f\}$ converges in $\mathscr{D}'$ as $j \to \infty$ to $\langle f, 1 \rangle \delta$. Using this result, we show that the Laplace transform of a distribution f can be defined by

$$\mathscr{L}[f](p) = \frac{1}{\phi(0)} \lim_{j \to \infty} \langle U_j T^{-p} f, \phi \rangle, \tag{1.1}$$

where ϕ is a testing function in $\mathscr{D}$ such that $\phi(0) \neq 0$. Theorem 2.3 tells us that if $T^{-p_1} f$ and $T^{-p_2} f$ are both in $\mathscr{S}'$, then we may use definition (1.1) for all complex numbers p with $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$. Thus we have all the machinery necessary to show that definition (1.1) is at least as general as Schwartz's definition of the Laplace transform for distributions, and that the two are equal whenever f has a Laplace transform in the Schwartz sense. This is done in § 4, after we have defined the transform by (1.1) and used the new definition to prove the standard properties of analyticity, invertibility and uniqueness of the Laplace transform.

In trying to determine the generality of the new definition of the Laplace transform, we characterize (in Theorem 3.2) those distributions h that are limits of sequences of the form $\{U_j f\}$ as linear combinations of $\delta(t)$ and p.v. $(1/t)$. This characterization gives an example (p.v. $(1/t)$) of a distribution f which is not in $\mathscr{B}'_0$, but for which the sequence $\{U_j f\}$ converges in $\mathscr{D}'$.

The question of generality is finally answered by Theorem 3.4 which says that if $\{U_j f\}$ converges, then f is in $\mathscr{S}'$. Thus, our definition of $\mathscr{L}[f](p)$ is valid only if $T^{-p} f$ is in $\mathscr{S}'$, and (1.1) can be no more general than Schwartz's definition of the transform. Hence the two definitions must be equivalent.

In § 5 we extend the results of §§ 3 and 4 to distributions in $\mathscr{D}'(R^n)$. The extensions are, for the most part, straightforward; and we prove only those which require basically new methods in n dimensions. The Appendix contains the construction of a partition of unity used several places in the text and a lemma used in the proof of Theorem 2.2.

2. The space $\mathscr{B}'_0$. Denote by $\mathscr{B}(R^n)$ (or, where R^n is understood, by $\mathscr{B}$) the space of all complex-valued functions of an n-dimensional real variable

$$t = (t_1, t_2, \cdots, t_n)$$

which possess continuous and bounded partial derivatives of all orders. For each multi-index j, define the seminorm q_j on $\mathscr{B}$ by

$$q_j(\phi) = \sup \{|\partial^j \phi(t)| : t \in R^n\} \tag{2.1}$$

and equip $\mathscr{B}$ with the locally convex topology determined by the family of seminorms $\{q_j\}$. (For convenience, hereafter, $\sup f(t)$ or $\sup_t f(t)$ will denote $\sup \{f(t) : t \in R^n\}$.) A sequence $\{\phi_k\}$ converges in $\mathscr{B}$ to a function ϕ with respect to this topology if and only if each derived sequence $\{\partial^j \phi_k\}$ converges uniformly to $\partial^j \phi$.

It is easy to see that $\mathscr{D} \subset \mathscr{B} \subset \mathscr{E}$. The subspace $\mathscr{D}$ is not dense in $\mathscr{B}$, however, for the constant function $1(t)$ is in $\mathscr{B}$ but cannot be uniformly approximated by functions in $\mathscr{D}$, since for any ϕ in $\mathscr{D}$,

$$q_0[\phi(t) - 1(t)] = \sup |\phi(t) - 1(t)| \geqq 1.$$

For this reason the dual space $\mathscr{B}'$ of $\mathscr{B}$ cannot be identified with a subspace of the space $\mathscr{D}'$ of distributions. In fact, Zemanian [14] demonstrates this by giving an example of a nonzero generalized function in $\mathscr{B}'$ whose restriction to $\mathscr{D}$ is the zero distribution.

Since we wish to work within the class $\mathscr{D}'$ of distributions, we will consider a subspace $\mathscr{B}_0$ of $\mathscr{B}$ consisting of those functions in $\mathscr{B}$ each of whose derivatives approach zero as $|t| \to \infty$. To be more specific, we say that a function ϕ is in $\mathscr{B}_0$ if and only if ϕ is in $\mathscr{B}$ and for each multi-index j and each positive number ε, there is a compact set $K_{j\varepsilon}$ such that if t is not in $K_{j\varepsilon}$, then $|\partial^j\phi(t)| < \varepsilon$.

Give $\mathscr{B}_0$ the topology induced by $\mathscr{B}$, which makes $\mathscr{B}_0$ a locally convex topological vector space. To see that $\mathscr{D}$ is dense in $\mathscr{B}_0$, let $\{\gamma_k\}$ be a sequence of functions in $\mathscr{D}$ such that

$$\gamma_k(t) = \begin{cases} 1, & |t| \leqq k, \\ 0, & |t| > k + 1, \end{cases}$$

$$\sup |\gamma_k^{(j)}(t)| \leqq \sup |\gamma_1^{(j)}(t)| \quad \text{for every multi-index } j.$$

If ϕ is a function in $\mathscr{B}_0$, then $\{\gamma_k\phi\}$ is a sequence in $\mathscr{D}$ that converges in $\mathscr{B}_0$ to ϕ, which shows that $\mathscr{D}$ is dense in $\mathscr{B}_0$. Therefore, the dual space $\mathscr{B}'_0$ of $\mathscr{B}_0$ is a subspace of $\mathscr{D}'$ and a distribution f in $\mathscr{B}'_0$ is completely determined by its values on $\mathscr{D}$. The following theorem is a useful characterization of distributions in $\mathscr{B}'_0$.

THEOREM 2.1. *A distribution f in $\mathscr{D}'$ is also in $\mathscr{B}'_0$ if and only if there is a number K such that*

$$|\langle f, \phi\rangle| \leqq K \max_{|j| \leqq K} \sup_t |\phi^{(j)}| \quad \textit{for every } \phi \textit{ in } \mathscr{D}. \tag{2.2}$$

Proof. To prove that condition (2.2) implies f belongs to $\mathscr{B}'_0$, let $\{\phi_k\}$ be a sequence in $\mathscr{D}$ that converges to zero in the topology of $\mathscr{B}_0$. Then $\sup |\phi_k^{(j)}| \to 0$ as $k \to \infty$ for every j and we have

$$\lim_{k\to\infty} |\langle f, \phi_k\rangle| \leqq \lim_{k\to\infty} K \max_{|j| \leqq K} \sup_t |\phi_k^{(j)}| = 0.$$

To show that f can be defined on all of $\mathscr{B}_0$, let ϕ be a function in $\mathscr{B}_0$ and $\{\phi_k\}$ a sequence in $\mathscr{D}$ that converges in $\mathscr{B}_0$ to ϕ. Then the set $\{K \max_{|j| \leqq K} \sup_t |\phi_k^{(j)}|\}$ is bounded above and so $\{|\langle f, \phi_k\rangle|\}$ is also bounded above. Since $\{\phi_k - \phi_l\}$ is a sequence in $\mathscr{D}$ that converges to zero in $\mathscr{B}_0$ as k and l tend to infinity independently,

$$\lim_{k,l\to\infty} \langle f, \phi_k - \phi_l\rangle = 0$$

and $\{\langle f, \phi_k\rangle\}$ is a Cauchy sequence with a finite limit.

Define $\langle f, \phi\rangle = \lim_{k\to\infty} \langle f, \phi_k\rangle$. If $\{\psi_k\}$ is another sequence in $\mathscr{D}$ that converges in $\mathscr{B}_0$ to ϕ, then $\{\phi_k - \psi_k\}$ converges in $\mathscr{B}_0$ to zero, so $\langle f, \phi\rangle$ is well-defined. Since $\mathscr{D}$ is dense in $\mathscr{B}_0$, f is extended to all of $\mathscr{B}_0$ and the extension is clearly linear. To see that f is continuous on $\mathscr{B}_0$, notice that

$$|\langle f, \phi\rangle| \leqq K \max_{|j| \leqq K} \sup |\phi^{(j)}|$$

holds even for $\phi \in \mathscr{B}_0$. Thus f is in $\mathscr{B}'_0$.

The proof that condition (2.2) holds if f is in $\mathscr{B}'_0$ proceeds by contradiction. Suppose that $\{\phi_k\}$ is a sequence in $\mathscr{B}_0$ such that for each k,

$$\begin{aligned} |\langle f, \phi_k\rangle| &> k \max_{|j| \leqq k} \sup_t |\phi_k^{(j)}(t)| \\ &= k \max_{|j| \leqq k} q_j(\phi_k), \end{aligned}$$

and define $\theta_k = \phi_k/[k \max_{|j| \leqq k} q_j(\phi_k)]$. Then θ_k is in $\mathscr{B}_0$ for every k and

$$\begin{aligned} q_m(\theta_k) &= \frac{q_m(\phi_m)}{k \max_{|j| \leqq k} q_j(\phi_k)} \\ &\leqq 1/k \quad \text{if } k \geqq m, \end{aligned}$$

so $\theta_k \to 0$ in $\mathscr{B}_0$ as $k \to \infty$. Since f is in $\mathscr{B}'_0$, this means that $\langle f, \theta_k\rangle \to 0$. However, by the definition of θ_k,

$$|\langle f, \theta_k\rangle| = \frac{|\langle f, \phi_k\rangle|}{k \max_{|j| \leqq k} q_j(\phi_k)} > 1.$$

This contradicts the fact that $\langle f, \theta_k\rangle \to 0$, and so there can be no such sequence $\{\phi_k\}$. Therefore, if f is in $\mathscr{B}'_0$, condition (2.2) holds, and the proof is complete.

Since $\mathscr{B}_0 \subset \mathscr{B}$ and the topology of $\mathscr{B}_0$ is that induced by $\mathscr{B}$, each element of $\mathscr{B}'$ has a restriction to $\mathscr{B}_0$ that is in $\mathscr{B}'_0$. The next theorem shows that a converse is also true, i.e., that each element of $\mathscr{B}'_0$ can be extended to all of $\mathscr{B}$.

THEOREM 2.2. *Each distribution f in $\mathscr{B}'_0$ has a unique extension $\hat{f}$ in $\mathscr{B}'$ with the property that $\langle \hat{f}, \phi_k\rangle$ converges to $\langle \hat{f}, \phi\rangle$ whenever $\{\phi_k\}$ is a uniformly bounded sequence in $\mathscr{B}$ that converges to ϕ with respect to the topology induced on $\mathscr{B}$ by $\mathscr{E}$.*

Proof. If f is in $\mathscr{B}'_0$, then by Theorem 2.1 there is a number K such that for every ψ in $\mathscr{B}_0$,

$$|\langle f, \psi\rangle| \leqq K \max_{|j| \leqq K} \sup_t |\partial^j \psi(t)|.$$

Let ϕ be in $\mathscr{B}$ and suppose I is a finite set of nonnegative integers. Let $\{\gamma_i\}_{i=1}^{\infty}$ be the partition of unity defined in the Appendix. Then

$$\begin{aligned} \left|\left\langle f, \sum_{i \in I} \gamma_i \phi\right\rangle\right| &\leqq K \max_{|j| \leqq K} \sup_t \left| \partial^j \left(\sum_{i \in I} \gamma_i \phi\right)(t)\right| \\ &\leqq K \max_{|j| \leqq K} \sup_t \left| \sum_{k \leqq j} \binom{j}{k} \left(\sum_{i \in I} \gamma_i\right)^{(k)} \phi^{(j-k)}(t)\right| \\ &\leqq K K^n (K!)^n \max_{|k| \leqq K} \sup_t |\gamma_0^{(k)}(t)| \max_{|j| \leqq K} \sup_t |\phi^{(j)}(t)| \\ &= B. \end{aligned}$$

Now B is independent of the choice of the set I, so by the lemma proved in the

Appendix we know that for any finite set I of nonnegative integers,

$$\sum_{i\in I} |\langle f, \gamma_i\phi\rangle| \leqq 4B.$$

Therefore

$$\sum_{i=0}^{\infty} |\langle f, \gamma_i\phi\rangle| \leqq 4B$$

and the series $\sum_{i=0}^{\infty} \langle f, \gamma_i\phi\rangle$ converges absolutely.

Define an extension $\hat{f}$ of f by

$$\langle \hat{f}, \phi\rangle = \sum_{i=0}^{\infty} \langle f, \gamma_i\phi\rangle$$

for every ϕ in $\mathscr{B}$. To see that $\hat{f} = f$ on $\mathscr{B}_0$, notice that if $\psi \in \mathscr{B}_0$, then $\sum_{i=0}^{\infty} \langle f, \gamma_i\psi\rangle$ is also absolutely convergent and the sequence $\{\sum_{i=0}^{k} \gamma_i\psi\}$ converges to ψ in $\mathscr{B}_0$ as $k \to \infty$. Thus

$$\langle \hat{f}, \psi\rangle = \sum_{i=0}^{\infty} \langle f, \gamma_i\psi\rangle = \left\langle f, \sum_{i=0}^{\infty} \gamma_i\psi\right\rangle = \langle f, \psi\rangle.$$

It remains to be shown that $\hat{f}$ is continuous on $\mathscr{B}$. This will follow from the second part of the proof which shows that $\hat{f}$ is continuous even with respect to a weaker topology than the one given to $\mathscr{B}$. To this end, let $\{\phi_k\}$ be a uniformly bounded sequence in $\mathscr{B}$ that converges to zero in the topology of $\mathscr{E}$, and let

$$C_K = \max_{|j|\leqq K} \sup_k \sup_t |\partial^j\phi_k(t)|,$$

where K is the constant defined for f by Theorem 2.1. Let I be a finite set of nonnegative integers and for each $i \in I$, let k_i be a positive integer. Then

$$\begin{aligned}\left|\left\langle f, \sum_{i\in I} \gamma_i\phi_{k_i}\right\rangle\right| &\leqq K \max_{|j|\leqq K} \sup_t \partial^j\left(\sum_{i\in I} \gamma_i\phi_{k_i}\right)\\ &\leqq KK^n(K!)^n C_K \max_{|j|\leqq K} \sup_t |\partial^j\gamma_0(t)|\\ &= B'.\end{aligned}$$

Therefore, for every finite subset I of nonnegative integers and every choice of the collection $\{k_i\}$ of positive integers,

$$\sum_{i\in I} |\langle f, \gamma_i\phi_{k_i}\rangle| \leqq 4B'. \tag{2.3}$$

Now we already know that for each k, $\sum_{i=0}^{\infty} \langle f, \gamma_i\phi_k\rangle$ converges absolutely. We will show that this convergence is uniform with respect to k. Let ε be a positive number. Then for each k the absolute convergence of $\sum_{i=0}^{\infty} \langle f, \gamma_i\phi_k\rangle$ guarantees the existence of a smallest positive integer N_k such that

$$\sum_{i=N_k+1}^{\infty} |\langle f, \gamma_i\phi_k\rangle| < \frac{\varepsilon}{2}. \tag{2.4}$$

Suppose that the set $\{N_k\}$ cannot be bounded above. (Assume $N_k > 1$ for every k, choosing, if necessary, a subsequence of $\{\phi_k\}$ for which this is true.)

Since N_k is the smallest positive integer that satisfies (2.4), there must also be positive integers $\{M_k\}$ such that for each k,

$$\sum_{i=N_k}^{M_k} |\langle f, \gamma_i \phi_k \rangle| \geqq \frac{\varepsilon}{2}. \tag{2.5}$$

Pick a sequence $\{\nu_k\}$ in the following way. Let $\nu_1 = N_1$. Since $M_1 < \infty$, we can pick ν_2 such that $N_{\nu_2} > M_1$. Similarly, for each k, pick ν_k such that $N_{\nu_k} > M_{\nu_{k-1}}$. Then if M is a positive integer which is larger than $8B'/\varepsilon$, we have by (2.5)

$$\sum_{k=1}^{M} \sum_{i=N_{\nu_k}}^{M_{\nu_k}} |\langle f, \gamma_i \phi_{\nu_k} \rangle| \geqq M \frac{\varepsilon}{2} > 4B'.$$

But this is a sum of the form

$$\sum_{i \in I} |\langle f, \gamma_i \phi_{k_i} \rangle|$$

given in (2.3), where the finite set

$$I = \bigcup_{1 \leqq k \leqq M} \bigcup_{N_{\nu_k} \leqq i \leqq M_{\nu_k}} \{i\},$$

and $k_i = \nu_k$ for $N_{\nu_k} \leqq i \leqq M_{\nu_k}$. The assumption that the set $\{N_k\}$ is unbounded has led us to a contradiction; so we may assume there is a positive integer N such that $N_k \leqq N$ for every k, and we have

$$\sum_{i=N+1}^{\infty} |\langle f, \gamma_i \phi_k \rangle| < \frac{\varepsilon}{2} \quad \text{for every } k. \tag{2.6}$$

Now $\{\phi_k\}$ converges to zero in the topology of $\mathscr{E}$, and the derivatives of γ_i are uniformly bounded for all i; so there must be a positive integer N' such that if $k > N'$,

$$K \max_{|j| \leqq K} \sup_t \left| \partial^j \left(\sum_{i=0}^{N} \gamma_i \phi_k \right) \right| < \frac{\varepsilon}{2}. \tag{2.7}$$

Then, by (2.6) and (2.7), if $k \geqq N'$ we have

$$\begin{aligned} |\langle \hat{f}, \phi_k \rangle| &= \left| \sum_{i=0}^{\infty} \langle f, \gamma_i \phi_k \rangle \right| \\ &\leqq \left| \left\langle f, \sum_{i=0}^{N} \gamma_i \phi_k \right\rangle \right| + \sum_{i=N+1}^{\infty} |\langle f, \gamma_i \phi_k \rangle| \\ &\leqq K \max_{|j| \leqq K} \sup_t \left| \partial^j \left(\sum_{i=0}^{N} \gamma_i \phi_k \right) \right| + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore $\lim_{k \to \infty} \langle \hat{f}, \phi_k \rangle = 0$. This proves that $\hat{f}$ is sequentially continuous. Since $\mathscr{B}$ is Hausdorff and the topology of $\mathscr{B}$ is defined by a countable family of seminorms, $\mathscr{B}$ is metrizable. Thus sequential continuity of $\hat{f}$ on $\mathscr{B}$ guarantees that $\hat{f}$ is in $\mathscr{B}'$.

The remarks at the beginning of this section show that there may be more than one way to extend a distribution f in $\mathscr{B}'_0$ to all of $\mathscr{B}$. However, $\mathscr{D}$ is dense in $\mathscr{E}$ and, therefore, $\mathscr{D}$ is dense in $\mathscr{B}$ with the topology induced by $\mathscr{E}$. Moreover, if ϕ is in $\mathscr{B}$, there is a uniformly bounded sequence $\{\phi_k\}$ in $\mathscr{D}$ that converges to ϕ in this topology. Thus any two extensions of f which satisfy the property of the theorem must also be equal on $\mathscr{B}$. That is, there can be only one such extension, and $\hat{f}$ is unique. The proof of Theorem 2.2 is now complete.

In the sequel, whenever we apply a distribution f in $\mathscr{B}'_0$ to a test function ϕ in $\mathscr{B}$, we understand this to mean $\langle \hat{f}, \phi\rangle$, where $\hat{f}$ is the particular (unique) extension of f defined in Theorem 2.2. In particular, the constant functions are in $\mathscr{B}$, so if f is in $\mathscr{B}'_0$, $\langle f, c\rangle = c\langle f, 1\rangle$ is then defined. If f happens to be a regular distribution in $\mathscr{B}'_0$ determined by an integrable function $f(t)$, then

$$\langle f, 1\rangle = \int_{R^n} f(t)\, dt.$$

For this reason, distributions in $\mathscr{B}'_0$ are frequently called integrable distributions.

There are two more results concerning $\mathscr{B}'_0$ which will be needed in later sections. Recall that if a and b are in R^n, $a < b$ means $a_i < b_i$, $i = 1, 2, \cdots, m$, and e^{at} is the function $\exp\left[\sum_{i=1}^n a_i t_i\right]$. Let C^n denote the linear space of n-tuples of complex numbers.

THEOREM 2.3. *If f is in $\mathscr{D}'(R^n)$ and a, b are in R^n with $a < b$ such that $e^{-at}f(t)$ and $e^{-bt}f(t)$ are both in $\mathscr{S}'$, then for every p in C^n with $a < \operatorname{Re} p < b$, $e^{-pt}f(t)$ is in $\mathscr{B}'_0$.*

Proof. Let p be in C^n with $a < \operatorname{Re} p < b$, and let ε be in R^n with $\varepsilon > 0$ such that

$$\varepsilon_i < \min\{\operatorname{Re} p_i - a_i, b_i - \operatorname{Re} p_i : i = 1, 2, \cdots, n\}.$$

If $\lambda(t) = e^{\varepsilon t} + e^{-\varepsilon t}$, then $\lambda(t)\, e^{-pt}f(t)$ is in $\mathscr{S}'$ and $1/\lambda(t)$ is in $\mathscr{S}$. Also, for every $\phi \in \mathscr{B}_0$, $(\phi/\lambda)(t)$ is in $\mathscr{S}$, so we may write

$$\langle e^{-pt}f(t), \phi(t)\rangle = \langle \lambda(t)\, e^{-pt}f(t), (\phi/\lambda)(t)\rangle.$$

This expression clearly identifies $e^{-pt}f(t)$ as a continuous linear transformation on $\mathscr{B}_0$ as long as $a < \operatorname{Re} p < b$, so the theorem is proved.

THEOREM 2.4. *If f and g are in $\mathscr{B}'_0(R^n)$, then their convolution can be defined and is also in $\mathscr{B}'_0(R^n)$.*

Proof. Recall that if f is in $\mathscr{D}'$, then $\check{f}$ is the distribution defined for every $\phi \in \mathscr{D}$ by

$$\langle \check{f}, \phi\rangle = \langle f(t), \phi(-t)\rangle.$$

Using the tensor product $\otimes$ to formally define $f * g$, we have

$$\begin{aligned}\langle f * g, \phi\rangle &= \langle f(t) \otimes g(\tau), \phi(t + \tau)\rangle \\ &= \langle f(t), \langle g(\tau), \phi(t + \tau)\rangle\rangle \\ &= \langle f(t), \langle \check{g}(\tau), \phi(t - \tau)\rangle\rangle \\ &= \langle f(t), (\check{g} * \phi)(t)\rangle.\end{aligned}$$

This string of inequalities will be justified and the convolution $f * g$ will be defined as a distribution in $\mathscr{B}'_0$ if we can show that $\check{g} * \phi$ is in $\mathscr{B}$ when ϕ is in $\mathscr{B}_0$ and g is in $\mathscr{B}'_0$, and that $\check{g} * \phi_k$ converges to zero in $\mathscr{B}$ whenever ϕ_k converges to zero in $\mathscr{B}_0$. To do this, consider

$$\begin{aligned}
\sup_t |\partial^j(\check{g} * \phi)(t)| &= \sup_t |\partial^j\langle \check{g}(\tau), \phi(t - \tau)\rangle| \\
&= \sup_t |\partial^j\langle g(\tau), \phi(t + \tau)\rangle| \\
&= \sup_t |\langle g(\tau), \phi^{(j)}(t + \tau)\rangle| \\
&\leqq \sup_t K \max_{|i| \leqq K} \sup_\tau |\partial^i \phi^{(j)}(t + \tau)| \\
&= K \max_{|r| \leqq K} \sup_t |\phi^{(i+j)}(t)| \\
&= B_{K,j},
\end{aligned}$$

where K is the constant defined for g by Theorem 2.1.

Therefore $\check{g} * \phi$ is in $\mathscr{B}_0$ and if ϕ_k converges to zero in $\mathscr{B}_0$, then $\sup_t |\phi_k^{(i+j)}(t)|$ converges to zero, and $\check{g} * \phi_k$ must converge to zero in $\mathscr{B}$. Thus $\langle f * g, \phi\rangle = \langle f, \check{g} * \phi\rangle$ defines $f * g$ as a distribution in $\mathscr{B}'_0$, and the theorem is proved.

In the sequel we shall frequently use the fact that $\mathscr{B}'_0$ is a subset of $\mathscr{S}'$. This is easily seen to be true, since $\mathscr{S} \subset \mathscr{B}_0$ and $\mathscr{S}$ is dense in $\mathscr{B}_0$ with respect to the topology of $\mathscr{B}_0$. Another way of verifying that $\mathscr{B}'_0 \subset \mathscr{S}'$ is to compare Theorem 2.1 with the corresponding result for $\mathscr{S}'$ (Zemanian [13, p. 111]).

3. The transformations U_j and T^{-p}. In this section we define and give some results concerning two linear transformations on the space $\mathscr{D}'(R^n)$. Although the definitions of the transformations will be stated for n-dimensional distributions, the theorems proved in this section will concern only distributions in $\mathscr{D}'(R)$. The generalizations to $\mathscr{D}'(R^n)$ of these results will be postponed until § 5.

If $a = (a_1, \cdots, a_n) > 0$ in R^n, define the linear transformation U_a on $\mathscr{D}'(R^n)$ by

$$\begin{aligned}
\langle U_a f(t), \phi(t)\rangle &= \langle a_1 a_2 \cdots a_n f(a_1 t_1, \cdots, a_n t_n), \phi(t_1, \cdots, t_n)\rangle \\
&= \left\langle f(t_1, \cdots, t_n), \phi\left(\frac{t_1}{a_1}, \frac{t_2}{a_2}, \cdots, \frac{t_n}{a_n}\right)\right\rangle
\end{aligned} \tag{3.1}$$

for every distribution f and every test function ϕ. By $a > 0$ where $a \in R^n$, we mean $\{a_i > 0 : 1 \leqq i \leqq n\}$ so $U_a f$ is well-defined as a distribution. Also, U_a is continuous and linear on $\mathscr{D}(R^n)$, as can be verified.

Another useful transformation on $\mathscr{D}'(R^n)$ is defined in the following way: For each $p = p_1, \cdots, p_n$, in C^n let T^{-p} be defined by

$$\begin{aligned}
\langle T^{-p} f(t), \phi(t)\rangle &= \langle e^{-pt} f(t), \phi(t)\rangle \\
&= \left\langle \exp\left(-\sum_i p_i t_i\right) f(t_1, \cdots, t_n), \phi(t_1, \cdots, t_n)\right\rangle
\end{aligned} \tag{3.2}$$

for each distribution f and test function ϕ. The transformation T^{-p} is clearly continuous and linear on $\mathscr{D}'(R^n)$.

The rest of this section will be devoted to proving results about these two transformations applied to $\mathscr{D}'(R)$, the one-dimensional distributions. We are primarily concerned with the convergence of the sequences of distributions $\{U_j f\}$ or $\{U_j T^{-p} f\}$ as $j \to \infty$. The first theorem is a direct corollary to Theorem 2.2.

THEOREM 3.1. *If f is in $\mathscr{B}'_0$, then*

$$\lim_{j \to \infty} U_j f = \langle f, 1 \rangle \delta,$$

where the limit is taken in $\mathscr{D}'$.

Proof. Let ϕ be in $\mathscr{D}$ and for each positive integer j let $\phi_j(t) = \phi(t/j)$. Then ϕ_j is also in $\mathscr{D}$ for each j and the sequence $\{\phi_j\}$ converges uniformly on compact sets as $j \to \infty$ to the function $\phi(0)1$. Also, if $k \geqq 1$, the sequence $\{\phi_j^{(k)}(t)\} = \{(1/j)^k \phi^{(k)}(t/j)\}$ converges uniformly on compact sets to zero as $j \to \infty$. Therefore, the uniformly bounded sequence $\{\phi_j\}$ in $\mathscr{B}$ converges with respect to the topology induced on $\mathscr{B}$ by $\mathscr{E}$, and by Theorem 2.2,

$$\begin{aligned} \lim_{j \to \infty} \langle U_j f, \phi \rangle &= \lim_{j \to \infty} \langle f, \phi_j \rangle \\ &= \langle f, \phi(0)1 \rangle \\ &= \langle \langle f, 1 \rangle \delta, \phi \rangle. \end{aligned}$$

Thus we have shown that

$$\lim_{j \to \infty} U_j f = \langle f, 1 \rangle \delta,$$

and the proof is complete.

An obvious question to ask is: Does the sequence $\{U_j f\}$ ever converge if f is not in $\mathscr{B}'_0$? The answer is given by demonstrating a distribution f which is not in $\mathscr{B}'_0$ but for which the sequence $\{U_j f\}$ does converge. This is done in the following examples.

Example 3.1. Let $f(t) = -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu)$. If ϕ is in $\mathscr{D}$, then

$$\langle f, \phi \rangle = \sum_{\nu=1}^{\infty} \phi^{(1)}(\nu);$$

and the sum is actually finite since ϕ has compact support. In fact, if the support of ϕ is contained in $\{t : t \leqq K\}$, we have

$$\begin{aligned} \lim_{j \to \infty} \langle U_j f, \phi \rangle &= \lim_{j \to \infty} \left\langle -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu), \phi\left(\frac{t}{j}\right) \right\rangle \\ &= \lim_{j \to \infty} \sum_{\nu=1}^{jK} \frac{1}{j} \phi^{(1)}\left(\frac{\nu}{j}\right) \\ &= \int_0^K \phi^{(1)}(t)\, dt = \phi(K) - \phi(0) \\ &= -\phi(0) = -\langle \delta, \phi \rangle. \end{aligned}$$

Therefore, $\lim_{j \to \infty} U_j f = -\delta$.

To see that f is not in $\mathscr{B}'_0$, look at the function $\phi(t) = (\sin 2\pi t)/t$. If we define $\phi(0) = \lim_{t\to 0} \phi(t)$, then ϕ is in $\mathscr{B}_0$ since it is infinitely differentiable and each derivative approaches zero like $1/t$ as $|t| \to \infty$. However, $\langle f, \phi\rangle$ is not defined in this case since

$$\langle f, \phi\rangle = 2\pi \sum_{\nu=1}^{\infty} \frac{1}{\nu},$$

and this series does not converge. Thus $f(t) = \sum_{\nu=1}^{\infty} \delta'(t - \nu)$ is a distribution not in $\mathscr{B}'_0$ for which the sequence $\{U_j f\}$ converges.

Example 3.2. Recall that the one-dimensional distribution p.v. $1/t$ is defined by

$$\left\langle \text{p.v.}\,\frac{1}{t}, \phi(t)\right\rangle = \lim_{\varepsilon\to 0}\left[\int_{-\infty}^{-\varepsilon} \frac{\phi(t)}{t}\,dt + \int_{\varepsilon}^{\infty} \frac{\phi(t)}{t}\,dt\right],$$

where ε is always positive. This distribution is not in $\mathscr{B}'_0$ (since it obviously cannot be extended to all of $\mathscr{B}$) but it is invariant on $\mathscr{D}$ under all transformations of the type U_a, where a is a positive real number.

The next theorem characterizes all distributions which are limits in $\mathscr{D}'(R)$ of sequences $\{U_j f\}$ as $j \to \infty$.

THEOREM 3.2. *If f is a one-dimensional distribution and $\{U_j f\}$ converges in $\mathscr{D}'$ to a distribution h, then*

$$h(t) = c_1\, \text{p.v.}\,(1/t) + c_2\,\delta(t),$$

where c_1 and c_2 are constants.

Proof. Since $U_j f \to h$ in $\mathscr{D}'$ as $j \to \infty$, it is easy to see that $U_a h = h$ for every positive real number a. Therefore, if $a \neq 0$,

$$h(at) = (1/a)h(t), \tag{3.3}$$

and we may differentiate (3.3) with respect to a and evaluate the result at the point $a = 1$, to get $th^{(1)}(t) = -h(t)$. Therefore $(th(t))^{(1)} = 0$, and by a familiar result on the differentiation of distributions (Horváth [8, p. 327]) there is a constant c_1 such that

$$th(t) = c_1. \tag{3.4}$$

But for any constant c_1, the constant distribution $c_1(t)$ satisfies

$$c_1(t) = tc_1\, \text{p.v.}\,(1/t). \tag{3.5}$$

So, from (3.4) and (3.5) we get

$$t[h(t) - c_1\, \text{p.v.}\,(1/t)] = 0,$$

which implies (Horváth [8, p. 352]) that there is a constant c_2 such that

$$h(t) = c_1\, \text{p.v.}\,(1/t) + c_2\,\delta(t).$$

The proof is now complete.

The next theorem will show that in the cases of interest to us, c_1 must be zero.

THEOREM 3.3. *If there are two complex numbers p_1 and p_2 with* $\operatorname{Re} p_1 \neq \operatorname{Re} p_2$ *such that $\{U_j T^{-p_1} f\}$ and $\{U_j T^{-p_2} f\}$ both converge in $\mathscr{D}'_t$ as $j \to \infty$, then for every* *complex number p for which the sequence converges there is a constant $c(p)$ such that*

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\,\delta(t).$$

Proof. We may assume without loss of generality that $p_1 = 0$ and that $p_2 = p$ has real part greater than zero. Let ϕ be a function in $\mathscr{D}$ whose support is contained in $(0, \infty)$, and for every positive integer j let $\phi_j(t) = e^{-pjt}\phi(t)$. Clearly, the sequence $\{\phi_j(t)\}$ converges to zero in $\mathscr{D}$ as $j \to \infty$.

By Theorem 3.2 we know that $\{U_j T^{-p} f\}$ converges to

$$c_1(p)\,\text{p.v.}\,(1/t) + c_2(p)\,\delta(t);$$

and since ϕ does not have support at the origin, $\langle \delta, \phi \rangle = 0$. Therefore

$$\lim_{j\to\infty} \langle U_j T^{-p} f, \phi \rangle = \langle c_1(p)\,\text{p.v.}\,(1/t), \phi(t) \rangle.$$

But

$$\begin{aligned}\lim_{j\to\infty} \langle U_j T^{-p} f, \phi \rangle &= \lim_{j\to\infty} \langle U_j f(t), e^{-pjt}\phi(t) \rangle \\ &= \lim_{j\to\infty} \langle U_j f, \phi_j \rangle = 0\end{aligned}$$

since $\phi_j \to 0$ in $\mathscr{D}$, $\{U_j f\}$ converges in $\mathscr{D}'$, and convergence in $\mathscr{D}'$ is uniform on bounded subsets of $\mathscr{D}$. Furthermore, the support of p.v. $(1/t)$ is the whole real line, and the only way $\langle c_1(p)\,\text{p.v.}\,(1/t), \phi(t) \rangle$ can equal zero for every ϕ with support contained in $(0, \infty)$ is for $c_1(p)$ to be zero. Thus $\lim_{j\to\infty} U_j T^{-p} f = c_2(p)\,\delta(t)$.

Now, let $g(t) = T^{-p} f(t)$; let σ be a function in $\mathscr{D}$ with support contained in $(-\infty, 0)$; and for every positive integer j, let $\sigma_j(t) = e^{pjt}\sigma(t)$. Then $\{\sigma_j\}$ converges in $\mathscr{D}$ to zero and as before we get

$$\begin{aligned}\lim_{j\to\infty} \langle U_j f, \sigma \rangle &= \lim_{j\to\infty} \langle U_j T^{p} g, \sigma \rangle \\ &= \lim_{j\to\infty} \langle U_j g, \sigma_j \rangle \\ &= 0.\end{aligned}$$

But $\lim_{j\to\infty} U_j f = c_1(0)\,\text{p.v.}\,(1/t) + c_2(0)\,\delta(t)$ and $\langle \delta, \sigma \rangle = 0$, so $\langle c_1(0)\,\text{p.v.}\,(1/t), \sigma(t) \rangle = 0$. As before, this can happen for all σ in $\mathscr{D}$ with support contained in $(-\infty, 0)$ only if $c_1(0)$ is zero. Therefore, $\lim_{j\to\infty} U_j f = c_2(0)\delta$. Thus for every complex number p where the sequence converges,

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\delta.$$

Corollary 3.1. *If f is a distribution and there exist real numbers α and β such that $\{U_j T^{-p} f\}$ converges in $\mathscr{D}'$ as long as $\alpha < \operatorname{Re} p < \beta$, then for each such complex number p,*

$$\lim_{j\to\infty} U_j T^{-p} f = c(p)\delta.$$

Corollary 3.2. *If $\{U_j f\}$ converges in $\mathscr{D}'$ to $c_1\,\text{p.v.}\,(1/t) + c_2\,\delta(t)$ where $c_1 \neq 0$, then the sequence $\{U_j T^{-p} f\}$ cannot converge in $\mathscr{D}'$ as long as* $\operatorname{Re} p \neq 0$.

Our purpose is to use sequences of the form $\{U_jT^{-p}f\}$ to define the Laplace transform of f at p. Therefore, we would like to strengthen Corollary 3.1 by showing that if there are two complex numbers p_1, p_2 such that $\{U_jT^{-p_1}f\}$ and $\{U_jT^{-p_2}f\}$ both converge, then as long as $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$, $\{U_jT^{-p}f\}$ also converges. This will follow from the next theorem which shows that whenever $\{U_jf\}$ converges in $\mathscr{D}'$, then f is in $\mathscr{S}'$.

THEOREM 3.4. *If f is a distribution such that the sequence $\{U_jf\}$ converges in $\mathscr{D}'$ as $j \to \infty$, then $f \in \mathscr{S}'$.*

Proof. It is easy to show that if $\{f_k\}$ is a sequence that converges in $\mathscr{D}'$ and K is a compact set in R, then there is a constant C and a positive integer r such that for every test function ϕ with support contained in K,

$$|\langle f_k, \phi\rangle| \leqq C \max_{|j| \leqq r} \sup |\phi^{(j)}|$$

is satisfied. Therefore there are constants C and r such that if the support of ϕ is in the interval $[-1, 1]$, then for every j,

$$|\langle U_jf, \phi\rangle| \leqq C \max_{|i| \leqq r} \sup |\phi^{(i)}|.$$

If ϕ is in $\mathscr{D}$ with support contained in the interval $[-k, k]$, then the support of $\phi(kt)$ is in $[-1, 1]$, so

$$\begin{aligned} |\langle f, \phi\rangle| &= |\langle U_kf, \phi(kt)\rangle| \\ &\leqq C \max_{|i| \leqq r} \sup_t |[\phi(kt)]^{(i)}| \qquad (3.6)\\ &\leqq Ck^r \max_{|i| \leqq r} \sup |\phi^{(i)}|. \end{aligned}$$

Now, let $\{\gamma_k\}$ be the partition of unity defined for R in the Appendix and let θ be a function in $\mathscr{S}$. Then the function $\gamma_k\theta$ has support contained in the set $\{t : k - 1 \leqq |t| \leqq k + 1\}$; so by the properties of γ_k and inequality (3.6) we have

$$\begin{aligned} |\langle f, \gamma_k\theta\rangle| &\leqq C(k + 1)^r \max_{|i| \leqq r} \sup |[\gamma_k\theta]^{(i)}| \qquad (3.7)\\ &\leqq CL(k + 1)^r \max_{|i| \leqq r} \sup \{|\theta^{(i)}(t)| : k - 1 < t < k + 1\}, \end{aligned}$$

where $L = rr! \max_{|i| \leqq r} \sup |\gamma^{(i)}|$. Since θ is in $\mathscr{S}$, there is a constant C' such that

$$\max_{|i| \leqq r} |\theta^{(i)}(t)| \leqq \frac{C'}{(1 + |t|^2)^{r+2}} \qquad (3.8)$$

for all t. So from (3.7) and (3.8) we get

$$\begin{aligned} |\langle f, \gamma_k\theta\rangle| &\leqq CLC' \sup \left\{\frac{(k + 1)^r}{(1 + |t|^2)^{r+2}} : k - 1 < t < k + 1\right\} \\ &\leqq CLC' \frac{(k + 1)^r}{(1 + |k - 1|^2)^{r+2}} \\ &\leqq \frac{CLC'}{(k + 1)^2} \quad \text{as long as } k \geqq 3. \end{aligned}$$

Therefore the series $\sum_{k=0}^{\infty} \langle f, \gamma_k\theta\rangle$ converges absolutely.

Since θ was an arbitrary function in $\mathscr{S}$, f may be extended to $\mathscr{S}$ by defining for any θ in $\mathscr{S}$

$$\langle f, \theta \rangle = \sum_{k=0}^{\infty} \langle f, \gamma_k \theta \rangle. \tag{3.9}$$

If f were already in $\mathscr{S}'$, then (3.9) would be satisfied for every θ in $\mathscr{S}$; so our definition is consistent. It is easy to see that (3.9) extends f to $\mathscr{S}$ in a linear and continuous fashion, so f is in $\mathscr{S}'$, and the proof is complete.

COROLLARY 3.3. *If there are two complex numbers p_1 and p_2 with $\operatorname{Re} p_1 < \operatorname{Re} p_2$ such that $\{U_j T^{-p_1} f\}$ and $\{U_j T^{-p_2} f\}$ both converge in $\mathscr{D}'$, then whenever $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$, $\{U_j T^{-p} f\}$ converges in $\mathscr{D}'$ to $\langle T^{-p} f, 1 \rangle \delta$.*

Proof. If $\{U_j T^{-p_1} f\}$ and $\{U_j T^{-p_2} f\}$ both converge in $\mathscr{D}'$, then by Theorem 3.4, $T^{-p_1} f$ and $T^{-p_2} f$ are both in $\mathscr{S}'$. Also, by Theorem 2.3, $T^{-p} f$ is in $\mathscr{B}'_0$ as long as $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$. Therefore, by Theorem 3.1,

$$\lim_{j \to \infty} U_j T^{-p} f = \langle T^{-p} f, 1 \rangle \delta,$$

and the corollary is proved.

4. The Laplace transform. In this section we give a new characterization of the Laplace transform for one-dimensional distributions. We use it to prove the standard theorems concerning analyticity, uniqueness, and invertibility of the transform, then show that the new characterization is equivalent to Schwartz's definition of the Laplace transform for distributions. However, the development given here is completely independent of Schwartz's treatment.

We say that a distribution f is Laplace transformable if there is an open interval (α, β) such that whenever p is a complex number with real part in (α, β), $T^{-p} f$ is a distribution in $\mathscr{B}'_0$. If (α, β) is the largest such open interval, then the set

$$\Omega = \{p : \operatorname{Re} p \in (\alpha, \beta)\}$$

is called the domain of definition of the Laplace transform for f. The existence of the set Ω follows from Theorem 2.3.

If f is a Laplace transformable distribution whose transform has domain of definition Ω, then for any $p \in \Omega$, we define the Laplace transform of f at p by

$$\mathscr{L}[f](p) = \frac{1}{\phi(0)} \lim_{j \to \infty} \langle U_j T^{-p} f, \phi \rangle, \tag{4.1}$$

where ϕ is a test function in $\mathscr{D}$ with $\phi(0) \neq 0$. Theorem 3.1 guarantees the existence of the limit in (4.1) and tells us what it is. Thus, we have another characterization

$$\mathscr{L}[f](p) = \langle T^{-p} f, 1 \rangle. \tag{4.2}$$

By (4.2) we see that $\mathscr{L}[f]$ is a complex-valued function of the complex variable p with domain Ω. It also follows from (4.2) that the mapping $\mathscr{L}$ is linear. For, if f and g are distributions that are transformable at p and a and b are complex numbers, then $af + bg$ is Laplace transformable at p and

$$\begin{aligned} \mathscr{L}[af + bg] &= \langle T^{-p}[af + bg], 1 \rangle \\ &= a\langle T^{-p} f, 1 \rangle + b\langle T^{-p} g, 1 \rangle \\ &= a\mathscr{L}[f](p) + b\mathscr{L}[g](p). \end{aligned}$$

The next theorem shows that if f is Laplace transformable in Ω, then $\mathscr{L}[f]$ is an analytic function of p in Ω.

THEOREM 4.1. *If f is a distribution that is Laplace transformable in Ω, then $\mathscr{L}[f]$ is analytic in Ω and*

$$\frac{d}{dp}\mathscr{L}[f](p) = \mathscr{L}[-tf(t)](p).$$

Proof. Suppose that $\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$; pick p_0 in Ω, and ε in $(0, 1)$ such that $\varepsilon < \min\{\operatorname{Re} p_0 - \alpha, \beta - \operatorname{Re} p_0\}$. If $\lambda(t) = e^{\varepsilon t} + e^{-\varepsilon t}$, then $1/\lambda$ is in $\mathscr{S} \subset \mathscr{B}_0$, and $\lambda T^{-p_0} f$ is in $\mathscr{B}'_0$. Also, as long as $|p - p_0| < \varepsilon$, we have

$$\begin{aligned}
\frac{\mathscr{L}[f](p) - \mathscr{L}[f](p_0)}{p - p_0} &= \left\langle \frac{e^{-pt} - e^{-p_0 t}}{p - p_0} f(t), 1(t) \right\rangle \\
&= \left\langle \lambda(t)\, e^{-p_0 t} f(t), \frac{1}{\lambda(t)}\left[\frac{e^{-(p-p_0)t} - 1}{p - p_0}\right] \right\rangle \\
&= \left\langle \lambda(t)\, e^{-p_0 t} f(t), \frac{-t}{\lambda(t)} + \frac{(p - p_0)t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p - p_0)t]^{j-2}}{j!} \right\rangle.
\end{aligned}$$

Now, each derivative of

$$\frac{t^2}{\lambda(t)} \sum_{j=2}^{\infty} \frac{[-(p - p_0)t]^{j-2}}{j!}$$

is bounded in absolute value by the corresponding derivative of $(t^2/\lambda(t))\, e^{|(p-p_0)t|}$, and is therefore in $\mathscr{S}$. Thus, as $p \to p_0$, $(1/\lambda(t))[(e^{-(p-p_0)t} - 1)/(p - p_0)]$ converges in $\mathscr{B}_0$ to $-t/\lambda(t)$, and we have

$$\begin{aligned}
\frac{d}{dp}\mathscr{L}[f](p_0) &= \lim_{p \to p_0} \frac{\mathscr{L}[f](p) - \mathscr{L}[f](p_0)}{p - p_0} \\
&= \left\langle \lambda(t) T^{-p_0} f(t), \frac{-t}{\lambda(t)} \right\rangle \\
&= \langle T^{-p_0}[-tf(t)], 1(t) \rangle \\
&= \mathscr{L}[-tf(t)](p_0).
\end{aligned}$$

This completes the proof of Theorem 4.1.

Much of the usefulness of the Laplace transform is a result of the way it treats the convolution of two distributions. This important property of the transform is given by the next theorem.

THEOREM 4.2. *If f and g are Laplace transformable distributions such that the domains of their respective transforms have intersection Ω, then $f * g$ is Laplace transformable in Ω and for every p in Ω,*

$$\mathscr{L}[f * g](p) = \mathscr{L}[f](p)\mathscr{L}[g](p).$$

Proof. For p in Ω, $T^{-p}f$ and $T^{-p}g$ are both in $\mathscr{B}'_0$; so by Theorem 2.4, $T^{-p}f * T^{-p}g = T^{-p}(f * g)$ is in $\mathscr{B}'_0$. Therefore $f * g$ is Laplace transformable at

p; and from (4.2) and the definition of convolution we get

$$\begin{aligned}\mathscr{L}[f * g](p) &= \langle T^{-p}(f * g), 1\rangle \\ &= \langle T^{-p}f * T^{-p}g, 1\rangle \\ &= \langle T^{-p}f(t) \otimes T^{-p}g(\tau), 1(t + \tau)\rangle \\ &= \langle T^{-p}f(t) \otimes T^{-p}g(\tau), 1(t)1(\tau)\rangle \\ &= \langle T^{-p}f, 1\rangle\langle T^{-p}g, 1\rangle \\ &= \mathscr{L}[f](p)\mathscr{L}[g](p),\end{aligned}$$

which completes the proof.

No theory of the Laplace transform would be useful without inversion and uniqueness theorems. The next theorem has these results as corollaries. In what follows we have as independent variables at various times the real variable t and the real and imaginary parts of the complex variable p. For this reason we sometimes indicate the particular independent variable for a space or an operation by a subscript, e.g., $\langle f(\tau), e^{-i\omega\tau}\rangle_\tau$, where $f(\tau)$ is in $\mathscr{B}'_{0\tau}$ and ω is a parameter.

THEOREM 4.3. *If f is a distribution in $\mathscr{B}'_{0_t}$, then*

$$f(t) = \frac{1}{2\pi} \lim_{r\to\infty} \int_{-r}^{r} e^{i\omega t}\langle f(\tau), e^{-i\omega\tau}\rangle_\tau \, d\omega, \tag{4.3}$$

where the limit is taken in $\mathscr{D}'_t$.

Proof. For each function γ_k in the partition of unity $\{\gamma_k\}$ defined in the Appendix, $\langle f(\tau), \gamma_k(\tau)\, e^{-i\omega\tau}\rangle$ is a continuous function of ω and, as shown in the proof of Theorem 2.2, the sum $\sum_{k=0}^{\infty} \langle f(\tau), \gamma_k(\tau)\, e^{-i\omega\tau}\rangle$ converges uniformly. Therefore $\langle f(\tau), e^{-i\omega\tau}\rangle$ is a continuous function of ω and the integral in (4.3) is well-defined. Let ϕ be in $\mathscr{D}_t$ and r be a positive real number. Then by standard theorems on the integration of distributions and test functions with respect to parameters, we have

$$\begin{aligned}\left\langle \int_{-r}^{r} e^{i\omega t}\langle f(\tau), e^{-i\omega\tau}\rangle_\tau \, d\omega, \phi(t)\right\rangle_t &= \int_{-r}^{r} \langle e^{i\omega t}\langle f(\tau), e^{-i\omega\tau}\rangle_\tau, \phi(t)\rangle_t \, d\omega \\ &= \int_{-r}^{r} \langle f(\tau), e^{-i\omega\tau}\rangle_\tau \langle e^{i\omega t}, \phi(t)\rangle_t \, d\omega \\ &= \int_{-r}^{r} \langle f(\tau), \langle e^{i\omega(t-\tau)}, \phi(t)\rangle_t\rangle_\tau \, d\omega \\ &= \left\langle f(\tau), \int_{-r}^{r} \langle e^{i\omega(t-\tau)}, \phi(t)\rangle_t \, d\omega \right\rangle_\tau \\ &= \left\langle f(\tau), \int_{-r}^{r} e^{-i\omega\tau} \int_{-\infty}^{\infty} e^{i\omega t}\phi(t)\, dt \, d\omega \right\rangle_\tau \\ &= \left\langle f(\tau), \int_{-r}^{r} e^{i\xi\tau}\tilde{\phi}(\xi)\, d\xi \right\rangle_\tau ,\end{aligned}$$

where $\xi = -\omega$ and $\tilde{\phi}(\xi)$ is the Fourier transform of $\phi(t)$. Clearly, as $r \to \infty$, $\int_{-r}^{r} e^{i\xi\tau}\tilde{\phi}(\xi)\,d\xi \to 2\pi\phi(\tau)$ uniformly with respect to τ, and similarly

$$\frac{d^k}{d\tau^k}\left[\int_{-r}^{r} e^{i\xi\tau}\tilde{\phi}(\xi)\,d\xi\right] = \int_{-r}^{r} (i\xi)^k e^{i\xi\tau}\tilde{\phi}(\xi)\,d\xi \to 2\pi\phi^{(k)}(\tau)$$

uniformly. So the limit in $\mathscr{B}_\tau$ of $\int_{-r}^{r} e^{i\xi\tau}\tilde{\phi}(\xi)\,d\xi$ as $r \to \infty$ is $2\pi\phi(\tau)$, and we have

$$\begin{aligned}\left\langle \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t}\langle f(\tau), e^{-i\omega\tau}\rangle_\tau\,d\omega, \phi(t)\right\rangle &= \frac{1}{2\pi}\left\langle f(\tau), \lim_{r\to\infty}\int_{-r}^{r}\langle e^{i\omega(t-\tau)}, \phi(t)\rangle_t\,d\omega\right\rangle_\tau \\ &= \frac{1}{2\pi}\langle f(\tau), 2\pi\phi(\tau)\rangle_\tau \\ &= \langle f(t), \phi(t)\rangle_t.\end{aligned}$$

Thus, as distributions,

$$f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t}\langle f(\tau), e^{-i\omega\tau}\rangle_\tau\,d\omega,$$

and the theorem is proved.

COROLLARY 4.1. *If σ is a real number such that $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$, then*

$$f(t) = \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} e^{pt}\langle e^{-p\tau}f(\tau), 1(\tau)\rangle_\tau\,dp,$$

where the limit is taken in $\mathscr{D}'_t$.

Proof. If $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$, then as long as $\operatorname{Re} p = \sigma$, $e^{-pt}f(t)$ is in $\mathscr{B}'_{0_t}$, and

$$e^{-\sigma t}f(t) = \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{i\omega t}\langle e^{-\sigma\tau}f(\tau), e^{-i\omega\tau}\rangle_\tau\,d\omega.$$

Therefore

$$\begin{aligned}f(t) &= \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} e^{\sigma t}e^{i\omega t}\langle e^{-\sigma\tau}f(\tau), e^{-i\omega\tau}\rangle_\tau\,d\omega \\ &= \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\sigma-ir}^{\sigma+ir} e^{pt}\langle e^{-p\tau}f(\tau), 1(\tau)\rangle_\tau\,dp,\end{aligned}$$

which proves the corollary.

COROLLARY 4.2 (Inversion theorem). *If f is Laplace transformable in*

$$\Omega = \{p : \alpha < \operatorname{Re} p < \beta\},$$

then, as long as $\alpha < \sigma < \beta$,

$$f(t) = \lim_{r\to\infty}\frac{1}{2\pi i}\int_{\sigma-ir}^{\sigma+ir} e^{pt}\mathscr{L}[f](p)\,dp,$$

where the limit is taken in $\mathscr{D}'_t$.

COROLLARY 4.3 (Uniqueness theorem). *If f and g are Laplace transformable distributions such that $\mathscr{L}[f](p) = \mathscr{L}[g](p)$ on some vertical line in the common domain of the transforms of f and g, then $f = g$ as distributions in $\mathscr{D}'_t$.*

The next theorem gives sufficient conditions that an analytic function $F(p)$ be the Laplace transform of a distribution $f(t)$, and characterizes the distribution f.

THEOREM 4.4. *If $F(p)$ is analytic for p in $\Omega = \{\sigma + i\omega : \alpha < \sigma < \beta\}$ and is bounded in Ω by a polynomial in ω (or in $|p|$), then $F(p) = \mathscr{L}[f](p)$, where the distribution $f(t)$ is defined by*

$$f(t) = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{\sigma - ir}^{\sigma + ir} e^{pt}F(p)\,dp \tag{4.4}$$

with the limit taken in $\mathscr{D}'_t$ for any fixed value of σ such that $\alpha < \sigma < \beta$.

Proof. The proof will be accomplished in four steps. It will be shown that (i) f is a distribution, (ii) f is independent of the value of σ chosen in (4.4) as long as $\alpha < \sigma < \beta$, (iii) $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$ as long as $\alpha < \sigma < \beta$, and (iv) $F(p) = \mathscr{L}[f](p) = \langle T^{-p}f, 1\rangle$ for every p in Ω.

(i) To see that f is a distribution, let $\alpha < \sigma < \beta$ and let ϕ be in $\mathscr{D}_t$. Then

$$\begin{aligned}\left\langle \frac{1}{2\pi i}\int_{\sigma - ir}^{\sigma + ir} e^{pt}F(p)\,dp, \phi(t)\right\rangle_t &= \left\langle \frac{1}{2\pi}\int_{-r}^{r} e^{\sigma t}e^{i\omega t}F(\sigma + i\omega)\,d\omega, \phi(t)\right\rangle_t \\ &= \frac{1}{2\pi}\int_{-r}^{r} F(\sigma + i\omega)\mathscr{F}[e^{-\sigma t}\phi(-t)](\omega)\,d\omega.\end{aligned} \tag{4.5}$$

Now $e^{-\sigma t}\phi(-t)$ is in $\mathscr{D}_t$, so its Fourier transform is certainly in $\mathscr{S}_\omega$. Also, since $F(\sigma + i\omega)$ is a function bounded by a polynomial in ω, it is a regular distribution in $\mathscr{S}'_\omega$. Therefore, the limit as $r \to \infty$ of the last integral in (4.5) is well-defined as the value of the regular distribution $F(\sigma + i\omega)$ at the testing function $\mathscr{F}[e^{-\sigma t}\phi(-t)]$, and we have

$$\begin{aligned}\langle f(t), \phi(t)\rangle &= \left\langle \frac{1}{2\pi i}\lim_{r\to\infty}\int_{\sigma - ir}^{\sigma + ir} e^{pt}F(p)\,dp, \phi(t)\right\rangle_t \\ &= \frac{1}{2\pi}\langle F(\sigma + i\omega), \mathscr{F}[e^{-\sigma t}\phi(-t)](\omega)\rangle_\omega.\end{aligned} \tag{4.6}$$

Clearly, if $\{\phi_k\}$ is a sequence that converges to zero in $\mathscr{D}_t$ as $k \to \infty$, then the sequence $\{\mathscr{F}[e^{-\sigma t}\phi_k(-t)]\}$ converges to zero in $\mathscr{S}_\omega$ as $k \to \infty$; so by (4.6), $\langle f, \phi_k\rangle \to 0$ also. Thus (4.4) defines f as a distribution in $\mathscr{D}'_t$.

(ii) To see that f is independent of the choice of σ, choose σ_1, σ_2 such that $\alpha < \sigma_1 < \sigma_2 < \beta$; and for every positive real number r, let Γ_r be the closed path in Ω defined by the lines $\operatorname{Re} p = \sigma_1$, $\operatorname{Re} p = \sigma_2$, and $\operatorname{Im} p = \pm r$. Since $F(p)$ is analytic in Ω, Cauchy's theorem says that $\int_{\Gamma_r} e^{pt}F(p)\,dp = 0$. Therefore

$$\int_{\sigma_1 - ir}^{\sigma_1 + ir} e^{pt}F(p)\,dp - \int_{\sigma_2 - ir}^{\sigma_2 + ir} e^{pt}F(p)\,dp = \int_{\sigma_1 - ir}^{\sigma_2 - ir} e^{pt}F(p)\,dp + \int_{\sigma_2 + ir}^{\sigma_1 + ir} e^{pt}F(p)\,dp. \tag{4.7}$$

But

$$\begin{aligned}\left\langle \int_{\sigma_1 \pm ir}^{\sigma_2 \pm ir} e^{pt}F(p)\,dp, \phi(t)\right\rangle_t &= \int_{\sigma_1}^{\sigma_2} \langle e^{(\sigma \pm ir)t}F(\sigma \pm ir), \phi(t)\rangle\,d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} F(\sigma \pm ir)\langle e^{(\sigma \pm ir)t}, \phi(t)\rangle\,d\sigma \\ &= \int_{\sigma_1}^{\sigma_2} F(\sigma \pm ir)\langle e^{\pm irt}, e^{\sigma t}\phi(t)\rangle\,d\sigma.\end{aligned} \tag{4.8}$$

Now $\langle e^{\pm irt}, e^{\sigma t}\phi(t)\rangle$ is a function in $\mathscr{S}_r$ for every value of σ, and the integral (4.8) is over a bounded interval; so as $r \to \infty$ the integral (4.8) approaches zero. Thus by (4.7) we see that

$$\lim_{r\to\infty}\int_{\sigma_1 - ir}^{\sigma_1 + ir} e^{pt}F(p)\,dp = \lim_{r\to\infty}\int_{\sigma_2 - ir}^{\sigma_2 + ir} e^{pt}F(p)\,dp$$

as long as $\alpha < \sigma_1 < \sigma_2 < \beta$.

(iii) In proving that $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$ whenever $\alpha < \sigma < \beta$, we use (4.6) and the fact that $F(\sigma + i\omega)$ is in $\mathscr{S}'_\omega$ to get bounds on $\langle e^{-\sigma t}f(t), \phi(t)\rangle$, where ϕ is in $\mathscr{D}_t$. We have

$$\begin{aligned}
|\langle e^{-\sigma t}f(t), \phi(t)\rangle| &= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + i\omega)\langle e^{i\omega t}, \phi(t)\rangle_t\,d\omega\right| \\
&\leqq C_1 \sup_\omega \left|(1 + \omega^2)^{r_1}\frac{d^{r_1}}{d\omega^{r_1}}\langle e^{i\omega t}, \phi(t)\rangle_t\right| \\
&= C_1 \sup_\omega \left|\sum_{k=0}^{r_1}\binom{r_1}{k}\omega^{2k}\langle e^{i\omega t}, (it)^{r_1}\phi(t)\rangle_t\right| \\
&\leqq C_1 \sup_\omega \sum_{k=0}^{r_1}\binom{r_1}{k}|i\omega|^{2k}|\langle e^{i\omega t}, (it)^{r_1}\phi(t)\rangle_t| \\
&\leqq C_1 r_1 r_1!\max_{|k|\leqq r_1}\sup_\omega |\langle e^{i\omega t}, [(it)^{r_1}\phi(t)]^{(2k)}\rangle_t| \\
&\leqq C_1 r_1 r_1!\max_{|k|\leqq r_1} C_2 \max_{|j|\leqq r_2}\sup_t |(1 + t^2)^{r_2}[(it)^{r_1}\phi(t)]^{(2k+j)}|,
\end{aligned} \tag{4.9}$$

where the last inequality follows from the fact that $e^{i\omega t}$ is a distribution in $\mathscr{S}'_t$ that is uniformly bounded with respect to ω. It is clear that we may expand the derivative of the product using Leibniz's rule and consolidate the various constants in (4.9) to get a positive number C and positive integer r which do not depend on ϕ such that

$$|\langle e^{-\sigma t}f(t), \phi(t)\rangle| \leqq C\max_{|j|\leqq r}\sup_t |(1 + t^2)^r\phi^{(j)}(t)|.$$

This bound means that $e^{-\sigma t}f(t)$ is in $\mathscr{S}'_t$ for all σ such that $\alpha < \sigma < \beta$, and so by Theorem 2.3, $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$ for all such σ.

(iv) Part (iv) of this proof can be verified by using the first three parts and the uniqueness theorem for the inverse Fourier transform. However, it will be proved here by actually showing that $\langle T^{-p}f, 1\rangle = F(p)$. Let $p = \sigma + i\tau$, where $\alpha < \sigma < \beta$, and let ϕ be a function in $\mathscr{D}_t$ with $\phi(0) = 1$ and such that the support of ϕ is contained in $(-1, 1)$. Then by Theorem 3.1, we see that

$$\begin{aligned}
\langle e^{-pt}f(t), 1(t)\rangle &= \lim_{j\to\infty}\langle U_j e^{-pt}f(t), \phi(t)\rangle \\
&= \lim_{j\to\infty}\left\langle e^{-pt}f(t), \phi\left(\frac{t}{j}\right)\right\rangle \\
&= \lim_{j\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\sigma + i\omega)\left\langle e^{i(\omega - \tau)t}, \phi\left(\frac{t}{j}\right)\right\rangle d\omega.
\end{aligned}$$

Let $F = F_1 + F_2$ where the support of F_2 is contained in $\{\sigma + i\omega : |\omega - \tau| < 1\}$ and $F_2 = F$ in $\{\sigma + i\omega : |\omega - \tau| < \frac{1}{2}\}$. Also choose $k \geqq 2$ large enough so that $G(\omega) = F_1(\sigma + i\omega)/[i(\omega - \tau)]^k$ is in L^1_ω, that is, an integrable function of ω. This can be done since $F(\sigma + i\omega)$ is bounded by some polynomial in ω for p in Ω. Then we have

$$\lim_{j\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\sigma + i\omega)\left\langle e^{i(\omega-\tau)t}, \phi\left(\frac{t}{j}\right)\right\rangle d\omega$$

$$\begin{aligned}
&= \lim_{j\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\langle [i(\omega - \tau)]^k e^{i(\omega-\tau)t}, \phi\left(\frac{t}{j}\right)\right\rangle_t d\omega \\
&= \lim_{j\to\infty} \frac{(-1)^k}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left\langle e^{i(\omega-\tau)t}, \frac{1}{j^k}\phi^{(k)}\left(\frac{t}{j}\right)\right\rangle_t d\omega \qquad (4.10)\\
&= \lim_{j\to\infty} \frac{(-1)^k}{j^{k-1}}\int_{-j}^{j} e^{-i\tau t}\frac{\phi^{(k)}(t/j)}{j}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\, e^{i\omega t}\, d\omega\, dt \\
&= \lim_{j\to\infty} \frac{(-1)^k}{j^{k-1}}\int_{-j}^{j} e^{-i\tau t}\frac{\phi^{(k)}(t/j)}{j}\mathscr{F}^{-1}[G(\omega)](t)\, dt.
\end{aligned}$$

Now since $G(\omega)$ is an L^1-function, its inverse Fourier transform is certainly bounded in absolute value, say by B. Therefore, the integrand in (4.10) is bounded in absolute value. We have

$$\begin{aligned}
\left|\int_{-j}^{j} e^{-i\tau t}\frac{\phi^{(k)}(t/j)}{j}\mathscr{F}^{-1}[G(\omega)](t)\, dt\right| &\leqq \frac{B}{j}\int_{-j}^{j}\left|\phi^{(k)}\left(\frac{t}{j}\right)\right| dt \\
&\leqq (B/j)(2j)\sup|\phi^{(k)}| \\
&= 2B\sup|\phi^{(k)}|.
\end{aligned}$$

Since $k \geqq 2$, we see that the limit in (4.10) must be zero.

The term we have neglected is

$$\begin{aligned}
&\lim_{j\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\sigma + i\omega)\left\langle e^{i(\omega-\tau)t}, \phi\left(\frac{t}{j}\right)\right\rangle_t d\omega \qquad (4.11)\\
&\qquad = \lim_{j\to\infty}\int_{-\infty}^{\infty} e^{-i\tau t}\phi\left(\frac{t}{j}\right)\mathscr{F}^{-1}[F_2(\sigma + i\omega)](t)\, dt.
\end{aligned}$$

Now $\mathscr{F}^{-1}[F_2(\sigma + i\omega)]$ is in $\mathscr{S}_t$, so as $j \to \infty$, $\{\phi(t/j)\mathscr{F}^{-1}[F_2(\sigma + i\omega)](t)\}$ converges in $\mathscr{S}_t$ to $\mathscr{F}^{-1}[F_2(\sigma + i\omega)]$. Also, $e^{-i\tau t}$ is a regular distribution in $\mathscr{S}'_t$, so the limit in (4.11) is

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-i\tau t}\mathscr{F}^{-1}[F_2(\sigma + i\omega)](t)\, dt &= F_2(\sigma + i\tau) = F(\sigma + i\tau) \\
&= F(p).
\end{aligned}$$

Thus we have shown that

$$\langle T^{-p}f, 1\rangle = F(p)$$

as long as $\alpha < \operatorname{Re} p < \beta$, and the proof of Theorem 4.4 is complete.

The Laplace transform has been developed so far without any reference to the extension of the classical Laplace transform to distributions as defined by Schwartz. However, it is easy to see that the development given here is equivalent to that of Schwartz. First, notice that by Theorem 2.3 and the fact that $\mathscr{B}'_0 \subset \mathscr{S}'$, a distribution f is Laplace transformable in our sense if and only if $e^{-pt}f(t)$ is in $\mathscr{S}'_t$ for every p in Ω. Therefore, the transformable distributions and domains of the transform are the same for both definitions of the transform. Next, we see that if there is an open interval (α, β) such that $e^{-\sigma t}f(t)$ is in $\mathscr{B}'_{0_t}$ whenever σ is in (α, β), then the Fourier transform of $e^{-\sigma t}f(t)$ is an ordinary function of ω defined by

$$\mathscr{F}[e^{-\sigma t}f(t)](\omega) = \langle e^{-\sigma t}f(t), e^{-i\omega t}\rangle. \tag{4.12}$$

The right-hand side of (4.12) makes sense as the application of a distribution in $\mathscr{B}'_0$ to a testing function in $\mathscr{B}$. To see that (4.12) is true, let ϕ be a function in $\mathscr{S}_\omega$. Then

$$\begin{aligned}\langle \mathscr{F}[e^{-\sigma t}f(t)](\omega), \phi(\omega)\rangle_\omega &= \langle e^{-\sigma t}f(t), \tilde{\phi}(t)\rangle_t \\ &= \left\langle e^{-\sigma t}f(t), \int_{-\infty}^{\infty} e^{-i\omega t}\phi(\omega)\, d\omega \right\rangle_t \\ &= \int_{-\infty}^{\infty} \langle e^{-\sigma t}f(t), e^{-i\omega t}\rangle_t \phi(\omega)\, d\omega \\ &= \langle \langle e^{-\sigma t}f(t), e^{-i\omega t}\rangle_t, \phi(\omega)\rangle_\omega.\end{aligned}$$

Thus, if f is Laplace transformable in Ω and $p = \sigma + i\omega$ is in Ω; by Schwartz's definition of the transform we have

$$\begin{aligned}\mathscr{L}[f](p) = \mathscr{F}[e^{-\sigma t}f(t)](\omega) &= \langle e^{-\sigma t}f(t), e^{-i\omega t}\rangle_t \\ &= \langle e^{-pt}f(t), 1(t)\rangle_t = \langle T^{-p}f, 1\rangle,\end{aligned}$$

and the two definitions of the Laplace transform are equivalent.

We will next derive some of the standard operation-transform formulas for the distributional Laplace transform using the characterization of the transform given in (4.1).

Let f be a Laplace transformable distribution whose transform has domain of definition $\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$. Then $f^{(1)}$ is also Laplace transformable in Ω. To compute the transform of $f^{(1)}$, let ϕ be a function in $\mathscr{D}$ such that $\phi(0) = 1$, $\phi'(0) \neq 0$, and let j be a positive integer. Then if p is in Ω, we have

$$\begin{aligned}\langle U_j T^{-p} f^{(1)}, \phi\rangle &= \left\langle f^{(1)}(t), e^{-pt}\phi\left(\frac{t}{j}\right)\right\rangle \\ &= \left\langle f(t), p\, e^{-pt}\phi\left(\frac{t}{j}\right) - \frac{1}{j} e^{-pt}\phi^{(1)}\left(\frac{t}{j}\right)\right\rangle \qquad (4.13) \\ &= p\langle U_j T^{-p} f, \phi\rangle - \frac{1}{j}\langle U_j T^{-p} f, \phi^{(1)}\rangle.\end{aligned}$$

As $j \to \infty$, the second term in the right-hand side of (4.13) converges to zero, and so by (4.1) we have

$$\mathscr{L}[f^{(1)}](p) = \lim_{j\to\infty} p\langle U_j T^{-p} f, \phi\rangle = p\mathscr{L}[f](p).$$

By an inductive argument it is easy to see that for every positive integer k

$$\mathscr{L}[f^{(k)}](p) = p\mathscr{L}[f^{(k-1)}](p) = p^k\mathscr{L}[f](p). \tag{4.14}$$

Another operational formula is furnished by Theorem 4.1, which says that

$$\mathscr{L}[-tf(t)](p) = \frac{d}{dp}\mathscr{L}[f](p).$$

This formula can be extended by induction to get, for every positive integer k,

$$\mathscr{L}[t^k f(t)](p) = (-1)^k \frac{d^k}{dp^k}\mathscr{L}[f](p). \tag{4.15}$$

If f is Laplace transformable in Ω, then $f(t-\tau)$ is transformable in Ω for every real number τ, and we have

$$\begin{aligned}\langle U_j T^{-p} f(t-\tau), \phi(t)\rangle &= \langle f(t-\tau), e^{-pt}\phi(t/j)\rangle \\ &= \langle f(t), e^{-p(t+\tau)}\phi((t+\tau)/j)\rangle \\ &= e^{-p\tau}\langle U_j T^{-p} f(t), \phi(t+\tau)\rangle.\end{aligned}$$

Now, $\phi(t+\tau)$ is in $\mathscr{D}$; and as long as $\phi(\tau) \neq 0$,

$$\lim_{j\to\infty} e^{-p\tau}\langle U_j T^{-p} f(t), \phi(t+\tau)\rangle = \frac{1}{\phi(\tau)} e^{-p\tau}\langle T^{-p} f, 1\rangle\langle\delta(t), \phi(t+\tau)\rangle,$$

so

$$\mathscr{L}[f(t-\tau)](p) = e^{-p\tau}\mathscr{L}[f](p). \tag{4.16}$$

If q is a fixed complex number and f is Laplace transformable in Ω, then $e^{-qt}f(t)$ is Laplace transformable in $\Omega' = \{p : \alpha - \operatorname{Re} q < \operatorname{Re} p < \beta - \operatorname{Re} q\}$, and we have

$$\langle U_j T^{-p}[e^{-qt}f(t)], \phi(t)\rangle = \langle U_j T^{-(p+q)} f, \phi\rangle.$$

Therefore, as long as p is in Ω',

$$\mathscr{L}[e^{-qt}f(t)](p) = \mathscr{L}[f](p+q). \tag{4.17}$$

If k is a fixed positive integer and f is Laplace transformable in Ω, then $U_k f$ is Laplace transformable in $\Omega'' = \{p : k\alpha < \operatorname{Re} p < k\beta\}$. For $p \in \Omega''$ we have

$$\begin{aligned}\langle U_j T^{-p}[U_k f], \phi\rangle &= \langle U_k f, e^{-pt}\phi(t/j)\rangle \\ &= \langle f(t), e^{-pt/k}\phi(t/jk)\rangle \\ &= \langle U_j T^{-p/k} f, \phi(t/k)\rangle.\end{aligned}$$

As $j \to \infty$, this converges to $\langle T^{-p/k} f, 1\rangle\langle\delta, \phi\rangle$, so we get the formula

$$\mathscr{L}[U_k f](p) = \mathscr{L}[f](p/k). \tag{4.18}$$

In order to demonstrate some of the theory developed so far, consider the distribution

$$f(t) = -\sum_{\nu=1}^{\infty} \delta^{(1)}(t - \nu).$$

Recall that in Example 3.1 we showed that f is not in $\mathscr{B}'_0$, but that $\{U_j f\}$ converges in $\mathscr{D}'$ to $-\delta$ as $j \to \infty$. Notice that if $\operatorname{Re} p > 0$, then $T^{-p}f$ is in $\mathscr{B}'_0$, so f has a Laplace transform defined in $\Omega = \{p : \operatorname{Re} p > 0\}$. Also notice that

$$f(t) = -\frac{d}{dt}\sum_{\nu=1}^{\infty} \delta(t - \nu),$$

so by (4.14) if $g(t) = \sum_{\nu=1}^{\infty} \delta(t - \nu)$,

$$\mathscr{L}[f](p) = -p\mathscr{L}[g](p) \quad \text{for every } p \text{ in } \Omega.$$

If $\lambda(t)$ is a function in $\mathscr{B}$ such that $\lambda(t) = 0$ for $t < 0$ and $\lambda(t) = 1$ for $t \geqq \frac{1}{2}$, then by (4.2),

$$\begin{aligned}\mathscr{L}[g](p) = \langle T^{-p}g, 1\rangle &= \left\langle \sum_{\nu=1}^{\infty} \delta(t - \nu), e^{-pt}\lambda(t)\right\rangle \\ &= \sum_{\nu=1}^{\infty} e^{-p\nu} = e^{-p}\sum_{\nu=0}^{\infty} (e^{-p})^{\nu} \\ &= 1/(e^{p} - 1), \qquad p \in \Omega.\end{aligned}$$

Therefore, for p in Ω,

$$\mathscr{L}[f](p) = -p\mathscr{L}[g](p) = p/(1 - e^{p}).$$

5. The N-dimensional Laplace transform. In this section the results proved in §§ 3 and 4 for distributions in $\mathscr{D}'(R)$ will be extended to $\mathscr{D}'(R^n)$. At the beginning of § 3, the linear transformations U_a and T^{-p} were defined, where a is in R^n with $a > 0$ and p is in C^n. Here, as in § 3, we are concerned with the limit of the sequence of distributions $\{U_j T^{-p} f\}$. However, in this section, j represents a multi-index, $j = j_1, j_2, \cdots, j_n$, instead of a positive integer-valued index. Let $j \to \infty$ mean that $j_1 \to \infty, j_2 \to \infty, \cdots, j_n \to \infty$, and for each i and k, $1 \leqq i \leqq n$ and $1 \leqq k \leqq n$, $j_i \to \infty$ independently of j_k. If $\{f_j\}$ is a "sequence" of distributions in $\mathscr{D}'(R^n)$ indexed by the multi-index j, then the statement

$$\lim_{j \to \infty} f_j = h$$

means that if ϕ is in $\mathscr{D}(R^n)$ and $\varepsilon > 0$, then there is a positive integer N such that whenever $j_k \geqq N$ for every k, $1 \leqq k \leqq n$, then $|\langle f_j, \phi\rangle - \langle h, \phi\rangle| < \varepsilon$.

The need for being very specific about what is meant by the limit of a sequence indexed by multi-indices will be demonstrated by the following example. Let the distribution h be defined by

$$h(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Then $h(x, y)$ is a rational function of x and y with a removable singularity at the origin and can be considered a distribution in $\mathscr{D}'(R^2)$. It is easy to see that for every positive integer k,

$$h(x, y) = k^2 h(kx, ky).$$

Therefore, if $U_k h$ is defined by

$$U_k h(x, y) = k^2 h(kx, ky),$$

then $\lim_{k\to\infty} U_k h = h$. However, it is not true that $\lim_{j\to\infty} U_j h = h$, where j represents a multi-index of order 2. To verify this, let $j_k = (2k, k)$ for every positive integer k. Then

$$\begin{aligned}\lim_{k\to\infty} U_{j_k} h(x, y) &= \lim_{k\to\infty} 2k^2 h(2kx, ky)\\ &= \lim_{k\to\infty} 2k^2\left[\frac{4k^2x^2 - k^2y^2}{(4k^2x^2 + k^2y^2)^2}\right]\\ &= \frac{2(4x^2 - y^2)}{(4x^2 + y^2)^2}\\ &\neq h(x, y).\end{aligned}$$

Thus, by our definition of the limit, h does not equal $\lim_{j\to\infty} U_j h$.

Since the results in this section are n-dimensional analogues of results already proved, we shall prove only those for which the one-dimensional proofs do not generalize immediately. In particular, Theorem 3.1 may be generalized to n-dimensions without changing the statement or the proof significantly, so we accept it as an n-dimensional result without proving it again here.

The next theorem has a corollary which is the analogue in n-dimensions of Theorem 3.2 and its converse.

THEORM 5.1. *If h is in $\mathscr{D}'(R^n)$, then $U_j h = h$ for every positive multi-index j if and only if*

$$h(t) = \sum_{\nu=1}^{2^n} c_\nu \left(\bigotimes_{\substack{i\in I_\nu \\ I_\nu\subset\{1,\cdots,n\}}} \text{p.v.}\frac{1}{t_i}\right) \otimes \left(\bigotimes_{i\notin I_\nu} \delta(t_i)\right), \tag{5.1}$$

for some constants c_ν, $1 \leqq \nu \leqq 2^n$.

Remark. In words, the theorem says that any distribution h in $\mathscr{D}'(R^n)$ which is invariant under each U_j is a linear combination of 2^n terms, each of which is the tensor product of n one-dimensional distributions of the form $\delta(t_i)$ or p.v. $(1/t_i)$. For example, if $n = 2$, then

$$\begin{aligned}h(t) = c_1 \,\text{p.v.}\frac{1}{t_1} \otimes \text{p.v.}\frac{1}{t_2} + c_2\,\text{p.v.}\frac{1}{t_1}\otimes\delta(t_2)\\ + c_3\,\delta(t_1)\otimes\text{p.v.}\frac{1}{t_2} + c_4\,\delta(t_1)\otimes\delta(t_2).\end{aligned}$$

Proof (of Theorem 5.1). The proof is by induction on n. If $n = 1$, then $U_j h = h$ for every positive multi-index j if and only if there is a distribution f such that $h = \lim_{j\to\infty} U_j f$. Therefore, the expansion (5.1) for k follows from Theorem 3.2 in this case. Let k be a positive integer and suppose that the theorem holds when $n = k - 1$. Let h be a distribution in $\mathscr{D}'(R^k)$ such that $U_j h = h$ for every positive multi-index j. If we let r_k denote the multi-index $(0, 0, \cdots, 0, r)$, where the r is in the kth position, then

$$U_{r_k} h = h \tag{5.2}$$

for every positive number r. Equation (5.2) may be differentiated with respect to r to get

$$\frac{d}{dr}(rh(t_1, \cdots, rt_k)) = \frac{d}{dr} h(t) = 0$$

or

$$h(t_1, \cdots, rt_k) + rt_k \frac{\partial h}{\partial t_k}(t_1, \cdots, rt_k) = 0. \tag{5.3}$$

Setting $r = 1$ in (5.3) gives

$$h(t) + t_k \frac{\partial h}{\partial t_k}(t) = \frac{\partial}{\partial t_k}(t_k h(t)) = 0. \tag{5.4}$$

Therefore, the distribution $t_k h(t)$ is independent of t_k, and there is a distribution h_k in $\mathscr{D}'(R^{k-1})$ such that

$$t_k h(t) = h_k(t_1, \cdots, t_{k-1}) \otimes 1(t_k).$$

Since $1(t_k) = t_k \, \text{p.v.}\,(1/t_k)$, we have

$$t_k[h(t) - h_k(t_1, \cdots, t_{k-1}) \otimes \text{p.v.}\,(1/t_k)] = 0.$$

By a standard result there must exist another distribution h'_k in $\mathscr{D}'(R^{k-1})$ such that

$$h(t) - h_k(t_1, \cdots, t_{k-1}) \otimes \text{p.v.}\,(1/t_k) = h'_k(t_1, \cdots, t_{k-1}) \otimes \delta(t_k)$$

or

$$h(t) = h_k \otimes \text{p.v.}\,(1/t_k) + h'_k \otimes \delta(t_k).$$

Now, since $U_j h = h$ for every multi-index j, $h_k \otimes \text{p.v.}\,(1/t_k) + h'_k \otimes \delta(t_k)$ must also be invariant under each U_j. Therefore, if we let $\bar{j} = j_1, \cdots, j_{k-1}$, we have

$$\begin{aligned} h_k \otimes \text{p.v.}\,(1/t_k) + h'_k \otimes \delta(t_k) &= U_j(h_k \otimes \text{p.v.}\,(1/t_k)) + U_j(h'_k \otimes \delta(t_k)) \\ &= U_{\bar{j}} h_k \otimes U_{j_k} \text{p.v.}\,(1/t_k) + U_{\bar{j}} h'_k \otimes U_{j_k} \delta(t_k) \\ &= U_{\bar{j}} h_k \otimes \text{p.v.}\,(1/t_k) + U_{\bar{j}} h'_k \otimes \delta(t_k) \end{aligned}$$

or

$$\text{p.v.}\,(1/t_k) \otimes (U_{\bar{j}} h_k - h_k) + \delta(t_k) \otimes (U_{\bar{j}} h'_k - h'_k) = 0.$$

This can happen for every multi-index $\bar{j}$ of order $k - 1$ if and only if $U_{\bar{j}} h_k = h_k$ and $U_{\bar{j}} h'_k = h'_k$ for every $\bar{j}$. Since h_k and h'_k are both in $\mathscr{D}'(R^{k-1})$, the induction hypothesis says that there must be constants b_ν and b'_ν, $1 \leqq \nu \leqq 2^{k-1}$, such that

$$h_k(t_1, \cdots, t_{k-1}) = \sum_{\nu=1}^{2^{k-1}} b_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1,\cdots,k-1\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right)$$

and

$$h'_k(t_1, \cdots, t_{k-1}) = \sum_{\nu=1}^{2^{k-1}} b'_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1,\cdots,k-1\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right).$$

Therefore

$$\begin{aligned} h(t) &= h_k \otimes \text{p.v.}\,(1/t_k) + h'_k \otimes \delta(t_k) \\ &= \sum_{\nu=1}^{2^k} c_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1,\cdots,k\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right), \end{aligned}$$

where the sequence $\{c_\nu\}_1^{2^k}$ is a rearrangement of the union of the two sequences $\{b_\nu\}_1^{2^{k-1}}$ and $\{b'_\nu\}_1^{2^{k-1}}$.

Thus for every positive integer n, a representation of the form (5.1) holds for h in $\mathscr{D}'(R^n)$ whenever $U_j h = h$ for every positive multi-index j.

Conversely, if h has a representation of the form (5.1), then it is easy to see that $U_j h = h$ for every multi-index j. This completes the proof of Theorem 5.1.

By observing that $h(t) = \lim_{j \to \infty} U_j f(t)$ for some distribution f in $\mathscr{D}'(R^n)$ if and only if $U_j h = h$ for every multi-index j, we get an important corollary to Theorem 5.1.

COROLLARY 5.1. *If h is in $\mathscr{D}'(R^n)$, then $h = \lim_{j \to \infty} U_j f$ for some distribution f in $\mathscr{D}'(R^n)$ if and only if there exist constants c_ν, $1 \leqq \nu \leqq 2^n$, such that*

$$h(t) = \sum_{\nu=1}^{2n} c_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1,\cdots,n\}}} \text{p.v.} \frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right).$$

The following theorem is an extension to n dimensions of Theorem 3.3.

THEOREM 5.2. *If f is in $\mathscr{D}'(R^n)$ and there are two complex numbers p_1, p_2 with $\operatorname{Re} p_1 \neq \operatorname{Re} p_2$ and a positive integer i, $1 \leqq i \leqq n$, such that $\{U_j e^{-p_1 t_i} f(t)\}$ and $\{U_j e^{-p_2 t_i} f(t)\}$ both converge in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then for every complex number q for which the sequence converges, there is a distribution $h(q)$ in $\mathscr{D}'(R^{n-1})$ such that*

$$\lim_{j \to \infty} U_j e^{-q t_i} f(t) = \delta(t_i) \otimes h(q). \tag{5.5}$$

Proof. We may assume, without loss of generality, that $p_1 = 0$ and that $p_2 = p$ has real part greater than zero. Define $h(0) = \lim_{j \to \infty} U_j f(t)$ and $h(p) = \lim_{j \to \infty} U_j e^{-p t_i} f(t)$. Let ϕ be a test function in $\mathscr{D}(R^n)$ with support of ϕ in $\{t : t_i > 0\}$. Then, if $j \to \infty$, clearly the sequence $\{e^{-p j_i t_i} \phi(t)\}$ converges to zero in $\mathscr{D}(R^n)$. Therefore we have

$$\begin{aligned} \lim_{j \to \infty} \langle U_j e^{-p t_i} f(t), \phi(t) \rangle &= \lim_{j \to \infty} \langle U_j f(t), e^{-p j_i t_i}, \phi(t) \rangle \\ &= \langle h(0), 0 \rangle \\ &= 0. \end{aligned}$$

But by Corollary 5.1, there are constants c_ν, $1 \leqq \nu \leqq 2^n$, such that

$$h(p) = \sum_{\nu=1}^{2n} c_\nu \left(\bigotimes_{\substack{i \in I_\nu \\ I_\nu \subset \{1,\cdots,n\}}} \text{p.v.}\frac{1}{t_i} \right) \otimes \left(\bigotimes_{i \notin I_\nu} \delta(t_i) \right). \tag{5.6}$$

A distribution of the form (5.6) can map every test function with support in $\{t : t_i > 0\}$ to zero only if the coefficient of every term in which the factor p.v. $(1/t_i)$ appears is zero. Therefore, $h(p) = \delta(t_i) \otimes h'(p)$, where $h'(p)$ is in $\mathscr{D}'(R^{n-1})$.

Using a similar argument, just as was done in the one-dimensional case, we can show that $h(0) = \delta(t_i) \otimes h'(0)$ for some distribution $h'(0)$ in $\mathscr{D}'(R^{n-1})$. Thus, for any q at which the sequence $\{U_j e^{-qt_i} f(t)\}$ converges, its limit is of the form given by (5.5), and the theorem is proved.

COROLLARY 5.2. *If $f \in \mathscr{D}'(R^n)$ is such that $\lim_{j\to\infty} U_j f = h(0)$ and for each i, $1 \leqq i \leqq n$, there is a complex p_i such that* $\operatorname{Re} p_i \neq 0$ *and* $\lim_{j\to\infty} U_j e^{-p_i t_i} f(t) = h(p_i)$, *then there is a constant c such that*

$$h(0) = c\delta(t) = c\delta(t_1, \cdots, t_n).$$

Proof. By Theorem 5.2 it can be seen that for each $i = 1, 2, \cdots, n$ there is a distribution $h_i(0)$ in $\mathscr{D}'(R^{n-1})$ such that $h(0) = \delta(t_i) \otimes h_i(0)$. This can happen only if $h(0) = c\delta(t)$.

COROLLARY 5.3. *Let Ω be an open set in C^n with the property that if p is in Ω, then the sequence $\{U_j T^{-p} f\}$ converges in $\mathscr{D}'(R^n)$ to a distribution $h(p)$ as $j \to \infty$. Then for every p in Ω there is a constant $c(p)$ such that*

$$h(p) = c(p)\delta(t_1, t_2, \cdots, t_n).$$

Proof. Let p be in Ω and pick $\varepsilon > 0$ such that the set $\{q : |q - p| < \varepsilon\}$ is also in Ω. Let $g(t) = e^{-pt} f(t)$. Then $\lim_{j\to\infty} U_j g(t) = h(p)$ and for $i = 1, 2, \cdots, n$, $\lim_{j\to\infty} U_j e^{-\varepsilon t_i/2} g(t) = h({}_ip)$, where, if $p = (p_1, p_2, \cdots, p_n)$, then

$${}_ip = (p_1, p_2, \cdots, p_i + \varepsilon/2, \cdots, p_n).$$

Therefore, by Corollary 5.2, $\lim_{j\to\infty} U_j g(t) = c(p)\delta(t)$, which completes the proof.

A generalization of Theorem 3.4 to $\mathscr{D}'(R^n)$ does not change the statement of the theorem significantly; however, it is included here for completeness.

THEOREM 5.3. *If f is a distribution such that the sequence $\{U_j f\}$ converges in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then f is in $\mathscr{S}'(R^n)$.*

The proof of Theorem 5.3 differs from that of Theorem 3.4 only in details which are obvious. In particular, sets of the form $\{t : |t| \leqq k\}$ must be substituted for intervals $[-k, k]$, and the value of the constant L introduced in (3.18) must be adjusted.

COROLLARY 5.4. *If p_1 and p_2 are in C^n with* $\operatorname{Re} p_1 < \operatorname{Re} p_2$ *and are such that $\{U_j T^{-p_1} f\}$ and $\{U_j T^{-p_2} f\}$ both converge in $\mathscr{D}'(R^n)$ as the multi-index $j \to \infty$, then whenever p is in C^n with* $\operatorname{Re} p_1 < \operatorname{Re} p < \operatorname{Re} p_2$,

$$\lim_{j\to\infty} U_j T^{-p} f = \langle T^{-p} f, 1 \rangle \delta.$$

The proof of Corollary 5.4 follows from Theorem 5.3, Theorem 2.3 and Theorem 3.1.

We come next to the extension of the Laplace transform to distributions in $\mathscr{D}'(R^n)$. Since the definitions and theorems in § 4 were based on the work done in previous sections, all of which has now been extended to n dimensions, the extensions of the results of § 4 are, for the most part, straightforward. We will state the n-dimensional results without proof but will comment on the differences caused by going to $\mathscr{D}'(R^n)$.

We say that a distribution f in $\mathscr{D}'(R^n)$ is Laplace transformable if there are two constants α, β in R^n such that whenever p is in C^n with $\alpha < \operatorname{Re} p < \beta$, then $T^{-p}f$ is in $\mathscr{B}'_0(R^n)$. If, for any other pair α', β' satisfying the same property, $\alpha' \geqq \alpha$ and $\beta' \leqq \beta$, then we call the subset of C^n

$$\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$$

the domain of definition of the Laplace transform for f. The existence of the set Ω again follows from Theorem 2.3.

The characterizations (4.1) and (4.2) of $\mathscr{L}[f]$ in one dimension are also valid in n dimensions, so we have

$$\mathscr{L}[f](p) = \frac{1}{\phi(0)} \lim_{j\to\infty} \langle U_j T^{-p} f, \phi\rangle, \tag{5.7}$$

where $p \in \Omega$ and ϕ is in $\mathscr{D}(R^n)$ with $\phi(0) \neq 0$, and

$$\mathscr{L}[f](p) = \langle T^{-p} f, 1\rangle. \tag{5.8}$$

Formulas (5.7) and (5.8) are exactly the same as (4.1) and (4.2) but are interpreted in n dimensions. Clearly, $\mathscr{L}[f]$ is a linear complex-valued function of the n-dimensional complex variable with domain Ω.

Theorem 4.1 on the analyticity of the transform may be extended to give the following theorem.

THEOREM 5.4. *If $f \in \mathscr{D}'(R^n)$ is Laplace transformable in Ω, then $\mathscr{L}[f]$ is analytic in Ω and*

$$\frac{\partial}{\partial p_i}\mathscr{L}[f](p) = \mathscr{L}[-t_i f(t)](p).$$

The proof of Theorem 5.4 requires the use of Hartog's theorem (Bochner and Martin [1]) which says that a complex-valued function of n complex variables is analytic if it is analytic in each variable separately with all other variables held constant. The proof that $\mathscr{L}[f]$ is analytic in each p_i separately is essentially the same as the proof of Theorem 4.1.

The convolution theorem requires no change.

THEOREM 5.5. *If f and g are Laplace transformable distributions in $\mathscr{D}'(R^n)$ and the domains of their respective transforms have intersection Ω, then $f * g$ is Laplace transformable in Ω and for every p in Ω,*

$$\mathscr{L}[f * g](p) = \mathscr{L}[f](p)\mathscr{L}[g](p).$$

THEOREM 5.6 (Inversion theorem). *If f is Laplace transformable in*

$$\Omega = \{p : \alpha < \operatorname{Re} p < \beta\},$$

then for any fixed $\sigma \in R^n$ such that $\alpha < \sigma < \beta$, we have

$$f(t) = \lim_{r \to \infty} \frac{1}{(2\pi i)^n} \int_{\sigma - ir}^{\sigma + ir} e^{pt} \mathscr{L}[f](p)\, dp, \tag{5.9}$$

where the limit is taken in $\mathscr{D}'_t(R^n)$ as $r \to \infty$ in R^n. The integral in (5.9) *is taken over the subset of n-dimensional complex space defined by*

$$\{p : \operatorname{Re} p_i = \sigma_i, |\operatorname{Im} p_i| < r_i, 1 \leqq i \leqq n\}.$$

THEOREM 5.7 (Uniqueness theorem). *If f and g are Laplace transformable distributions in $\mathscr{D}'(R^n)$ such that the domains of their transforms have intersection $\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$, and there is a fixed $\sigma \in R^n$ with $\alpha < \sigma < \beta$ such that whenever* $\operatorname{Re} p = \sigma$ *we have $\mathscr{L}[f](p) = \mathscr{L}[g](p)$; then $f = g$ as distributions.*

THEOREM 5.8. *If $F(p)$ is analytic for p in $\Omega = \{p : \alpha < \operatorname{Re} p < \beta\}$ and is bounded in Ω by a polynomial in $|\omega|$ (or in $|p|$), then $F(p) = \mathscr{L}[f](p)$, where the distribution f is defined as a limit in $\mathscr{D}'_t(R^n)$ by*

$$f(t) = \lim_{r \to \infty} \frac{1}{(2\pi i)^n} \int_{\sigma - ir}^{\sigma + ir} e^{pt} F(p)\, dp \tag{5.10}$$

for any fixed $\sigma \in R^n$ such that $\alpha < \sigma < \beta$.

Theorem 4.4, which is the one-dimensional analogue of Theorem 5.8, was proved in four steps, one of which required Cauchy's theorem. An n-dimensional analogue of Cauchy's theorem can be found in Fuks [5].

The transform formulas developed in § 4 also have n-dimensional analogues. For completeness, we list them here. In the following formulas, k is a multi-index, t and τ are in R^n, and p and q are in C^n. Recall that $t^k = t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}$,

$$\partial^k = \frac{\partial^{k_1 + k_2 + \cdots + k_3}}{\partial p_1^{k_1} \partial p_2^{k_2} \cdots \partial p_n^{k_n}} \quad \text{and} \quad \frac{p}{k} = \frac{p_1}{k_1}, \frac{p_2}{k_2}, \cdots, \frac{p_n}{k_n}.$$

$$\mathscr{L}[f^{(k)}](p) = p^k \mathscr{L}[f](p), \tag{5.11}$$

$$\mathscr{L}[t^k f(t)](p) = (-1)^{|k|} \partial^k \mathscr{L}[f](p), \tag{5.12}$$

$$\mathscr{L}[f(t - \tau)](p) = e^{-p\tau} \mathscr{L}[f](p), \tag{5.13}$$

$$\mathscr{L}[e^{-qt} f(t)](p) = \mathscr{L}[f](p + q), \tag{5.14}$$

$$\mathscr{L}[U_k f](p) = \mathscr{L}[f](p/k). \tag{5.15}$$

Appendix. This Appendix contains a lemma which is used in the proof of Theorem 2.2, along with the construction of a partition of unity for R^n which satisfies certain special properties. In order to construct such a partition of unity, let $\xi(t)$ be a function in $\mathscr{D}(R)$ that satisfies the following properties:

$$\xi(t) \geqq 0 \quad \text{for every } t, \tag{A.1}$$

$$\text{support of} \quad \xi(t) \subset [-\tfrac{1}{2}, \tfrac{1}{2}], \tag{A.2}$$

$$\xi(t) = \xi(-t) \quad \text{for every } t, \tag{A.3}$$

$$\int_{-1/2}^{1/2} \xi(t)\, dt = 1. \tag{A.4}$$

An example of such a function is

$$\xi(t) = \begin{cases} \dfrac{1}{A} \exp\left[\dfrac{1}{4t^2 - 1}\right], & |t| < \frac{1}{2}, \\ 0, & |t| \geqq \frac{1}{2}, \end{cases}$$

where

$$A = \int_{-1/2}^{1/2} \exp\left[\frac{1}{4t^2 - 1}\right] dt.$$

Let the function $\sigma(t)$ be defined by

$$\sigma(t) = \int_{-\infty}^{t} [\xi(\tau + \tfrac{1}{2}) - \xi(\tau - \tfrac{1}{2})]\, d\tau.$$

Then $\sigma \in \mathscr{D}(R)$, $\sigma(0) = 1$, $\sigma^{(j)}(0) = 0$ as long as $j \geqq 1$, $\sigma(t) = \sigma(-t)$ for all t, and support $\sigma \subset [-1, 1]$. Also, if $t \in (0, 1)$,

$$\begin{aligned} \sigma(t) + \sigma(t - 1) &= \int_{-\infty}^{t} \left[\xi\left(\tau + \frac{1}{2}\right) - \xi\left(\tau - \frac{1}{2}\right)\right] \mathrm{d}\tau + \int_{-\infty}^{t-1} \xi\left(\tau + \frac{1}{2}\right) d\tau \\ &= 1 - \int_0^t \xi\left(\tau - \frac{1}{2}\right) d\tau + \int_{-1}^{t-1} \xi\left(\tau + \frac{1}{2}\right) d\tau \\ &= 1 - \int_0^t \xi\left(\tau - \frac{1}{2}\right) d\tau + \int_0^t \xi\left(s - \frac{1}{2}\right) ds \\ &= 1, \quad \text{where } s = \tau + 1. \end{aligned}$$

Now, for $t \in R^n$, define $\gamma_0(t) = \sigma(|t|)$. Clearly, γ_0 is infinitely differentiable as long as $t \neq 0$. If we define $\gamma_0^{(j)}(0) = 0$ for every multi-index j with $|j| > 0$, then γ_0 is in $\mathscr{D}(R^n)$. For every positive integer k, define the function γ_k by

$$\gamma_k(t) = \sigma(|t| - k).$$

Then the support of γ_k is contained in $\{t : k - 1 \leqq |t| \leqq k + 1\}$, and γ_k is in $\mathscr{D}(R^n)$ for every k. Also, if $k < |t| \leqq k + 1$, then

$$\begin{aligned} \sum_{v=0}^{\infty} \gamma_v(t) &= \gamma_k(t) + \gamma_{k+1}(t) = \sigma(|t| - k) + \sigma(|t| - k - 1) \\ &= 1, \end{aligned}$$

since $|t| - k$ is in $(0, 1)$. Therefore $\{\gamma_k\}_{k=0}^{\infty}$ is a locally finite partition of unity which has the additional property that

$$\sup_t \left| \partial^j \left[\sum_{v \in I} \gamma_v \right] \right| \leqq \sup_t |\partial^j \gamma_0|$$

for any multi-index j and any subset I of nonnegative integers.

Next, we prove as a lemma a fact about complex numbers which is used in the proof of Theorem 2.2.

LEMMA. *If* $\{\alpha_j\}_{j\in J}$ *is a set of complex numbers with the property that there is a number B such that for every finite subset I of J we have*

$$\left|\sum_{j\in I}\alpha_j\right| \leqq B,$$

then it is also true that

$$\sum_{j\in I}|\alpha_j| \leqq 4B$$

for every finite subset I of J.

Proof. Suppose that there is a finite subset I' of J such that $\sum_{j\in I'}|\mathrm{Re}\,\alpha_j| > 2B$. Then there must be a subset I'' of I' such that all the numbers $\mathrm{Re}\,\alpha_j$ with $j\in I''$ have the same sign and

$$\left|\sum_{j\in I''}\mathrm{Re}\,\alpha_j\right| = \sum_{j\in I''}|\mathrm{Re}\,\alpha_j| > B.$$

But by the hypothesis of the lemma,

$$\left|\sum_{j\in I''}\mathrm{Re}\,\alpha_j\right| \leqq \left|\sum_{j\in I''}\alpha_j\right| \leqq B,$$

so we have reached a contradiction. Therefore, for every finite subset I of J, $\sum_{j\in I}|\mathrm{Re}\,\alpha_j| \leqq 2B$, and similarly $\sum_{j\in I}|\mathrm{Im}\,\alpha_j| \leqq 2B$. Thus

$$\begin{aligned}\sum_{j\in I}|\alpha_j| &\leqq \sum_{j\in I}|\mathrm{Re}\,\alpha_j| + \sum_{j\in I}|\mathrm{Im}\,\alpha_j| \\ &\leqq 4B,\end{aligned}$$

and the lemma is proved.

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TOTAL POSITIVITY PROPERTIES OF GENERATING FUNCTIONS*

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Abstract. In this note, we strengthen results obtained in Keilson (1972). Our main result is: Let $P_0(z) = \sum_{i=0}^{\infty} p_i z^i$ be the generating function of the sequence $\{p_i\}_{i=0}^{\infty}$, with p_i real for $i = 0, 1, \cdots, N-1$, $p_N > 0$ and $p_i = 0$ for $i = N+1, N+2, \cdots$. Let $p_i(t)$ be defined by $P(z+t) = \sum_{i=0}^{\infty} p_i(t) z^i$. Then (a) there exists a smallest nonnegative value t_r^* such that $p_{i+j}(t_r^*)$ has the sign reverse rule property of order r (RR_r) in $i, j = 0, 1, 2, \cdots$ (see Karlin (1968)) for $r = 1, 2, \cdots$; (b) $p_{i+j}(t)$ is RR_r in $i, j = 0, 1, 2, \cdots$ for each fixed $t \geqq t_r^*$, $r = 1, 2, \cdots$; and (c) $t_1^* \leqq t_2^* \leqq \cdots$. Binomial moment inequalities for Pólya frequency functions are an immediate consequence.

1. Introduction. Keilson (1972) considers the generating function

$$P_0(z) = \sum_{i=0}^{n} p_i z^i$$

for a set of nonnegative masses on the integers $0, 1, 2, \cdots, N$, with $p_N > 0$. From this, he defines the family of related generating functions

$$P_t(z) = P(z+t) = \sum_{k=0}^{n} p_k(t) z^k.$$

He shows that the semi-infinite interval $[0, \infty)$ has precisely one value t^*, such that $\{p_k(t)\}_0^N$ is log-concave for $t \geqq t^*$, and is not log-concave for $t < t^*$.

One interesting interpretation of the result is that starting with an arbitrary set of nonnegative masses, a family of log-concave sequences is *produced*; this differs from the usual situation in which log-concavity is shown to be *preserved* under some standard mathematical operation such as convolution or integration with respect to a kernel. Also, as a direct consequence of his result, Keilson obtains inequalities on the binomial moments:

$$B_{r+1}^{1/(r+1)} \leqq B_r^{1/r}, \qquad r = 1, 2, \cdots, \tag{1.1}$$

where $B_r = \sum p_j \binom{j}{r}$ is the binomial moment of order r.

Log-concave and log-convex functions have been shown to have many applications in analysis, statistics, reliability theory, inventory theory, and many other fields. (See, for example, Karlin (1968), Barlow and Proschan (1965), Arrow, Karlin and Scarf (1958), and Keilson and Gerber (1971).) Actually, such functions are special cases of the class of totally positive functions, which are treated definitively in Karlin (1968); totally positive functions constitute a powerful tool in developing inequalities, and have uses in many theoretical and applied fields.

In this note we show that total positivity properties for generating functions hold not only for order two (corresponding to log-concavity treated by Keilson),

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but more generally for each positive integer order. As a direct consequence, we obtain inequalities on the binomial moments generalizing those of (1.1).

To state and prove our results, we need to recall relevant total positivity definitions; see Karlin (1968). p_{ij}, defined for integer i and j, is TP_r if the mth order determinant $|p_{a_ib_j}| \geqq 0$ for each choice of integer $a_1 \leqq \cdots \leqq a_m$, $b_1 \leqq \cdots \leqq b_m$, $m = 1, \cdots, r$; we also call the infinite matrix (p_{ij}) TP_r. If p_{ij} is TP_r and of the special form p_{i-j}, then we say p_i is PF_r. We say $p_{i,j}$ is RR_r if the mth order determinant $|p_{a_ib_j}| \geqq 0$ for each choice of integer $a_1 \leqq \cdots \leqq a_m$, $b_m \leqq \cdots \leqq b_1$, $m = 1, \cdots, r$. We also note that p_i PF_r implies that p_{i+j} is RR_r. In the special case $r = 2$ (corresponding to Keilson's results), p_{i+j} RR_2 is equivalent to p_i PF_2, i.e., log concave.

In this note, we confine attention to sequences $\{p_0, p_1, p_2, \cdots\}$; by definition, $p_{-1} = p_{-2} = \cdots = 0$.

2. Total positivity results. The following generalization of Theorem 1 of Keilson (1972) is the main result of this section.

THEOREM 2.1. *Let $P_0(z) = \sum_{i=0}^{\infty} p_i z^i$ be the generating function of the sequence $\{p_i\}_{i=0}^{\infty}$, with p_i real for $i = 0, 1, \cdots, N-1$, $p_N > 0$, and $p_i = 0$ for $i = N+1, N+2, \cdots$. Let the sequence $\{p_i(t)\}_0^{\infty}$ be defined for all $t \geqq 0$ by*

$$P_t(z) = P_0(z+t) = \sum_0^{\infty} p_i(t) z^i, \tag{2.1}$$

so that

$$P_i(t) = \sum_{k=0}^{\infty} \binom{k}{i} t^{k-1} p_k. \tag{2.2}$$

Then (a) *there exists a smallest nonnegative value t_r^* such that $p_{i+j}(t_r^*)$ is RR_r in $i, j = 0, 1, 2, \cdots$, for each $r = 1, 2, \cdots$;* (b) *$p_{i+j}(t)$ is RR_r in $i, j = 0, 1, 2, \cdots$ for each fixed $t \geqq t_r^*$, $r = 1, 2, \cdots$; and* (c) *$t_1^* \leqq t_2^* \leqq t_3^* \leqq \cdots$.*

To prove (a), it will be helpful to first establish Lemmas 2.2 and 2.3 below.

LEMMA 2.2. *Under the hypotheses of Theorem 2.1, there exists a finite value t_r such that $p_i(t)$ is PF_r in integer i for each $t \geqq t_r$, $r = 1, 2, \cdots$.*

Proof. Write

$$P_0(z) = p_N \prod_i (z - r_i) \prod_j [(z - w_j)(z - \bar{w}_j)], \tag{2.3}$$

where the r_i are real zeros of $P_0(z)$ and the w_j and $\bar{w}_j$ are complex zeros taken in conjugate pairs. Hence

$$P_0(z+t) = p_N \prod_i [z + (t - r_i)] \prod_j [(z + \zeta_j)(z + \bar{\zeta}_j)], \tag{2.4}$$

where, corresponding to $w_j = x_j + iy_j$, $\zeta_j = (t - x_j) + iy_j$. Let $A_i = \max_{i,j}\{r_i, x_j\}$. Then for $t \geqq A_1$, $P_0(z+t)$ is a polynomial in z with nonnegative coefficients.

Consider the quadratic $Q_j(z) = z^2 + (\zeta_j + \bar{\zeta}_j)z + |\zeta_j|^2 = z^2 + a_j z + b_j$, where $a_j = 2(t - x_j)$ and $b_j = (t - x_j)^2 + y_j^2$. To show that the coefficients of $Q_j(z)$

constitute a PF_r-sequence, it suffices to show that the infinite matrix

$$B = \begin{Vmatrix} 1 & a_j & b_j & 0 & 0 & \cdots \\ 0 & 1 & a_j & b_j & 0 & \cdots \\ 0 & 0 & 1 & a_j & b_j & \cdots \\ 0 & 0 & 0 & 1 & a_j & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{Vmatrix}$$

is TP_r. (See Karlin (1968, p. 393).) As in Karlin (1968, p. 117), B is TP_r if $a_j/(2\sqrt{b_j}) \geqq c_r$, where $c_r = \cos(\pi/(r+1))$. Simplifying, B is TP_r if

$$t \geqq x_j + y_j c_r(1 - c_r^2)^{-1/2}.$$

Let $A_2 = \max_j \{x_j + y_j c_r(1 - c_r^2)^{-1/2}\}$. Then for each $t \geqq A_2$, the coefficients of each $Q_j(z)$ constitute a PF_r-sequence. Since the convolution of PF_r-sequences is PF_r, we conclude that $p_i(t)$ is PF_r in integer i for each $t \geqq t_r \overset{\text{def}}{=} \max(A_1, A_2)$. Q.E.D.

LEMMA 2.3. *Let p_{i+j} be RR_j in $i, j = 0, 1, 2, \cdots$. Then $p_{i+j}(t)$ is RR_r in $i, j = 0, 1, 2, \cdots$ for each $t > 0$.*

Proof. First note that the function $\phi(k+1, i+1) \overset{\text{def}}{=} \binom{k}{i} t^{k-i}$ is TP_∞ in $i, k = 0, 1, 2, \cdots$, and also obeys the semigroup property

$$\phi(k, i+j) = \sum_h \phi(h, i)\phi(k-h, j). \tag{2.5}$$

(See Karlin (1968, p. 142).) From the representation (2.2) and Theorem 5.4 of Karlin (1968, p. 130), the conclusion now follows. Q.E.D.

Remark. Note that Lemma 2.3 holds even when the support of $\{p_i\}$ is $\{0, 1, 2, \cdots\}$.

With the aid of Lemmas 2.2 and 2.3, we may now prove Theorem 2.1.

Proof of Theorem 2.1. (a) By taking t_r^* as the infimum of nonnegative t_r for which $p_{i+j}(t_r)$ is RR_r in $i, j = 0, 1, 2, \cdots$, the desired conclusion is an immediate consequence of Lemmas 2.2, 2.3, and the fact that a convergent RR_r-sequence has an RR_r-limit.

(b) follows directly from Lemma 2.3.

(c) is an obvious consequence of the fact that RR_{r+1} implies RR_r for $r = 1, 2, \cdots$. Q.E.D.

The moment inequalities of Theorem 2 of Karlin, Proschan and Barlow (1961) may now be obtained from Theorem 2.1 above; the argument is similar to that used in proving Theorem 2 of Keilson (1972).

THEOREM 2.4 (Karlin, Proschan, Barlow). *Let $\{p_m\}_0^\infty$ be a PF_r-sequence, with binomial moments $B_m = \sum p_j \binom{j}{m}$. Then B_{m+n} is RR_r in $m = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots$.*

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ON EXISTENCE AND NONEXISTENCE IN THE LARGE OF SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS WITH A NONLINEAR BOUNDARY CONDITION*

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Abstract. This paper deals with solutions $u(t, x)$ of parabolic differential inequalities (a) $u_t \leqq Lu$, or (b) $u_t \geqq Lu$, respectively, where L is a linear, weakly elliptic differential operator of second order. The behavior of u for large t is studied under the assumption, that on the lateral boundary a nonlinear boundary condition of the form (a) $\partial u/\partial v \leqq f(u)$, or (b) $\partial u/\partial v \geqq f(u)$, is imposed, where $f(z) \to \infty$ as $z \to \infty$. It is shown that the value of the integral $\int^\infty dz/f(z)f'(z)$ is crucial for the growth properties of u. If this integral is infinite, then we have the case of global existence, i.e., any solution u of (a) is bounded in bounded sets. If, on the other hand, the integral is finite, then all solutions u of (b) with large initial values become infinite in finite time.

1. Introduction. Let $D \subset \mathbb{R}^n$ be a bounded open set and let $G = (0, T) \times D$, $R_0 = \{0\} \times \bar{D}$, $R_1 = (0, T) \times \partial D$. We consider functions $u(t, x)$, $x = (x_1, \cdots, x_n)$, satisfying inequalities

$$u_t \leqq Lu \quad \text{in } G, \qquad \frac{\partial u}{\partial v} \leqq f(u) \quad \text{on } R_1 \tag{1}$$

or

$$u_t \geqq Lu \quad \text{in } G, \qquad \frac{\partial u}{\partial v} \geqq f(u) \quad \text{on } R_1, \tag{2}$$

where

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t, x)u_{x_ix_j} + \sum_{i=1}^{n} b_i(t, x)u_{x_i} + c(t, x)u + d(t, x). \tag{3}$$

It is assumed throughout that the matrix $a = (a_{ij})$ is positive semidefinite in G and that $f:\mathbb{R} \to \mathbb{R}$ is continuous. Let Z be the class of functions u which are continuous in $[0, T) \times \bar{D}$, which have continuous derivatives u_t, $u_x = (u_{x_i})$, $u_{xx} = (u_{x_ix_j})$ in G and for which the *outer* normal derivative

$$\frac{\partial u}{\partial v} = u_v(t, x) = \lim_{\alpha \to +0} [u(t, x) - u(t, x - \alpha v)]/\alpha, \qquad (t, x) \in R_1,$$

exists (we assume that an outer normal $v \in \mathbb{R}^n$, $|v| = 1$, satisfying $x - \alpha v \in D$ for small $\alpha > 0$ exists at every point $x \in \partial D$).

The functions $f(z)$ we have in mind are positive and tend to infinity as $z \to \infty$. Thus, in the heat flow interpretation, the condition $u_v = f(u)$ is an absorption law which makes heat flow into the body. Our objective is (i) to find growth conditions on f under which solutions of (1) are bounded in any bounded set G, and (ii) to find conditions on f such that solutions of (2) become infinite in finite time.

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In particular if solutions of a boundary value problem

$$(4) \qquad u_t = Lu \quad \text{in } G, \quad u_\nu = f(u) \quad \text{on } R_1, \quad u = u_0(x) \quad \text{on } R_0$$

are considered, case (i) leads to global existence (for all positive t), while in case (ii) there is no global existence.

Our research was initiated by a recent paper of Levine and Payne [1]. In this paper it is proved that in the case of the heat equation, $u_t = \Delta u$, (4) has no global solution if $f(z) = |z|^{1+\varepsilon}h(z)$, h increasing, $\varepsilon > 0$, and if u_0 is sufficiently large. Similar results are given or indicated for several other types of parabolic and hyperbolic boundary value problems.

By using the theory of parabolic differential inequalities, we are able to give a rather complete characterization of the two cases (i) and (ii), depending on growth properties of f. It turns out that for a wide class of parabolic differential equations (strong parabolicity is not assumed, and "heating" or "cooling" terms are permitted) the behavior of solutions for large t can be described in terms of the integral

$$\int^\infty \frac{dz}{f(z)f'(z)}.$$

If this integral diverges, we have case (i) (global existence), if it converges, we have case (ii). To illustrate the result, the function $f(z) = z\sqrt{\log z}$ belongs to case (i), while $f(z) = z(\log z)^{(1+\varepsilon)/2}$, $\varepsilon > 0$, belongs to case (ii).

2. A monotonicity theorem. The following theorem on parabolic differential inequalities is basic for our treatment. It is a special case of Theorem 31.IV in [2]. There are no assumptions on the coefficients in L or on f (except that the matrix (a_{ij}) be positive semidefinite).

THEOREM. *Let $v, w \in Z$ and*

(a) $v - Lv < w - Lw$ *in* G,

(b) $\partial v/\partial \nu - f(v) < \partial w/\partial \nu - f(w)$ *on* R_1,

(c) $v < w$ *on* R_0.

Then

$$v < w \quad \textit{in } G.$$

3. Upper bounds. We shall use the above monotonicity theorem in order to obtain lower bounds $v(t, x)$ for solutions $u(t, x)$ of (2) and upper bounds $w(t, x)$ for solutions $u(t, x)$ of (1). In this section, we are dealing with the latter case, i.e., we assume that u satisfies inequalities (1). We try to find an upper bound w of the form

$$w(t, x) = \psi(s), \quad \text{where } s = g(t) + h(x).$$

Using the notation $h_i = \partial h/\partial x_i$, $h_{ij} = \partial^2 h/\partial x_i \partial x_j$, $h_\nu = \partial h/\partial \nu$, we are led to the following inequalities:

(a) $g' > \sum_{i,j} a_{ij}\left(h_{ij} + \dfrac{\psi''}{\psi'}h_i h_j\right) + \sum_i b_i h_i + \dfrac{c\psi + d}{\psi'}$ in G,

(b) $\psi' h_\nu > f(\psi)$ on R_1,

(c) $\psi(g(0) + h(x)) > u(0, x)$ on R_0.

The functions ψ we consider are positive with positive derivative ψ' (thus the division by ψ' in (a) is justified).

THEOREM 1. *Let $f(z)$ and $f'(z)$ be continuous, positive and increasing for $z \geqq z_0$ and let*

$$\int_{z_0}^{\infty} \frac{dz}{f(z)f'(z)} = \infty. \tag{5}$$

Furthermore, let

$$|a_{ij}| \leqq A, \quad |b_i| \leqq A, \quad c \leqq A, \quad d \leqq A$$

and let condition (H) *be satisfied*:

(H) *There exists a function $h(x) \in C^1(\bar{D}) \cap C^2(D)$ such that*

$$\sum_{i,j} a_{ij}h_{ij} \leqq A \quad \text{in } G \quad \text{and} \quad \frac{\partial h}{\partial \nu} \geqq \delta > 0 \quad \text{on } \partial D.$$

Then there exists a function $w(t, x)$, continuous in $[0, \infty) \times \bar{D}$ and tending to ∞ as $t \to \infty$ uniformly in x, such that for any $u \in Z$ satisfying inequalities (1),

$$u(0, x) < w(t_0, x) \quad (t_0 \geqq 0) \quad \text{implies } u(t, x) < w(t + t_0, x) \quad \text{in } G.$$

The function $w(t, x)$ depends only on f, A, δ and $\max |h|$, $\max |h_x|$.

Proof. The proof is very simple if $f(z) = Bz$ or $f(z) = Bz\sqrt{\log z}$. In the first case conditions (a)–(c) are satisfied by putting $\psi(s) = e^{\alpha s}$, $g(t) = \beta t + \gamma$ with an appropriate choice of α, β, γ; in the second case one might take $\psi(s) = e^{\alpha s^2}$, $g(t) = \beta e^{\gamma t}$.

In the general case we assume that $h(x) > 0$ in D and $h_\nu > 1$ on ∂D. This is justified since h may be replaced by $Bh + C$. We define $\psi(s)$ by

$$\psi' = f(\psi), \quad \psi(0) = z_0,$$

thus satisfying condition (b). Since f' has a positive lower bound, it follows from (5) that the integral from z_0 to ∞ of the function $1/f$ is divergent. Therefore $\psi(s)$ exists for $0 \leqq s < \infty$ and grows at least like $e^{\alpha s}$, $\alpha > 0$. Now we consider the function $p(s) = \psi''/\psi'$. From the differential equation for ψ we get $p(s) = f'(\psi(s))$ and hence (substitution $\psi(\sigma) = z$)

$$\int_0^s \frac{d\sigma}{p(\sigma)} = \int_0^s \frac{d\sigma}{f'(\psi(\sigma))} = \int_{z_0}^{\psi(s)} \frac{dz}{f(z)f'(z)} \to \infty \quad \text{as } s \to \infty. \tag{6}$$

Therefore the solution $g(t)$ of

$$g' = B + Cp(g + N) \tag{a_1}$$

with initial value $g(0) = 0$ exists for $0 \leqq t < \infty$ for any choice of B, C, $N > 0$. If we choose these constants in such a way that

$$\sum_{i,j} a_{ij}h_{ij} + \sum_i b_i h_i + (c\psi + d)/\psi' \leqq B, \qquad 0 \leqq \sum_{i,j} a_{ij}h_i h_j < C, \quad 0 \leqq h < N,$$

then the function $g(t)$ obtained from (a_1) satisfies (a), and the same is true for the function $g(t + t_0)$, where $t_0 > 0$. Note that $p(s)$ is increasing, hence $p(g(t) + h(x)) < p(g(t) + N)$, and that $\psi/\psi' \leqq \psi/f(\psi) \leqq 1/f'(z_0)$.

If, for a given $u \in Z$ satisfying (1), t_0 is determined in such a way that

$$u(0,x) < \psi(g(t_0) + h(x)) \quad \text{for } x \in \bar{D},$$

then the function $W = \psi(g(t_0 + t) + h(x))$ is an upper bound for u. This follows from the monotonicity theorem, applied to the functions u and W (instead of v and w). The conditions (a)–(c) are satisfied. Hence, the function $w = \psi(g(t) + h(x))$ has the properties stated in Theorem 1.

4. Lower bounds. It is assumed that $u \in Z$ satisfies inequalities (2). Using again the monotonicity theorem, we shall construct lower bounds v for u, which tend to infinity as $t \to T_0 < \infty$. The next theorem gives conditions under which such a construction is feasible.

THEOREM 2. *Let f, f' be continuous, positive and increasing for $z \geqq z_0$ and let*

$$\int_{z_0}^{\infty} \frac{dz}{f(z)f'(z)} < \infty. \tag{7}$$

Assume that there exist a unit vector $\gamma \in \mathbb{R}^n$ and two positive constants δ, A such that

$$\sum_{i,j=1}^{n} a_{ij}\gamma_i\gamma_j \geqq \delta \quad \text{in } G, \qquad |b_i| \leqq A, \qquad c \geqq -A, \qquad d \geqq -A \quad \text{in } G.$$

Then there exists a function $v(t,x)$, continuous in $[0, T_0) \times \bar{D}$ $(0 < T_0 < \infty)$ and tending to infinity as $t \to T_0$, uniformly in x, such that for any $u \in Z$ satisfying (2),

$$u(0,x) > v(t_0, x) \quad (0 < t_0 < T_0)$$

implies

$$u(t,x) > v(t + t_0, x) \quad \text{in } G \quad \text{and} \quad T \leqq T_0 - t_0.$$

The function v depends only on δ, A, f and the diameter of D.

Proof. The function

$$v(t,x) = \psi(s), \quad s = g(t) + h(x),$$

is, according to the monotonicity theorem of § 2, a lower bound for u if the three inequalities (a)–(c) of § 3 are satisfied, but with $>$ replaced by $<$.

Let

$$h(x) = \beta + \varepsilon \sum_{i=1}^{n} \gamma_i x_i,$$

where γ is given by the hypothesis on (a_{ij}) and where β and $\varepsilon > 0$ are determined in such a way that $h > 0$ in D and $h_\nu < 1$ on ∂D. The function $\psi(s)$ is defined exactly as in the proof of Theorem 1, as solution of $\psi' = f(\psi)$, $\psi(0) = z_0$. Then (b) is satisfied. Let us assume for the moment that the solution ψ exists for $0 \leqq s < \infty$, i.e., that the integral from z_0 to ∞ of $1/f(z)$ is divergent. Again, we have $p(s) = \psi''/\psi' = f(\psi)$, but this time the integral from 0 to ∞ of the function $1/p$ is finite, according to (6), (7). Therefore the solution $g(t)$ of

$$g' = Cp(g) - B, \qquad (B, C > 0) \tag{a_2}$$

with an initial value $g(0) = g_0 > 0$ satisfying $Cp(g_0) - B \geqq 1$, exists only in a finite interval $0 \leqq t < T_0$, and $g(t) \to \infty$ as $t \to T_0$. Since in the present case (a) reads

$$g' < \frac{\varepsilon^2 \psi''}{\psi'} \sum_{i,j} a_{ij} \gamma_i \gamma_j + \varepsilon \sum_i b_i \gamma_i + \frac{c\psi + d}{\psi'},$$

we choose $C = \delta\varepsilon^2$ and $B > 0$ in such a way that

$$\varepsilon \sum b_i \gamma_i + (c\psi + d)/\psi' > -B.$$

Using these constants, the solution $g(t)$ of (a_2) satisfies condition (a) with the $<$ sign, and the same is true for $g(t + t_0)$, $0 < t_0 < T_0$. It follows as in the proof of Theorem 1 that the function $v(t, x) = \psi(g(t) + h(x))$ has the properties stated in Theorem 2.

If the integral from z_0 to ∞ of $1/f$ is convergent, we may replace f by a function $f_1 \leqq f$ in such a way that the integral of $1/f_1$ is divergent and the integral of $1/f_1 f'_1$ is convergent, and proceed as above. Or, what amounts roughly to the same, we determine $\psi(s)$ from the inequality $\psi' \leqq f(\psi)$ for $0 \leqq s < \infty$, $\psi(0) = z_0$ in such a way that the integral from 0 to ∞ of $1/p(s) = \psi'/\psi''$ is convergent. Then (b) is again satisfied, and the rest of the above proof goes through.

5. Remarks. The above theorems can be generalized in various ways, using essentially the same method of proof.

(a) The boundary condition involving the normal derivative may be prescribed only on part of the lateral boundary of G. To fix our ideas, let $R'_1 = (0, T) \times D_1$, where D_1 is a subset of ∂D, and let R'_0 be the union of R_0 and $R_1 - R'_1$. The monotonicity theorem of § 2 remains true if R_0, R_1 are replaced by R'_0, R'_1. The same holds for Theorem 1. Naturally, the condition involving the initial values of u, $u(0, x) < w(t_0, x)$, has to be modified. In the present case, it reads

$$u(t, x) < w(t_0 + t, x) \quad \text{on } R'_0.$$

Also, Theorem 2 remains true if the condition involving $u(0, x)$ is replaced by $u(t, x) > v(t_0 + t, x)$ on R'_0. But the theorem obtained in this way is of little use since the condition of R'_0 cannot be satisfied if the boundary values of u on R'_0 are bounded. We mention only that a "useful" theorem on lower bounds can be proved if, e.g., $f(z) = z(\log z)^{1+\varepsilon}$ $(\varepsilon > 0)$ for large z. Here, one works with a lower bound of the form $v(t, x) = \psi(g(t)h(x))$, where $h(x) < 0$ on $\partial D - D_1$ and $h(x) > 0$, $h_\nu > 0$ on D_1, and $g(t) = 1/(T_0 - t)$.

(b) The monotonicity theorem, and, as a consequence, the theorems on upper and lower bounds can be carried over to infinite regions D. In this case, growth conditions on the coefficients a_{ij}, b_i, c, d for large values of $|x|$ have to be imposed. See, e.g., [2, 31.XIII]. Once the monotonicity theorem is established, the proofs of Theorems 1 and 2 carry over with minor changes.

(c) It is clear that the above results apply also to nonlinear equations. For example, let us consider the quasilinear case,

$$Lu = \sum_{i,j=1}^{n} a_{ij}(t, x, u, u_x) u_{x_i x_j} + b(t, x, u, u_x).$$

If we assume that

$$|b(t, x, z, p)| \leqq L(1 + |z| + |p|), \qquad z \in \mathbb{R}, \quad p \in \mathbb{R}^n,$$

and that the matrix $(\bar{a}_{ij})$, where $\bar{a}_{ij}(t, x) = a_{ij}(t, x, u, u_x)$, satisfies the conditions of Theorem 1 [Theorem 2], then Theorem 1 [Theorem 2] holds with respect to the quasilinear operator L.

(d) Condition (H) used in Theorem 1 is not very restrictive. If, for example, the a_{ij} are bounded and D is convex with $0 \in D$, then $h(x) = x^2$ $(= x \cdot x)$ serves our purpose. Also, if $u(t, x)$ is any solution of $u_t = \sum_{i,j} a_{ij} u_{x_i x_j}$ in G, $u_\nu > 0$ on R_1, then one may take $h(x) = u(t_0, x)$, t_0 fixed.

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A BOUNDARY VALUE PROBLEM FOR A TWO-DIMENSIONAL SYSTEM WITH A PARAMETER*

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Abstract. We consider the problem $Jz' = [Q(t) + \lambda A(t) - \lambda F(t, z)]z$, $a \cdot z(0, \lambda) = 0$, $b \cdot z(1, \lambda) = 0$, where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, λ is a real parameter, a, b and z are 2-vectors, while Q, A and F are 2×2 real matrices. We assume the linear problem ($F \equiv 0$) has a countable sequence of real eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots$ accumulating at infinity, and that the corresponding eigenfunction has a certain nodal property. λ_k is said to be a bifurcation point for the nonlinear problem if there are nontrivial solutions for λ near λ_k. We establish sufficient conditions for each λ_j, $j = 1, 2, \cdots$, to be a bifurcation point. We also establish global results. We use a relatively simple polar coordinate argument.

1. Introduction. Recently there has been a great deal of mathematical interest in nonlinear eigenvalue problems for both ordinary and partial differential equations. Such problems occur in a variety of applications (see the collection of articles in [3]). Most of the work in ordinary differential equations has been concerned with second order scalar equations, for example, Crandall and Rabinowitz [1], Hartman [2], Rabinowitz [5], Turner [6] and Wolkowisky [7]. Such problems are also of interest for arbitrary systems of equations and have been studied by Keller in an article in [3]. In an earlier paper [4] the authors investigated a nonlinear eigenvalue problem for a second order scalar equation, using relatively simple geometric arguments in contrast to the degree theoretic approach of Crandall and Rabinowitz and the fixed-point arguments of Wolkowisky in the papers cited above. The elementary nature of our arguments makes the theory accessible to those in applied areas, without requiring a command of sophisticated mathematical tools.

In this paper we use these geometric arguments to obtain results for two-dimensional systems, analogous to those in [4]. Although we have no specific application in mind, this appears to be the next logical step in the development of a theory of nonlinear eigenvalue problems for systems of ordinary differential equations. All of the conditions are stated as matrix conditions, i.e., positive definiteness of certain matrices replaces positivity of corresponding coefficients in the scalar case. The nature of these conditions is such that the results of [4] are not directly included (see Remark 2 below the proof of Theorem 1).

2. Preliminaries. For given 2×2 real matrices $Q(t)$, $A(t)$ and $F(t, z)$, mapping from $[0, 1]$ and $[0, 1] \times R^2$ respectively, and given nontrivial 2-vectors a

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and b, with respective components $a_i, b_i, i = 1, 2$, we consider the problem

(1) $$Jz' = [Q(t) + \lambda A(t) - \lambda F(t, z)]z, \qquad z = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

(2a) $$a \cdot z(0, \lambda) = 0 \quad (\cdot \text{ is the scalar product}),$$

(2b) $$b \cdot z(1, \lambda) = 0.$$

Here Greek letters represent real-valued functions or constants, lower-case italic letters represent 2-vectors and $\|\cdot\|$ will denote both the Euclidean norm of a vector and the associated operator norm of a matrix. $A > 0$ will mean that the matrix A is positive definite. As in [4], we introduce polar coordinates $\rho = (\xi^2 + \eta^2)^{1/2}$,

$$z(t, \lambda) = \rho(t, \lambda)\begin{bmatrix} \cos \theta(t, \lambda) \\ \sin (t, \lambda) \end{bmatrix} \equiv \rho(t, \lambda)u(t, \lambda).$$

Our basic hypotheses are as follows.

(H1) The linear problem ($F \equiv 0$) is regular, in the sense that its eigenvalues form a sequence $0 < \lambda_0 < \lambda_1 < \cdots$ and if (ξ_k, η_k) is the eigenfunction (unique up to a constant factor) corresponding to λ_k, then ξ_k has exactly k zeros in $(0, 1)$.

(H2) The matrix functions A, Q and F are continuous on the respective domains $[0, 1]$ and $[0, 1] \times R^2$, and solutions to the IVP (initial value problem) (1), (2a) are unique.

(H3) $A(t) > 0, F(t, 0) \equiv 0, F(t, z) > 0$ for $0 \neq \|z\|$ small.

Remark. Wolkowisky has pointed out the uniqueness of the trivial solution of (1), in the sense that $\xi^2(t, \lambda) + \eta^2(t, \lambda) > 0$ for $\lambda \in R$, $t \in [0, 1]$, for any nontrivial solution. This implies in particular that $\rho > 0$ is differentiable in t on $[0, 1]$.

If we substitute the polar coordinate functions into (1) and (2), and note that $u' = \theta' Ju$, then some simple matrix manipulations yield:

(3) $$\theta'(t, \lambda) = -u^*Mu, \quad \rho' = -\rho u^*JMu, \quad M = Q + \lambda A - \lambda F,$$

(4a) $$\theta(0, \lambda) = \arctan(-a_1/a_2) \equiv \alpha, \quad \arctan \in (-\pi/2, \pi/2],$$

(4b) $$\theta(1, \lambda) = \arctan(-b_1/b_2) - n\pi \equiv \beta_n,$$

where n is an integer. If either of a_2 or b_2 is zero, then the corresponding arctan is $\pi/2$; the star denotes transpose.

LEMMA 1. *If* (H2) *holds and λ is restricted to a compact interval $I = [0, \Lambda]$, then there are constants $\delta(\Lambda) > 0$, $K(\Lambda) > 0$ such that for any solution of* (3) *with $\lambda \in I$,*

$$\rho(0, \lambda) \leqq \delta \Rightarrow \rho(t, \lambda) \leqq K\rho(0, \lambda), \quad t \in [0, 1].$$

In particular, solutions with sufficiently small initial data are extendable.

Proof. Since $\rho' \leqq \|-JM(t, \lambda, \rho, \theta)\|\rho$,

$$\rho(t, \lambda) \leqq \rho(0, \lambda) \exp\left[\int_0^t (\lambda\|A(s)\| + \|Q(s)\| + \lambda\|F(s, z(s, \lambda))\|)\, ds\right].$$

Since F is continuous, there is an m such that $\|F\| \leqq m$ for $t \in [0, 1]$, $\|z(s, \lambda)\| \leqq 1$. Then as long as $\rho(s, \lambda) \leqq 1$ on $[0, t]$, we have

$$\rho(t, \lambda) \leqq \rho(0, \lambda) \exp\left[\int_0^1 (\Lambda\|A\| + \|Q\| + m\Lambda)\, ds\right].$$

Clearly we can choose $\rho(0, \lambda)$ so small that this last implies $\rho(t, \lambda) \leqq 1$ for $t \in [0, 1]$, $\lambda \in [0, \Lambda]$, which in turn implies the desired conclusion.

Let (ζ, ϕ) be the polar radius and angle, respectively, for the associated linear problem. The equation for ϕ ((3) with $F \equiv 0$) does not involve ζ, so ϕ does not depend on $\zeta(0, \lambda)$. For the nonlinear problem, however, θ is a function of (t, λ, μ), $\mu \equiv \rho(0, \lambda)$.

LEMMA 2. *Let* (H1), (H2) *and* (H3) *hold. For each* $\lambda > \lambda_j$ *there exists* $\rho_1(\lambda) > 0$ *such that*

$$0 < \mu < \rho_1 \Rightarrow \theta(1, \lambda, \mu) < \phi(1, \lambda_j) \equiv \beta_j$$

for every solution of (3) *with* $\rho(0, \lambda) = \mu$, $\theta(0, \lambda, \mu) = \alpha$.

Proof. Let $\lambda > \lambda_j$ be fixed, and consider any solution of (3) with $\rho(0, \lambda, \mu) = \alpha$. By further restricting ρ_1 if necessary, we can assert that if $\mu < \rho_1$, then $F(t, z(t, \lambda)) \geqq 0$ and $\|F\| < \varepsilon$, where $\varepsilon(\lambda)$ is so small that $(\lambda - \lambda_j)A - \lambda F > 0$ on $[0, 1]$. Then

$$\theta'(t, \lambda, \mu) = -u^*[Q + \lambda A - \lambda F]u < -u^*[Q + \lambda_j A]u.$$

Since $\phi'(t, \lambda_j) = -u^*[Q + \lambda_j A]u$ (with θ replaced by ϕ in the vector u) and $\theta(0, \lambda, \mu) = \alpha = \phi(0, \lambda_j)$, a standard comparison theorem gives

$$\theta(1, \lambda, \mu) < \phi(1, \lambda_j) = \beta_j.$$

3. The local and global theorems.

THEOREM 1. *Let* (H1), (H2) *and* (H3) *hold. Then for each* λ_j, *there is a non-degenerate interval* $I_j = (\lambda_j, \Lambda_j)$ *such that for each* $\lambda \in I_j$, *the problem* (1), (2) *has at least two solutions. The first component of each solution vanishes at least j times in* (0, 1).

Proof. Lemma 1 implies that solutions of (1), (2a) are extendable if μ is sufficiently small, say $\mu < \mu_0$, and the uniqueness assumption in (H2) implies that extendable solutions of (3), (4a) depend continuously on (μ, λ). We assume $\mu < \mu_0$.

According to (H3), for μ sufficiently small we have

$$\theta'(t, \lambda_j, \mu) = -u^*[Q + \lambda_j A - \lambda_j F]u > -u^*[Q + \lambda_j A]u.$$

If (ζ, ϕ) corresponds to the jth eigenfunction pair of the linear problem, then $\phi' = -u^*(Q + \lambda_j A)u$; therefore, for μ_0 sufficiently small and $0 < \mu < \mu_0$,

$$\theta(1, \lambda_j, \mu) > \phi(1, \lambda_j) = \beta_j.$$

Continuity in (λ, μ) implies $\theta(1, \lambda, \mu) > \beta_j$ in some interval $\lambda_j < \lambda < \Lambda(\mu)$ for fixed μ. We define $I_j = \bigcup_{0<\mu<\mu_0} (\lambda_j, \Lambda(\mu)) = (\lambda_j, \Lambda_j)$.

If $\lambda \in I_j$ is fixed, then $\lambda \in (\lambda_j, \Lambda(\mu))$ for some μ, $0 < \mu < \mu_0$, hence $\theta(1, \lambda, \mu) > \beta_j$. On the other hand, Lemma 2 implies there is a μ_1 such that $\theta(1, \lambda, \mu_1) < \beta_j$. Continuity of $\theta(1, \lambda, \mu)$ in (λ, μ) implies the existence of $\mu_2, \mu_1 \leqq \mu_2 \leqq \mu$, such that $\theta(1, \lambda, \mu_2) = \beta_j$, and this in turn implies the existence of a solution (ξ_j, η_j) of the original problem (1), (2).

Noting that $\xi_j(t, \lambda) = 0$ if and only if $\theta_j(t, \lambda, \mu) = \pi/2 - k\pi$, $k = 0, \pm 1, \cdots$, the nodal properties follow easily from a counting argument. The second solution is obtained by taking $\arctan \in (\pi/2, 3\pi/2]$; which is equivalent to using the new initial values $-\xi_j(0, \lambda)$, $-\eta_j(0, \lambda)$ in (1), (2) (or, equivalently, replacing α and β_0 by $\alpha + \pi$, $\beta_0 + \pi$).

(H4) $q_{22}(t) + [a_{22}(t) - f_{22}(t, z)] > 0$.

Remark 1. If (H4) holds for all z, $t \in [0, 1]$, $\lambda \geqq \lambda_0$, then $\xi_j(t, \lambda)$ will have *exactly* j zeros in $(0, 1)$. This follows from the fact that $\xi = 0$ if and only if $\theta = \pi/2 - k\pi$, which in turn implies

$$\theta' = -u^*[Q + \lambda(A - F)]u < 0 \quad \text{for } u = \begin{bmatrix} 0 \\ (-1)^k \end{bmatrix}.$$

Thus $\theta(t, \lambda, \mu)$ can cross each line $\theta = \pi/2 - k\pi$ at most once. One can also estimate the number of zeros of $\eta_j(t, \lambda)$ using similar arguments.

Remark 2. The above theorem does not reduce to the corresponding result in [1] or [4] when the second order scalar equation

$$-y'' = [q(t) + \lambda a(t) - \lambda f(t, y, y')]y$$

is written as a system in the usual way. Even if $a(t)$ and $f(t, y, y')$ are positive for $y^2 + (y')^2$ small and $t \in [0, 1]$, the matrices A and F will only be semidefinite. The reason one can still obtain the desired conclusion is that A and F interact in a special way to make the right side of the differential equation for $\theta(t)$:

$$\theta' = -\sin^2 \theta - [q + \lambda(a - f)] \cos^2 \theta,$$

"nearly positive definite" in the same sense as in the proof of the above theorem. (See [4] for details.)

We now turn to the question of a global theorem, that is, an assertion that for every $\lambda > \lambda_j$, there is a solution $z_j(t, \lambda)$ of the original problem such that ξ_j has j zeros (equivalently, $\Lambda_j = +\infty$ for $j = 0, 1, 2, \cdots$).

(H5) For each $\lambda > \lambda_0$, there is a solution of (1) such that $\theta(0, \lambda, \mu) = \alpha$, $\theta(1, \lambda, \mu) \geqq \beta_0$.

THEOREM 2. *Let* (H1), (H2), (H3) *and* (H4) *hold and assume that solutions of* (1), (2a) *extend to* [0, 1]. *Then* (H5) *is a necessary and sufficient condition that there exist, for each k and each* $\lambda > \lambda_k$, $k = 0, 1, 2, \cdots$, *two solutions of* (1), (2) *such that* $\xi(t, \lambda)$ *has exactly k zeros in* $(0, 1)$.

Proof. Let k and $\lambda > \lambda_k$ be fixed, and suppose (H5) holds. Let ρ, θ solve (3), (4a). By Lemma 2, there is a μ_1 such that $\theta(1, \lambda, \mu_1) < \beta_k$. By (H5), there is a μ_2 such that

$$\theta(1, \lambda, \mu_2) \geqq \beta_0 \geqq \beta_0 - k\pi = \beta_k > \theta(1, \lambda, \mu_1).$$

The assumption that solutions to (1), (2a) are unique and extendable implies continuity of θ in μ; therefore there exists a $\tilde{\mu}$ such that $\theta(1, \lambda, \tilde{\mu}) = \beta_k$, and this solution will generate the desired solution of (1), (2).

Now suppose (H5) does not hold and let $\lambda > \lambda_k \geqq \lambda_0$. Then there is no solution of (1), (2a) for which $\theta(1, \lambda, \mu) = \beta_0$. Thus each solution of (1), (2) satisfies $\theta(1, \lambda, \mu) = \beta_k$, $k \geqq 1$, which implies that $\theta(t, \lambda, \mu)$ crosses the line $\theta = -\pi/2$. Therefore there is no zero-free eigenfunction, yet $\lambda > \lambda_0$.

Remark 3. The continuity of $\theta(1, \lambda, \mu)$ in (λ, μ), used several times in this paper, is the sole reason for the extendability assumption in the above theorem and the uniqueness assumption in (H2) (for small μ, extendability follows from Lemma 1). One can completely remove the uniqueness assumption at the expense of more complicated arguments, by using funnel sections, as in [4].

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A NOTE ON THE VAN WIJNGAARDEN TRANSFORMATION*

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Abstract. Theory and applications are given of the van Wijngaarden transformation by means of which slowly convergent or even divergent series may be converted into rapidly convergent series. Using the technique of generating functions it is shown that by the van Wijngaarden transformation the Borel sum is kept invariant. The van Wijngaarden transformation is shown to be the Laplace transform of the Euler transformation.

Introduction. In a paper on a transformation of formal series van Wijngaarden [3] suggested the following method for summing a slowly convergent or even divergent series $\sum a_k$.[1] Introduce a sequence of nonvanishing multipliers λ_k and transform the series $\sum \lambda_k a_k$ by means of the Euler method. If the transformed series is denoted by $\sum b_k$, there exists a sequence of conjugate multipliers μ_k such that $\sum \mu_k b_k$ has the same (generalized) sum as $\sum a_k$. In particular, van Wijngaarden takes the special case $\lambda_k = \lambda^k/k!$ and finds that

$$\mu_k = 2^{k+1}\lambda \int_0^\infty e^{-\lambda t}(1+t)^{-k-1}t^k\,dt.$$

In a number of interesting cases the new series turns out to be rapidly convergent which makes this process particularly well adapted to numerical computations.

In view of recent interest the problem is taken up again and considered from a different point of view. The first section deals with the main properties of the Euler transformation of a formal series $\sum a_k$. The analysis becomes very transparent by using the generating functions $\sum a_k x^k$ and $\sum a_k x^k/k!$. The second section shows the importance of the Borel summation and the concept of the Borel sum

$$\int_0^\infty e^{-x}\left(\sum a_k x^k/k!\right)dx.$$

For convergent series the Borel sum equals the ordinary sum. For a wide class of divergent series the Borel sum exists and can be taken as the generalized sum. The importance of the Borel sum is that it is invariant for an Euler transformation. In the third section it is shown that the use of generating functions permits a rather simple treatment of the van Wijngaarden transformation. It is shown also that this transformation leaves the Borel sum invariant. The properties of the multipliers μ_k of the special van Wijngaarden transformation given above are the subject of a paper [2] by N. M. Temme who concentrates in particular on their numerical computation.

In many applications one wishes to compute a Laplace integral

$$\omega \int_0^\infty e^{-\omega x}F(x)\,dx$$

by termwise integration of some series expansion of the integrand function $F(x)$.

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[1] In all summations denoted by Σ, the index runs from 0 to infinity unless stated otherwise.

Sometimes the resulting series is slowly convergent or even divergent. One might consider subjecting this series to an Euler transformation or a van Wijngaarden transformation. However, the same result can be obtained in a much more direct and simpler way. It suffices to expand $F(x) = \sum a_k x^k$ as suggested by the generating function of $\sum a_k$ when subjected to an Euler transformation. We write

$$F(x) = 2^{k+1} \sum b_k \frac{x^k}{(1+x)^{k+1}},$$

where $\sum b_k$ is the Eulerized series of $\sum a_k$. Termwise integration of the latter expansion gives at once $\sum \mu_k a_k$. This result may be formulated by saying that the special van Wijngaarden transformation is the Laplace transformation of the Euler transformation.

1. The Euler transformation. We consider a formal series $\sum a_k$ and introduce the forward shift operator S and the weighted mean operator M by means of

$$Sa_k = a_{k+1},$$

$$M = p + qS \quad \text{with } |p| < 1 \quad \text{and} \quad q = 1 - p.$$

Then we have formally

$$\sum a_k = \sum S^k a_0 = \frac{a_0}{1 - S} = \frac{qa_0}{1 - M} = q \sum M^k a_0.$$

This suggests the so-called Euler transformation $E(q)$,

$$b_k = qM^k a_0,$$

or explicitly

$$b_k = \sum_{j=0}^{k} \binom{k}{j} p^{k-j} q^{j+1} a_j. \tag{1.1}$$

In numerical practice one uses the Euler method preferably with $p = q = \frac{1}{2}$. According to the folklore of the numerical analyst the Euler transformation turns slowly convergent series into rapidly convergent series and transforms divergent series into less divergent or even convergent series.

In order to get a better insight into what is really going on we consider the generating functions

$$a(z) = \sum a_k z^{k+1}, \qquad b(z) = \sum b_k z^{k+1}. \tag{1.2}$$

We restrict our discussion at first to those series for which $a(z)$ has a nonvanishing radius of convergence R_a. This enables us to handle divergent series such as $1 - 2 + 3 - 4 + \cdots$ but a series like $1! - 2! + 3! - 4! + \cdots$ falls outside this class.

It is easily seen by comparing equal powers of z that the relation (1.1) is equivalent to

$$b(z) = a\left(\frac{qz}{1 - pz}\right). \tag{1.3}$$

The radius of convergence of $a(w)$, where

$$w = \frac{qz}{1 - pz}, \qquad z = \frac{w}{q + pw}, \tag{1.4}$$

is determined by the singularities $w = s$ of the holomorphic function $a(w)$ as

$$R_a = \inf|s|. \tag{1.5}$$

Then the radius of convergence of $b(z)$ is given by

$$R_b = \inf\left|\frac{s}{q + ps}\right|. \tag{1.6}$$

The Euler method is most effective if R_b/R_a is as large as possible.

Example 1.1. If $a(w)$ is singular at $w = -1$ and $w = \infty$, then $b(z)$ is singular at $z = (p - q)^{-1}$ and $z = p^{-1}$. The ordinary method with $p = q = \frac{1}{2}$ gives $R_a = 1$ and $R_b = 2$. However, the method with $p = \frac{1}{3}$ gives even $R_b = 3$. If this is applied to $a(w) = w(1 + w)^{-1/2}$, for example, we find indeed that $b(z) = 2z(9 - z^2)^{-1/2}$.

Example 1.2. The ordinary Euler transformation changes the divergent series $1 - 2 + 3 - 4 + \cdots$ into the convergent series $\frac{1}{2} - \frac{1}{4} + 0 + 0 + \cdots$ of which only the first two terms differ from zero. This rather surprising phenomenon is explained by the fact that $a(z) = z(1 + z)^{-2}$ is singular only at $z = -1$ which gives $R_b = \infty$. Indeed, $b(z) = \frac{1}{2}z(1 - \frac{1}{2}z)$.

If $\sum a_k$ converges with sum A, then we know that $R_a \geqq 1$ and that $a(1) = A$. It follows from (1.3) also that $b(1) = A$. Hence it is tempting to conclude that $\sum b_k$ converges with the same sum A. However, it is not a priori obvious that $\sum b_k$ is convergent. But if this series converges, Abel's theorem states that its sum must be A. When the singularities of $a(z)$ are distributed in such a way that $R_b > 1$ there is no problem. However, when $R_b = 1$ also some further analysis is needed.

A summation method which sums every convergent series to its ordinary sum is called *regular*. Hardy [1] gives necessary and sufficient conditions for the regularity of a wide class of summation methods. The regularity of the Euler transformation then follows by checking the conditions. We shall give here a direct proof.

THEOREM 1.1. *The Euler method $E(q)$ is regular.*

Proof. Let $A_n = \sum_{k=0}^{n-1} a_k$, $B_n = \sum_{k=0}^{n-1} b_k$, $A_n \to A$. Since (1.4) implies $z/(1 - z) = w/[q(1 - w)]$, the relation (1.3) may be replaced by

$$\frac{z}{1 - z} b(z) = \frac{1}{q}\,\frac{w}{1 - w} a(w).$$

Expanding both sides into a power series we have

$$\sum B_k z^{k+1} = q^{-1} \sum A_j w^{j+1} = q^{-1} \sum A_j \left(\frac{qz}{1 - pz}\right)^{j+1}.$$

Taking the coefficient of z^{k+1} we obtain

$$B_k = \sum_{j=1}^{k} \binom{k}{j} p^{k-j} q^j A_j. \tag{1.7}$$

If $A_j = A$ for all j, then $B_k = (1 - p^k)A$. Thus also $B_k \to A$ since $|p| < 1$. Henceforth we may assume $A = 0$. The relation (1.7) will be written as

$$B_k = \sum_{j=1}^{m} + \sum_{j=m+1}^{k} = U + V.$$

Given an $\varepsilon > 0$ we can choose $m = m(\varepsilon)$ so that $|A_j| < \varepsilon$ for $j > m$. Then

$$|V| \leqq \varepsilon \sum_{j=0}^{k} \binom{k}{j} p^{k-j} q^j = \varepsilon.$$

For fixed m we have $U \to 0$ as $k \to \infty$ since each term of U tends to zero. Hence for k sufficiently large $|U| < \varepsilon$ so that $|B_k| < 2\varepsilon$. This means that $B_k \to 0$ which proves the theorem.

2. Borel summation. In an alternative way the Euler method may be discussed by considering the generating functions

$$\alpha(z) = \sum a_k z^k/k!, \qquad \beta(z) = \sum b_k z^k/k!. \tag{2.1}$$

The relation (1.1) is easily seen to be equivalent to the functional equation

$$e^{-z}\beta(z) = qe^{-qz}\alpha(qz). \tag{2.2}$$

It clearly suffices to consider the coefficient of z^k in the expansions of $q\alpha(qz) \exp(pz)$.

From (2.2) we obtain the following interesting result.

THEOREM 2.1. *The Euler methods $E(q)$ form a commutative semigroup with*

$$E(q_1)E(q_2) = E(q_1 q_2).$$

We shall now extend the discussion of the Euler method to those series $\sum a_k$ for which $\alpha(z)$ is holomorphic in a domain which contains the positive real axis. If

$$A = \int_0^\infty e^{-x}\alpha(x)\,dx \tag{2.3}$$

exists, then the series is said to be Borel summable with A as its Borel sum. From (2.2) and (2.3) the next theorem follows at once.

THEOREM 2.2. *The Euler transformation $E(q)$ with q real and positive does not change the Borel sum.*

Further we have the following property which is proved in Hardy [1, § 8.5] in a more general context.

THEOREM 2.3. *The Borel method* (2.3) *is regular.*

Proof. We put for $k = 0, 1, 2, \cdots$,

$$\phi_k(x) = \frac{1}{k!}\int_x^\infty e^{-t}t^k\,dt = e^{-x}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}\right)$$

and

$$\psi_k(x) = e^{-x}x^k/k!.$$

If $\sum a_k = A$ and $\sum_{j=k}^{\infty} a_k = A_k$, then

$$\int_0^x e^{-t}\alpha(t)\,dt = \sum \frac{a_k}{k!}\int_0^x e^{-t}t^k\,dt = \sum a_k(1-\phi_k)$$
$$= A - \sum a_k\phi_k = A - \sum (A_k - A_{k+1})\phi_k = A - \sum A_k\psi_k.$$

Hence it remains to prove that

$$\lim_{x\to\infty} e^{-x}\sum A_k x^k/k! = 0,$$

but this is an elementary matter.

In many applications a series which has to be summed is derived from some integral expression. We consider in particular the integral

$$\int_0^\infty e^{-x}F(x)\,dx, \tag{2.4}$$

where $F(x)$ is holomorphic in $x = 0$, say $F(x) = \sum a_k x^k/k!$. One may have the idea of applying an Euler transformation to the series $\sum a_k$ which is obtained by termwise integration of the power series expansion of $F(x)$. However, the preceding analysis shows that the final result could be obtained in a shorter way by writing the integral in the form

$$\int_0^\infty e^{-\mu x}\{e^{-(1-\mu)x}F(x)\}\,dx$$

and termwise integrating the power series expansion of $F(x)\exp\{-(1-\mu)x\}$. Comparing this expression with (2.2) it appears that implicitly an Euler transformation with $q = 1/\mu$ has been applied.

3. The van Wijngaarden method. In his paper [3], van Wijngaarden advocates the following method of summing $\sum a_k$. Introduce nonvanishing multipliers λ_k and subject $\sum \lambda_k a_k$ to an Euler transformation. Let the resulting series be $\sum b_k$. Then there exist conjugate multipliers μ_k such that $\sum \mu_k b_k$ has the same sum as $\sum a_k$. The formal analysis is very simple. According to (1.3) and (1.4) we have

$$\sum b_k\left(\frac{t}{q+pt}\right)^{k+1} = \sum \lambda_k a_k t^{k+1}. \tag{3.1}$$

We suppose that there exists a moment generating function $\phi(t)$ such that

$$\lambda_k^{-1} = \int_0^\infty \phi(t)t^k\,dt. \tag{3.2}$$

Then formal integration of (3.1) after multiplication by $\phi(t)/t$ gives

$$\sum \mu_k b_k = \sum a_k, \tag{3.3}$$

where

$$\mu_k = \int_0^\infty \frac{t^k}{(q+pt)^{k+1}}\phi(t)\,dt. \tag{3.4}$$

van Wijngaarden considers, in particular, the multiplier set $\lambda_k = s^k/k!$. Then $\phi(t) = s\exp(-st)$ so that

$$\mu_k = \frac{s}{p^{k+1}}\int_0^\infty \frac{t^k}{(1+t)^{k+1}}\exp\left(-\frac{qst}{p}\right)dt. \tag{3.5}$$

The general case of an arbitrary Euler transformation with the multiplier parameter s is seen to be equivalent to the special Euler transformation with $p = q = \frac{1}{2}$ and the multiplier parameter qs/p. Therefore without loss of generality we may put $p = q = \frac{1}{2}$.

If this method is applied to the power series

$$\sum a_k\omega^{-k} \tag{3.6}$$

with ω either real or complex with $\operatorname{Re}\omega > 0$, we take $\lambda_k = \omega^k/k!$ and compute

$$\sum 2^{-k-1}s_k(\omega)b_k, \tag{3.7}$$

where

$$s_k(\omega) = \omega\int_0^\infty e^{-\omega t}\frac{t^k}{(1+t)^{k+1}}dt. \tag{3.8}$$

The functions $s_k(\omega)$ are studied from a computational point of view in a publication [2] by N. M. Temme.

THEOREM 3.1. *The functions $s_k(\omega)$ have for $k \to \infty$, $\omega/k \ll 1$ the following asymptotic behavior*:

$$s_k(\omega) \sim \pi^{1/2}k^{-1/4}\omega^{3/4}\exp\left(\tfrac{1}{2}\omega - 2\sqrt{k\omega}\right). \tag{3.9}$$

Proof. Writing

$$s_k(\omega) = \int_0^\infty e^{-kf(t)}\omega(1+t)^{-1}\,dt$$

with $f(t) = \log[(1+t)/t] + (\omega/k)t$ we apply the saddle-point method. The positive real axis is the line of steepest descent with a saddle point t_0 determined by $c^2t(1+t) = 1$, where $c^2 = \omega/k$, $c > 0$. Explicitly,

$$t_0 = -\tfrac{1}{2} + (1/c^2 + \tfrac{1}{4})^{1/2} = c^{-1} - \tfrac{1}{2} + O(c).$$

This gives

$$f(t_0) = 2c - \tfrac{1}{2}c^2 + O(c^3)$$

and

$$f''(t_0) = 2c^3 + O(c^4).$$

Using these expressions it follows that $s_k(w)$ is asymptotically equivalent to

$$s_k(\omega) \sim c\omega\exp\{-k(2c - \tfrac{1}{2}c^2)\}\int_{-\infty}^{\infty} e^{-kc^3u^2}\,du,$$

which can be written in the form stated above.

THEOREM 3.2. *The van Wijngaarden transformation with $\lambda_k = s^k/k!$ does not change the Borel sum.*

Proof. The relation between the generating functions of $\sum a_k$ and $\sum \lambda_k a_k$ is given by

$$\alpha(x) = \int_0^\infty f(xt)\phi(t)\,dt,$$

where

$$f(x) = \sum \frac{\lambda_k a_k x^k}{k!}.$$

According to (2.2) the generating function of $\sum b_k$ is given by

$$\beta(x) = q\, e^{px} f(qx).$$

Thus we have

$$\alpha(x) = \int_0^\infty e^{-ptx}\beta(xt)\phi(qt)\,dt.$$

On the other hand, we have for the generating function

$$q(x) = \sum \frac{\mu_k b_k x^k}{k!}$$

the expression

$$g(x) = \int_0^\infty \frac{1}{q+pt}\sum \frac{b_k}{k!}\left(\frac{xt}{q+pt}\right)^k \phi(t)\,dt$$

or

$$g(x) = \int_0^\infty \frac{1}{1+pt}\beta\left(\frac{xt}{1+pt}\right)\phi(qt)\,dt.$$

Substitution of the expressions derived above for $\alpha(x)$ and $g(x)$ into $\int_0^\infty e^{-x}\alpha(x)\,dx$ and $\int_0^\infty e^{-x}g(x)\,dx$ shows their equality. The formalities are certainly justified if $\sum a_k$ is such that

$$\sum \frac{a_k x^k}{k!k!} = O(x^m)$$

for $x \to \infty$ with some constant m. The general problem can be treated in a similar way.

Example 3.1. The rapidly divergent series $\sum(-1)^k k!\omega^{-k}$ is treated by the multipliers $\lambda_k = \omega^k/k!$. This gives the formal series $\sum(-1)^k$. If the latter series is subjected to the ordinary Euler method $E(\frac{1}{2})$ we find

$$\tfrac{1}{2} + 0 + 0 + 0 + \cdots.$$

Accordingly, the Borel sum of the given series equals simply $\frac{1}{2}\mu_0$ or

$$\omega\int_0^\infty \frac{e^{-\omega t}}{1+t}\,dt.$$

Indeed, the Borel sum of the original series as derived from the generating series $\alpha(x) = \sum(-1)^k(x/\omega)^k = \omega(x+\omega)^{-1}$ agrees with the above given integral expression.

If we wish to compute an integral of the type

$$\int_0^\infty \phi(x)F(x)\,dx, \tag{3.10}$$

where $F(x)$ is holomorphic in $x = 0$, say $F(x) = \sum a_k x^k$, we may subject the series which is obtained by termwise integration to a van Wijngaarden transformation with multipliers (3.2). This means that an Euler transformation is applied to $\sum a_k$. According to (1.3) the result of expanding $xF(x)$ in powers of $x(q+px)^{-1}$ is

$$F(x) = \sum b_k \frac{x^k}{(q+px)^{k+1}}. \tag{3.11}$$

In view of (3.4), the effect of the van Wijngaarden transformation is merely the substitution of (3.11) into (3.10) followed by termwise integration. Thus

$$\int_0^\infty \phi(x)F(x)\,dx = \sum \mu_k b_k. \tag{3.12}$$

Of particular interest is the case of a Laplace integral where (3.10) takes the form

$$f(\omega) = \omega \int_0^\infty e^{-\omega x}F(x)\,dx. \tag{3.13}$$

The usual treatment of termwise Laplace transformation of the power series $F(x) = \sum a_k x^k$ gives the asymptotic expansion

$$f(\omega) \sim \sum k!a_k\omega^{-k}. \tag{3.14}$$

If, however, $F(x)$ is expanded as in (3.11) we obtain the expansion

$$f(\omega) = \frac{1}{q}\sum b_k p^{-k} s_k(\omega q/p), \tag{3.15}$$

where $\sum b_k$ is the Euler transform of $\sum a_k$ and the s_k are given by (3.8). The same expansion would be obtained by applying the special van Wijngaarden transformation with $\lambda_k = \omega^k/k!$ to the asymptotic expansion (3.14). This result may be phrased as follows.

THEOREM 3.3. *The special van Wijngaarden transformation is the Laplace transformation of the Euler transformation.*

The expansion (3.15) may be convergent even if (3.14) is divergent for all ω. If, for example, the generating function $\sum a_k x^k$ and $\sum b_k x^k$ both have a finite radius of convergence, the asymptotic behavior (3.9) shows that (3.15) converges for all ω. On the other hand, it shares with (3.14) the asymptotic character for $\omega \to \infty$ since for fixed k,

$$s_k(\omega) \sim k!\omega^{-k}. \tag{3.16}$$

Example 3.2. We consider the Laplace integral

$$\sqrt{\pi\omega}\, e^{\omega} \operatorname{erfc} \sqrt{\omega} = \omega \int_0^\infty e^{-\omega t}(1 + t)^{-1/2}\, dt.$$

To the integrand function $(1 + t)^{-1/2}$ we apply the Euler transformation $E(2/3)$ since by this choice the radius of convergence of $\sum b_k x^k$ takes the optimal value of 3. In fact, $\sum b_k x^k = \frac{2}{3}(1 - x^2/9)^{-1/2}$ so that finally

$$\sqrt{\pi\omega}\, e^{\omega} \operatorname{erfc} \sqrt{\omega} = \sum \frac{(\frac{1}{2})k}{k!} s_{2k}(2\omega)$$

is convergent for all ω.

Example 3.3. An interesting generalization is indicated in the following integral for the modified Bessel function:

$$e^{\omega/2} K_0(\tfrac{1}{2}\omega) = \int_0^\infty e^{-\omega t} t^{-1/2}(1 + t)^{-1/2}\, dt.$$

Applying the same transformation as in the previous example, given by

$$(1 + t)^{-1/2} = \sum \frac{3^{k+1} b_k t^k}{(2 + t)^{k+1}},$$

we obtain the expansion

$$e^{\omega/2} K_0(\tfrac{1}{2}\omega) = 2^{-1/2} \sum b_k 3^{k+1} \int_0^\infty e^{-2\omega t} \frac{t^{k-1/2}}{(1 + t)^{k+1}}\, dt.$$

Introducing the following variant of (3.8):

$$\sigma_k(\omega) = \omega \int_0^\infty e^{-\omega t} \frac{t^{k-1/2}}{(1 + t)^{k+1}}\, dt,$$

we see that the final result may be written as

$$\omega \sqrt{2}\, e^{\omega/2} K_0(\tfrac{1}{2}\omega) = \sum \frac{(\frac{1}{2})_k}{k!} \sigma_{2k}(2\omega).$$

The asymptotic behavior of $\sigma_k(\omega)$ for $k \to \infty$ can be obtained in the same way as for $s_k(\omega)$. The (obvious) result is

$$\sigma_k(\omega) \sim \pi^{1/2} k^{-3/4} \omega^{5/4} \exp\left(\tfrac{1}{2}\omega - 2\sqrt{k\omega}\right).$$

This shows that also in this case a convergent expansion is obtained.

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ZERO DISTRIBUTION AND BEHAVIOR OF ORTHOGONAL POLYNOMIALS IN THE SOBOLEV SPACE $W^{1,2}[-1, 1]$*

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Abstract. The distribution of the zeros of the polynomials orthogonal in the Sobolev space $W^{1,2}[-1, 1]$ with constant weights is established within a certain range of values of the parameters by relating the zeros to the zeros of the Legendre polynomials. In addition, as the weights vary, properties of the expansion of one set of orthogonal polynomials in terms of the other set are established. Monotonicity of the roots as the weights vary and interlacing properties of the zeros are established in several cases.

1. Introduction. In recent years, there has been some interest in studying the nature of polynomials which are orthogonal in the Sobolev space whose norm is given by

$$\|f\|^2 = \int_a^b p_0(x)[f(x)]^2\,dx + \int_a^b p_1(x)[f'(x)]^2\,dx, \tag{1}$$

where a and b may be either finite or infinite and $p_0(x)$ and $p_1(x)$ are nonnegative weights. Althammer [1], Gröbner [7] and Schäfke [11] have studied the properties of these polynomials in some detail when $a = -1$, $b = 1$, and p_0 and p_1 are constants. In this case, the polynomials can be thought of as generalizations of the classical Legendre polynomials. Lesky [8] and Brenner [4] have, in addition, considered the weights $p_0(x) = \exp(-x)$, $p_1(x) = \gamma \exp(-x)$, $a = 0$, $b = +\infty$, with γ a positive constant. In both of these special cases, the zeros of the orthogonal polynomials are shown to lie within the open interval (a, b). The present paper is devoted in part to a study of the distribution of the zeros for the case when $a = -1$, $b = 1$, $p_0 = 1$ and $p_1 = \gamma \geqq 0$, and to their behavior as λ varies. In addition, as the weight is varied, properties of the expansion of one set of orthogonal polynomials in terms of the other set will be established. Wilson [15] and Askey [2], [3] have established results of a similar nature for classical orthogonal polynomials.

2. Preliminary results and notation. The polynomials to be studied are those of degree n orthogonal in the sense of the inner product

$$(f, g) \equiv \int_{-1}^{1} f(x)g(x)\,dx + \lambda \int_{-1}^{1} f'(x)g'(x)\,dx. \tag{2}$$

We shall denote these polynomials by $R_n(x;\lambda)$ to indicate their dependence on both n and λ. It will be convenient for our purposes to think of $R_n(x;\lambda)$ as a perturbation of the integral of the Legendre polynomial $P_{n-1}(x)$:

$$R_n(x) \equiv \int_{-1}^{x} P_{n-1}(t)\,dt, \qquad n \geqq 2. \tag{3}$$

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We shall also define $R_0(x) \equiv 1$ and $R_1(x) \equiv x$. As noted by Schäfke [11], since

$$\begin{aligned} R_n(x) &= S_n(x)/(2n-1) \\ &= (P_n(x) - P_{n-2}(x))/(2n-1), \end{aligned}$$

such polynomials are almost orthogonal in the sense that $(R_n(x), R_m(x)) = 0$ when $|n-m| \neq 2$ and $n \neq m$. In a manner parallel to his construction of the orthogonal polynomials, we form

$$\begin{aligned} R_0(x;\lambda) &= R_0(x) = 1, \\ R_1(x;\lambda) &= R_1(x) = x, \\ R_n(x;\lambda) &= R_n(x) + \alpha_n(\lambda) R_{n-2}(x;\lambda), \qquad n \geqq 2. \end{aligned} \tag{4}$$

One finds directly from the definition (2) that $\alpha_2(\lambda) = 1/3$. By comparing (4) with Schäfke's form (1.7), one sees that

$$\alpha_n(\lambda) = \frac{a_{n-2}(2n-5)}{a_n(2n-1)}, \qquad n \geqq 3. \tag{5}$$

Schäfke is also able to show that

$$a_n = \sum_{j=0}^{[(n-1)/2]} \left(\frac{\lambda}{4}\right)^j \frac{1}{(2j)!} \frac{(n+2j-1)!}{(n-2j-1)!}, \qquad n \geqq 1, \tag{6}$$

and that a_{n+2}/a_n is a strictly increasing function of λ on $[0, \infty)$. From (5) and (6), it follows that $\alpha_n(\lambda)$ is $O(1/\lambda)$ for large λ, so that $\alpha_n(\lambda)$ tends monotonically to 0 as $\lambda \to +\infty$. Also, from (5) and (6),

$$\alpha_n(0) = (2n-5)/(2n-1), \qquad n \geqq 3. \tag{7}$$

It is clear that $\alpha_n(\lambda)$ is to be determined from the condition

$$(R_n(x;\lambda), R_{n-2}(x;\lambda)) = 0.$$

This leads directly to the recurrence relation

$$\alpha_n(\lambda) = \frac{-(R_n(x), R_{n-2}(x))}{[(R_{n-2}(x), R_{n-2}(x)) + \alpha_{n-2}(\lambda)(R_{n-2}(x), R_{n-4}(x))]}, \quad n \geqq 4. \tag{8}$$

Schäfke's equations (1.4) and (1.5) lead immediately to the relations

$$(R_n(x), R_n(x)) = \frac{2[\lambda + 2/(2n+1)(2n-3)]}{2n-1}, \qquad n \geqq 2, \tag{9}$$

and

$$(R_n(x), R_{n-2}(x)) = \frac{-2}{(2n-1)(2n-3)(2n-5)}, \qquad n \geqq 3. \tag{10}$$

From (7), (8), (9) and (10), it follows that, when $\lambda > 0$ and $n \geqq 3$,

$$0 < \alpha_n(\lambda) < \frac{1}{\lambda(2n-1)(2n-3)}. \tag{11}$$

Inequality (11) shows the behavior of $\alpha_n(\lambda)$ for large n and λ.

Using (8), (9) and (10), one can also show that

$$\frac{2n-9}{(2n-3)(2n-5)(2n-9)\lambda+2(2n-3)}-\frac{1}{\lambda(2n-1)(2n-3)} \tag{12}$$
$$<\alpha_{n-1}(\lambda)-\alpha_n(\lambda)<\frac{1}{\lambda(2n-3)(2n-5)},\qquad n\geqq 6.$$

Inequality (12) shows, in particular, that, for a given λ and for all $n \geqq n_0(\lambda)$, $\alpha_n(\lambda)$ must decrease with n. It is clear that, as λ decreases, $n_0(\lambda)$ increases, since $\alpha_n(0)$ is an increasing function of n. From (7) and (11), we see that

$$\lim_{n\to\infty}\alpha_n(\lambda)=\begin{cases}0, & \lambda>0,\\ 1, & \lambda=0.\end{cases}$$

From the definition, we have seen that $\alpha_2 = 1/3$, and from either (5) or (8), it follows that

$$\alpha_3=\frac{1}{5(3\lambda+1)},$$
$$\alpha_4=\frac{3}{7(15\lambda+1)},$$
$$\alpha_5=\frac{5(3\lambda+1)}{9(105\lambda^2+45\lambda+1)}.$$

It is important to realize that $R_n(x)$ is directly related to $dP_{n-1}(x)/dx$ and thus to the ultraspherical polynomial $P_{n-2}^{(1,1)}(x)$. To see this, integrate the differential equation satisfied by the Legendre polynomial of degree $n-1$ [10]:

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+n(n-1)y=0 \tag{13}$$

to obtain

$$n(n-1)R_n(x)=(x^2-1)\frac{d^2R_n(x)}{dx^2}=(x^2-1)\frac{dP_{n-1}(x)}{dx}. \tag{14}$$

Therefore [12, 4.21.7, p. 62],

$$R_n(x)=\frac{(x^2-1)P_{n-2}^{(1,1)}(x)}{2(n-1)},\qquad n\geqq 2. \tag{15}$$

In particular, $R_n(x)$ satisfies the following homogeneous second order differential equation:

$$d^2y/dx^2+n(n-1)(1-x^2)^{-1}y=0. \tag{16}$$

3. Zero distribution. This section is devoted to establishing a result on distribution of zeros of orthogonal polynomials. We have the following theorem.

THEOREM 1. *The zeros of the polynomials $R_n(x;\lambda)$ orthogonal with respect to the inner product* (2) *are interlaced with the zeros of the Legendre polynomials of degree $n-1$ whenever $\lambda \geqq 2/n$.*

Proof. We first demonstrate, for all n and $\lambda \geqq 0$, that

$$|R_n(x;\lambda)| \leqq 1 \tag{17}$$

whenever $x \in [-1, 1]$. From the definition (4), one sees that the statement is valid for $n = 0, 1, 2$. It is well known [10, p. 200] that

$$|R_n(x)| \leqq \frac{2}{\pi^{1/2}(n-1)^{3/2}}, \qquad n \geqq 2. \tag{18}$$

Assume now that

$$|R_{n-2}(x;\lambda)| \leqq 1.$$

From (4), (7), (18) and the fact that $\alpha_n(\lambda)$ is maximum for $\lambda = 0$,

$$|R_n(x;\lambda)| \leqq \frac{2}{\pi^{1/2}(n-1)^{3/2}} + \frac{2n-5}{2n-1}, \qquad n \geqq 3.$$

Therefore, the assertion is correct for all n; and, from (4), (11) and (18), it follows that

$$|R_n(x;\lambda) - R_n(x)| \leqq \alpha_n(\lambda) < \frac{1}{\lambda(2n-1)(2n-3)}, \qquad n \geqq 3. \tag{19}$$

Inequalities (18) and (19) allow us to improve the bound on $R_n(x;\lambda)$. We have

$$|R_n(x;\lambda)| < \frac{2}{\pi^{1/2}(n-1)^{3/2}} + \frac{1}{\lambda(2n-1)(2n-3)}, \qquad n \geqq 3. \tag{20}$$

Reapplying (4) and using (11) and (20), one finds that

$$\begin{aligned} &|R_n(x;\lambda) - R_n(x)| \\ &\quad < \frac{1}{\lambda(2n-1)(2n-3)}\left[\frac{2}{\pi^{1/2}(n-3)^{3/2}} + \frac{1}{\lambda(2n-5)(2n-7)}\right], \qquad n \geqq 5. \end{aligned} \tag{21}$$

From (15), one sees that all the zeros of $R_n(x)$ are real and simple and lie in $[-1, 1]$, from which it follows that the relative maxima and minima of $R_n(x)$ must alternate in sign. Applying Sonin's theorem [12, 7.31, p. 161] to (16), we find that the relative maxima of $|R_n(x)|$ decrease as x increases from 0 to 1. Thus the smallest maximum in [0, 1] is that relative maximum which is closest to $x = 1$. Furthermore, by definition of $R_n(x)$, at any of its critical points, $P_{n-1}(x) = 0$. The distribution of the zeros of the Legendre polynomials is well known. For example, we can use the well-known inequalities of Markov and Stieltjes for the positive zeros $x_j = \cos\theta_j$ of $P_{n-1}(x)$ [12, 6.21, p. 118]:

$$(j - \tfrac{1}{2})\pi/(n-1) < \theta_j < j\pi/n, \qquad 1 \leqq j \leqq [(n-1)/2]. \tag{22}$$

Buell [5] has shown that no such interval as given by (22) can contain a zero of $dP_{n-1}(x)/dx$, hence, by (14), a zero of $R_n(x)$. By Sonin's theorem, it is clear that

$x_1 = \cos\theta_1$ is the abscissa of the smallest maximum of $|R_n(x)|$, so that

$$|R_n(x_j)| \geqq |R_n(x_1)| > |R_n(\cos(\pi/n))|. \tag{23}$$

Therefore, to obtain a lower bound on $|R_n(x_j)|$, we shall bound $|R_n(\cos(\pi/n))|$ from below. To obtain this bound, we employ the method of Liouville–Steklov [12, 8.63, pp. 208–210], which has been used by Gatteschi [6] to obtain information about the Legendre polynomials. From (15), it follows that

$$R_n\left(\cos\frac{\pi}{n}\right) = \frac{-\sin(\pi/n)\sin(\pi/2n)\cos(\pi/2n)P_{n-2}^{(1,1)}(\cos(\pi/n))}{n-1}. \tag{24}$$

By (4.24.2) of Szegö [12, p. 66], the differential equation

$$\frac{d^2u}{d\theta^2} + \left\{N^2 - 3\frac{\sin^{-2}(\theta/2) + \cos^{-2}(\theta/2)}{16}\right\}u = 0, \tag{25}$$

where $N \equiv n - \frac{1}{2}$, is satisfied by

$$u = [\sin(\theta/2)]^{3/2}[\cos(\theta/2)]^{3/2}P_{n-2}^{(1,1)}(\cos\theta).$$

By (1.8.9) of Szegö [12, p. 17], the differential equation

$$\frac{d^2u}{d\theta^2} + \left(N^2 - \frac{3}{4\theta^2}\right)u = 0 \tag{26}$$

is satisfied by

$$u = \theta^{1/2}J_1(N\theta)$$

and by

$$u = \theta^{1/2}Y_1(N\theta),$$

where J_1 is the Bessel function of first kind and order 1 and Y_1 is a Bessel function of the second kind and order 1. Now (25) can be written as

$$\frac{d^2u}{d\theta^2} + \left(N^2 - \frac{3}{4\theta^2}\right)u = \left[\frac{3}{16\cos^2(\theta/2)} - \frac{3}{4}\left(\frac{1}{\theta^2} - \frac{1}{4\sin^2(\theta/2)}\right)\right]u. \tag{27}$$

Using an argument given by Szegö [12, 8.63, pp. 208–210], one sees that

$$\begin{aligned}
&[\sin(\theta/2)]^{3/2}[\cos(\theta/2)]^{3/2}P_{n-2}^{(1,1)}(\cos\theta) \\
&\quad = 2^{-1/2}N^{-1}(n-1)\theta^{1/2}J_1(N\theta) \\
&\qquad - \frac{\pi\theta^{1/2}}{2}\int_0^\theta [J_1(N\theta)Y_1(Nt) - Y_1(N\theta)J_1(Nt)]t^{1/2} \\
&\qquad\quad \cdot\left\{\frac{3}{16\cos^2(t/2)} - \frac{3}{4}\left[\frac{1}{t^2} - \frac{1}{4\sin^2(t/2)}\right]\right\} \\
&\qquad\quad \cdot [\sin(t/2)]^{3/2}[\cos(t/2)]^{3/2}P_{n-2}^{(1,1)}(\cos t)\,dt.
\end{aligned} \tag{28}$$

Suppose now that $\theta = \pi/n$ in (28), and define

$$\Delta(t, n) \equiv J_1(N\pi/n)Y_1(Nt) - Y_1(N\pi/n)J_1(Nt).$$

Then the important quantity to appraise is

$$\varepsilon_1(n) = -\frac{3\pi \cdot 2^{1/2}}{32}\int_0^{\pi/n} \Delta(t,n)t^{1/2}\left[\frac{1}{4\cos^2(t/2)} - \left(\frac{1}{t^2} - \frac{1}{4\sin^2(t/2)}\right)\right] \cdot (\sin t)^{3/2} P_{n-2}^{(1,1)}(\cos t)\, dt$$

$$= -\frac{3\pi \cdot 2^{1/2}}{32}\int_0^{\pi/n} \Delta(t,n)\left[\left(\frac{t}{\sin t}\right)^2 - 1\right]\left(\frac{\sin t}{t}\right)^{3/2} \cdot P_{n-2}^{(1,1)}(\cos t)\, dt.$$

By (1.71.4) of Szegö [12, p. 15],

$$Y_1(x) = 2\frac{[\log(x/2) + \gamma]J_1(x) - 1/x}{\pi} - \frac{x}{2\pi}$$

$$\tag{29} -\frac{1}{\pi}\sum_{i=1}^{\infty} \frac{(-1)^i (x/2)^{2i+1}}{i!(i+1)!}[1/1 + 1/2 + \cdots + 1/i + 1/1 + 1/2 + \cdots + 1/(i+1)].$$

The series in (29) is an alternating series, and the absolute value of the ratio of its ith to its $(i-1)$st term is given by

$$r_i = \frac{x^2}{4i(i+1)}\left[1 + \frac{1/i + 1(i+1)}{1/1 + 1/2 + \cdots + 1/(i-1) + 1/1 + \cdots + 1/i}\right].$$

The ratio r_i is seen to be a decreasing function of i, so it attains its maximum when $i = 2$. One finds that $r_2 = x^2/18 < 1, 0 \leqq x \leqq \pi$. From (1.71.1) of Szegö [12, p.14], it is seen that the series representation for J_1 is also alternating and that its terms are decreasing in $0 \leqq x \leqq \pi$ whenever $v \geqq 1$. We have, for example, the following inequality, valid for $0 \leqq x \leqq \pi$:

$$\tag{30} \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} \leqq J_1(x) \leqq \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} - \frac{x^7}{18432} + \frac{x^9}{1474560}.$$

It has been pointed out by R. Askey in a private communication that such inequalities indeed hold for all $x \geqq 0$. We shall suppose now that $n \geqq 6$, so that $11\pi/12 \leqq N\pi/n \leqq \pi$. Since J_1 decreases in this range, we have, using tables of Bessel functions [13],

$$\tag{31} J_1(N\pi/n) \leqq J_1(11\pi/12) \leqq 0.384$$

and

$$\tag{32} J_1(N\pi/n) \geqq J_1(\pi) \geqq 0.284.$$

Also, when $n \geqq 6$, and $0 < t \leqq \pi/n$, one sees that

$$\tag{33} (t/\sin t)^2 - 1 < 0.359t^2,$$

since $(\sin t)^{-2} - t^{-2}$ is an increasing function for $0 \leqq t \leqq \pi/2$. Finally, it is well known [12, 7.32, p. 163] that on $[-1, 1]$,

$$\tag{34} |P_{n-2}^{(1,1)}(x)| \leqq n - 1.$$

If we use the first few terms of the expansions for $J_1(x)$ and $Y_1(x)$ to bound $|\Delta(t, n)|$ and apply (31), (33), (34), and elementary formulas for integrals of powers of t and of the type

$$\int_0^{\pi/n} t^k \log(nt/\pi)\, dt,$$

we find that

$$|\varepsilon_1(n)| \leqq 3.036/n^2.$$

From (28), we have that

$$\begin{aligned}
&\sin(\pi/2n)\cos(\pi/2n)P_{n-2}^{(1,1)}(\cos(\pi/n)) \\
(35)\qquad &= (n-1)(n-\tfrac{1}{2})^{-1}\left(\frac{\pi/n}{\sin(\pi/n)}\right)^{1/2} J_1\left[\frac{(n-\frac{1}{2})\pi}{n}\right] + 2^{1/2}\left(\frac{\pi/n}{\sin(\pi/n)}\right)^{1/2}\varepsilon_1(n).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\pi/n}{\sin(\pi/n)} &\leqq \frac{6}{6-(\pi/n)^2} \\
&\leqq 1.048, \qquad n \geqq 6.
\end{aligned}$$

Defining

$$\varepsilon(n) \equiv 2^{1/2}\left(\frac{\pi/n}{\sin(\pi/n)}\right)^{1/2}\varepsilon_1(n),$$

one finds that

$$(36)\qquad |\varepsilon(n)| \leqq 4.397/n^2.$$

Therefore, using (32), (35) and (36), one finds, for $n \geqq 6$, that

$$(37)\qquad |\sin(\pi/2n)\cos(\pi/2n)P_{n-2}^{(1,1)}(\cos(\pi/n))| \geqq 0.1359.$$

From (24), we conclude that

$$(38)\qquad |R_n(\cos(\pi/n))| \geqq 0.407/n^2, \qquad n \geqq 6.$$

Now one can verify that the second member of inequality (38) will exceed that of (21) for $n \geqq 6$ and $\lambda \geqq 2/n$, provided that

$$(39)\quad \frac{n}{\pi^{1/2}(n-3)^{1/2}(2n-3)} \leqq \frac{0.407(n-3)(2n-1)}{n^2} - \frac{n^2(n-3)}{4(2n-3)(2n-5)(2n-7)}.$$

That (39) is valid for $n = 6$ can be verified by substitution. Furthermore, it is seen that the left member is a decreasing function of n, the first term in the right member an increasing function of n, and the second term in the right member a decreasing function of n. It follows that the inequality holds for $n > 6$ as well.

Putting (21), (23), (38) and (39) together, we see that

$$(40)\qquad |R_n(x_i;\lambda) - R_n(x_i)| < |R_n(x_i)|, \qquad n \geqq 6, \quad \lambda \geqq 2/n,$$

at every critical value x_i, $1 \leqq i \leqq n-1$, of $R_n(x)$. Thus, at the maxima and minima of $R_n(x)$, $R_n(x;\lambda)$ and $R_n(x)$ are of the same sign. Since we have already shown that the maxima and minima of $R_n(x)$ are of alternating sign, it follows that there is a zero of $R_n(x;\lambda)$ between any two zeros of $P_{n-1}(x)$, the Legendre polynomial of degree $n-1$. This accounts for $n-2$ of the zeros of $R_n(x;\lambda)$. We can account for the remaining two roots as follows: Since $R_n(1) = 0$ and $R_n'(1) = P_{n-1}(1) = 1$, it follows that $R_n(x_1) < 0$ and by (40) that $R_n(x_1;\lambda) < 0$ also. But it can be seen from the work of Althammer [1] that $R_n(1;\lambda) > 0$. This fact can also be established by applying (4), from which we conclude that, when $n \geqq 2$,

$$R_n(1;\lambda) = \begin{cases} \alpha_n\alpha_{n-2}\cdots\alpha_2 > 0, & n \text{ even}, \\ \alpha_n\alpha_{n-2}\cdots\alpha_3 > 0, & n \text{ odd}. \end{cases}$$

It follows that $R_n(x;\lambda)$ has a zero between $x = x_1$ and $x = 1$. By symmetry, there must also be a zero between $x = x_{n-1}$ and $x = -1$. We want now to show that (40) remains true when $n < 6$.

First of all, for $n = 0$ and 1, it is vacuously satisfied. For $n = 2$, we find that

$$R_2(x;\lambda) = (3x^2 - 1)/6, \tag{41}$$

a multiple of the Legendre polynomial of degree 2. In contrast,

$$R_2(x) = (x^2 - 1)/2.$$

In this case, the origin is the only critical point, and

$$|R_2(0;\lambda) - R_2(0)| = \tfrac{1}{3} < |R_2(0)| = \tfrac{1}{2}.$$

When $n = 3$, one finds that

$$R_3(x;\lambda) = \frac{x^3}{2} - \frac{3x(5\lambda + 1)}{10(3\lambda + 1)} \tag{42}$$

and that

$$R_3(x) = x(x^2 - 1)/2.$$

The latter has critical values at $x = \pm\sqrt{3}/3$.

By symmetry, we need only check what happens at the positive value. We find

$$|R_3(\sqrt{3}/3;\lambda) - R_3(\sqrt{3}/3)| = \frac{\sqrt{3}}{15(3\lambda + 1)} < \sqrt{3}/9 = |R_3(\sqrt{3}/3)|, \qquad \lambda \geqq 0.$$

For $n = 4$, it is seen that

$$R_4(x;\lambda) = \frac{5x^4}{8} - \frac{3x^2}{2}\left(\frac{1}{2} - \frac{1}{7(15\lambda + 1)}\right) + \frac{1}{8} - \frac{1}{14(15\lambda + 1)} \tag{43}$$

and that

$$R_4(x) = (5x^4 - 6x^2 + 1)/8.$$

We must check the critical points $x = 0$ and $x = \sqrt{3/5}$. It is found that

$$|R_4(0;\lambda) - R_4(0)| = \frac{1}{14(15\lambda + 1)} < \tfrac{1}{8} = |R_4(0)|, \qquad \lambda \geqq 0.$$

Also,

$$|R_4(\sqrt{\tfrac{3}{5}};\lambda) - R_4(\sqrt{\tfrac{3}{5}})| = \frac{2}{35(15\lambda+1)} < \tfrac{1}{10} = |R_4(\sqrt{\tfrac{3}{5}})|, \qquad \lambda \geqq 0.$$

The last case to be considered is that for $n = 5$. For this case,

$$\begin{aligned} R_5(x;\lambda) &= \frac{7x^5}{8} - \frac{5x^3}{2}\left[\frac{1}{2} - \frac{3\lambda+1}{9(105\lambda^2+45\lambda+1)}\right] \\ &\quad + \frac{x}{2}\left[\frac{3}{4} - \frac{5\lambda+1}{3(105\lambda^2+45\lambda+1)}\right] \end{aligned} \tag{44}$$

and

$$R_5(x) = x(7x^4 - 10x^2 + 3)/8.$$

We have critical values at $x^2 = (15 \pm 2\sqrt{30})/35$. It turns out that (40) is again valid. This proves the theorem.

4. Positivity results. The second major result is the relation between the sets of orthogonal polynomials as the parameter λ varies. We have indeed the following.

THEOREM 2. *Whenever* $\lambda < \mu$,

$$R_n(x;\lambda) = \sum_{k=0}^{[n/2]} a(k,n)R_{n-2k}(x;\mu),$$

where $a(k,n) > 0$ *for all values of* k *when* n *is odd. For even* n, $a(n/2,n) = 0$ *and* $a(k,n) > 0$ *whenever* $0 \leqq k \leqq n/2 - 1$. *For* $\lambda > \mu$, $a(0,n) > 0$ *and* $a(k,n) < 0$ *whenever* n *is odd and* $1 \leqq k \leqq (n-1)/2$. *In case* n *is even,* $a(k,n) < 0$ *for* $1 \leqq k \leqq n/2 - 1$ *and* $a(n/2,n) = 0$.

Proof. It is clear, first of all, that for any λ, the set $\{R_i(x;\lambda)\}$ of orthogonal polynomials forms a basis. Thus there exists a unique representation of any polynomial $R_n(x;\lambda)$ in terms of the set $\{R_i(x;\mu)\}$. Since $R_n(x;\lambda)$ is an even function for even n and an odd function for odd n, only those polynomials of even degree or odd degree respectively enter the summation. Furthermore, one can use (4) to expand $R_{n-2k}(x;\mu)$ and $R_n(x;\lambda)$ in terms of the set $\{R_i(x)\}$ of the integrals of the Legendre polynomials. Clearly, from (4), $a(0,n) = 1$.

Also, using (4), we have

$$\begin{aligned} R_n(x;\lambda) &= \sum_{k=0}^{[n/2]} a(k,n)\Big[R_{n-2k}(x) + \alpha_{n-2k}(\mu)R_{n-2k-2}(x) \\ &\quad + \alpha_{n-2k}(\mu)\alpha_{n-2k-2}(\mu)R_{n-2k-4}(x) + \cdots \\ &\quad + \alpha_{n-2k}(\mu)\alpha_{n-2k-2}(\mu)\cdots\alpha_{n-2k-2j+2}(\mu)R_{n-2k-2j}(x) \\ &\quad + \cdots + \alpha_{n-2k}(\mu)\alpha_{n-2k-2}(\mu)\cdots \\ &\quad \cdot \alpha_{n-2k-2[(n-2k)/2]+2}(\mu)R_{n-2k-2[(n-2k)/2]}(x)\Big] \\ &= R_n(x) + \alpha_n(\lambda)R_{n-2}(x) + \alpha_n(\lambda)\alpha_{n-2}(\lambda)R_{n-4}(x) \\ &\quad + \cdots + \alpha_n(\lambda)\alpha_{n-2}(\lambda)\cdots\alpha_{n-2i+2}(\lambda)R_{n-2i}(x) \\ &\quad + \cdots + \alpha_n(\lambda)\alpha_{n-2}(\lambda)\cdots\alpha_{n-2[n/2]+2}(\lambda)R_{n-2[n/2]}(x). \end{aligned} \tag{45}$$

We can now compute the coefficients $a(k, n)$ recursively. For example, from (45), since $\{R_i(x)\}$ also forms a basis, we find, on comparing coefficients of $R_{n-2}(x)$,

$$a(1, n) = \alpha_n(\lambda) - \alpha_n(\mu). \tag{46}$$

Since $\alpha_n(\lambda)$ is strictly decreasing in λ for $n \geqq 3$, (46) is positive for $n \geqq 3$ when $\lambda < \mu$ and negative when $n \geqq 3$ and $\lambda > \mu$. Similarly we obtain, comparing coefficients of $R_{n-4}(x)$,

$$\alpha_n(\mu)\alpha_{n-2}(\mu) + a(1, n)\alpha_{n-2}(\mu) + a(2, n) = \alpha_n(\lambda)\alpha_{n-2}(\lambda). \tag{47}$$

Using (46) in (47), we see that

$$a(2, n) = \alpha_n(\lambda)[\alpha_{n-2}(\lambda) - \alpha_{n-2}(\mu)], \tag{48}$$

which is positive for $n \geqq 5$ when $\lambda < \mu$ and negative when $\lambda > \mu$ and $n \geqq 5$.

We now prove by induction that, when $k \geqq 2$,

$$a(k, n) = \alpha_n(\lambda)\alpha_{n-2}(\lambda) \cdots \alpha_{n-2k+4}(\lambda)[\alpha_{n-2k+2}(\lambda) - \alpha_{n-2k+2}(\mu)]. \tag{49}$$

First of all, the result is valid for $k = 2$ as is seen from (48). Suppose that (49) is correct for $j < k$. If we form the coefficient of $R_{n-2k}(x)$ on both sides of (45), we see that

$$\begin{aligned} &\alpha_n(\mu)\alpha_{n-2}(\mu) \cdots \alpha_{n-2k+2}(\mu) + a(1, n)\alpha_{n-2}(\mu)\alpha_{n-4}(\mu) \cdots \alpha_{n-2k+2}(\mu) \\ &\quad + a(2, n)\alpha_{n-4}(\mu)\alpha_{n-6}(\mu) \cdots \alpha_{n-2k+2}(\mu) + \cdots \\ &\quad + a(k-1, n)\alpha_{n-2k+2}(\mu) + a(k, n) = \alpha_n(\lambda)\alpha_{n-2}(\lambda) \cdots \alpha_{n-2k+2}(\lambda). \end{aligned} \tag{50}$$

By the inductive hypothesis, we may substitute into (50) for $a(j, n)$, when $j < k$, the expression given by (49), and we find that (49) follows for $j = k$. From (49), it follows that, for $k \geqq 1$, $a(k, n)$ is positive when $\lambda < \mu$ and negative when $\lambda > \mu$ unless $k = n/2$, when it vanishes on account of the fact that $\alpha_2(\lambda) = \frac{1}{3}$ for all λ.

We say that, for $\lambda < \mu$, the polynomials $R_n(x; \lambda)$ are above the polynomials $R_n(x; \mu)$ and that, for $\lambda > \mu$, the polynomials $R_n(x; \lambda)$ are below the polynomials $R_n(x; \mu)$. For results of this nature in another context, see Wilson [15]. There are several immediate corollaries of this theorem.

COROLLARY 1. *For* $\lambda \neq \mu$, $R_n(x; \lambda) \not\equiv R_n(x; \mu)$, $n \geqq 3$.

COROLLARY 2. $\partial R_n(x; \lambda)/\partial x$ *lies above* $\{R_n(x; \mu)\}$ *for every value of* μ.

Proof. Expanding $R_n(x; \lambda)$ in terms of the integrals of the Legendre polynomials and differentiating the sum with respect to x, we have the derivative expressed as a finite positive expansion in terms of Legendre polynomials. However, from Theorem 2, the Legendre polynomials lie above $\{R_n(x; \mu)\}$ for any μ, from which fact the result follows.

COROLLARY 3. $-\partial R_n(x; \lambda)/\partial \lambda$ *lies above* $\{R_n(x; \mu)\}$ *when* $\lambda < \mu$.

Proof. From (46) and (49), $a(k, n)$ is a product of nonnegative decreasing functions of λ, so the result is immediate.

5. Concluding remarks and observations. There are several interesting open questions. These are:

1. For any $\lambda > 0$, are the zeros of $R_n(x; \lambda)$ interlaced with the zeros of $R_{n+1}(x; \lambda)$?

2. Are the positive zeros of $R_n(x;\lambda)$, when $n \geqq 3$, monotone functions of λ, passing from the zeros of the Legendre polynomials to the zeros of the integrals of the Legendre polynomials as λ increases from 0 to ∞?

3. There exists a reproducing kernel for all polynomials whose degree does not exceed n, that is,

$$\sum_{i=0}^{n} Q_i(x;\lambda)Q_i(y;\lambda),$$

where $Q_i(x;\lambda)$ is the orthonormal polynomial in (2) with positive leading coefficient. In this regard, see Lewis [9]. Is there a Christoffel identity here as there is in L^2-spaces?

The first question could be answered in the affirmative, provided it could be shown that

$$R'_{n+1}(x;\lambda)R_n(x;\lambda) - R_{n+1}(x;\lambda)R'_n(x;\lambda) > 0 \tag{51}$$

in the closed interval $[-1,1]$. In this connection, see Videnskii [14] and Szegö [12, 3.3, pp. 43–45]. For $n = 1$, (51) is obvious, since $R_1(x;\lambda)$ is the Legendre polynomial of degree 1 and $R_2(x;\lambda)$ a positive multiple of the Legendre polynomial of degree 2. Suppose that $n = 2$. Then, by (41) and (42), it is seen that (51) becomes

$$\frac{x^4}{4} - \frac{x^2}{10(3\lambda+1)} + \frac{1}{12} - \frac{1}{30(3\lambda+1)}. \tag{52}$$

The term (52) has a negative discriminant, so that it is positive for all x. Inequality (51) cannot be justified, as it is in L^2, by considering the limit of the reproducing kernel as $y \to x$, as simple examples show.

The second question is also unresolved; although, for a few such values of n, the positive zeros have been shown to be monotone. Suppose, first of all, that $n = 3$, so that there is only one positive root, $x = x(\lambda)$, to be considered. If $x(\lambda)$ were not monotone in λ, there would exist two different values of λ, say λ_0 and λ_1, such that $x(\lambda_0) = x(\lambda_1)$. Since there are only three roots and the zeros are symmetric with respect to the origin, it would follow that $R_3(x;\lambda_0) \equiv R_3(x;\lambda_1)$. By Corollary 1 to Theorem 2, however, this is impossible. When $n = 4$, there is the possibility that, for two different values of λ, one of the positive roots could be the same. By (43), the roots of $R_4(x;\lambda)$ are given by

$$x^2(\lambda) = \frac{4}{5}\left\{\frac{3}{2}\left[\frac{1}{2} - \frac{1}{7(15\lambda+1)}\right] \pm \left[\frac{1}{4} - \frac{420\lambda+19}{196(15\lambda+1)^2}\right]^{1/2}\right\}. \tag{53}$$

From (53), one sees, by differentiation, that both positive roots are strictly increasing functions of λ. The roots of the quintic polynomial (44) are $x = 0$ and

$$\begin{aligned} x^2(\lambda) = \frac{4}{7}\Bigg\{&\frac{5}{2}\left[\frac{1}{2} - \frac{3\lambda+1}{9(105\lambda^2+45\lambda+1)}\right] \\ &\pm \left[\frac{1}{4} + \frac{28350\lambda^3 + 8595\lambda^2 - 1200\lambda - 11}{324(105\lambda^2+45\lambda+1)^2}\right]^{1/2}\Bigg\}. \end{aligned} \tag{54}$$

Upon differentiation and a laborious computation, it can be shown that both positive roots, as given by (54), are strictly monotone increasing.

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ON A CLASS OF WEIESTRASS ELLIPTIC FUNCTIONS AT HALF AND QUARTER PERIODS*

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Abstract. The evaluation of a class of Weierstrass elliptic functions and their allied functions at half and quarter periods is considered in this paper. The double periods of the functions are restricted to 1 and $2^k ci$, where k is an integer and $c = 1, \sqrt{3}$ or $1/\sqrt{3}$. The functions are found expressible in closed form in terms of two special coefficients. The results are tabulated for $k = -1, 0, 1$.

1. Introduction. In the present paper, we are concerned with the evaluation of a class of Weierstrass elliptic functions and their allied functions, namely, the derivative of the functions and the Weierstrass zeta function, at half and quarter periods or their combinations. The double periods 2ω and $2\omega'$ of the functions are restricted to 1 and $2^k ci$, respectively, where k is a positive or negative integer including zero and $c = 1, \sqrt{3}$ or $1/\sqrt{3}$.

It is known [1] that the Weierstrass elliptic function $W(z)$ at the half periods ω, ω' and $\omega + \omega'$ satisfies the cubic equation

$$W^3(z) - 15\sigma_4^* W(z) - 35\sigma_6^* = 0, \tag{1}$$

where σ_4^* and σ_6^* are the particular cases of the following double series:

$$\sigma_{2s}^* = \sum_{n,m=-\infty}^{\infty}{}' \frac{1}{(2m\omega + 2n\omega')^{2s}}, \qquad s \geqq 2. \tag{2}$$

The prime on the summation sign denotes the omission of simultaneous zeros of m and n from the double summation. In this paper, we are concerned with the case $2\omega = 1$ only. When there is a need to emphasize the period $2\omega'$, we write

$$W(z) = W(z|2\omega'), \qquad \sigma_{2s}^* = \sigma_{2s}^*(2\omega'). \tag{3}$$

Further, define two special coefficients σ_4 and σ_6 by

$$\sigma_4 = \sigma_4^*(i), \qquad \sigma_6 = \sigma_6^*(e^{\pi i/3}). \tag{4}$$

The values of these two coefficients are given in a recent paper by the author [2]. It is found that when $2\omega' = 2^k ci$, the class of Weierstrass elliptic functions and their allied functions, at half and quarter periods or their combinations, can be expressed in closed form in terms of σ_4 when $c = 1$ and in terms of σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$. A method of evaluation is first described and then the functions are evaluated and tabulated for $k = -1, 0, 1$.

It may be noted that some relations used in evaluation are existing relations but some are new. The present investigation is motivated by a need raised in summation of series involving hyperbolic functions. It is thought that it deserves a separate consideration on its own merit.

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2. σ_4^* and σ_6^* and W at half periods. Denote the three roots of the cubic equation in (1) when $2\omega = 1$ by

$$(5) \qquad e_1(2\omega') = W(\tfrac{1}{2}|2\omega'), \quad e_2(2\omega') = W(\omega'|2\omega'), \quad e_3(2\omega') = W(\tfrac{1}{2} + \omega'|2\omega').$$

When $2\omega' = 2^k ci$, the coefficients σ_4^* and σ_6^* and the functions e_1, e_2 and e_3 in the case $k = 0$ have been evaluated in terms of σ_4 for $c = 1$ and in terms of σ_6 for $c = \sqrt{3}$ or $1/\sqrt{3}$ [2]. Furthermore, it can be shown [1, p. 379] that for any integral k including zero,

$$(6) \qquad \begin{aligned} \sigma_4^*(2^k ci) &= \tfrac{1}{16}e_1^2(2^{k-1}ci) - \tfrac{1}{4}\sigma_4^*(2^{k-1}ci), \\ \sigma_6^*(2^k ci) &= \tfrac{3}{32}\sigma_4^*(2^{k-1}ci)e_1(2^{k-1}ci) + \tfrac{11}{32}\sigma_6^*(2^{k-1}ci). \end{aligned}$$

Similarly, it can also be shown that

$$(7) \qquad \begin{aligned} \sigma_4^*(2^k ci) &= e_2^2(2^{k+1}ci) - 4\sigma_4^*(2^{k+1}ci), \\ \sigma_6^*(2^k ci) &= 6\sigma_4^*(2^{k+1}ci)e_2(2^{k+1}ci) + 22\sigma_6^*(2^{k+1}ci). \end{aligned}$$

The first pair of recurrence relations can be used to evaluate the coefficients for $k \geqq 1$ from those of $k = 0$, while the second pair can be used to evaluate the coefficients for $k \leqq -1$.

It is mentioned that when $2\omega = 1$ and $2\omega' = 2^k ci$, the cubic equation in (1) possesses three distinct real roots such that $e_1 > e_3 > e_2$. To facilitate the solution of the cubic equation in this *irreducible* case, it is found that one of the roots is given by

$$(8) \qquad e_2(2^k ci) = -\tfrac{1}{2}e_1(2^{k-1}ci)$$

or

$$(9) \qquad e_1(2^k ci) = -2e_2(2^{k+1}ci),$$

so that the cubic equation can be solved without difficulty.

3. W at quarter periods. The Weierstrass elliptic function W at quarter periods can be found from the following relation [1, p. 376]:

$$(10) \qquad \begin{aligned} W(z) = W(2z) &+ \{W(2z) - e_2\}^{1/2}\{W(2z) - e_3\}^{1/2} \\ &+ \{W(2z) - e_3\}^{1/2}\{W(2z) - e_1\}^{1/2} \\ &+ \{W(2z) - e_1\}^{1/2}\{W(2z) - e_2\}^{1/2}. \end{aligned}$$

At quarter periods when $z = \frac{1}{4}, \frac{1}{2}\omega'$ and $\frac{1}{4} \pm \frac{1}{2}\omega'$, respectively, we have

$$(11) \qquad \begin{aligned} W(\tfrac{1}{4}) &= e_1 + (e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}, \\ W(\tfrac{1}{2}\omega') &= e_2 - (e_1 - e_2)^{1/2}(e_3 - e_2)^{1/2}, \\ W(\tfrac{1}{4} \pm \tfrac{1}{2}\omega') &= e_3 \mp i(e_1 - e_3)^{1/2}(e_3 - e_2)^{1/2}. \end{aligned}$$

Here the amplitudes of the functions under radicals are considered in the manner as described by MacRobert [3, pp. 13–15] so that the resulting functions under radicals are all positive.

4. W' at quarter periods. The function $W'(z)$ or the derivative of $W(z)$ vanishes identically at half periods. At quarter periods, it can be found from the following relation [1, p. 367]:

$$W'(z) = -2\{W(z) - e_1\}^{1/2}\{W(z) - e_2\}^{1/2}\{W(z) - e_3\}^{1/2}. \tag{12}$$

We have at quarter periods by similarly considering the amplitudes of the functions under radicals,

$$\begin{aligned} W'(\tfrac{1}{4}) &= -2(e_1 - e_2)(e_1 - e_3)^{1/2} - 2(e_1 - e_3)(e_1 - e_2)^{1/2}, \\ W'(\tfrac{1}{2}\omega') &= -2i(e_3 - e_2)(e_1 - e_2)^{1/2} - 2i(e_1 - e_2)(e_3 - e_2)^{1/2}, \\ W'(\tfrac{1}{4} \pm \tfrac{1}{2}\omega') &= 2(e_1 - e_3)(e_3 - e_2)^{1/2} \pm 2i(e_3 - e_2)(e_1 - e_3)^{1/2}. \end{aligned} \tag{13}$$

5. Weierstrass zeta function at half periods. From the results in a previous paper [2], the Weierstrass zeta function at the half period $\frac{1}{2}$ for the three values of c is, respectively,

(14)

c	1	$\sqrt{3}$	$1/\sqrt{3}$
$\zeta\left(\frac{1}{2}\,\middle\vert ci\right)$	$\frac{1}{2}\pi$	$\frac{\sqrt{3}}{6}\pi + \frac{1}{8}(35\sigma_6)^{1/3}$	$\frac{\sqrt{3}}{2}\pi - \frac{3}{8}(35\sigma_6)^{1/3}$

The function at other half periods is given by

$$\begin{aligned} \zeta(\tfrac{1}{2}ci|ci) &= ci\zeta(\tfrac{1}{2}|ci) - \pi i, \\ \zeta(\tfrac{1}{2} + \tfrac{1}{2}ci|ci) &= (1 + ci)\zeta(\tfrac{1}{2}|ci) - \pi i. \end{aligned} \tag{15}$$

It can be shown from the double series definition of the function that

$$\zeta(\tfrac{1}{2}|2^k ci) = \tfrac{1}{2}\zeta(\tfrac{1}{2}|2^{k-1}ci) + \tfrac{1}{8}e_1(2^{k-1}ci), \tag{16}$$

so that the function at the half period $\frac{1}{2}$ can be evaluated recurrently. This relation is convenient to use when $k \geqq 1$. When $k \leqq -1$, the following relation may be used instead:

$$\zeta(\tfrac{1}{2}|2^k ci) = 2\zeta(\tfrac{1}{2}|2^{k+1}ci) + \tfrac{1}{2}e_2(2^{k+1}ci). \tag{17}$$

The function at other half periods can be found from (15) by replacing c with $2^k c$.

6. Weierstrass zeta function at quarter periods. From the pseudo-addition theorem of the Weierstrass zeta function [4, p. 446], we find that when two of the arguments are equal,

$$\{2\zeta(z) - \zeta(2z)\}^2 = 2W(z) + W(2z). \tag{18}$$

Consequently, by taking the positive sign,

$$\zeta(z) = \tfrac{1}{2}\zeta(2z) + \tfrac{1}{2}\{2W(z) + W(2z)\}^{1/2}. \tag{19}$$

Here the positive sign is chosen on a consideration of the behavior of the function near the origin. Hence, we have at quarter periods by similarly considering the amplitudes of the functions under radicals,

$$
\begin{aligned}
\zeta(\tfrac{1}{4}) &= \tfrac{1}{2}\zeta(\tfrac{1}{2}) + \tfrac{1}{2}\{3e_1 + 2(e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}\}^{1/2}, \\
\zeta(\tfrac{1}{2}\omega') &= \tfrac{1}{2}\zeta(\omega') - \frac{i}{2}\{-3e_2 + 2(e_1 - e_2)^{1/2}(e_3 - e_2)^{1/2}\}^{1/2}, \\
\zeta(\tfrac{1}{4} + \tfrac{1}{2}\omega') &= \tfrac{1}{2}\zeta(\tfrac{1}{2} + \omega') + \tfrac{1}{2}\{3e_3 - 2i(e_1 - e_3)^{1/2}(e_3 - e_2)^{1/2}\}^{1/2}, \\
\zeta(\tfrac{1}{4} - \tfrac{1}{2}\omega') &= \tfrac{1}{2}\zeta(\tfrac{1}{2} + \omega') - \zeta(\omega') + \tfrac{1}{2}\{3e_3 + 2i(e_1 - e_3)^{1/2}(e_3 - e_2)^{1/2}\}^{1/2}.
\end{aligned} \tag{20}
$$

The last relation is obtained on account of pseudo-periodicity of the function.

7. Functions at combined half and quarter periods. From the following relations [1, p. 365],

$$
\begin{aligned}
\{W(z + \tfrac{1}{2}) - e_1\}\{W(z) - e_1\} &= (e_1 - e_2)(e_1 - e_3), \\
\{W(z + \omega') - e_2\}\{W(z) - e_2\} &= (e_2 - e_1)(e_2 - e_3),
\end{aligned} \tag{21}
$$

we find when z is at quarter periods $\frac{1}{2}\omega'$ and $\frac{1}{4}$, respectively,

$$
\begin{aligned}
W(\tfrac{1}{2} + \tfrac{1}{2}\omega') &= e_2 + (e_3 - e_2)^{1/2}(e_1 - e_2)^{1/2}, \\
W(\tfrac{1}{4} + \omega') &= e_1 - (e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}.
\end{aligned} \tag{22}
$$

Likewise, from (12) and (19), we find

$$
\begin{aligned}
W'(\tfrac{1}{2} + \tfrac{1}{2}\omega') &= 2i(e_1 - e_2)(e_3 - e_2)^{1/2} - 2i(e_3 - e_2)(e_1 - e_2)^{1/2}, \\
W'(\tfrac{1}{4} + \omega') &= 2(e_1 - e_2)(e_1 - e_3)^{1/2} - 2(e_1 - e_3)(e_1 - e_2)^{1/2}, \\
\zeta(\tfrac{1}{2} + \tfrac{1}{2}\omega') &= \zeta(\tfrac{1}{2}) + \tfrac{1}{2}\zeta(\omega') - \frac{i}{2}\{-3e_2 - 2(e_1 - e_2)^{1/2}(e_3 - e_2)^{1/2}\}^{1/2}, \\
\zeta(\tfrac{1}{4} + \omega') &= \tfrac{1}{2}\zeta(\tfrac{1}{2}) + \zeta(\omega') + \tfrac{1}{2}\{3e_1 - 2(e_1 - e_2)^{1/2}(e_1 - e_3)^{1/2}\}^{1/2}.
\end{aligned} \tag{23}
$$

8. The results. As a check of the amplitude of the function in the foregoing expressions, a direct method is to compare the resulting value of the function with that given by the series expansion of the function whenever practicable. In case no such series expansion is available, an indirect check may be made with the aid of the following relations:

$$
\begin{aligned}
\zeta(z|i/c) &= ic\zeta(ciz|ci), \\
W(z|i/c) &= -c^2 W(ciz|ci), \\
W'(z|i/c) &= -ic^3 W'(ciz|ci),
\end{aligned} \tag{24}
$$

where c is written in place of $2^k c$ for convenience.

From the foregoing results, it is seen that the functions under consideration are expressed in terms of e_1, e_2 and e_3. It follows therefore that when $2\omega = 1$ and $2\omega' = 2^k ci$, they can be expressed in terms of σ_4 for $c = 1$ and in terms of σ_4 for $c = \sqrt{3}$ or $1/\sqrt{3}$. For convenience of reference, the values of the functions are shown for $k = -1, 0, 1$ in the accompanying tables, where the following abbreviations are used:

$$
u = (15\sigma_4)^{1/2}, \qquad v = (35\sigma_6)^{1/3}, \tag{25}
$$

and

$$
\begin{aligned}
&A_1 = 6\sqrt{2} + 2\sqrt{3} + 2\sqrt{6} + 1, && A_2 = 6\sqrt{2} - 2\sqrt{3} + 2\sqrt{6} - 1,\\
&A_3 = 6\sqrt{2} + 2\sqrt{3} - 2\sqrt{6} - 1, && A_4 = 6\sqrt{2} - 2\sqrt{3} - 2\sqrt{6} + 1,\\
&B_1 = (6\sqrt{3} + 6\sqrt{2} + 2\sqrt{6} + 3)^{1/2}, && B_2 = (6\sqrt{3} + 6\sqrt{2} - 2\sqrt{6} - 3)^{1/2},\\
&B_3 = (6\sqrt{3} - 6\sqrt{2} + 2\sqrt{6} - 3)^{1/2}, && B_4 = (6\sqrt{3} - 6\sqrt{2} - 2\sqrt{6} + 3)^{1/2},\\
&C_1 = 2\sqrt{3}(7\sqrt{6} + 8\sqrt{3} + 9\sqrt{2} + 16)^{1/2}, && C_2 = 2\sqrt{3}(7\sqrt{6} - 8\sqrt{3} - 9\sqrt{2} + 16)^{1/2},\\
&C_3 = 2\sqrt{3}(7\sqrt{6} + 8\sqrt{3} - 9\sqrt{2} - 16)^{1/2}, && C_4 = 2\sqrt{3}(7\sqrt{6} - 8\sqrt{3} + 9\sqrt{2} - 16)^{1/2},\\
&D_1 = 2(3\sqrt{2} + \sqrt{6})^{1/2}, && D_2 = 2(3\sqrt{2} - \sqrt{6})^{1/2}.
\end{aligned}
\tag{26}
$$

9. Checking formulas. The following relations may be used for further checking purposes: for $2\omega = 1$,

$$
\begin{aligned}
&\sigma_4^*(i/c) = c^4\sigma_4^*(ci),\\
&\sigma_6^*(i/c) = -c^6\sigma_6^*(ci),\\
&e_1 + e_2 + e_3 = 0,\\
&e_1^2 + e_2^2 + e_3^2 = 30\sigma_4^*,\\
&e_1e_2e_3 = 35\sigma_6^*,\\
&\zeta(\tfrac{1}{2}) + \zeta(\omega') - \zeta(\tfrac{1}{2} + \omega') = 0,
\end{aligned}
$$

$$
W(\tfrac{1}{4}) + W(\tfrac{1}{2}\omega') + W(\tfrac{1}{4} + \tfrac{1}{2}\omega') + W(\tfrac{1}{4} - \tfrac{1}{2}\omega') + W(\tfrac{1}{2} + \tfrac{1}{2}\omega') + W(\tfrac{1}{4} + \omega') = 0,
$$

$$
\begin{aligned}
-W'(\tfrac{1}{4})W'(\tfrac{1}{4} + \omega') &= W'(\tfrac{1}{2}\omega')W'(\tfrac{1}{2} + \tfrac{1}{2}\omega')\\
&= W'(\tfrac{1}{4} + \tfrac{1}{2}\omega')W'(\tfrac{1}{4} - \tfrac{1}{2}\omega')\\
&= 4(e_1 - e_2)(e_1 - e_3)(e_3 - e_2),
\end{aligned}
\tag{27}
$$

$$
W(\tfrac{1}{4}) + W(\tfrac{1}{2}\omega') + W(\tfrac{1}{4} \pm \tfrac{1}{2}\omega') = \{\zeta(\tfrac{1}{4}) \pm \zeta(\tfrac{1}{2}\omega') - \zeta(\tfrac{1}{4} \pm \tfrac{1}{2}\omega')\}^2,
$$

$$
\begin{aligned}
&W(\tfrac{1}{2} + \tfrac{1}{2}\omega') + W(\tfrac{1}{4} + \omega') + W(\tfrac{1}{4} + \tfrac{1}{2}\omega')\\
&\qquad = \{\zeta(\tfrac{1}{2} + \tfrac{1}{2}\omega') + \zeta(\tfrac{1}{4} + \omega') + \zeta(\tfrac{1}{4} + \tfrac{1}{2}\omega') - 2\zeta(\tfrac{1}{2} + \omega')\}^2,
\end{aligned}
$$

$$
\begin{vmatrix}
W'(\tfrac{1}{4}) & W(\tfrac{1}{4}) & 1\\
\pm W'(\tfrac{1}{2}\omega') & W(\tfrac{1}{2}\omega') & 1\\
-W'(\tfrac{1}{4} \pm \tfrac{1}{2}\omega') & W(\tfrac{1}{4} \pm \tfrac{1}{2}\omega') & 1
\end{vmatrix} = 0
$$

and

$$
\begin{vmatrix}
W'(\tfrac{1}{2} + \tfrac{1}{2}\omega') & W(\tfrac{1}{2} + \tfrac{1}{2}\omega') & 1\\
W'(\tfrac{1}{4} + \omega') & W(\tfrac{1}{4} + \omega') & 1\\
W'(\tfrac{1}{4} + \tfrac{1}{2}\omega') & W(\tfrac{1}{4} + \tfrac{1}{2}\omega') & 1
\end{vmatrix} = 0.
$$

Note that in the first two expressions, c stands for 2^kc in general.

TABLE 1

Values of functions when $2\omega = 1$ and $2\omega' = 2^k ci$: for $c = 1$

Function	$k = -1$	$k = 0$	$k = 1$
σ_4^*/u^2	$11/15$	$1/15$	$11/240$
σ_6^*/u^3	$-2/5$	0	$1/160$
e_1/u	2	1	$(2\sqrt{2} + 1)/4$
e_2/u	$-2\sqrt{2} - 1$	-1	$-1/2$
e_3/u	$2\sqrt{2} - 1$	0	$-(2\sqrt{2} - 1)/4$
$W(\frac{1}{4})/u$	3	$\sqrt{2} + 1$	$(2\sqrt{2} + 1)/4 + (\sqrt{2} + 1)2^{1/4}/2$
$W(\frac{1}{2}\omega')/u$	$-2\sqrt{2} - 1 - 2(\sqrt{2} + 1)2^{1/4}$	$-\sqrt{2} - 1$	$-3/4$
$W(\frac{1}{4} \pm \frac{1}{2}\omega')/u$	$2\sqrt{2} - 1 \mp 2i(\sqrt{2} - 1)2^{1/4}$	$\mp i$	$-(2\sqrt{2} - 1)/4 \mp i(\sqrt{2} - 1)2^{1/4}/2$
$W(\frac{1}{2} + \frac{1}{2}\omega')/u$	$-2\sqrt{2} - 1 + 2(\sqrt{2} + 1)2^{1/4}$	$\sqrt{2} - 1$	$-1/4$
$W(\frac{1}{4} + \omega')/u$	1	$-\sqrt{2} + 1$	$(2\sqrt{2} + 1)/4 - (\sqrt{2} + 1)2^{1/4}/2$
$W'(\frac{1}{4})/u^{3/2}$	$-4\sqrt{2}$	$-2(2 + \sqrt{2})$	$-2 - \sqrt{2} - (3 + 2\sqrt{2})2^{1/4}/2$
$W'(\frac{1}{2}\omega')/u^{3/2}$	$-4i\{4 + 2\sqrt{2} + (3 + 2\sqrt{2})2^{1/4}\}$	$-2i(2 + \sqrt{2})$	$-i\sqrt{2}/2$
$W'(\frac{1}{4} \pm \frac{1}{2}\omega')/u^{3/2}$	$4(3 - 2\sqrt{2})2^{1/4} \pm 8i(2 - \sqrt{2})$	$2(1 \pm i)$	$2 - \sqrt{2} \pm i(3 - 2\sqrt{2})2^{1/4}/2$
$W'(\frac{1}{2} + \frac{1}{2}\omega')/u^{3/2}$	$-4i\{4 + 2\sqrt{2} - (3 + 2\sqrt{2})2^{1/4}\}$	$2i(2 - \sqrt{2})$	$i/2$
$W'(\frac{1}{4} + \omega')/u^{3/2}$	4	$2(2 - \sqrt{2})$	$-2 - \sqrt{2} + (3 + 2\sqrt{2})2^{1/4}/2$

$\zeta(\tfrac{1}{2})$	$(2\pi - u)/2$	$\pi/2$	$(2\pi + u)/8$
$\zeta(\omega')$	$-i(2\pi + u)/4$	$-\pi i/2$	$-i(2\pi - u)/4$
$\zeta(\tfrac{1}{2} + \omega')$	$\dfrac{2-i}{2}\pi - \dfrac{2+i}{4}u$	$\dfrac{1-i}{2}\pi$	$\dfrac{1-2i}{4}\pi + \dfrac{1+2i}{8}u$
$\zeta(\tfrac{1}{4})$	$\dfrac{2\pi - u}{4} + \sqrt{2u}$	$\dfrac{\pi}{4} + \dfrac{\sqrt{2}+1}{2}\sqrt{u}$	$\dfrac{2\pi + u}{16} + \dfrac{\sqrt{u}}{4}(2^{1/4} + 1)^2$
$\zeta(\tfrac{1}{2}\omega')$	$-\dfrac{2\pi + u}{8}i - \dfrac{i\sqrt{u}}{2}(2^{1/4} + 1)^2$	$-\dfrac{\pi i}{4} - \dfrac{\sqrt{2}+1}{2}i\sqrt{u}$	$-\dfrac{2\pi - u}{8}i - \dfrac{\sqrt{2}}{2}i\sqrt{u}$
$\zeta(\tfrac{1}{4} \pm \tfrac{1}{2}\omega')$	$\dfrac{2 \mp i}{4}\pi - \dfrac{2 \pm i}{8}u \mp \dfrac{i\sqrt{u}}{2}(2^{1/4} \pm i)^2$	$\dfrac{1 \mp i}{4}(\pi + 2\sqrt{u})$	$\dfrac{1 \mp 2i}{8}\pi + \dfrac{1 \pm 2i}{16}u + \dfrac{\sqrt{u}}{4}(2^{1/4} \mp i)^2$
$\zeta(\tfrac{1}{2} + \tfrac{1}{2}\omega')$	$\dfrac{4-i}{4}\pi - \dfrac{4+i}{8}u - \dfrac{i\sqrt{u}}{2}(2^{1/4} - 1)^2$	$\dfrac{2-i}{4}\pi - \dfrac{\sqrt{2}-1}{2}i\sqrt{u}$	$\dfrac{1-i}{4}\pi + \dfrac{1+i}{8}u - \dfrac{i}{2}\sqrt{u}$
$\zeta(\tfrac{1}{4} + \omega')$	$\dfrac{1-i}{2}\pi - \dfrac{1+i}{4}u + \sqrt{u}$	$\dfrac{1-2i}{4}\pi + \dfrac{\sqrt{2}-1}{2}\sqrt{u}$	$\dfrac{1-4i}{8}\pi + \dfrac{1+4i}{16}u + \dfrac{\sqrt{u}}{4}(2^{1/4} - 1)^2$

TABLE 2
Values of functions when $2\omega = 1$ and $2\omega' = 2^k ci$: for $c = \sqrt{3}$

Function	$k=-1$	$k=0$	$k=1$
σ_4^*/v^2	$(9-4\sqrt{3})/16$	$1/16$	$(9+4\sqrt{3})/256$
σ_6^*/v^3	$-(210\sqrt{3}-347)/1120$	$11/1120$	$27(210\sqrt{3}+347)/71680$
e_1/v	$(2\sqrt{3}-1)/2$	$(2\sqrt{3}+1)/4$	$A_1/16$
e_2/v	$-A_3/4$	$-(2\sqrt{3}-1)/4$	$-(2\sqrt{3}+1)/8$
e_3/v	$A_4/4$	$-1/2$	$-A_2/16$
$W(\frac{1}{4})/v$	$(6\sqrt{3}+1)/4$	$(\sqrt{6}+2\sqrt{3}+3\sqrt{2}+1)/4$	$(C_1+A_1)/16$
$W(\frac{1}{2}\omega')/v$	$-(C_2+A_3)/4$	$-(-\sqrt{6}+2\sqrt{3}+3\sqrt{2}-1)/4$	$-(6\sqrt{3}-1)/16$
$W(\frac{1}{4}\pm\frac{1}{2}\omega')/v$	$(A_4\mp iC_3)/4$	$-(2\pm\sqrt{3}i)/4$	$(-A_2\mp iC_4)/16$
$W(\frac{1}{2}+\frac{1}{2}\omega')/v$	$(C_2-A_3)/4$	$-(\sqrt{6}+2\sqrt{3}-3\sqrt{2}-1)/4$	$-(5+2\sqrt{3})/16$
$W(\frac{1}{4}+\omega')/v$	$-(5-2\sqrt{3})/4$	$-(\sqrt{6}-2\sqrt{3}+3\sqrt{2}-1)/4$	$-(C_1-A_1)/16$
$W'(\frac{1}{4})/v^{3/2}$	$-(3+2\sqrt{3})(B_2+B_3)/4$	$-(\sqrt{2}+1)(3+\sqrt{6})3^{1/4}/2$	$-(B_1+D_1)C_1/32$
$W'(\frac{1}{2}\omega')/v^{3/2}$	$-i(B_2+D_2)C_2/4$	$-i(\sqrt{2}+1)(3+\sqrt{6})3^{1/4}/2$	$-i(2\sqrt{3}-3)(B_1+B_4)/32$
$W'(\frac{1}{4}\pm\frac{1}{2}\omega')/v^{3/2}$	$(B_3\pm iD_2)C_3/4$	$\sqrt{2}\{3+\sqrt{3}\pm i(3-\sqrt{3})\}3^{1/4}/8$	$(D_1\pm iB_4)C_4/32$
$W'(\frac{1}{2}+\frac{1}{2}\omega')/v^{3/2}$	$i(B_2-D_2)C_2/4$	$i(\sqrt{2}+1)(3-\sqrt{6})3^{1/4}/2$	$i(2\sqrt{3}-3)(B_1-B_4)/32$
$W'(\frac{1}{4}+\omega')/v^{3/2}$	$(3+2\sqrt{3})(B_2-B_3)/4$	$(\sqrt{2}-1)(3-\sqrt{6})3^{1/4}/2$	$(B_1-D_1)C_1/32$

$\zeta(\frac{1}{2})$	$\frac{\sqrt{3}}{3}\pi - \frac{2\sqrt{3}-3}{8}v$	$\frac{\sqrt{3}}{6}\pi + \frac{v}{8}$	$\frac{\sqrt{3}}{12}\pi + \frac{2\sqrt{3}+3}{32}v$
$\zeta(\omega')$	$-\frac{\pi i}{2} - \frac{6-3\sqrt{3}}{16}iv$	$-\frac{\pi i}{2} + \frac{\sqrt{3}}{8}iv$	$-\frac{\pi i}{2} - \frac{6+3\sqrt{3}}{16}iv$
$\zeta(\frac{1}{2}+\omega')$	$\frac{2\sqrt{3}-3i}{6}\pi - \frac{2\sqrt{3}+3i}{16}(2-\sqrt{3})v$	$\frac{\sqrt{3}-3i}{6}\pi + \frac{1+\sqrt{3}i}{8}v$	$\frac{\sqrt{3}-6i}{12}\pi + \frac{\sqrt{3}+6i}{32}(2+\sqrt{3})v$
$\zeta(\frac{1}{4})$	$\frac{\sqrt{3}\pi}{6} - \frac{2\sqrt{3}-3}{16}v + 3^{1/4}\sqrt{v}$	$\frac{\sqrt{3}\pi}{12} + \frac{v}{16} + \frac{B_1}{4}\sqrt{v}$	$\frac{\sqrt{3}\pi}{24} + \frac{2\sqrt{3}+3}{64}v + \frac{B_1+D_1}{8}\sqrt{v}$
$\zeta(\frac{1}{2}\omega')$	$-\frac{\pi i}{4} - \frac{6-3\sqrt{3}}{32}iv - \frac{B_2+D_2}{4}i\sqrt{v}$	$-\frac{\pi i}{4} + \frac{\sqrt{3}iv}{16} - \frac{B_2}{4}i\sqrt{v}$	$-\frac{\pi i}{4} + \frac{6+3\sqrt{3}}{32}iv - \frac{3^{1/4}}{2}i\sqrt{v}$
$\zeta(\frac{1}{4}\pm\frac{1}{2}\omega')$	$\frac{2\sqrt{3}\mp 3i}{12}\pi - \frac{2\sqrt{3}\pm 3i}{32}(2-\sqrt{3})v$ $+ \frac{D_2 \mp iB_3}{4}\sqrt{v}$	$\frac{\sqrt{3}\mp 3i}{12}\pi + \frac{1\pm\sqrt{3}i}{16}v$ $+ \frac{\sqrt{3}-1\mp i(\sqrt{3}+1)}{8}3^{1/4}\sqrt{2v}$	$\frac{\sqrt{3}\mp 6i}{24}\pi + \frac{\sqrt{3}\pm 6i}{64}(2+\sqrt{3})v$ $+ \frac{B_4 \mp iD_1}{8}\sqrt{v}$
$\zeta(\frac{1}{2}+\frac{1}{2}\omega')$	$\frac{4\sqrt{3}-3i}{12}\pi - \frac{4\sqrt{3}+3i}{32}(2-\sqrt{3})v$ $- \frac{B_2-D_2}{4}i\sqrt{v}$	$\frac{2\sqrt{3}-3i}{12}\pi + \frac{2+\sqrt{3}i}{16}v$ $- \frac{B_3}{4}i\sqrt{v}$	$\frac{\sqrt{3}-3i}{12}\pi + \frac{\sqrt{3}+3i}{32}(2+\sqrt{3})v$ $- \frac{\sqrt{3}+1}{8}3^{1/4}i\sqrt{2v}$
$\zeta(\frac{1}{4}+\omega')$	$\frac{\sqrt{3}-3i}{6}\pi - \frac{\sqrt{3}+3i}{16}(2-\sqrt{3})v$ $+ \frac{\sqrt{3}-1}{4}3^{1/4}\sqrt{2v}$	$\frac{\sqrt{3}-6i}{12}\pi + \frac{1+2\sqrt{3}i}{16}v$ $+ \frac{B_4}{4}\sqrt{v}$	$\frac{\sqrt{3}-12i}{24}\pi + \frac{\sqrt{3}+12i}{64}(2+\sqrt{3})v$ $+ \frac{B_1-D_1}{8}\sqrt{v}$

TABLE 3

Values of functions when $2\omega = 1$ and $2\omega' = 2^k ci$; for $c = 1/\sqrt{3}$

Function	$k=-1$	$k=0$	$k=1$
σ_4^*/v^2	$9(9+4\sqrt{3})/16$	$9/16$	$9(9-4\sqrt{3})/256$
σ_6^*/v^3	$-27(210\sqrt{3}+347)/1120$	$-297/1120$	$27(210\sqrt{3}-347)/71680$
e_1/v	$3(2\sqrt{3}+1)/2$	$3(2\sqrt{3}-1)/4$	$3A_3/16$
e_2/v	$-3A_1/4$	$-3(2\sqrt{3}+1)/4$	$-3(2\sqrt{3}-1)/8$
e_3/v	$3A_2/4$	$3/2$	$-3A_4/16$
$W(\frac{1}{4})/v$	$3(6\sqrt{3}-1)/4$	$3(-\sqrt{6}+2\sqrt{3}+3\sqrt{2}-1)/4$	$3(C_2+A_3)/16$
$W(\frac{1}{4}\omega')/v$	$-3(C_1+A_1)/4$	$-3(\sqrt{6}+2\sqrt{3}+3\sqrt{2}+1)/4$	$-3(6\sqrt{3}+1)/16$
$W(\frac{1}{4}\pm\frac{1}{2}\omega')/v$	$3(A_2\mp iC_4)/4$	$3(2\mp i\sqrt{3})/4$	$3(-A_4\mp iC_3)/16$
$W(\frac{1}{2}+\frac{1}{2}\omega')/v$	$3(C_1-A_1)/4$	$3(\sqrt{6}-2\sqrt{3}+3\sqrt{2}-1)/4$	$3(5-2\sqrt{3})/16$
$W(\frac{1}{4}+\omega')/v$	$3(5+2\sqrt{3})/4$	$3(\sqrt{6}+2\sqrt{3}-3\sqrt{2}-1)/4$	$-3(C_2-A_3)/16$
$W'(\frac{1}{4})/v^{3/2}$	$-9(2-\sqrt{3})(B_1+B_4)/4$	$-9(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})3^{1/4}/2$	$-3\sqrt{3}(B_2+D_2)C_2/32$
$W'(\frac{1}{2}\omega')/v^{3/2}$	$-3\sqrt{3}i(B_1+D_1)C_1/4$	$-9i(\sqrt{2}+1)(\sqrt{3}+\sqrt{2})3^{1/4}/2$	$-9i(2+\sqrt{3})(B_2+B_3)/32$
$W'(\frac{1}{4}\pm\frac{1}{2}\omega')/v^{3/2}$	$3\sqrt{3}(B_4\pm iD_1)C_4/4$	$9\sqrt{2}\{\sqrt{3}-1\pm i(\sqrt{3}+1)\}3^{1/4}/8$	$3\sqrt{3}(D_2\pm iB_3)C_3/32$
$W'(\frac{1}{2}+\frac{1}{2}\omega')/v^{3/2}$	$3\sqrt{3}i(B_1-D_1)C_1/4$	$9i(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})3^{1/4}/2$	$9i(2+\sqrt{3})(B_2-B_3)/32$
$W'(\frac{1}{4}+\omega')/v^{3/2}$	$9(2-\sqrt{3})(B_1-B_4)/4$	$9(\sqrt{2}+1)(\sqrt{3}-\sqrt{2})3^{1/4}/2$	$3\sqrt{3}(B_2-D_2)C_2/32$

$\zeta(\frac{1}{2})$	$\sqrt{3}\pi - \frac{6\sqrt{3}+9}{8}v$	$\frac{\sqrt{3}\pi}{2} - \frac{3v}{8}$	$\frac{\sqrt{3}\pi}{4} + \frac{6\sqrt{3}-9}{32}v$
$\zeta(\omega')$	$-\frac{\pi i}{2} - \frac{6+3\sqrt{3}}{16}iv$	$-\frac{\pi i}{2} - \frac{\sqrt{3}iv}{8}$	$-\frac{\pi i}{2} + \frac{6-3\sqrt{3}}{16}iv$
$\zeta(\frac{1}{2}+\omega')$	$\frac{2\sqrt{3}-i}{2}\pi - \frac{2\sqrt{3}+i}{16}(6+3\sqrt{3})v$	$\frac{\sqrt{3}-i}{2}\pi - \frac{3+\sqrt{3}i}{8}v$	$\frac{\sqrt{3}-2i}{4}\pi + \frac{\sqrt{3}+2i}{32}(6-3\sqrt{3})v$
$\zeta(\frac{1}{4})$	$\frac{\sqrt{3}\pi}{2} - \frac{6\sqrt{3}+9}{16}v + 3^{1/4}\sqrt{3v}$	$\frac{\sqrt{3}\pi}{4} - \frac{3v}{16} + \frac{B_2}{4}\sqrt{3v}$	$\frac{\sqrt{3}\pi}{8} + \frac{6\sqrt{3}-9}{64}v + \frac{B_2+D_2}{8}\sqrt{3v}$
$\zeta(\frac{1}{2}\omega')$	$-\frac{\pi i}{4} - \frac{6+3\sqrt{3}}{32}iv - \frac{B_1+D_1}{4}i\sqrt{3v}$	$-\frac{\pi i}{4} - \frac{\sqrt{3}iv}{16} - \frac{B_1}{4}i\sqrt{3v}$	$-\frac{\pi i}{4} + \frac{6-3\sqrt{3}}{32}iv - \frac{3^{1/4}}{2}i\sqrt{3v}$
$\zeta(\frac{1}{4} \pm \frac{1}{2}\omega')$	$\frac{2\sqrt{3}\mp i}{4}\pi - \frac{2\sqrt{3}\pm i}{32}(6+3\sqrt{3})v$ $+ \frac{D_1 \mp iB_4}{4}\sqrt{3v}$	$\frac{\sqrt{3}\mp i}{4}\pi - \frac{3\pm\sqrt{3}i}{16}v$ $+ \frac{\sqrt{3}+1\mp i(\sqrt{3}-1)}{8}3^{1/4}\sqrt{6v}$	$\frac{\sqrt{3}\mp 2i}{8}\pi + \frac{\sqrt{3}\pm 2i}{64}(6-3\sqrt{3})v$ $+ \frac{B_3 \mp iD_2}{8}\sqrt{3v}$
$\zeta(\frac{1}{2}+\frac{1}{2}\omega')$	$\frac{4\sqrt{3}-i}{4}\pi - \frac{4\sqrt{3}+i}{32}(6+3\sqrt{3})v$ $- \frac{B_1-D_1}{4}i\sqrt{3v}$	$\frac{2\sqrt{3}-i}{4}\pi - \frac{6+\sqrt{3}i}{16}v$ $- \frac{B_4}{4}i\sqrt{3v}$	$\frac{\sqrt{3}-i}{4}\pi + \frac{\sqrt{3}+i}{32}(6-3\sqrt{3})v$ $- \frac{\sqrt{3}-1}{8}3^{1/4}i\sqrt{6v}$
$\zeta(\frac{1}{4}+\omega')$	$\frac{\sqrt{3}-i}{2}\pi - \frac{\sqrt{3}+i}{16}(6+3\sqrt{3})v$ $+ \frac{\sqrt{3}+1}{4}3^{1/4}\sqrt{6v}$	$\frac{\sqrt{3}-2i}{4}\pi - \frac{3+2\sqrt{3}i}{16}v$ $+ \frac{B_3}{4}\sqrt{3v}$	$\frac{\sqrt{3}-4i}{8}\pi + \frac{\sqrt{3}+4i}{64}(6-3\sqrt{3})v$ $+ \frac{B_2-D_2}{8}\sqrt{3v}$

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ON SUMMATION OF SERIES OF HYPERBOLIC FUNCTIONS. II*

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Abstract. This paper extends the method of summation of series of hyperbolic functions presented in the previous paper to two alternating series of an even degree and also to four series of an odd degree, two positive and two alternating. The series are likewise summed in closed form in terms of two special coefficients σ_4 and σ_6 when the parameter c involved in the series takes on the special values 1, $\sqrt{3}$ or $1/\sqrt{3}$.

1. Introduction. In a previous paper of the same title [1], the author presented a method of summation of four positive series of hyperbolic functions of an even degree. The method is based partly on partial fraction decompositions of hyperbolic functions and partly on values of the Weierstrass elliptic function at half periods of double periods 1 and ci.

The purpose of the present paper is to extend the method of summation first to the following two alternating series of an even degree:

$$\text{(1)} \qquad \mathrm{I}_{2s}^*(c) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sinh^{2s} n\pi c}, \qquad \mathrm{II}_{2s}^*(c) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\cosh^{2s} n\pi c},$$

and second to the following four series of an odd degree, two positive and two alternating:

$$\text{(2)} \qquad \begin{aligned} \mathrm{II}_{2s-1}(c) &= \sum_{n=1}^{\infty} \frac{1}{\cosh^{2s-1} n\pi c}, & \mathrm{IV}_{2s-1}(c) &= \sum_{n=1}^{\infty} \frac{1}{\cosh^{2s-1} (2n-1)\pi c/2}, \\ \mathrm{II}_{2s-1}^*(c) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\cosh^{2s-1} n\pi c}, & \mathrm{III}_{2s-1}^*(c) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sinh^{2s-1} (2n-1)\pi c/2}, \end{aligned}$$

where $s \geqq 1$ and $c = 1, \sqrt{3}$ or $1/\sqrt{3}$.

To sum the series in (1), essentially the same method of summation is employed. However, the sums involve values of the Weierstrass elliptic function at half periods of double periods 1 and $2ci$ instead. To sum the series in (2), different partial fraction decompositions of hyperbolic functions are used. It turns out that the sums involve values of the Weierstrass zeta and elliptic functions at quarter and half periods. The functions involved in both cases have been evaluated in a recent paper [2]. With these values, the sums of the series can also be expressed in closed form in terms of the coefficient σ_4 when $c = 1$ and in terms of the coefficient σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$. The reader may consult the author's previous paper [1] for the values of these two coefficients.

2. Summation of the series when $s = 1$. Using the partial fraction decompositions in the previous paper [1], and further decomposing the resulting alternating

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$$\begin{aligned}
\mathrm{I}_2^*(c) &= \frac{1}{12c^2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^2 - 4n^2c^2}{(m^2 + 4n^2c^2)^2} \\
&\quad - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{m^2 - (2n-1)^2c^2}{\{m^2 + (2n-1)^2c^2\}^2}, \\
\mathrm{II}_2^*(c) &= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - 4(2n-1)^2c^2}{\{(2m-1)^2 + 4(2n-1)^2c^2\}^2} \\
&\quad - \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(2m-1)^2 - 16n^2c^2}{\{(2m-1)^2 + 16n^2c^2\}^2}.
\end{aligned} \tag{3}$$

When the double series are expressed in terms of the Weierstrass elliptic function, the preceding relations become

$$\begin{aligned}
\mathrm{I}_2^*(c) &= -\frac{1}{6} - \frac{1}{2\pi^2} e_2(2ci), \\
\mathrm{II}_2^*(c) &= \frac{1}{2} - \frac{1}{2\pi^2} e_1(2ci) + \frac{1}{2\pi^2} e_3(2ci),
\end{aligned} \tag{4}$$

where $e_1(2ci)$, $e_2(2ci)$ and $e_3(2ci)$ are the Weierstrass elliptic function of double periods 1 and $2ci$ at the half periods, $\frac{1}{2}$, ci and $(1 + 2ci)/2$, respectively.

Again, from the following partial fraction decompositions [3],

$$\begin{aligned}
\frac{1}{\sinh \pi x} &= \frac{1}{\pi x} + \frac{i}{\pi} \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{m + ix} - \frac{1}{m - ix} \right), \\
\frac{1}{\cosh \pi x} &= -\frac{2}{\pi} \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{2m - 1 + 2ix} + \frac{1}{2m - 1 - 2ix} \right),
\end{aligned} \tag{5}$$

we similarly find

$$\begin{aligned}
\mathrm{II}_1(c) &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{2m - 1 + 2nci} + \frac{1}{2m - 1 - 2nci} \right), \\
\mathrm{IV}_1(c) &= -\frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^m \left\{ \frac{1}{2m - 1 + (2n-1)ci} + \frac{1}{2m - 1 - (2n-1)ci} \right\}, \\
\mathrm{II}_1^*(c) &= \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+n} \left(\frac{1}{2m - 1 + 2nci} + \frac{1}{2m - 1 - 2nci} \right), \\
\mathrm{III}_1^*(c) &= \frac{1}{2c} - \frac{2i}{\pi} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+n} \left\{ \frac{1}{2m + (2n-1)ci} - \frac{1}{2m - (2n-1)ci} \right\}.
\end{aligned} \tag{6}$$

The double series involved can be expressed in terms of the Weierstrass zeta function. Denote the function by $\zeta(z|2\omega')$, where $2\omega'$ signifies one of the two periods of the function whereas the other period is $2\omega = 1$. We find from the usual double series definition of the function that when $2\omega' = ci$,

$$\zeta(\tfrac{1}{4}|ci) = \pi + \tfrac{1}{2}\zeta(\tfrac{1}{2}|ci) - 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{2m-1+4nci} + \frac{1}{2m-1-4nci}\right),$$

$$\zeta(\tfrac{1}{4} + \tfrac{1}{2}ci|ci) = \tfrac{1}{2}(1 + 2ci)\zeta(\tfrac{1}{2}|ci) - 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left\{\frac{1}{2m-1+(4n-2)ci} + \frac{1}{2m-1-(4n-2)ci}\right\},$$

$$\zeta(\tfrac{1}{4}ci|ci) = -\frac{\pi i}{c} + \frac{ci}{2}\zeta(\tfrac{1}{2}|ci) - 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n\left\{\frac{1}{4m+(2n-1)ci} - \frac{1}{4m-(2n-1)ci}\right\}, \tag{7}$$

$$\zeta(\tfrac{1}{2} + \tfrac{1}{4}ci|ci) = \tfrac{1}{2}(2 + ci)\zeta(\tfrac{1}{2}|ci) - 4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n\left\{\frac{1}{4m-2+(2n-1)ci} - \frac{1}{4m-2-(2n-1)ci}\right\}.$$

By writing $c/2$ in place of c, the first two relations give, respectively,

$$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{2m-1+2nci} + \frac{1}{2m-1-2nci}\right) = \pi + \tfrac{1}{2}\zeta(\tfrac{1}{2}|\tfrac{1}{2}ci) - \zeta(\tfrac{1}{4}|\tfrac{1}{2}ci),$$

$$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left\{\frac{1}{2m-1+(2n-1)ci} + \frac{1}{2m-1-(2n-1)ci}\right\} = \tfrac{1}{2}(1 + ci)\zeta(\tfrac{1}{2}|\tfrac{1}{2}ci) - \zeta(\tfrac{1}{4} + \tfrac{1}{4}ci|\tfrac{1}{2}ci). \tag{8}$$

The differences of the first two relations and the last two relations give, respectively,

$$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+n}\left(\frac{1}{2m-1+2nci} + \frac{1}{2m-1-2nci}\right) = \pi - ci\zeta(\tfrac{1}{2}|ci) + \zeta(\tfrac{1}{4} + \tfrac{1}{2}ci) - \zeta(\tfrac{1}{4}|ci),$$

$$4\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+n}\left\{\frac{1}{2m+(2n-1)ci} - \frac{1}{2m-(2n-1)ci}\right\} = -\frac{\pi i}{c} - \zeta(\tfrac{1}{2}|ci) + \zeta(\tfrac{1}{2} + \tfrac{1}{4}ci|ci) - \zeta(\tfrac{1}{4}ci|ci). \tag{9}$$

These relations lead to

$$\begin{aligned}
\mathrm{II}_1(c) &= -\frac{1}{2} - \frac{1}{4\pi}\zeta(\tfrac{1}{2}|\tfrac{1}{2}ci) + \frac{1}{2\pi}\zeta(\tfrac{1}{4}|\tfrac{1}{2}ci),\\
\mathrm{IV}_1(c) &= -\frac{1}{4\pi}(1 + ci)\zeta(\tfrac{1}{2}|\tfrac{1}{2}ci) + \frac{1}{2\pi}\zeta(\tfrac{1}{4} + \tfrac{1}{4}ci|\tfrac{1}{2}ci),\\
\mathrm{II}_1^*(c) &= \frac{1}{2} - \frac{ci}{2\pi}\zeta(\tfrac{1}{2}|ci) + \frac{1}{2\pi}\zeta(\tfrac{1}{4} + \tfrac{1}{2}ci|ci) - \frac{1}{2\pi}\zeta(\tfrac{1}{4}|ci),\\
\mathrm{III}_1^*(c) &= \frac{i}{2\pi}\zeta(\tfrac{1}{2}|ci) - \frac{i}{2\pi}\zeta(\tfrac{1}{2} + \tfrac{1}{4}ci|ci) + \frac{i}{2\pi}\zeta(\tfrac{1}{4}ci|ci).
\end{aligned} \tag{10}$$

It is noted that the order of summation of the various double series in this section is not interchangeable. With the values of the Weierstrass zeta and elliptic functions at quarter and half periods given in the paper [2], the sums of the six series in (4) and (10) can be expressed in closed form in terms of σ_4 when $c = 1$ and in terms of σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$. The results are shown in Table 1, in which the following abbreviations are used:

$$
\begin{aligned}
& u = (15\sigma_4)^{1/2}, && v = (35\sigma_6)^{1/3}, \\
(11)\quad & B_1 = (6\sqrt{3} + 6\sqrt{2} + 2\sqrt{6} + 3)^{1/2}, && B_2 = (6\sqrt{3} + 6\sqrt{2} - 2\sqrt{6} - 3)^{1/2}, \\
& B_3 = (6\sqrt{3} - 6\sqrt{2} + 2\sqrt{6} - 3)^{1/2}, && B_4 = (6\sqrt{3} - 6\sqrt{2} - 2\sqrt{6} + 3)^{1/2}.
\end{aligned}
$$

TABLE 1

Values of series in (1) *and* (2) *for* $s = 1$

c	$\mathrm{II}_1(c)$	$\mathrm{IV}_1(c)$	$\mathrm{II}_1^*(c)$
1	$-\frac{1}{2} + \frac{1}{2\pi}\sqrt{2u}$	$\frac{1}{2\pi}\sqrt{u}$	$\frac{1}{2} - \frac{1}{2\pi}\sqrt{u}$
$\sqrt{3}$	$-\frac{1}{2} + \frac{1}{2\pi}3^{1/4}\sqrt{v}$	$\frac{\sqrt{3}-1}{8\pi}3^{1/4}\sqrt{2v}$	$\frac{1}{2} - \frac{1}{8\pi}(B_1 - B_4)\sqrt{v}$
$\frac{1}{\sqrt{3}}$	$-\frac{1}{2} + \frac{1}{2\pi}3^{1/4}\sqrt{3v}$	$\frac{\sqrt{3}+1}{8\pi}3^{1/4}\sqrt{6v}$	$\frac{1}{2} - \frac{1}{8\pi}(B_2 - B_3)\sqrt{3v}$

c	$\mathrm{III}_1^*(c)$	$\mathrm{I}_2^*(c)$	$\mathrm{II}_2^*(c)$
1	$\frac{1}{2\pi}\sqrt{u}$	$-\frac{1}{6} + \frac{1}{4\pi^2}u$	$\frac{1}{2} - \frac{\sqrt{2}}{2\pi^2}u$
$\sqrt{3}$	$\frac{1}{8\pi}(B_2 - B_3)\sqrt{v}$	$-\frac{1}{6} + \frac{2\sqrt{3}+1}{16\pi^2}v$	$\frac{1}{2} - \frac{\sqrt{2}(3+\sqrt{3})}{8\pi^2}v$
$\frac{1}{\sqrt{3}}$	$\frac{1}{8\pi}(B_1 - B_4)\sqrt{3v}$	$-\frac{1}{6} + \frac{2\sqrt{3}-1}{16\pi^2}3v$	$\frac{1}{2} - \frac{\sqrt{2}(3-\sqrt{3})}{8\pi^2}3v$

3. Summation of the series when $s \geqq 2$. Define, for double periods $2\omega = 1$ and $2\omega' = ci$,

$$
(12)\qquad
\begin{aligned}
\sigma_{2s}^*(ci) &= \sum_{n,m=-\infty}^{\infty}{}' \frac{1}{(m + nci)^{2s}}, && s \geqq 2, \\
W_s(z|ci) &= \sum_{n,m=-\infty}^{\infty} \frac{1}{(z - m - nci)^s}, && s \geqq 3,
\end{aligned}
$$

where the prime on the first summation sign denotes the omission of simultaneous zeros of m and n from the double summation. The function W_s thus defined is an elliptic function of double periods 1 and ci. Furthermore, denote for $s \geqq 1$,

$$
\begin{aligned}
S_{2s} &= \sum_{n=1}^{\infty} \frac{1}{n^{2s}} = \frac{(2\pi)^{2s}}{2(2s)!} B_{2s}, \\
S_{2s}^{*} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{2s}} = \frac{(2^{2s-1}-1)\pi^{2s}}{(2s)!} B_{2s}, \\
U_{2s} &= \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2s}} = \left(1 - \frac{1}{2^{2s}}\right) S_{2s}, \\
U_{2s-1}^{*} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^{2s-1}} = \frac{1}{2}\left(\frac{\pi}{2}\right)^{2s-1} \frac{E_{2s-2}}{(2s-2)!},
\end{aligned} \tag{13}
$$

where B_{2s} and E_{2s-2} are Bernoulli and Euler's numbers, respectively. The first few values are $B_2 = 1/6, B_4 = 1/30, B_6 = 1/42$; and $E_0 = 1, E_2 = 1, E_4 = 5$. Note that B_2 and B_4 in (11) have different meanings.

To sum the desired series in (1), the same method as in the paper [1] is employed. The resulting alternating double series is further decomposed into positive double series. We find for $s \geqq 2$,

(14)

$$
\begin{aligned}
\sum_{k=1}^{s} A_{2s,2k} \mathrm{I}_{2k}^{*}(c) &= \frac{S_{2s}^{*}}{(\pi c)^{2s}} - \frac{(-1)^s}{\pi^{2s}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left\{ \frac{1}{(m+2nci)^{2s}} + \frac{1}{(m-2nci)^{2s}} \right\} \\
&\quad + \frac{(-1)^s}{\pi^{2s}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{1}{\{m+(2n-1)ci\}^{2s}} + \frac{1}{\{m-(2n-1)ci\}^{2s}} \right],
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^{s} (-1)^{k+1} A_{2s,2k} \mathrm{II}_{2k}^{*}(c) \\
&= \frac{(-1)^s 2^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left\{ \frac{1}{(2m-1+4nci)^{2s}} + \frac{1}{(2m-1-4nci)^{2s}} \right\} \\
&\quad - \frac{(-1)^s 2^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left[\frac{1}{\{2m-1+2(2n-1)ci\}^{2s}} + \frac{1}{\{2m-1-2(2n-1)ci\}^{2s}} \right],
\end{aligned}
$$

where the coefficients $A_{2s,2k}$ have been tabulated in the paper [1]. In particular, $A_{2s,2s} = 1$. Similarly, it can be shown that the preceding double series can be expressed in terms of the function $W_{2s}(z|2ci)$ at half periods. The results are, for $s \geqq 2$,

$$
\begin{aligned}
\mathrm{I}_{2s}^{*}(c) &= \frac{(-1)^s}{\pi^{2s}} S_{2s} + \frac{(-1)^s}{2\pi^{2s}} \{W_{2s}(ci|2ci) - \sigma_{2s}^{*}(2ci)\} - \sum_{k=1}^{s-1} A_{2s,2k} \mathrm{I}_{2k}^{*}(c), \\
\mathrm{II}_{2s}^{*}(c) &= \frac{2^{2s}}{\pi^{2s}} U_{2s} + \frac{1}{2\pi^{2s}} \{W_{2s}(\tfrac{1}{2}+ci|2ci) - W_{2s}(\tfrac{1}{2}|2ci)\} \\
&\quad - \sum_{k=1}^{s-1} (-1)^{s+k} A_{2s,2k} \mathrm{II}_{2k}^{*}(c),
\end{aligned} \tag{15}
$$

in which σ_{2s}^* can be expressed in terms of W_{2s} as follows: For $s \geqq 2$,

$$\sigma_{2s}^*(2ci) = \frac{1}{2^{2s}-1}\{W_{2s}(\tfrac{1}{2}|2ci) + W_{2s}(ci|2ci) + W_{2s}(\tfrac{1}{2}+ci|2ci)\}. \tag{16}$$

To proceed further, let

$$\begin{aligned} \frac{1}{(2s-2)!}\frac{d^{2s-2}}{dx^{2s-2}}\frac{\pi}{\sinh \pi x} &= \pi^{2s-1}\sum_{k=1}^{s}\frac{A_{2s-1,2k-1}}{\sinh^{2k-1}\pi x}, \\ \frac{1}{(2s-2)!}\frac{d^{2s-2}}{dx^{2s-2}}\frac{\pi}{\cosh \pi x} &= \pi^{2s-1}\sum_{k=1}^{s}(-1)^{k+1}\frac{A_{2s-1,2k-1}}{\cosh^{2k-1}\pi x}, \end{aligned} \tag{17}$$

where $A_{2s-1,2k-1}$ are coefficients to be determined. It is easy to see that $A_{2s-1,2s-1} = 1$. Using the partial fraction decompositions in (5), we similarly find for $s \geqq 2$,

$$\begin{aligned} &\sum_{k=1}^{s}(-1)^{k+1}A_{2s-1,2k-1}\mathrm{II}_{2k-1}(c) \\ &\quad = \frac{(-1)^s 2^{2s-1}}{\pi^{2s-1}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left\{\frac{1}{(2m-1+2nci)^{2s-1}} + \frac{1}{(2m-1-2nci)^{2s-1}}\right\}, \\ &\sum_{k=1}^{s}(-1)^{k+1}A_{2s-1,2k-1}\mathrm{IV}_{2k-1}(c) \\ &\quad = \frac{(-1)^s 2^{2s-1}}{\pi^{2s-1}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left[\frac{1}{\{2m-1+(2n-1)ci\}^{2s-1}}\right. \\ &\qquad\qquad \left. + \frac{1}{\{2m-1-(2n-1)ci\}^{2s-1}}\right], \\ &\sum_{k=1}^{s}(-1)^{k+1}A_{2s-1,2k-1}\mathrm{II}_{2k-1}^*(c) \\ &\quad = -\frac{(-1)^s 2^{2s-1}}{\pi^{2s-1}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+n}\left\{\frac{1}{(2m-1+2nci)^{2s-1}} + \frac{1}{(2m-1-2nci)^{2s-1}}\right\}, \\ &\sum_{k=1}^{s}A_{2s-1,2k-1}\mathrm{III}_{2k-1}^*(c) = \left(\frac{2}{\pi c}\right)^{2s-1}U_{2s-1}^* \\ &\quad + \frac{i(-1)^s 2^{2s-1}}{\pi^{2s-1}}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{m+n}\left[\frac{1}{\{2m+(2n-1)ci\}^{2s-1}}\right. \\ &\qquad\qquad \left. - \frac{1}{\{2m-(2n-1)ci\}^{2s-1}}\right]. \end{aligned} \tag{18}$$

These double series can be expressed in terms of the function W_{2s-1}. By decomposing the double series representing this function, we have for $s \geqq 2$,

$$\frac{1}{4^{2s-1}} W_{2s-1}(\tfrac{1}{4}|ci) = U^*_{2s-1} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m\left\{\frac{1}{(2m-1+4nci)^{2s-1}} + \frac{1}{(2m-1-4nci)^{2s-1}}\right\},$$

$$\frac{1}{4^{2s-1}} W_{2s-1}(\tfrac{1}{4}+\tfrac{1}{2}ci|ci) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^m \cdot\left[\frac{1}{\{2m-1+(4n-2)ci\}^{2s-1}} + \frac{1}{\{2m-1-(4n-2)ci\}^{2s-1}}\right],$$

(19)
$$\frac{1}{4^{2s-1}} W_{2s-1}(\tfrac{1}{4}ci|ci) = \frac{i(-1)^s}{c^{2s-1}} U^*_{2s-1} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n\left[\frac{1}{\{4m+(2n-1)ci\}^{2s-1}} - \frac{1}{\{4m-(2n-1)ci\}^{2s-1}}\right],$$

$$\frac{1}{4^{2s-1}} W_{2s-1}(\tfrac{1}{2}+\tfrac{1}{4}ci|ci) = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^n \cdot\left[\frac{1}{\{4m-2+(2n-1)ci\}^{2s-1}} - \frac{1}{\{4m-2-(2n-1)ci\}^{2s-1}}\right].$$

Consequently, the following relations are obtained for $s \geqq 2$:

$$\mathrm{II}_{2s-1}(c) = -\frac{2^{2s-1}}{\pi^{2s-1}} U^*_{2s-1} + \frac{W_{2s-1}(\frac{1}{4}|\frac{1}{2}ci)}{(2\pi)^{2s-1}} - \sum_{k=1}^{s-1}(-1)^{s+k}A_{2s-1,2k-1}\mathrm{II}_{2k-1}(c),$$

$$\mathrm{IV}_{2s-1}(c) = \frac{W_{2s-1}(\frac{1}{4}+\frac{1}{4}ci|\frac{1}{2}ci)}{(2\pi)^{2s-1}} - \sum_{k=1}^{s-1}(-1)^{s+k}A_{2s-1,2k-1}\mathrm{IV}_{2k-1}(c),$$

(20)
$$\mathrm{II}^*_{2s-1}(c) = \frac{2^{2s-1}}{\pi^{2s-1}} U^*_{2s-1} - \frac{W_{2s-1}(\frac{1}{4}|ci)}{(2\pi)^{2s-1}} + \frac{W_{2s-1}(\frac{1}{4}+\frac{1}{2}ci|ci)}{(2\pi)^{2s-1}} - \sum_{k=1}^{s-1}(-1)^{s+k}A_{2s-1,2k-1}\mathrm{II}^*_{2k-1}(c),$$

$$\mathrm{III}^*_{2s-1}(c) = -\frac{i(-1)^s}{(2\pi)^{2s-1}}\{W_{2s-1}(\tfrac{1}{4}ci|ci) - W_{2s-1}(\tfrac{1}{2}+\tfrac{1}{4}ci|ci)\} - \sum_{k=1}^{s-1}A_{2s-1,2k-1}\mathrm{III}^*_{2k-1}(c).$$

It is noted that, unlike those in the previous section, the order of summation of the various double series in this section is interchangeable. To evaluate the coefficients $A_{2s-1,2k-1}$, we differentiate both sides of either equation in (17) twice

TABLE 2
Values of series in (1) *and* (2) *for* $s = 2$ *and* 3

c	$\mathrm{II}_3(c)$	$\mathrm{II}_5(c)$
1	$-\frac{1}{2}+\frac{\sqrt{2}}{4\pi}\sqrt{u}+\frac{\sqrt{2}}{4\pi^3}u^{3/2}$	$-\frac{1}{2}+\frac{3\sqrt{2}}{16\pi}\sqrt{u}+\frac{5\sqrt{2}}{24\pi^3}u^{3/2}+\frac{3\sqrt{2}}{16\pi^5}u^{5/2}$
$\sqrt{3}$	$-\frac{1}{2}+\frac{3^{1/4}}{4\pi}\sqrt{v}+\frac{3+2\sqrt{3}}{64\pi^3}(B_2+B_3)v^{3/2}$	$-\frac{1}{2}+\frac{3\cdot 3^{1/4}}{16\pi}\sqrt{v}+\left\{\frac{15+10\sqrt{3}}{384\pi^3}+\frac{39+20\sqrt{3}}{1024\pi^3}v\right\}(B_2+B_3)v^{3/2}$
$\frac{1}{\sqrt{3}}$	$-\frac{1}{2}+\frac{3^{1/4}}{4\pi}\sqrt{3v}+\frac{2-\sqrt{3}}{64\pi^3}9(B_1+B_4)v^{3/2}$	$-\frac{1}{2}+\frac{3\cdot 3^{1/4}}{16\pi}\sqrt{3v}+\left\{\frac{15(2-\sqrt{3})}{128\pi^3}+\frac{27(13\sqrt{3}-20)}{1024\pi^5}v\right\}(B_1+B_4)v^{3/2}$

c	$\mathrm{IV}_3(c)$	$\mathrm{IV}_5(c)$
1	$\frac{1}{4\pi}\sqrt{u}-\frac{1}{4\pi^3}u^{3/2}$	$\frac{3}{16\pi}\sqrt{u}-\frac{5}{24\pi^3}u^{3/2}-\frac{1}{16\pi^5}u^{5/2}$
$\sqrt{3}$	$\frac{\sqrt{3}-1}{16\pi}3^{1/4}\sqrt{2v}-\frac{3+2\sqrt{3}}{64\pi^3}(B_2-B_3)v^{3/2}$	$\frac{3(\sqrt{3}-1)}{64\pi}3^{1/4}\sqrt{2v}-\left\{\frac{5(3+2\sqrt{3})}{384\pi^3}-\frac{3+4\sqrt{3}}{1024\pi^5}v\right\}(B_2-B_3)v^{3/2}$
$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{3}+1}{16\pi}3^{1/4}\sqrt{6v}-\frac{9(2-\sqrt{3})}{64\pi^3}(B_1-B_4)v^{3/2}$	$\frac{3(\sqrt{3}+1)}{64\pi}3^{1/4}\sqrt{6v}-\left\{\frac{15(2-\sqrt{3})}{128\pi^3}+\frac{27(4-\sqrt{3})}{1024\pi^5}v\right\}(B_1-B_4)v^{3/2}$

c	$\mathrm{II}_3^*(c)$	$\mathrm{II}_5^*(c)$
1	$\frac{1}{2}-\frac{1}{4\pi}\sqrt{u}-\frac{1}{2\pi^3}u^{3/2}$	$\frac{1}{2}-\frac{3}{16\pi}\sqrt{u}-\frac{5}{12\pi^3}u^{3/2}-\frac{1}{4\pi^5}u^{5/2}$
$\sqrt{3}$	$\frac{1}{2}-\frac{B_1-B_4}{16\pi}\sqrt{v}-\frac{3\sqrt{2}+\sqrt{6}}{16\pi^3}3^{1/4}v^{3/2}$	$\frac{1}{2}-\frac{B_1-B_4}{64\pi}3\sqrt{v}-\frac{5(3\sqrt{2}+6)}{96\pi^3}3^{1/4}v^{3/2}-\frac{2\sqrt{6}+3\sqrt{2}}{32\pi^5}3^{1/4}v^{5/2}$
$\frac{1}{\sqrt{3}}$	$\frac{1}{2}-\frac{B_2-B_3}{16\pi}\sqrt{3v}-\frac{9(\sqrt{6}-\sqrt{2})}{16\pi^3}3^{1/4}v^{3/2}$	$\frac{1}{2}-\frac{B_2-B_3}{64\pi}3\sqrt{3v}-\frac{15(\sqrt{6}-\sqrt{2})}{32\pi^3}3^{1/4}v^{3/2}-\frac{27(2\sqrt{2}-\sqrt{6})}{32\pi^5}3^{1/4}v^{5/2}$

c	$\mathrm{III}_3^*(c)$	$\mathrm{III}_5^*(c)$
1	$-\frac{1}{4\pi}\sqrt{u}+\frac{1}{2\pi^3}u^{3/2}$	$\frac{3}{16\pi}\sqrt{u}-\frac{5}{12\pi^3}u^{3/2}+\frac{1}{4\pi^5}u^{5/2}$
$\sqrt{3}$	$-\frac{B_2-B_3}{16\pi}\sqrt{v}+\frac{3\sqrt{2}-\sqrt{6}}{16\pi^3}3^{1/4}v^{3/2}$	$\frac{B_2-B_3}{64\pi}3\sqrt{v}-\frac{15\sqrt{2}-5\sqrt{6}}{96\pi^3}3^{1/4}v^{3/2}+\frac{2\sqrt{6}-3\sqrt{2}}{32\pi^5}3^{1/4}v^{5/2}$
$\frac{1}{\sqrt{3}}$	$-\frac{B_1-B_4}{16\pi}\sqrt{3v}+\frac{9(\sqrt{6}+\sqrt{2})}{16\pi^3}3^{1/4}v^{3/2}$	$\frac{B_1-B_4}{64\pi}3\sqrt{3v}-\frac{15(\sqrt{6}+\sqrt{2})}{32\pi^3}3^{1/4}v^{3/2}+\frac{27(2\sqrt{2}+\sqrt{6})}{32\pi^5}3^{1/4}v^{5/2}$

c	$\mathrm{I}_4^*(c)$	$\mathrm{I}_6^*(c)$
1	$\frac{11}{90}-\frac{u}{6\pi^2}-\frac{u^2}{80\pi^4}$	$-\frac{191}{1890}+\frac{2u}{15\pi^2}+\frac{u^2}{80\pi^4}+\frac{u^3}{160\pi^6}$
$\sqrt{3}$	$\frac{11}{90}-\frac{2\sqrt{3}+1}{24\pi^2}v-\frac{4\sqrt{3}+1}{256\pi^4}v^2$	$-\frac{191}{1890}+\frac{2\sqrt{3}+1}{30\pi^2}v+\frac{4\sqrt{3}+1}{256\pi^4}v^2+\frac{42\sqrt{3}-1}{14336\pi^6}v^3$
$\frac{1}{\sqrt{3}}$	$\frac{11}{90}-\frac{2\sqrt{3}-1}{8\pi^2}v+\frac{9(4\sqrt{3}-1)}{256\pi^4}v^2$	$-\frac{191}{1890}+\frac{2\sqrt{3}-1}{10\pi^2}v-\frac{9(4\sqrt{3}-1)}{256\pi^4}v^2+\frac{27(42\sqrt{3}+1)}{14336\pi^6}v^3$

c	$\mathrm{II}_4^*(c)$	$\mathrm{II}_6^*(c)$
1	$\frac{1}{2}-\frac{\sqrt{2}}{3\pi^2}u-\frac{\sqrt{2}}{4\pi^4}u^2$	$\frac{1}{2}-\frac{4\sqrt{2}}{15\pi^2}u-\frac{\sqrt{2}}{4\pi^4}u^2-\frac{11\sqrt{2}}{80\pi^6}u^3$
$\sqrt{3}$	$\frac{1}{2}-\frac{3+\sqrt{3}}{12\pi^2}\sqrt{2v}-\frac{7\sqrt{3}+9}{64\pi^4}\sqrt{2v^2}$	$\frac{1}{2}-\frac{3+\sqrt{3}}{15\pi^2}\sqrt{2v}-\frac{7\sqrt{3}+9}{64\pi^4}\sqrt{2v^2}-\frac{9(13+7\sqrt{3})}{1024\pi^6}\sqrt{2v^3}$
$\frac{1}{\sqrt{3}}$	$\frac{1}{2}-\frac{3-\sqrt{3}}{4\pi^2}\sqrt{2v}-\frac{9(7\sqrt{3}-9)}{64\pi^4}\sqrt{2v^2}$	$\frac{1}{2}-\frac{3-\sqrt{3}}{5\pi^2}\sqrt{2v}-\frac{9(7\sqrt{3}-9)}{64\pi^4}\sqrt{2v^2}-\frac{243(13-7\sqrt{3})}{1024\pi^6}\sqrt{2v^3}$

and equate the coefficients. The following recurrence relation is obtained for $s \geqq 1$ and $1 \leqq k \leqq s$:

(21)

$$A_{2s+1,2k-1} = \frac{1}{2s(2s-1)} \{(2k-2)(2k-3)A_{2s-1,\ k-3} + (2k-1)^2 A_{2s-1,2k-1}\}.$$

In particular, for $s \geqq 1$,

(22) $$A_{2s-1,2s-1} = 1.$$

It is thus seen that the desired series in (1) and (2) can be evaluated recurrently from (15) and (20) by using the values of the series in Table 1 and the values of $W_{2s}(z|2ci)$ at half periods or the values of $W_{2s-1}(z|ci)$ and $W_{2s-1}(z|\frac{1}{2}ci)$ at quarter and half periods. In general, the function W_s can be evaluated successively from the values of W_2, W_3 and W_4 by using the following recurrence relation:

(23)

$$\tfrac{1}{6}(s-2)(s-3)F_s = F_2F_{s-2} + F_3F_{s-3} + F_4F_{s-4} + \cdots + F_{s-2}F_2, \qquad s \geqq 5,$$

where

(24) $$F_s = (s-1)W_s(z|2\omega'), \qquad s \geqq 2.$$

Note that W_2 is the Weierstrass elliptic function, W_3 is equal to $-W_2'/2$ and W_4 is given by

(25) $$W_4(z|2\omega') = W_2^2(z|2\omega') - 5\sigma_4^*(2\omega').$$

With the values of W_2, W_2' and σ_4^* given in the previous paper [2], all the desired functions of W_{2s} and W_{2s-1} can be evaluated. Note that at half periods, $W_{2s-1}(z|2\omega') = 0$ for $s \geqq 2$. Therefore, the sums of the desired series in (1) and (2) can be expressed in closed form in terms of σ_4 when $c = 1$ and in terms of σ_6 when $c = \sqrt{3}$ or $1/\sqrt{3}$. The values of the series for $s = 2$ and $s = 3$ are shown in Table 2, in which the same abbreviations as in (11) are used. Table 3 shows the values of $A_{2s-1,2k-1}$.

Postscript. It is noted that the expressions of $B_1 \pm B_4$ and $B_2 \pm B_3$ involved

TABLE 3

Values of $A_{2s-1,2k-1}$

$s \backslash k$	1	2	3	4	5	6
6	1/3,62880	7381/18,14400	2497/15120	121/120	11/6	1
5	1/40320	41/1008	161/280	3/2	1	
4	1/720	91/360	7/6	1		
3	1/24	5/6	1			
2	1/2	1				
1	1					

in some summations of the series as shown in Tables 1 and 2 can further be simplified to the following form:

$$\begin{aligned} B_1 + B_4 &= 4 \cdot 3^{1/4}, \qquad B_1 - B_4 = 3^{1/4}2^{1/2}(\sqrt{3} + 1), \\ B_2 + B_3 &= 4 \cdot 3^{1/4}, \qquad B_2 - B_3 = 3^{1/4}2^{1/2}(\sqrt{3} - 1). \end{aligned} \tag{26}$$

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ASYMPTOTIC EQUIVALENCE OF NONLINEAR DIFFERENTIAL EQUATIONS IN BANACH SPACES*

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Abstract. New concepts of asymptotic equivalence of differential equations in Banach spaces are introduced. Our techniques use the comparison theorem and a result on asymptotic equilibrium in Banach spaces. As an application we extend a nonlinear perturbation result of A. G. Kartsatos.

1. Introduction. Many results have been obtained on the asymptotic relationship between the solutions of a differential equation and a perturbation of that equation (see the references). One technique that is often employed in connection with this problem is to utilize a variation of parameters formula in conjunction with a fixed-point theorem. The articles [4], [5], [7]–[9], [12], [13] treat linear perturbation problems using this approach, while the papers [1]–[3], [11] employ the Alekseev formula as a tool for discussing nonlinear perturbation problems.

Another procedure employed in asymptotic behavior problems for differential equations is the comparison principle. It has also been used coupled with fixed-point theorems; see [4], [7], [9], [12]. In this paper, we present a new approach to the asymptotic equilibrium problem in differential equations. Our technique uses the comparison principle and employs a known result on asymptotic equilibrium in a Banach space. This observation coupled with the new concepts introduced in the definitions presents a new view of asymptotic equivalence.

As an application of our work, we present an extension of a nonlinear perturbation result of A. G. Kartsatos [8]. Recent extensions of Kartsatos' work in different directions may be found in [14] where a Lyapunov-like function approach is used and in [6] where the concept of admissibility is used in conjunction with the Schauder fixed-point theorem.

2. Definitions and preliminaries. Let R_+ denote the half-line $[0, \infty)$ and B denote a real Banach space with norm $\|\cdot\|$. We consider the differential equations

$$dx/dt = F(t, x), \tag{1}$$

$$dy/dt = G(t, y), \tag{2}$$

where F and G are in $C[R_+ \times B, B]$. It will be tacitly assumed that solutions of (1) and (2) exist locally. If B is locally compact, then local existence follows from the continuity of F and G.

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We now introduce the terminology associated with the equilibrium problem. Let A and Y_0 denote subsets of B.

DEFINITION 1. The differential equations (1) and (2) possess an *A-terminal correspondence with respect to Y_0* provided for each $a \in A$ and $y_0 \in Y_0$, there exists a $\tau_0 = \tau_0(a, y_0)$ with the property that corresponding to any solution $y = y(t, \tau_0, y_0)$ of (2), there is an $x_0 \in B$ such that the solution $x(t, \tau_0, x_0)$ of (1) is defined on $[\tau_0, \infty)$ and satisfies

$$\lim_{t\to\infty} x(t, \tau_0, x_0) - y(t, \tau_0, y_0) = a.$$

(This is the strong limit in B.)

DEFINITION 2. The equations (1) and (2) possess an *A-convergent correspondence with respect to Y_0* provided for each $a \in A$ and $y_0 \in Y_0$, there exists a $T_0 = T_0(a, y_0) \geqq 0$ with the property that corresponding to any solution $y(t, t_0, y_0)$ of (2) with $t_0 \geqq T_0$, there is a solution $x(t, t_0, x_0)$ of (1) valid for $t \in [t_0, \infty)$, with $x_0 - y_0 = a$, and which satisfies

$$\lim_{t\to\infty} x(t, t_0, x_0) - y(t, t_0, y_0) = c \quad \text{for some } c \in B.$$

If $Y_0 = B$, the qualifying phrase—with respect to Y_0—will be omitted.

DEFINITION 3. The equations (1) and (2) are in *A-asymptotic equilibrium with respect to Y_0* if (1) and (2) possess an A-terminal correspondence with respect to Y_0 and an A-convergent correspondence with respect to Y_0.

DEFINITION 4. If (1) and (2) possess an A-terminal correspondence with respect to Y_0 and $\tau_0 = \tau_0(y_0)$ for all $a \in A$, then (1) and (2) possess a *uniform A-terminal correspondence, with respect to Y_0*.

DEFINITION 5. If (1) and (2) possess an A-convergent correspondence with respect to Y_0 and $T_0 = T_0(y_0)$ for all $a \in A$, then (1) and (2) possess a *uniform A-convergent correspondence with respect to Y_0*.

DEFINITION 6. If (1) and (2) possess a uniform A-terminal correspondence with respect to Y_0 and a uniform A-convergent correspondence with respect to Y_0, then (1) and (2) are in *uniform A-asymptotic equilibrium with respect to Y_0*.

The above definitions are "eventual" in the sense that τ_0 and T_0 need not be zero. However, we will not clutter the definitions by adding this qualifier.

Some examples are given to demonstrate that the above concepts are distinct.

Example. Let $n > 1$ denote an odd positive integer. Consider the differential equation

$$dx/dt = a(t)x^n; \tag{3}$$

in (3), $x \in R$, $t \in R_+$ and $a(t)$ is a positive continuous function defined for $t \in R_+$ with $\int_0^\infty a(t)\,dt < \infty$. The structure of the solution space of equation (3) for initial positions in $Y_0 = R_+ - \{0\}$ is as follows. There is a solution $\phi = \phi(t)$ of (3) that is valid on R_+ with the property that $\lim_{t\to\infty} \phi(t) = \infty$. This function ϕ separates the bounded solutions of (3) from the solutions with a finite escape time.

If A is any compact subset of $R_+ - \{0\}$, then (3) and

$$dw/dt = 0 \tag{4}$$

are in uniform A-asymptotic equilibrium with respect to Y_0. To verify this let

$x_0 \in Y_0$; then, choose $T_0(x_0) = \tau_0(x_0)$ such that $\phi(t) - x_0 > d(A)$, $t \geqq T_0(x_0)$, where $d(A)$ denotes the diameter of A. This choice of T_0 is possible because $\lim_{t\to\infty} \phi(t) = \infty$. The solution $x(t, t_0, x_1)$, where $t_0 \geqq T_0$, $x_1 = x_0 + a$, $a \in A$, of (3) satisfies $\lim_{t\to\infty} x(t, t_0, x_1) = x_\infty$ for some x_∞. Thus, (3) and (4) possess a uniform A-convergence correspondence with respect to Y_0.

Given any terminal value $x_\infty \in Y_0$, there is a solution $x(t, \tau_0, x_0)$ of (3) with $\lim_{t\to\infty} x(t, \tau_0, x_0) = x_\infty$. If $a \in A$ is prescribed, then any solution w of (4) can be written as $w = w(t, \tau_0, x_\infty + a) \equiv x_\infty + a$ for some x_∞. Therefore, corresponding to any solution w of (4), there exists an $x_0 \in R$ such that

$$\lim_{t\to\infty} x(t, \tau_0, x_0) - w(t, \tau_0, x_\infty + a) = a.$$

This shows that (3) and (4) possess a uniform A-terminal correspondence, which in turn shows that (3) and (4) are in uniform A-asymptotic equilibrium.

If $A = R_+ - \{0\}$, then, as may be demonstrated by using similar arguments as those above, equations (3) and (4) are in A-asymptotic equilibrium with respect to Y_0; however, neither the A-terminal nor the A-convergence correspondence involved in the equilibrium is uniform.

3. Main results. For a prescribed set A, it is convenient to define the set $A_\rho = \{\rho \in R : \rho = \|a\| \text{ for } a \in A\}$ and to consider A_ρ as a subset of the Banach space R. We will denote by B_{ρ_0} the ball,

$$B_{\rho_0} = \{b \in B : \|b\| \leqq \rho_0\}.$$

We will discuss the A-convergence correspondence. The basic tool used here is Theorem 5.6.1 of [10, p. 161].

THEOREM 1. *Assume that for each* $\rho > 0$:

(1i) *There exists a* $T = T(\rho) \geqq 0$ *such that*

$$\|F(t, x)\| \leqq f_\rho(t, \|x\|) \quad (t \in [T(\rho), \infty), \quad x \in B_\rho,$$

where $f_\rho \in C[[T(\rho), \infty) \times [0, \rho), R_+]$ *and* $f_\rho(t, r)$ *is monotonically nondecreasing in* r *for each fixed* $t \in [T(\rho), \infty)$; *and*

(1ii) *For each* r_0, $0 < r_0 < \rho$, *there exists a* $t_0 = t_0(r_0) \geqq T(\rho)$ *such that the maximal solution* $r(t, t_0, r_0)$ *of the initial value problem*

$$dr/dt = f_\rho(t, r), \qquad r(t_0) = r_0 \tag{5_ρ}$$

satisfies $r(t, t_0, r_0) < \rho$, $t \geqq t_0$.

Then for any set A, *equations* (1) *and*

$$du/dt = 0 \quad (t \in R_+, u \in B) \tag{6}$$

possess an

(1iii) *A-convergence correspondence and*

(1iv) *a uniform A-convergence correspondence.*

Proof. We will establish (1iii) first. Let $u_0 \in B$ and $a \in A$ be given and define $\rho = \|u_0\| + \|a\| + 1$. Select $T_0 = t_0(\|u_0\| + \|a\|)$, where $t_0(\cdot)$ is defined in hypothesis (1ii).

To establish that (1) and (6) possess an A-convergent correspondence, it suffices to show that $\lim_{t\to\infty} x(t, t_0, x_0) \equiv x_\infty$ exists where $t_0 \geqq T_0$ and $x_0 = u_0 + a$. Once the existence of x_∞ is demonstrated, then it follows that the asymptotic limit c is determined by $c = u_0 - x_\infty$.

Theorem 5.6.1 of [10, p. 161] implies that x_∞ exists; this completes the proof of conclusion (1iii).

Conclusion (1iv) is obtained in an analogous fashion as (1iii). In this situation, $x_0 \in B$ and $a \in A$ are prescribed. Result (1iv) can be demonstrated by choosing $T_0 = t_0(\|x_0\|)$ and using the relationships $u_0 = x_0 + a$ and $c = x_0 + a - x_\infty$. This completes the proof of Theorem 1.

Next, we obtain some sufficient conditions for A-terminal correspondences. A terminal analogue (Theorem 5.6.2 of [10, p. 164]) of Theorem 5.6.1 will be employed in the proof.

THEOREM 2. *Assume that*

(2i) *hypotheses* (1i) *and* (1ii) *are satisfied,*

(2ii) *F maps bounded sets into relatively compact sets.*

Then, for any set A,

(2iii) *equations* (1) *and* (6) *possess an A-terminal correspondence, and*

(2iv) *equations* (6) *and* (1) *possess a uniform A-terminal correspondence.*

Proof. First, we indicate the proof of (2iii). Let $a \in A$ and $u_0 \in B$ be given; consider $\rho = 2[\|u_0\| + \|a\|]$. Corresponding to $r_0 = \|u_0\| + \|a\|$, there exists a function $t_0 = t_0(r_0)$ such that the maximal solution $r(t, t_0, r_0)$ of (5_ρ) is bounded on $[t_0, \infty)$. Let $r_\infty \equiv \lim_{t\to\infty} r(t, t_0, r_0)$.

Choose J sufficiently large so that $\int_J^\infty f_\rho(t, 2r_\infty)\, dt < r_\infty$. The existence of such a J is demonstrated in the proof of Theorem 5.6.2 of [10, p. 164].

We now select $\tau_0 = \tau_0(a, u_0) = \max\,[t_0(r_0), J]$. For such a τ_0, the proof of Theorem 5.6.2 can again be used to show that there exists an $x_0 \in B$ with the property that $\lim_{t\to\infty} x(t, \tau_0, x_0) = u_0 + a$. This establishes the conclusion (2iii).

The conclusion (2iv) is obtained through an application of Theorem 5.6.1. The details are omitted.

We now turn to equations (1) and (2) and the A-asymptotic equilibrium problem.

THEOREM 3. *Assume that*

(3i) *$F, G \in C[R_+ \times B, B]$ and maps bounded sets into relatively compact sets;*

(3ii) *for each $\rho > 0$ there exists a $T = T(\rho) \geqq 0$ such that for $t \geqq T$, $v \in B_\rho$, and each $z(t) \in B$, $t \in [T, \infty)$,*

$$\|F(t, z(t) + v) - G(t, z(t))\| \leqq h_\rho(t, \|v\|), \tag{7}$$

where $h_\rho \in C[[T, \infty) \times [0, \rho], R_+]$, $h_\rho(t, r)$ is monotonic nondecreasing in r for each $t \in R_+$;

(3iii) *for each $\rho > 0$ and each $r_0, r_0 < \rho$, there exists a $t_0 = t_0(r_0) \geqq T(\rho)$ such that the maximal solution $r(t, t_0, r_0)$ of the initial value problem*

$$\begin{aligned} dr/dt &= h_\rho(t, r), \\ r(t_0) &= r_0 \end{aligned} \tag{8_ρ}$$

satisfies $r(t, t_0, r_0) < \rho$, $t \in [t_0, \infty)$. Then, for any set A, equations (1) *and* (2) *are in A-asymptotic equilibrium.*

Proof. For any solution $y(t)$ of (2), we consider $z = x - y(t)$. If x satisfies (1), it follows that z satisfies the equation

$$dz/dt = F(t, z + y(t)) - G(t, y(t)). \tag{9}$$

The equations (8_ρ) and (9) are of the type considered in Theorems 1 and 2. By virtue of the hypotheses (3i), (3ii) and (3iii), Theorems 1 and 2 imply that the equations (9) and (6) are in A-asymptotic equilibrium. Recalling that $z = x - y(t)$, a direct verification shows that (1) and (2) are in A-asymptotic equilibrium.

We now consider the situation when the hypotheses of Theorems 1 and 2 are not satisfied globally. The restrictions that this specification places upon the sets A and Y_0 are of special interest.

THEOREM 4. *Assume that*

(4i) *for (fixed)* $\rho_1 > 0$ *and* $T > 0$,

$$\|F(t, x)\| \leqq f_{\rho_1}(t, \|x\|) \qquad (t \geqq T, x \in B_{\rho_1}),$$

where $f_{\rho_1} \in C[[T, \infty) \times [0, \rho_1], R_+]$ *and* $f_{\rho_1}(t, r)$ *is nondecreasing in* r *for fixed* $t \in [T, \infty)$;

(4ii) *Let* $0 < \rho_0 < \rho_1$; *for each* r_0, $0 < r_0 < \rho_0$, *there exists a* $t_0 = t_0(r_0) \geqq T$ *such that the maximal solution* $r(t, t_0, r_0)$ *of the initial value problem* (5_{ρ_1}) *with* $r(t_0) = r_0$ *satisfies* $r(t, t_0, r_0) < \rho_0$.

Then, for the sets $A = B_{\alpha_1}$ *and* $Y_0 = B_{\alpha_2}$ *where* $\alpha_1 + \alpha_2 < \rho_0$, *equations* (1) *and* (6) *possess an A-convergence correspondence with respect to* Y_0. *Furthermore, if, in addition to the above hypotheses, F maps bounded sets into relatively compact sets, then* (1) *and* (6) *possess an A-terminal correspondence with respect to* Y_0.

Proof. The proof follows readily from the arguments given in Theorems 3 and 4. One difference is in the choice of J in Theorem 2. In this instance choose J sufficiently large so that

$$\int_J^\infty f(s, \rho_1)\, ds < \rho_1 - \rho_0.$$

An analogue of Theorem 3 with restricted domains can be readily composed.

THEOREM 5. *Assume that*

(5i) *condition* (3i) *is satisfied;*

(5ii) *for (fixed)* $\rho_1 > 0$ *and* $T \geqq 0$, *the inequality*

$$\|F(t, z(t) + v) - G(t, z(t))\| \leqq h_{\rho_1}(t, \|v\|)$$

is satisfied for $t \geqq T$, $v \in B_{\rho_1}$, *and each* $z(t) \in B$, $t \geqq T$. *The function* $h_{\rho_1} \in C[[T, \infty) \times [0, \rho_1], R_+]$ *and* $h_{\rho_1}(t, r)$ *is monotone nondecreasing in* r *for each* $t \in R_+$.

(5iii) *Let* $0 < \rho_0 < \rho_1$; *suppose that for each* r_0, $0 < r_0 < \rho_0$, *there exists a* $t_0 = t_0(r_0) \geqq T$ *such that the maximal solution* $r(t, t_0, r_0)$ *of* (8_{ρ_1}) *satisfies* $r(t, t_0, r_0) < \rho_0$.

Then, for $A = B_{\alpha_1}$ *and* $Y_0 = B_{\alpha_2}$ *where* $\alpha_1 + \alpha_2 < \rho_0$, *equations* (1) *and* (2) *are in A-asymptotic equilibrium with respect to* Y_0.

4. Application. As a corollary to Theorem 5, we shall indicate a result on the nonlinear perturbation of a nonlinear system of differential equations. Consider $B = R^n$ and $F(t, x) \equiv G(t, x) + H(t, x)$. Suppose that $y = y(t)$, a solution of (2) defined on $[T, \infty)$, and ρ_1, a positive constant, are given. Let

$$\|H(t, y(t) + v)\| \leqq \lambda(t) \qquad (t \geqq T, \|v\| \leqq \rho_1)$$

and

$$\|G(t, y(t) + v) - G(t, y(t))\| \leqq \sigma(t)g(\|v\|) \qquad (t \geqq T, \|v\| \leqq \rho_1).$$

We assume that λ and σ are in $C[[T, \infty), R_+] \cap L^1[[T, \infty), R_+]$. Define for $0 \leqq r \leqq \rho_1$, $G(r) = \sup_{0 \leqq u \leqq r} g(u)$ and $K \equiv G(\rho_1)$.

Choose τ sufficiently large so that for some ρ_0, $0 < \rho_0 < \rho_1$, we have $\int_\tau^\infty [K\sigma(s) + \lambda(s)]\, ds < \rho_1 - \rho_0$. This property implies that the maximal solution $r(t, t_0, r_0)$ of

$$dr/dt = \sigma(t)G(r) + \lambda(t)$$

exists and is bounded on $[\tau, \infty)$ by ρ_1 provided $r_0 < \rho_0$. For $\alpha_1 \geqq 0$ and $\alpha_2 \geqq 0$ with $\alpha_1 + \alpha_2 < \rho_0$, Theorem 5 implies that equations (1) and (2) are B_{α_1}-asymptotically equivalent with respect to B_{α_2}.

The above application extends the previously mentioned result of Kartsatos [8] who considered an A-terminal correspondence with $A = B_0 = \{0\}$.

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BOUNDS FOR THE SOLUTIONS OF POISSON PROBLEMS AND APPLICATIONS TO NONLINEAR EIGENVALUE PROBLEMS*

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Abstract. Pointwise upper bounds for the solutions of Poisson problems are given. These bounds together with the method of lower and upper solutions are then used to estimate the spectrum for a class of nonlinear eigenvalue problems.

1.1. Introduction. The first part of this paper deals with two-dimensional Poisson problems. Sharp pointwise bounds for solutions satisfying different kinds of boundary conditions are constructed. The results extend a theorem by Pólya and Szegö [9, p. 115] for the torsion problem $\Delta u = -2$ in $\mathscr{D}$, $u = 0$ on $\partial\mathscr{D}$. It states that $0 \leqq 2\pi u \leqq A$, where A stands for the area of the domain. Equality holds on the right if and only if $\mathscr{D}$ is a circle and u is taken at the center. Other results in this direction were also obtained by Payne [8]. Our results are based on geometrical isoperimetric inequalities by Alexandrow [1] and some generalizations announced in [5]. Theorem 2.1 is already mentioned in [5] without proof.

The second part is concerned with nonlinear boundary value problems of the type $\Delta u + \lambda\rho(x)f(u) = 0$. This kind of problem arises in the theory of thermal ignition of a chemically active mixture of gases. With the help of the method of lower and upper solutions [10] and the results of §§ 1 and 2, we give estimates for the least upper bound λ_{cr} of the values of λ for which the nonlinear problem has a solution. The value λ_{cr} is the critical explosion parameter for the unsteady problem, that is, for $\lambda < \lambda_{\mathrm{cr}}$ there exists a stable solution of the time-dependent equation [7]. The problem of estimating λ_{cr} has been studied by many authors [3], [4], [6], [7], [11]. For other properties of λ_{cr}, especially those concerning the uniqueness of solutions and their stability, we refer to [6], [7]; see also [3].

1.2. Bounds for the solutions of Poisson problems. Let $\mathscr{D}$ be a simply connected domain in the plane whose boundary $\partial\mathscr{D}$ consists of piecewise analytic arcs. Let $x = (x_1, x_2)$ stand for a generic point in R^2. Consider in $\overline{\mathscr{D}}$ a positive function $\rho(x)$ satisfying the differential inequality

$$\Delta \log \rho + 2C\rho \geqq 0 \quad \text{in } \overline{\mathscr{D}}. \tag{1.1}$$

Here, C denotes an arbitrary real number. We define

$$M = \int_{\mathscr{D}} \rho(x)\,dx \quad (dx = dx_1\,dx_2).$$

This section deals with the Poisson problem

$$\begin{aligned} \Delta\varphi(x) &= -\rho(x) \quad \text{in } \mathscr{D}, \\ \varphi(x) &= 0 \qquad\quad \text{on } \partial\mathscr{D}. \end{aligned} \tag{1.2}$$

Next, we shall introduce the complex variable $z = x_1 + ix_2$ and interpret problem (1.2) in the complex z-plane. We set $\varphi(x) = \varphi(z)$. Let $w(z)$ be an analytic function

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which maps the domain $\mathscr{D}$ conformally onto the domain $\tilde{\mathscr{D}} = w(\mathscr{D})$ of the complex w-plane. By $z(w)$ we denote the inverse function of $w(z)$. Problem (1.2) is equivalent to the problem

$$\begin{aligned} \Delta_w \tilde{\varphi}(w) &= -\tilde{\rho}(w) \quad \text{in } \tilde{\mathscr{D}}, \\ \tilde{\varphi}(w) &= 0 \qquad\quad \text{on } \partial\tilde{\mathscr{D}}, \end{aligned} \tag{1.2'}$$

where

$$\tilde{\varphi}(w) = \varphi(z(w)) \quad \text{and} \quad \tilde{\rho}(w) = \rho(z(w))|\,dz/dw|^2.$$

Since $\log|dz/dw|^2$ is a harmonic function, it follows that $\tilde{\rho}(w)$ satisfies the differential inequality $\Delta_w \log\tilde{\rho}(w) + 2C\tilde{\rho}(w) \geqq 0$ in $\tilde{\mathscr{D}}$.

DEFINITION. We shall say that $(\mathscr{D}, \rho)$ is *conformally equivalent* to $(\tilde{\mathscr{D}}, P)$ if there exists a conformal mapping $z(w){:}\tilde{\mathscr{D}} \to \mathscr{D}$ such that $\mathscr{D} = z(\tilde{\mathscr{D}})$ and $P(w) = \rho(z(w))|dz/dw|^2$.

Remark. Let $\mathscr{M}$ be a piece of a Riemann manifold given in the following isothermic representation. In the domain D of the x-plane, a Riemann metric is defined by the line element $d\sigma^2 = \rho(x)[dx_1^2 + dx_2^2]$. In this parameter system the Beltrami operator is given by $L = (1/\rho(x))\Delta$. For the Gaussian curvature of $\mathscr{M}$ we have $K(x) = -[\Delta\log\rho(x)]/2\rho(x)$. Problem (1.2) can now be written as $L[\varphi(x)] = -1$ in $\mathscr{M}$, $\varphi(x) = 0$ on $\partial\mathscr{M}$. Condition (1.1) means that $K(x) \leqq C$ on $\bar{\mathscr{M}}$.

THEOREM 1.1. *If* $CM < 4\pi$, *then*

$$\varphi(x) \leqq \frac{1}{C}\log\frac{4\pi}{4\pi - CM}. \tag{1.3}$$

Equality holds if and only if $(\mathscr{D}, \rho)$ *is conformally equivalent to* $(S, \hat{\rho})$, *where* S *is the circle* $r < R$ $(r, \theta$ *polar coordinates) and* $\hat{\rho} = (1 + Cr^2/4)^{-2}$. *In this case* $\varphi(x)$ *takes its maximum at the center.*

Proof. Let $D(t) = \{x \in D;\, \varphi(x) \geqq t\}$ and let $\Gamma(t) = \{x \in D;\, \varphi(x) = t\}$. Because of the boundary condition it is clear that $\partial D(t) = \Gamma(t)$. We shall write $A(t) = \int_{\mathscr{D}(t)} \rho\, dx$ and $L(t) = \int_{\Gamma(t)} \sqrt{\rho}\, ds$, where ds denotes the Euclidean line element $\sqrt{dx_1^2 + dx_2^2}$. By the Schwarz inequality we have

$$\int_{\Gamma(t)} \left|\frac{\partial\varphi}{\partial n}\right| ds \int_{\Gamma(t)} \frac{\rho}{|\partial\varphi/\partial n|}\, ds \geqq \left[\int_{\Gamma(t)} \sqrt{\rho}\, ds\right]^2. \tag{1.4}$$

Here n is the outer normal. Because of (1.2) it follows that

$$\int_{\Gamma(t)} \left|\frac{\partial\varphi}{\partial n}\right| ds = A(t). \tag{1.5}$$

Observing that

$$\int_{\Gamma(t)} \frac{\rho}{|\partial\varphi/\partial n|}\, ds = -\frac{dA}{dt}(t),$$

we conclude that

$$-A(t)\frac{dA(t)}{dt} \geqq L^2(t). \tag{1.6}$$

Applying Alexandrow's geometrical inequality [1, p. 514],

$$L^2(t) \geqq (4\pi - CA(t))A(t), \tag{1.7}$$

we get

$$-\frac{dA(t)}{dt} \geqq 4\pi - CA(t). \tag{1.8}$$

Therefore, $e^{-Ct}[A(t) - 4\pi/C]$ is a nonincreasing function of t. If $CM < 4\pi$, it follows that

$$\frac{1}{C}\log\frac{4\pi}{4\pi - CM} \geqq \max_{x\in\mathscr{D}} \varphi(x),$$

which yields (1.3). We now observe that in (1.7) the equality sign holds if and only if $(D(t), \rho(x))$ is conformally equivalent (isometric) to a geodesic circle on a surface of constant curvature C. A parametric representation of the extremal case is given in the following form. Let $S(t)$ be the circle

$$r^2 \leqq \frac{4A(t)}{4\pi - CA(t)}$$

and

$$d\hat{\sigma}^2 = \hat{\rho}(r)\, ds^2$$

be the Riemannian metric. If $D = S = S(0)$ and $\rho(x) = \hat{\rho}(r)$, a straightforward calculation yields for the solution of (1.2)

$$\hat{\varphi}(r) = \frac{1}{C}\log\frac{4\pi}{(4\pi - CM)(1 + Cr^2/4)},$$

where $M = \pi R^2(1 + \frac{1}{4}CR^2)^{-1}$. At the origin we have

$$\hat{\varphi}(0) = \frac{1}{C}\log\frac{4\pi}{4\pi - CM},$$

which completes the proof of the theorem.

A slightly different proof of this theorem is found in [3]. If $C = 0$, we obtain $\varphi(x) \leqq M/4\pi$, which is an extension of results in [8], [9]. For other properties of (1.2), especially energy estimates, we refer to [2].

2. Generalizations. Let $\mathscr{D}$ and ρ satisfy the same assumptions as in § 1. We now suppose that $\partial\mathscr{D}$ is given in its parametric representation $x(s)$, where s denotes the length of $\partial\mathscr{D}$ between the points $x(0)$ and $x(s)$. $\partial\mathscr{D}$ is oriented in the positive sense. By $\kappa(s)$ we denote the curvature of $\partial\mathscr{D}$. It is defined everywhere except at the corners where it has to be interpreted as a Dirac measure. n stands for the outer normal to $\partial\mathscr{D}$. $\partial\mathscr{D}$ is subdivided into two connected arcs

$$\Gamma_1 = \{x(s)\,;\, 0 \leqq s \leqq a\}$$

and

$$\Gamma_0 = \partial\mathscr{D} - \Gamma_1.$$

Let

$$\mu(\eta) = \int_\eta \left[\kappa(s) + \frac{1}{2}\frac{\partial}{\partial n}\log\rho\right] ds$$

for each set $\eta \subseteq \Gamma_1$. Clearly, $\mu(\eta)$ defines an additive measure on Γ_1. We write $\mu^+(\Gamma_1) = \max_{\eta\subseteq\Gamma_1}\mu(\eta)$ for its positive component.

The geometric interpretation of μ is as follows. Consider the Riemann manifold $\mathscr{M}$ described in § 1. Then $(1/\sqrt{\rho})\{\kappa + \frac{1}{2}(\partial/\partial n)\log\rho\}$ represents the geodesic curvature of the boundary arc $\Gamma_1 \subseteq \partial\mathscr{M}$.

Consider the Poisson problem with mixed boundary conditions

$$\begin{aligned} \Delta\varphi &= -\rho \quad \text{in } \mathscr{D}, \\ \frac{\partial\varphi}{\partial n} &= 0 \quad \text{on } \Gamma_1, \\ \varphi &= 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{2.1}$$

For this problem the following theorem holds.

THEOREM 2.1. *Let $\mu^+(\Gamma_1) \equiv \pi - \alpha \leqq \pi$ and let $CM < 2\alpha$. Then*

$$\varphi(x) \leqq \frac{1}{C}\log\frac{2\alpha}{2\alpha - CM}. \tag{2.2}$$

Equality holds if and only if $(\mathscr{D}, \rho)$ is conformally equivalent to $(\hat{S}, \hat{\rho})$, where $\hat{S}$ is the circular sector $\{0 < r < R, 0 < \theta < \alpha\}$ with the boundary arcs $\Gamma_1 = \partial\hat{S}_1 \equiv \{\theta = 0$ and $\theta = \alpha\}$ and $\Gamma_0 = \partial\hat{S}_0 \equiv \{r = R\}$, $R^2 = 4M/(2\alpha - CM)$, and $\hat{\rho} = (1 + Cr^2/4)^{-2}$. In this case $\varphi(x)$ achieves its maximum at the origin.

Proof. Using the same notation as in the proof of Theorem 1.1, we show that (1.6) holds also in this case. Here, $\Gamma(t)$ is not necessarily a closed level line. Alexandrow's inequality (1.7) has therefore to be replaced by the inequality [5]

$$L^2(t) \geqq (2\alpha - CA(t))A(t). \tag{2.3}$$

Equality holds if and only if $(D(t), \rho)$ is conformally equivalent to $(\hat{S}(t), \hat{\rho})$, where

$$\hat{S}(t) = \left\{ r \leqq R(t) = \left[\frac{4A(t)}{2\alpha - CA(t)}\right]^{1/2}, 0 \leqq \theta \leqq \alpha \right\}$$

with $\Gamma(t) = \partial\hat{S}_0(t) \equiv \{r = R(t)\}$. The remaining part of the proof is the same as for Theorem 1.1. It will therefore be omitted.

If $C = 0$, inequality (2.2) yields the estimate $\varphi(x) \leqq M/2\alpha$. If $\Gamma_1 = \varnothing$, then $\alpha = \pi$, and (2.2) leads to $\varphi(x) \leqq (1/C)\log 2\pi/(2\pi - CM)$, which is weaker than (1.3)

3. Nonlinear eigenvalue problems. In this section we are concerned with a class of nonlinear Dirichlet problems of the following type:

$$\begin{aligned} \Delta u + \lambda\rho(x)f(u) &= 0 \quad \text{in } \mathscr{D}, \\ u &= 0 \quad \text{on } \partial\mathscr{D}. \end{aligned} \tag{3.1}$$

$\mathscr{D}$ and ρ are defined as in § 1. We suppose $f(t)$ to satisfy the following conditions:

H − 0 $f(t)$ is continuous and positive for $t \in R^+$,

H − 1 $f(0) > 0$,

H − 2 $f_t(t) > 0$ and is continuous for $t \in R^+$.

Several authors [6], [7] were able to show that problem (3.1) has a positive solution for all $\lambda \in (0, \lambda_{cr})$. λ_{cr} is the least upper bound such that problem (3.1) has a solution for each $\lambda < \lambda_{cr}$ whereas it is not solvable for $\lambda > \lambda_{cr}$. λ_{cr} is called the *critical explosion parameter* [11].

Let

$$\Delta\phi + \nu\rho\phi = 0 \quad \text{in } \mathscr{D}, \qquad \phi = 0 \quad \text{on } \partial\mathscr{D} \tag{3.2}$$

be the problem of an inhomogeneous membrane with mass density $\rho(x)$. With ν we denote its lowest eigenvalue. As Hudjaev [6] showed, Barta's simple argument using Green's identity can be used here and yields

$$\lambda_{cr} \leqq \frac{\nu}{\inf_{t>0} [f(t)/t]}. \tag{3.3}$$

Let

$$m = \inf_{t>0} [f(t)/t] < \infty.$$

Then we have the following result.

THEOREM 3.1. *Under the assumption $CM < 4\pi$, λ_{cr} is bounded from below by the expression*

$$\frac{C}{m}\left\{\log \frac{4\pi}{4\pi - CM}\right\}^{-1} \leqq \lambda_{cr}. \tag{3.4}$$

Proof. It suffices to prove the existence of a solution of (3.1) for the value $\lambda_0 = (C/m)\{\log 4\pi/(4\pi - CM)\}^{-1}$. For this purpose we use the method of upper and lower solutions [6], [10]. $\psi(x)$ is called an *upper solution* to problem (3.1) if

$$\Delta\psi + \lambda\rho f(\psi) \leqq 0 \quad \text{in } \mathscr{D}, \qquad \psi \geqq 0 \quad \text{on } \partial\mathscr{D}. \tag{3.5}$$

If there exists a positive upper solution then problem (3.1) has a solution [6], [10] such that $u \leqq \psi$. Consider the problem $\Delta\phi + \lambda\rho(x)f(t) = 0$ in $\mathscr{D}$, $\phi = 0$ on $\partial\mathscr{D}$, where t is an arbitrary, but fixed positive real number. By Theorem 1.1 we have

$$\phi(x) \leqq \frac{\lambda f(t)}{C} \log\left(\frac{4\pi}{4\pi - CM}\right) \equiv B(t).$$

If $B(t) \leqq t$, i.e., if $\lambda \leqq (t \cdot C/f(t)) \cdot \{\log 4\pi/(4\pi - CM)\}^{-1}$, then $\phi(x)$ satisfies (3.5) and there exists a solution of (3.1). Since t was arbitrary, we can set $t/f(t) = m^{-1}$, which is the optimal choice. By the maximum principle, $\phi(x)$ is positive. For each $\lambda \leqq \lambda_0$ we have constructed a positive upper solution. The assertion is therefore established.

Example. Let $\mathscr{D} = \{x; |x| \leqq 2/\sqrt{C},\ C > 0\}$ and $\rho = (1 + Cr^2/4)^{-2}$. In this case $CM = 2\pi$ and $v = 2C$. It then follows from (3.3) and (3.4) that $1.442_7(C/m) \leqq \lambda_{cr} \leqq (2C/m)$.

Remark. If $f(t)$ is convex, then λ_0 cannot coincide with λ_{cr}. We shall prove this statement by contradiction. Let us assume that $\lambda_0 = \lambda_{cr}$. Then by the previous remarks on upper solutions, u_{cr} (= solution of (3.1) corresponding to λ_{cr}) exists and $u_{cr} \leqq \phi$. It is known [7], [6], [3] that in this case λ_{cr} is equal to the lowest eigenvalue p of the linearized problem

$$\Delta v + p\rho f_t(u_{cr})v = 0 \quad \text{in } \mathscr{D}, \quad v = 0 \text{ on } \partial\mathscr{D}.$$

The eigenfunction v corresponding to $p = \lambda_{cr}$ does not change the sign. We shall assume that $v(x) \geqq 0$ in $\overline{\mathscr{D}}$. From the construction of ϕ it follows that the domain where $\Delta\phi + \lambda_{cr}\rho f(\phi) < 0$ has a positive Lebesgue measure. By Green's identity we have

$$\begin{aligned} 0 &= \int_{\mathscr{D}} v\Delta(\phi - u_{cr})\,dx - \int_{\mathscr{D}} (\phi - u_{cr})\Delta v\,dx \\ &< -\lambda_{cr}\int_{\mathscr{D}} v\rho(f(\phi) - f(u_{cr}))\,dx + \lambda_{cr}\int_{\mathscr{D}} v\rho f_t(u_{cr})(\phi - u_{cr})\,dx < 0. \end{aligned}$$

For the last inequality we used the convexity of f. We have obtained a contradiction. Hence $\lambda_0 \neq \lambda_{cr}$.

4. Generalizations. We now suppose that $\mathscr{D}$, ρ, Γ_0 and Γ_1 satisfy the assumptions of § 2. Consider the nonlinear problem

$$\begin{aligned} \Delta u + \lambda\rho(x)f(u) &= 0 \quad \text{in } \mathscr{D}, \\ u &= 0 \quad \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} + \sigma(x)u &= 0 \quad \text{on } \Gamma_1, \end{aligned} \tag{4.1}$$

where $f(t)$ satisfies the conditions H–0, H–1 and H–2, and where $\sigma(x)$ is a nonnegative Hölder continuous function on Γ_1. This problem, too, is solvable in an interval $I = (0, \tilde{\lambda}_{cr})$ but for no $\lambda > \tilde{\lambda}_{cr}$. Let $\tilde{v}$ denote the lowest eigenvalue of the linear problem

$$\begin{aligned} &\Delta\tilde{\phi} + \tilde{v}\rho\tilde{\phi} = 0 \quad \text{in } \mathscr{D}, \quad \tilde{\phi} = 0 \text{ on } \Gamma_0, \\ &\frac{\partial\tilde{\phi}}{\partial n} + \sigma\tilde{\phi} = 0 \quad \text{on } \Gamma_1. \end{aligned}$$

In analogy to (3.3) we have $\tilde{\lambda}_{cr} \leqq \tilde{v}/m$.

In exactly the same way as Theorem 3.1, except that we use inequality (2.2) instead of (1.3), we prove the following.

THEOREM 4.1. *Let* $\mu^+(\Gamma_1) = \pi - \alpha \leqq \pi$ *and* $CM < 2\alpha$. *Then the following estimate holds for* $\tilde{\lambda}_{cr}$:

$$\frac{C}{m}\left\{\log \frac{2\alpha}{2\alpha - CM}\right\}^{-1} \leqq \tilde{\lambda}_{cr}.$$

The remark at the end of § 3 applies also to Theorem 4.1.

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DEPENDENCE OF A NONLINEAR INTEGRODIFFERENTIAL SYSTEM ON PARAMETERS*

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Abstract. In this paper we study the dependence on parameters of the solutions of a class of nonlinear systems of integrodifferential equations arising in nuclear reactor dynamics. Continuity with respect to a small parameter (physically, the thermal conductivity), a large parameter and initial conditions is established.

1. Introduction. We consider the following real, nonlinear integrodifferential system for $0 \leqq x \leqq c; 0 \leqq t < \infty$:

$$
\begin{aligned}
u'(t) &= -\int_0^c \alpha(x)T(x, t)\,dx, \\
T_t(x, t) &= (b(x)T_x(x, t))_x - q(x)T(x, t) + \eta(x)\sigma(u(t)),
\end{aligned} \tag{1.1}
$$

together with initial and boundary conditions

$$
u(0) = u_0, \quad T(x, 0) = f(x), \tag{1.2}
$$

$$
a_1 T(0, t) + a_2 T_x(0, t) = b_1 T(c, t) + b_2 T_x(c, t) = 0 \tag{1.3}
$$

in a number of settings. In this system α, η, f, σ, b and q are given functions; u_0, a_1, a_2, b_1, b_2, c are constants with u_0 arbitrary, $c > 0$, $|a_1| + |a_2| > 0$, $|b_1| + |b_2| > 0$, the prime indicates differentiation with respect to t. This system is of physical interest as a dynamic model of a one-dimensional continuous medium nuclear reactor. For the physical significance of the quantities in (1.1), the reader should consult the references and their bibliographies.

Systems similar to (1.1) have been studied in a variety of settings by Levin and Nohel [4], [5], [6], [7], Bronikowski [1], [2] and more recently by Bronikowski, Hall and Nohel [3]. Most results in these papers concerned the existence and asymptotic behavior of solutions as $t \to \infty$. Here our prime concern is the dependence of solutions on various quantities appearing in the system.

We will assume throughout this paper that $b, b', q \in C[0, c]$, $b(x) > 0, q(x) \geqq 0$ for all $x \in [0, c]$, and that $a_1 a_2 \leqq 0$, $b_1 b_2 \geqq 0$. Then the Sturm–Liouville problem

$$
\begin{aligned}
&(by')' + (\lambda - q)y = 0, \\
&a_1 y(0) + a_2 y'(0) = 0, \qquad b_1 y(c) + b_2 y'(c) = 0,
\end{aligned} \tag{1.4}
$$

which is associated with (1.1b) in a natural way, has a countable set of simple nonnegative eigenvalues, λ_n, $n = 0, 1, 2, \cdots$, satisfying $\lambda_n = n^2\pi^2 L^{-2} + O(1)$ $(n \to \infty)$, where $L = \int_0^c b(x)^{-1/2}\,dx$. We will assume that the corresponding eigenfunctions, $y_n(x)$, are normalized and will denote by g_n the Fourier coefficients of any suitable function g; thus $g_n = \int_0^c g y_n$. The smallest eigenvalue will be positive

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if either $q \not\equiv 0$ or $|a_1| + |b_1| > 0$. If neither condition is true, then 0 is an eigenvalue, denoted by λ_0, with corresponding $y_0(x) = c^{-1/2}$. In any event, λ_1 denotes the smallest positive eigenvalue. The principal effect of 0 as an eigenvalue is in the asymptotic behavior of solutions as $t \to \infty$. It is of only very minor concern here.

Regarding existence and uniqueness of solutions of (1.1), we have the following theorem [3, first conclusions of Thms. 1 and 2, based on Lemma 1].

THEOREM A. *Let the following conditions be satisfied*:

$$\sigma \in C'(-\infty, \infty), \quad u\sigma(u) > 0 \textit{ if } u \neq 0, \tag{1.5}$$
$$S(x) = \int_0^x \sigma(u)\,du \to \infty \quad \textit{as } |x| \to \infty,$$

$$\alpha, \eta, f, (b\eta')', (bf')' \in L_2(0, c), \quad f \in C(0, c), \tag{1.6}$$
$$f, \eta \quad \textit{satisfy the boundary condition } (1.4\text{b}),$$

$$\alpha_n \eta_n \geqq 0, \qquad n = 0, 1, 2, \cdots, \tag{1.7}$$

there exist constants $\hat{c} \leqq 1$, $\tilde{c} \geqq 1$ *such that*

$$\hat{c} \leqq \alpha_n/\eta_n \leqq \tilde{c} \quad \textit{for all } n \textit{ for which } \alpha_n\eta_n > 0, \tag{1.8}$$

$$\alpha_n \eta_n = 0 \to \alpha_n = \eta_n = 0. \tag{1.9}$$

Then (1.1) *has a unique solution* $u(t)$, $T(x, t)$ *existing for* $0 \leqq t < \infty$, $0 \leqq x \leqq c$ *satisfying* (1.2) *and* (1.3). *Moreover if* (1.4) *does not have* 0 *as an eigenvalue, then for some constant* $K > 0$,

$$|u(t)| \leqq K, \qquad \sup_{0 \leqq x \leqq c} |T(x, t)| \leqq K \qquad (0 \leqq t < \infty). \tag{1.10}$$

If 0 *is an eigenvalue of* (1.4), *then for some constant* $K > 0$, $u(t)$ *satisfies* (1.10a) *and*

$$\sup_{0 \leqq x \leqq c} |T(x, t)| \leqq K\left(1 + \left|\int_0^t \sigma(u(s))\,ds\right|\right). \tag{1.11}$$

There are certain modifications of these hypotheses under which (1.10) and (1.11) may be established [3, Lemma 2], [8]. We will not utilize them, however, to obtain our results.

The continuity of solutions of (1.1) with respect to u_0, α, η and f (the functions regarded as points in the appropriate L_2 space) is established in § 2. All that is required is to point out several modifications in the proof of our former result for the linear system [1, Thm. 4]. In § 3, in order to relate some of our results to those in [4], we consider (1.1) where $b(x) \equiv b > 0$, $q(x) \equiv 0$. In the physical setting b, the thermal conductivity of the reactor medium may be small, and one is thus led to consider the behavior of solutions as $b \to 0^+$. The linear version of Theorem 2, which describes this behavior was established in [2]. Finally, in § 4 we study the dependence of solutions as $c \to \infty$ in the linear case and relate limits of these solutions to those of [4] in Theorem 3. The extension of this result to the nonlinear system is discussed after the proof is given.

In [3] it was shown that $u(t)$ and $T(x, t)$ satisfy certain relations which we list here for future reference. In these equations,

$$k_{1n} = \alpha_n f_n, \quad k_{2n} = \alpha_n \eta_n,$$

and the sums are from $n = 1$ to ∞ unless otherwise noted. First, $u(t)$ satisfies the Volterra equation

$$u(t) = u_0 + K_1(t) + \int_0^t K_2(t - \tau)\sigma(u(\tau))\, d\tau, \tag{1.12}$$

where, if (1.4) does not have 0 as an eigenvalue,

$$K_i(t) = \sum k_{in}\lambda_n^{-1}(e^{-\lambda_n t} - 1). \tag{1.13}$$

If 0 is an eigenvalue of (1.4), the formula for $K_i(t)$ must be modified by the addition of the term $-k_{i0}t$. The uniform convergence of the formally differented series in (1.13) for $t \geqq 0$ shows that $u(t)$ satisfies the following integrodifferential equation:

$$u'(t) = K_1'(t) + \int_0^t K_2'(t - \tau)\sigma(u(\tau))\, d\tau, \tag{1.14}$$

where

$$K_i'(t) = -\sum k_{in}\, e^{-\lambda_n t}. \tag{1.15}$$

It has also been found convenient to express the temperature component of a solution in the form $T(x, t) = T_H(x, t) + T_I(x, t)$, where

$$T_H(x, t) = \sum f_n y_n(x)\, e^{-\lambda_n t}, \tag{1.16}$$

$$T_I(x, t) = \int_0^t \sum \eta_n y_n(x)\, e^{-\lambda_n(t-\tau)}\sigma(u(\tau))\, d\tau. \tag{1.17}$$

2. Dependence on u_0, α, η and f. Using the notation of [1], we let Z denote the product space $(-\infty, \infty) \times L_2(0, c)^3$. The norm of a point

$$P = (u_0, \alpha, \eta, f) \in Z \quad \text{is } \|P\| = |u_0| + \|\alpha\|_2 + \|\eta\|_2 + \|f\|_2,$$

where the subscript denotes the usual L_2-norm. If for $i = 1, 2$, $P_i = (u_{i0}, \alpha_i, \eta_i, f_i) \in Z$, then $\|\Delta\alpha\|_2$ denotes $\|\alpha_1 - \alpha_2\|_2$, and

$$\|P_1 - P_2\| = |u_{10} - u_{20}| + \|\Delta\alpha\|_2 + \|\Delta\eta\|_2 + \|\Delta f\|_2.$$

Also, $u_i(t)$, $T_i(x, t) = T_{iH}(x, t) + T_{iI}(x, t)$ will denote solutions of (1.1)–(1.3) corresponding to $u_{i0}, \alpha_i, \eta_i, f_i$.

THEOREM 1. *Let b, σ, α_i, η_i, f_i satisfy the hypotheses of Theorem* A. *Let u_{10} and u_{20} be arbitrary real numbers, and let $0 < \delta \neq 3/4$, $T > 0$, $0 < r < \lambda_1$. Then for some constant $K > 0$,*

$$|u_1(t) - u_2(t)| \leqq K\|P_1 - P_2\| \qquad (\textit{uniformly for } 0 \leqq t \leqq T), \tag{2.1}$$

$$|T_{1H}(x, t) - T_{2H}(x, t)| \leqq g(t)\|\Delta f\|_2 \qquad (0 \leqq t, \textit{ uniformly for } 0 \leqq x \leqq c), \tag{2.2}$$

$$\begin{gathered} |T_{1I}(x, t) - T_{2I}(x, t)| \leqq h(t)\|P_1 - P_2\| \\ (\textit{uniformly for } 0 \leqq x \leqq c, 0 \leqq t \leqq T), \end{gathered} \tag{2.3}$$

where, if 0 *is not an eigenvalue of* (1.4),

$$g(t) = K e^{-rt} t^{-(1/4+\delta)}, \qquad h(t) = K t^{3/4-\delta}, \tag{2.4}$$

whereas, if 0 *is an eigenvalue of* (1.4),

$$g(t) = c^{-1/2} + K e^{-rt} t^{-(1/4+\delta)}, \qquad h(t) = tc^{-1/2} + K t^{3/4-\delta}. \tag{2.5}$$

Discussion of proof. The proof is nearly the same as that of [1, Thm. 4]. It is necessary to replace $u_i(\tau)$ by $\sigma(u_i(\tau))$ in the integrals and to utilize the fact that, by (1.10a), both $u_1(t)$ and $u_2(t)$ are bounded. If B denotes an upper bound for $|u_i(t)|$ ($i = 1, 2, t \geqq 0$) and if $L = \sup_{|y| \leqq B} |\sigma'(y)|$, then

$$|\sigma(u_1(t)) - \sigma(u_2(t))| \leqq L|u_1(t) - u_2(t)|.$$

The insertion of this Lipschitz condition on σ at the appropriate step in the calculations yields (2.1), where the constant K now depends on L. There is no change in the derivation of (2.2), but to obtain (2.3), the introduction of the above estimate must be made as before.

3. Dependence on *b*. In this section we consider (1.1) when $b(x) \equiv b > 0$, $q(x) \equiv 0$, and we determine the limiting behavior of solutions as $b \to 0^+$. We explicitly indicate the dependence on b in our notation $u(t, b)$, $T(x, t, b)$ for a solution. Thus the system we study is

$$\begin{aligned} u'(t, b) &= -\int_0^c \alpha(x) T(x, t, b)\, dx, \\ T_t(x, t, b) &= b T_{xx}(x, t, b) + \eta(x)\sigma(u(t)), \end{aligned} \tag{3.1}$$

subject to initial conditions (1.2) and boundary conditions (1.3) in which $a_2 = b_2 = 0$. In this case, then,

$$\lambda_n = bn^2\pi^2 c^{-2}, \qquad y_n(x) = \sqrt{\frac{2}{c}}\, \frac{\sin n\pi x}{c}, \qquad n = 1, 2, \cdots.$$

Our starting point is the "degenerate" system obtained by setting $b = 0$ in (3.1), solutions of which will be $V(t)$, $S(x, t)$. Thus

$$\begin{aligned} V'(t) &= -\int_0^c \alpha(x) S(x, t)\, dx, \\ S_t(x, t) &= \eta(x)\sigma(V(t)). \end{aligned} \tag{3.2}$$

The following lemma is immediate.

LEMMA. *Let* (1.5a) *and* (1.6) *be satisfied. Then* (3.2) *has a unique solution satisfying* (1.2) *and* (1.3).

Proof. Let $V(t)$ be the unique solution of the initial value problem

$$V'' + \langle \alpha, \eta \rangle \sigma(V) = 0, \qquad V(0) = u_0, \qquad V'(0) = -\langle \alpha, f \rangle, \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ is the usual L_2-inner-product, and let

$$S(x, t) = f(x) + \eta(x) \int_0^t \sigma(V(\tau))\, d\tau. \tag{3.4}$$

In view of

$$V'(t) = -\langle \alpha, f \rangle - \langle \alpha, \eta \rangle \int_0^t \sigma(V(\tau))\, d\tau, \tag{3.5}$$

the pair $V(t)$, $S(x, t)$ is easily verified to be the desired solution of (3.2).

Clearly the smoothness hypotheses were not necessary to establish this lemma. We also note that if σ satisfies (1.5b) and if $\langle \alpha, \eta \rangle > 0$, then it is well known that the solution $V(t)$ of (3.3) is periodic. In particular, if the system is linear, letting $\omega = \langle \alpha, \eta \rangle^{1/2}$, we obtain

$$V(t) = -\frac{\langle \alpha, f \rangle}{\omega} \sin \omega t + u_0 \cos \omega t,$$

$$S(x, t) = f(x) + \eta(x)\left[\frac{\langle \alpha, f \rangle}{\omega^2}(\cos \omega t - 1) + \frac{\omega_0}{\omega} \sin \omega t\right].$$

This is, incidentally, the same as the solution of the degenerate system in [4] where, of course, the inner product is an integral over $(-\infty, \infty)$.

We now describe the limiting behavior of solutions of (3.1) as $b \to 0^+$. Note that since $b > 0$, the hypotheses concerning (1.4) are still satisfied.

THEOREM 2. *Let σ, α, η, f satisfy the hypotheses of Theorem* A. *Then*

$$\lim_{b \to 0^+} u(t, b) = V(t), \qquad \lim_{b \to 0^+} T(x, t, b) = S(x, t) \tag{3.6}$$

for each $t \leqq 0$ and $0 \leqq x \leqq c$. If $0 < T < \infty$, $0 < x_1 < x_2 < c$, the limits are uniform over $[0, T]$ and $[x_1, x_2] \times [0, T]$, respectively.

Proof. Noting that $\langle \alpha, f \rangle = \sum k_{1n}$ and $\langle \alpha, \eta \rangle = \sum k_{2n}$ by Parseval's theorem, we obtain from (1.14), (1.15), (3.5),

$$u'(t, b) - V'(t) = \sum k_{1n}(1 - e^{-\lambda_n t}) + J_1 + J_2, \tag{3.7}$$

where

$$\begin{aligned} J_1 &= \int_0^t \sum k_{2n}(1 - e^{-\lambda_n(t-\tau)})\sigma(V(\tau))\, d\tau, \\ J_2 &= \int_0^t \sum k_{2n} e^{-\lambda_n(t-\tau)}(\sigma(V(\tau)) - \sigma(u(\tau, b))\, d\tau. \end{aligned} \tag{3.8}$$

Let $\varepsilon > 0$, $T > 0$. Because of (1.6) the series $\sum k_{in}$ both converge absolutely. Let N be such that

$$\sum_{n=N+1}^{\infty} |k_{in}| < \varepsilon, \qquad i = 1, 2.$$

Using this estimate, the inequality $1 - e^{-x} \leqq \min(1, x)$ $(x \geqq 0)$, and the formula for λ_n, we obtain for $i = 1, 2$,

$$|\sum k_{in}(1 - e^{-\lambda_n t})| \leqq K_1 b + \varepsilon \qquad (0 \leqq t \leqq T), \tag{3.9}$$

where $K_1 = \max_{i=1,2} \sum_{n=1}^{N} |k_{in}| N^2 \pi^2 c^{-2} T$. If $M > 0$ denotes a bound on $|\sigma(V(t))|$

$(0 \leqq t \leqq T)$ and $K_2 = \sum |k_{2n}|$, then from (3.8) and (3.9) we have

$$|J_1| \leqq (K_1 b + \varepsilon) M T,$$
$$|J_2| \leqq K_2 \int_0^t |\sigma(V(\tau)) - \sigma(u(\tau, b))| \, d\tau. \tag{3.10}$$

Now, in [3, Lemmas 1 and 4, Thm. 1(i)] it was shown that the bound on $|u(t)|$ in (1.10) depends only on σ, u_0, α, η and f but is independent of the function $b(x)$ (the constant b in our setting). In particular $|u(t, b)| \leqq 2\Omega$, $0 \leqq t < \infty$, $b > 0$, where

$$\Omega = \max\left[\sqrt{\frac{2W_0}{\hat{c}}}, \max\left[|x| : S(x) \leqq W_0\right]\right],$$
$$W_0 = \max(S(u_0), S(-u_0)) + \frac{\tilde{c}}{2}\sum f_n^2. \tag{3.11}$$

($S(u) = \int_0^u \sigma$, $\hat{c}$ and $\tilde{c}$ were defined in (1.8).) Now let $L = \sup|\sigma'(y)|$ for

$$|y| \leqq \max(M, 2\Omega).$$

Then for all $t \geqq 0$, $b > 0$,

$$|\sigma(V(t)) - \sigma(u(t, b))| \leqq L|V(t) - u(t, b)|. \tag{3.12}$$

Combining (3.7)–(3.12) we obtain for $0 \leqq t \leqq T$, $b > 0$,

$$|u'(t, b) - V'(t)| \leqq K_1(1 + MT)b + (1 + MT)\varepsilon$$
$$+ K_2 L \int_0^t |u(\tau, b) - V(\tau)| \, d\tau. \tag{3.13}$$

Integrating (3.13) and applying Gronwall's inequality we obtain for such t and b,

$$|u(t, b) - V(t)| \leqq (C_1\varepsilon + C_2 b) \exp(K_2 L T^2), \tag{3.14}$$

where $C_1 = (1 + MT)T$, $C_2 = K_1 C_1$. Since ε is arbitrary, (3.14) implies (3.6a).

Turning our attention to $T(x, t, b)$ we first recall that under hypothesis (1.6),

$$\lim_{t \to 0^+} T_H(x, t, b) = \lim_{t \to 0^+} \sum f_n y_n(x) \exp(-n^2\pi^2 c^{-2} bt) = f(x) \tag{3.15}$$

for all $x \in (0, c)$, uniformly on any closed subinterval. Interchanging the roles of b and t in (3.15), an examination of the quantifiers involved establishes

$$\lim_{b \to 0^+} T_H(x, t, b) = f(x) \qquad (0 < x < c, t > 0), \tag{3.16}$$

the limit being uniform for $0 \leqq t \leqq T$ and $0 < x_1 \leqq x \leqq x_2 < c$. In the same way,

$$\lim_{b \to 0^+} \sum \eta_n y_n(x) \, e^{-\lambda_n t} = \eta(x). \tag{3.17}$$

In view of (3.16) and (3.4), the proof of (3.6b) depends only on showing

$$\lim_{b \to 0^+} T_I(x, t, b) = \eta(x) \int_0^t \sigma(V(\tau)) \, d\tau. \tag{3.18}$$

To this end we write

$$T_I(x, t, b) - \eta(x) \int_0^t \sigma(V(\tau))\, d\tau = I_1 + I_2, \tag{3.19}$$

where

$$I_1 = \int_0^t \sum \eta_n y_n(x)\, e^{-\lambda_n(t-\tau)}[\sigma(u(\tau, b)) - \sigma(V(\tau)]\, d\tau,$$

$$I_2 = \int_0^t [\sum \eta_n y_n(x)\, e^{-\lambda_n(t-\tau)} - \eta(x)]\sigma(V(\tau))\, d\tau.$$

Let $0 \leqq t \leqq T$, $0 < x_1 \leqq x \leqq x_2 < c$, and let $\varepsilon > 0$ be given. Let $M > 0$ denote a bound again for $|\sigma(V(t))|$ on $[0, T]$ and also for $|\eta(x)|$ on $[x_1, x_2]$. Because of (3.6a) and (3.17) we can find a $\delta > 0$ so that the bracketed differences in both I_1 and I_2 are, in absolute value, smaller than ε when $0 < b < \delta$ uniformly in t and x as described. Also, the series in I_1 is majorized by $M + \varepsilon$ and so we obtain

$$|I_1| \leqq (M + \varepsilon)T\varepsilon, \qquad |I_2| \leqq MT\varepsilon,$$

which establishes (3.18) and the proof is complete.

4. Dependence on *c*. Levin and Nohel have studied linear and nonlinear versions of (1.1) on the x-interval $(-\infty, \infty)$ in [4], [5], [6]. In this section we will show that the solutions on the infinite interval are, in fact, pointwise limits of solutions on the finite interval as the length increases. One difficulty is that our interval has been $(0, c)$. There are two possibilities: to restate our results for $(-c/2, c/2)$ or to restate theirs for $(0, \infty)$. We have chosen the latter approach.

Because of the nature of the problem considered in [4], namely constant coefficients in the heat equation and linearity, we consider the same problem on the finite interval and state Theorem 3 for this special case. The extension to the nonlinear case will be discussed after the proof of this theorem.

It will be noted that our hypotheses are much milder than in the nonlinear case. We will denote by $W(t)$, $R(x, t) = R_H(x, t) + R_I(x, t)$ a solution of the system on $(0, \infty)$. The system and side conditions now are

$$\begin{aligned} &W'(t) = -\int_0^\infty \alpha(x)R(x, t)\, dx, \\ &R_t(x, t) = bR_{xx}(x, t) + \eta(x)W(t), \end{aligned} \tag{4.1}$$

$$W(0) = u_0, \quad T(x, 0) = f(x) \qquad (0 < x < \infty), \tag{4.2}$$

$$T(0, t) = 0 \qquad (0 < t < \infty). \tag{4.3}$$

It should be noted that a positive constant a which occurred in (4.1b) in [4] has been incorporated into b and η. This has simplified all of our expressions, and nothing essential has been lost. The following theorem concerning (4.1) is true, the proof being essentially as in [4].

THEOREM B. *Let*

$$\begin{aligned} &\alpha, \eta, f \in L_2(0, \infty), \quad f \in C(0, \infty), \\ &\eta \quad \textit{satisfy a local Holder condition on } (0, \infty). \end{aligned} \tag{4.4}$$

Then (4.1) *has a unique solution satisfying* (4.2) *and* (4.3).

In the proof of this theorem certain relations satisfied by $W(t)$ and $R(x, t)$, analogous to (1.12)–(1.17), can be derived. One writes down a formal solution of (4.1b) using the fundamental solution of the heat equation for $0 < t$, $x < \infty$,

$$G(x, \xi, t) = (4bt)^{-1/2}\left(\exp\frac{-(x-\xi)^2}{4bt} - \exp\frac{-(x+\xi)^2}{4bt}\right),$$

or equivalently, and better suited for our purposes,

$$G(x, \xi, t) = \frac{2}{\pi}\int_0^\infty \sin y\xi \sin yx\, e^{-by^2 t}\, dy.$$

This formal expression for $R(x, t)$, (4.7) below, is substituted into (4.1a), and the resulting equation integrated. These formal calculations eventually are justified. We now list these equations in forms suitable for future use.

$$W(t) = u_0 + \int_0^\infty K_1(y, t)\, d\rho(y) + \int_0^t \int_0^\infty K_2(y, t-\tau)\, d\rho(y) W(\tau)\, d\tau, \tag{4.5}$$

where, if $\hat{g}_S(y) = \int_0^\infty g(\xi) \sin y\xi\, d\xi$, the Fourier sine transform of $g \in L_2(0, \infty)$,

$$\begin{gathered} K_1(y, t) = \hat{\alpha}_S(y)\hat{f}_S(y)\sigma(y, t), \qquad K_2(y, t) = \hat{\alpha}_S(y)\hat{\eta}_S(y)\sigma(y, t), \\ \sigma(y, t) = (e^{-by^2 t} - 1)/by^2 \quad \text{if } y > 0, \quad \sigma(0, t) = -t, \\ \rho(y) = 2y/\pi. \end{gathered} \tag{4.6}$$

Moreover, the temperature components R_H and R_I are given by

$$\begin{aligned} R_H(x, t) &= \int_0^\infty H_1(y, x, t)\, d\rho(y), \\ R_I(x, t) &= \int_0^t \int_0^\infty H_2(y, x, t-\tau)\, d\rho(y) W(\tau)\, d\tau, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} H_1(y, x, t) &= \hat{f}_S(y) \sin yx\, e^{-by^2 t}, \\ H_2(y, x, t) &= \hat{\eta}_S(y) \sin yx\, e^{-by^2 t}. \end{aligned} \tag{4.8}$$

Our notation for a solution of (1.1) will be $u(t, c)$, $T(x, t, c)$. In the special setting of this section, the original system now takes on the form

$$\begin{aligned} u'(t, c) &= -\int_0^c \alpha(x) T(x, t, c)\, dx, \\ T_t(x, t, c) &= bT_{xx}(x, t, c) + \eta(x) u(t, c). \end{aligned} \tag{4.9}$$

The side conditions are still (1.2) and (1.3) with $a_2 = b_2 = 0$. Note the exact analogy between hypotheses (4.10) and (4.4) in the following theorem proved by Bronikowski in [1].

THEOREM C. *Let*

$$\begin{gathered} \alpha, \eta, f \in L_2(0, c), \quad f \in C(0, c), \\ \eta \quad \textit{satisfy a local Holder condition on } (0, c). \end{gathered} \tag{4.10}$$

Then (4.9) *has a unique solution satisfying* (1.2) *and* (1.3).

In order to compare solutions of (4.9) with those of (4.1), it is convenient to rewrite the series representations of u and T which occur in (1.12), (1.16) and (1.17) as Riemann–Stieltjes integrals. To this end we recall the formulas for λ_n and $y_n(x)$ after relation (3.1) and define

$$
\begin{aligned}
&f_c(y) = \int_0^c f(\xi)\sin y\xi\, d\xi,\\
&K_{1c}(y,t) = \alpha_c(y) f_c(y)\sigma(y,t), \qquad K_{2c}(y,t) = \alpha_c(y)\eta_c(y)\sigma(y,t),\\
&\rho_c(y) = \begin{cases} \dfrac{2n}{c}, & \dfrac{n\pi}{c} < y \leqq (n+1)\dfrac{\pi}{c}, \quad n = 0,1,2,\cdots,\\ 0, & y = 0,\end{cases}\\
&H_{1c}(y,x,t) = f_c(y)(\sin yx)\, e^{-by^2t},\\
&H_{2c}(y,x,t) = \eta_c(y)(\sin yx)\, e^{-by^2t}.
\end{aligned}
\tag{4.11}
$$

Then (1.12), (1.16) and (1.17) can be rewritten as

$$
\begin{aligned}
u(t,c) = u_0 &+ \int_0^\infty K_{1c}(y,t)\, d\rho_c(y)\\
&+ \int_0^t \int_0^\infty K_{2c}(y,t-\tau)u(\tau,c)\, d\rho_c(y)\, d\tau,
\end{aligned}
\tag{4.12}
$$

$$
T_H(x,t,c) = \int_0^\infty H_{1c}(y,x,t)\, d\rho_c(y), \tag{4.13}
$$

$$
T_I(x,t,c) = \int_0^t \int_0^\infty H_{2c}(y,x,t-\tau)u(\tau,c)\, d\rho_c(y)\, d\tau. \tag{4.14}
$$

We can now state and prove the following theorem.

THEOREM 3. *Let* α, η, f *satisfy* (4.4) *on* $(0,\infty)$ *so that their restrictions to* $(0,c)$ *automatically satisfy* (4.10) *there. Let* $u(t,c)$, $T(x,t,c)$ *be the solutions of* (4.9) *which result when the restrictions of* α, η, f *appear in* (4.9) *and* (1.2). *Let* $0 < t_0$, $T < \infty$, $0 < x_1 < x_2 < \infty$. *Then*

$$
\lim_{c\to\infty} u(t,c) = W(t) \qquad (t \geqq 0,\ \textit{uniformly for } 0 \leqq t \leqq T), \tag{4.15}
$$

$$
\begin{gathered}
\lim_{c\to\infty} T_H(x,t,c) = R_H(x,t)\\
(0 < x, t < \infty,\ \textit{uniformly for } x_1 \leqq x \leqq x_2,\ t_0 \leqq t < \infty),
\end{gathered}
\tag{4.16}
$$

$$
\begin{gathered}
\lim_{c\to\infty} T_I(x,t,c) = R_I(x,t)\\
(0 < x, 0 \leqq t,\ \textit{uniformly for } x_1 \leqq x \leqq x_2,\ 0 \leqq t \leqq T).
\end{gathered}
\tag{4.17}
$$

The proof of this theorem follows the sequence of lemmas below. The lemmas are much simpler if in addition to (4.4) (and, therefore, (4.10)) we further assume that

$$\alpha, \eta, f \in L_1(0, \infty) \tag{4.18}$$

and thus the restrictions of these functions to $(0, c)$ are in $L_1(0, c)$. These hypotheses will not be stated explicitly in each lemma for brevity. The first lemma lists some elementary properties of the functions listed in (4.6) and (4.11).

LEMMA 1. *Let Q be the quadrant $0 \leqq y,\ t < \infty$, $S(T)$ the strip $0 \leqq y < \infty$, $0 \leqq t \leqq T$. Then*

(4.19) *$\lim_{c\to\infty} f_c(y) = \hat{f}_S(y)$ uniformly for $0 \leqq y < \infty$. Moreover, f_c and $\hat{f}_S$ are continuous and bounded (by $\|f\|_1$) for $0 \leqq y < \infty$. Similar statements hold for $\alpha_c, \hat{\alpha}_S, \eta_c, \hat{\eta}_S$.*

(4.20) *$\lim_{c\to\infty} \rho_c(y) = \rho(y)$ uniformly for $0 \leqq y < \infty$ since $0 \leqq \rho(y) - \rho_c(y) \leqq 2/c$ for such y.*

(4.21) *$\sigma(y, t)$ is continuous on Q and bounded on any $S(T)$ since $-t \leqq \sigma(y, t) \leqq 0$.*

(4.22) *For $i = 1, 2$ it follows from* (4.19) *and* (4.21) *that $K_i(y, t), K_{ic}(y, t)$ are continuous on Q for all $c > 0$. Moreover $\lim_{c\to\infty} K_{ic}(y, t) = K_i(y, t)$ on Q uniformly on any $S(T)$ since, for example, if $(y, t) \in S(T)$, using* (4.21),

$$|K_{1c}(y, t) - K_1(y, t)| \leqq T|\alpha_c(y) f_c(y) - \hat{\alpha}_S(y)\hat{f}_S(y)|.$$

The result now follows from (4.19).

(4.23) *For $i = 1, 2$ it is easy to show that $H_i(y, x, t)$, $H_{ic}(y, x, t)$ are continuous for $0 \leqq y, x, t < \infty$, uniformly continuous on $0 \leqq y \leqq Y < \infty$, $0 \leqq x_1 \leqq x \leqq x_2 < \infty, 0 \leqq t < \infty$. Moreover $\lim_{c\to\infty} H_{ic}(y, x, t) = H_i(y, x, t)$ uniformly for $0 \leqq y, x, t < \infty$ since, for example,*

$$|H_{1c}(y, x, t) - H_1(y, x, t)| \leqq |f_c(y) - \hat{f}_S(y)|,$$

and now use (4.19).

LEMMA 2. *For $i = 1, 2$,*

$$\int_0^\infty K_i(y, t)\, d\rho(y), \qquad \int_0^\infty K_{ic}(y, t)\, d\rho_c(y) \tag{4.24}$$

converge uniformly for $0 \leqq t \leqq T, 0 < c$.

$$\lim_{c\to\infty} \int_0^\infty K_{ic}(y, t)\, d\rho_c(y) = \int_0^\infty K_i(y, t)\, d\rho(y) \tag{4.25}$$

$$(t \geqq 0, \textit{uniformly for } 0 \leqq t \leqq T).$$

Proof. We consider the case $i = 1$. From the definitions in (4.6) and (4.11) and from the boundedness conclusion of (4.19), we have

$$|K_1|, |K_{1c}| \leqq b^{-1}\|\alpha\|_1 \|f\|_1 y^{-2} \qquad (0 < y, c < \infty, 0 \leqq t). \tag{4.26}$$

This together with (4.21), (4.22) and the definition of $\sigma(0, t)$ establish (4.24) for K_1. To establish this result for K_{1c} we use (4.26), integration by parts and

$\rho_c(y) \leqq (2/\pi)y$ to calculate for any $0 < y_1 < y_2$,

$$\left|\int_{y_1}^{y_2} K_{1c}\, d\rho_c\right| \leqq b^{-1}\|\alpha\|_1\|f\|_1\left(\left.\frac{\rho_c(y)}{y^2}\right|_{y_1}^{y_2} + 2\int_{y_1}^{y_2} \rho_c(y)y^{-3}\, dy\right)$$
$$\leqq \frac{2}{\pi b}\|\alpha\|_1\|f\|_1\left(\frac{3}{y_1} - \frac{1}{y_2}\right).$$

This establishes the result. The proof is the same if $i = 2$ with η replacing f.

To prove (4.25), let $\varepsilon > 0$ be given. Now because of (4.24), we can choose Y so large that

$$\left|\int_Y^\infty K_i\, d\rho\right|, \left|\int_Y^\infty K_{ic}\, d\rho_c\right| < \varepsilon \quad \text{for } 0 \leqq t \leqq T, \quad 0 < c.$$

Then

$$\left|\int_0^\infty K_i\, d\rho - \int_0^\infty K_{ic}\, d\rho_c\right| \leqq \left|\int_0^Y K_i\, d\rho - \int_0^Y K_{ic}\, d\rho_c\right| + \left|\int_0^Y (K_i - K_{ic})\, d\rho_c\right| + 2\varepsilon.$$

Using the uniform continuity of K_i for $0 \leqq y \leqq Y$, $0 \leqq t \leqq T$, and a slight modification of a well-known convergence property of Riemann–Stieltjes integrals, there exists $c_1 > 0$ such that if $c \geqq c_1$, the first term on the right side of the estimate above is smaller than ε uniformly for $0 \leqq t \leqq T$. From (4.22) and the fact that the total variation of $\rho_c(y)$ over $[0, Y]$ is no more than $2Y/\pi$, there exists c_2 such that the second term is also smaller than ε for $c \geqq c_2$. Then for $c \geqq \max(c_1, c_2)$, the left side is less than 4ε uniformly for $0 \leqq t \leqq T$.

LEMMA 3. *For* $0 \leqq t$, *uniformly for* $0 \leqq t \leqq T$,

$$\lim_{c\to\infty} u(t, c) = W(t). \tag{4.27}$$

Proof. Let $\varepsilon > 0$. Using (4.5) and (4.12), we may write

$$W(t) - u(t, c) = J_1 + J_2 + J_3,$$

where

$$J_1 = \int_0^\infty K_1\, d\rho - \int_0^\infty K_{1c}\, d\rho_c,$$

$$J_2 = \int_0^t \int_0^\infty K_2\, d\rho - \int_0^\infty K_{2c}\, d\rho_c W(t - \tau)\, d\tau,$$

$$J_3 = \int_0^t \int_0^\infty K_{2c}\, d\rho_c (W(t - \tau) - u(t - \tau, c))\, d\tau.$$

By (4.25) if c is sufficiently large, $c \geqq c_1$, say, and if $M > 0$ is a bound for $|W(t)|$ on $[0, T]$, then $|J_1| < \varepsilon$, $|J_2| \leqq MT\varepsilon$ for $0 \leqq t \leqq T$. From (4.22) and (4.24) it follows that $\int_0^\infty K_2\, d\rho$ is continuous for $0 \leqq t < \infty$ and thus bounded, say by $M > 0$, on

$[0, T]$. Using (4.25) again it follows that for $c \geqq c_2$, say,

$$\left| \int_0^\infty K_{2c}\, d\rho_c \right| \leqq M + \varepsilon = M_1.$$

Thus

$$|J_3| \leqq M_1 \int_0^t |W(\tau) - u(\tau, c)|\, d\tau$$

for such c and $0 \leqq t \leqq T$. Combining these estimates on the J's we obtain

$$|W(t) - u(t, c)| \leqq (1 + MT)\varepsilon + M_1 \int_0^t |W(\tau) - u(\tau, c)|\, d\tau,$$

which implies (4.27) after Gronwall's inequality is applied.

LEMMA 4. *Let* $0 < x_1 < x_2 < \infty$ *and* $0 < t_0 < \infty$. *Then for* $i = 1, 2$,

$$\int_0^\infty H_i(y, x, t)\, d\rho(y), \qquad \int_0^\infty H_{ic}(y, x, t)\, d\rho_c(y) \tag{4.28}$$

converge uniformly for $0 < x, c < \infty$, $t_0 \leqq t$.

$$\left| \int_0^\infty H_i(y, x, t)\, d\rho(y) \right|, \left| \int_0^\infty H_{ic}(y, x, t)\, d\rho_c(y) \right| \leqq M t^{-1/2} \tag{4.29}$$

for $0 < x, t, c < \infty$, *for some constant* $M > 0$.

$$\lim_{c \to \infty} \int_0^\infty H_{ic}(y, x, t)\, d\rho_c(y) = \int_0^\infty H_i(y, x, t)\, d\rho(y) \tag{4.30}$$

for $0 < x, t < \infty$, *uniformly for* $x_1 \leqq x \leqq x_2$, $t_0 \leqq t$.

Proof. Relation (4.28) is immediate from the estimate

$$|H_{ic}|, |H_i| \leqq M_i e^{-by^2 t} \qquad (0 \leqq y, t < \infty, 0 < x, c < \infty), \tag{4.31}$$

where $M_1 = \|f\|_1$, $M_2 = \|\eta\|_1$. To obtain (4.29) for K_i, we use (4.31) and the change of variable $Z = \sqrt{bt}\, y$ so that

$$\left| \int_0^\infty H_i\, d\rho \right| \leqq \frac{2M_i}{\pi\sqrt{bt}} \int_0^\infty e^{-Z^2}\, dZ.$$

We get (4.29) for H_{ic} using (4.31), integration by parts and the inequalities $\rho_c(y) \leqq (2/\pi)y$, $e^{-x} \leqq x^{-1/2}$ for $0 < x, c, y < \infty$. The details, which we omit, show that M can be taken as the larger of

$$\left(2 + 4M_i \int_0^\infty Z^2 e^{-Z^2}\, dZ\right) \Big/ \pi b, \qquad i = 1, 2.$$

The convergence relation (4.30) is established just as (4.25) in Lemma 2. The procedure is again to "chop off the tails" of each integral and use the uniform continuity of H_i and H_{ic} as described in (4.23) together with the convergence property of Riemann–Stieltjes integrals.

Taking $i = 1$ in (4.30) and recalling the representations of $R_H(x, t)$ in (4.7) and $T_H(x, t, c)$ in (4.13) we see that conclusion (4.16) of Theorem 3 is true (but still with the L_1 hypothesis (4.19)). The next lemma establishes (4.17) with this restriction.

LEMMA 5. *Let $0 < T$, $0 < x_1 < x_2$. Then convergence relation* (4.17) *is true as described.*

Proof. From (4.7) and (4.14) we may write $T_I(x, t, c) - R_I(x, t) = J_1 + J_2$ when

$$J_1 = \int_0^t \int_0^\infty H_{2c}\, d\rho_c(u(t-\tau, c) - W(t-\tau))\, d\tau,$$

$$J_2 = \int_0^t \left(\int_0^\infty H_{2c}\, d\rho_c - \int_0^\infty H_2\, d\rho \right) W(t-\tau)\, d\tau.$$

Let $\varepsilon > 0$, and, without loss of generality, we assume $\varepsilon < \sqrt{T}$. From (4.27) there exists c_1 such that $|u(t, c) - W(t)| < \varepsilon$ if $c \geqq c_1$ for $0 \leqq t \leqq T$. This, together with (4.29), yields

$$|J_1| \leqq M\varepsilon \int_0^t \tau^{-1/2}\, d\tau \leqq 2MT^{1/2}\varepsilon \tag{4.32}$$

uniformly for $0 < x$, $0 \leqq t \leqq T$.

To estimate J_2, let $0 < t_0 \leqq \varepsilon^2$ and break up the τ interval of integration into two parts: from 0 to t_0 and from t_0 to t, letting J_{21} denote the integral over the first interval and J_{22} the integral over the second. If M denotes a bound on $|W(t)|$, $0 \leqq t \leqq T$, as well as the constant in (4.29) we obtain by (4.29),

$$|J_{21}| \leqq 2M^2 \int_0^{t_0} \tau^{-1/2}\, d\tau = 4M^2 t_0^{1/2} \leqq 4M^2\varepsilon \tag{4.33}$$

uniformly for $0 < x$, $c < \infty$, $0 \leqq t \leqq T$. Now in the case of J_{22} we apply (4.30) and pick c_2 so that $c \geqq c_2$ implies

$$\left| \int_0^\infty H_{2c}\, d\rho_c - \int_0^\infty H_2\, d\rho \right| < \varepsilon \quad \text{for } x_1 \leqq x \leqq x_2, \quad t_0 \leqq t.$$

Then for such c, x, t,

$$|J_{22}| \leqq M(T - t_0)\varepsilon. \tag{4.34}$$

Combining (4.32), (4.33) and (4.34) for $c \geqq \max(c_1, c_2)$ now produces the desired result.

Before proceeding to the proof of Theorem 3 we must state another result of Levin and Nohel proved in [5] for the case where $-\infty < x < \infty$, but easily shown to be true if $0 \leqq x < \infty$. It is closely related to our Theorem 2.

THEOREM D. *For $i = 1, 2$, let α_i, η_i, f_i satisfy* (4.4), *let u_{10} and u_{20} be arbitrary, and let $W_i(t)$, $R_i(x, t)$ be the corresponding solutions of* (4.1). *Then if $T > 0$, relations* (2.1)–(2.4) *are true with $r = \delta = 0$, and the estimates are uniform for $0 < x < \infty$.*

Proof of Theorem 3. We describe a sequence of steps which establishes (4.15)–(4.17) without the L_1 hypothesis (4.18). Let $\varepsilon > 0$, and let

$$\|g\|_{2,c} = \left(\int_0^c g^2\right)^{1/2}, \qquad \|g\|_{2,\infty} = \left(\int_0^\infty g^2\right)^{1/2}.$$

Given α, η, f satisfying (4.4) on $(0, \infty)$ we first pick $\bar{\alpha}, \bar{\eta}, \bar{f}$ continuously differentiable and of compact support on $(0, \infty)$ so that $\|\alpha - \bar{\alpha}\|_{2,\infty} < \varepsilon$ and similarly for η, $\bar{\eta}$, $f, \bar{f}$. This is possible because, as is well known, the set of such functions is dense in $L_2(0, \infty)$. Each such function is obviously in $L_1(0, \infty)$. If $\bar{W}$, $\bar{R}$ denote the solution of (4.1) corresponding to the barred functions, then by Theorem D we have $|W(t) - \bar{W}(t)|$ and $|R(x, t) - \bar{R}(x, t)|$ smaller than a constant multiple of ε as described in (2.1)–(2.4). Now, we restrict $\bar{\alpha}$, $\bar{\eta}$, $\bar{f}$ to $(0, c)$, and, letting $\bar{u}(t, c)$, $\bar{T}(x, t, c)$ denote the corresponding solution of (4.9), we can apply Lemmas 3, 4 and 5 so that for c sufficiently large we have $|\bar{W}(t) - \bar{u}(t, c)|, |\bar{R}(x, t) - \bar{T}(x, t, c)| < \varepsilon$ as described. Observing that $\|g\|_{2,c} \leqq \|g\|_{2,\infty}$ and therefore $\|\alpha - \bar{\alpha}\|_{2,c} < \varepsilon$ and similarly for $\eta, \bar{\eta}, f, \bar{f}$, we now apply Theorem 2 to $\bar{u}$, $\bar{T}$, and u, T, the solution of (4.9) corresponding to α, η, f restricted to $(0, c)$. Thus $|u(t, c) - \bar{u}(t, c)|, |T(x, t, c) - \bar{T}(x, t, c)|$ are smaller than a constant multiple of ε. The conclusions of Theorem 3 now follow by the triangle inequality. For example,

$$\begin{aligned} |u(t, c) - W(t)| &\leqq |u(t, c) - \bar{u}(t, c)| + |\bar{u}(t, c) - \bar{W}(t)| \\ &\quad + |\bar{W}(t) - W(t)| \leqq M_1\varepsilon + M_2\varepsilon + M_3\varepsilon \end{aligned}$$

uniformly for $0 \leqq t \leqq T$, where $M_1 = 3K$ (from (2.1), Theorem 2), $M_2 = 1$ (from 4.27), Lemma 3), $M_3 = 3K$ (from (2.1), Theorem D). The $T(x, t, c)$ and $R(x, t)$ estimates are done in a similar way. This completes the proof.

Theorem 3 can be extended to the nonlinear case as well. Here, the systems whose solutions are compared are (4.1) and (4.9) with $\sigma(W(t))$ replacing $W(t)$, $\sigma(u(t, c))$ replacing $u(t, c)$ in each heat equation. The proof of existence and asymptotic decay to 0 of solutions in the nonlinear infinite interval case was done in [6] with $\sigma \in C(-\infty, \infty)$ as the only smoothness requirement on σ. The additional assumption that $\sigma \in C'(-\infty, \infty)$ or that σ satisfies a Lipschitz condition permits the proof of Theorem D in this case. Then Theorem 3 can be established in the nonlinear case by modifying the proofs of Lemmas 1–5 above in the manner described in § 2 regarding Theorem 1. The full set of hypotheses will not be listed here, but the reader is referred to [6] for the details.

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ASYMPTOTIC BRANCH POINTS AND MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS*

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Abstract. We study the large positive solutions of the nonlinear operator equation $u = A_\lambda u$ in a partially ordered Banach space, where A_λ is a positive, asymptotically linear operator depending on a real parameter λ. We show that large positive solutions exist for λ near a number μ determined by the asymptotic derivatives A_λ; more specific assumptions about the asymptotic behavior of A_λ enable us to ascertain that the large solutions exist only for $\lambda > \mu$ or only for $\lambda < \mu$. The latter result is applied to prove the existence of at least two positive solutions of certain problems with isotone operators for which $A_\lambda 0 > 0$.

1. Introduction. In this paper, we discuss the behavior of the positive fixed points of large norm of a family $\{A(\lambda):\lambda \in J\}$ (where J is an interval of real numbers) of asymptotically linear positive operators $A(\lambda)$ on a partially ordered Banach space $\mathscr{E}$ with a positive cone $\mathscr{K}$. We are interested in the existence of a number μ such that, for each λ sufficiently close to μ, the operator $A(\lambda)$ has a positive fixed point $u(\lambda)$ such that $\|u(\lambda)\| \to \infty$ as $\lambda \to \mu$. We call such a number an *asymptotic branch point*; as is known, under certain conditions on the asymptotic derivatives of the $A(\lambda)$, the number μ is determined by the eigenvalues of these asymptotic derivatives. We are specifically interested in circumstances under which it can be asserted that the large fixed points $u(\lambda)$ exist *only for λ on one side of μ.*

Very roughly, our results are as follows: Suppose that the operators $A(\lambda) \equiv A_\lambda$ have the following behavior on positive vectors of large norm:

$$A_\lambda u = A'_\lambda(\infty)u + C_\lambda u + o(\|u\|^s) \quad \text{as } \|u\| \to \infty, \quad u > 0,$$

where $A'_\lambda(\infty)$ is a positive, compact operator which is differentiable with respect to λ at $\lambda = \mu$, and C_λ is homogeneous of degree $s < 1$. Then μ is an asymptotic branch point (if and) only if $A'_\mu(\infty)$ has an eigenvalue 1 corresponding to a positive eigenvector ϕ. (We shall prove the "if" assertion only when $C_\lambda u = b_\lambda$; i.e., $s = 0$.) Moreover, the large fixed points exist only for $\lambda > \mu$ if $C_\mu\phi < 0$ and only for $\lambda < \mu$ if $C_\mu\phi > 0$, if $A'_\lambda(\infty)$ increases with λ.

These results will be applied to prove the existence of at least two fixed points in certain cases when $\{A(\lambda):\lambda \in J\}$ is an increasing family of forced, isotone operators. The method used is the following: Suppose there is an increasing family $\{u^0(\lambda):\lambda \in \Lambda\}$ of positive fixed points in an interval Λ and that there is an asymptotic branch point $\mu \in \Lambda$. If there exist fixed points $u(\lambda)$ for $\lambda > \mu$ such that $\|u(\lambda)\| \to \infty$ as $\lambda \to \mu$, then clearly $u(\lambda) \neq u^0(\lambda)$, since $\{u^0(\lambda):\lambda \in \Lambda\}$ is an increasing family.

In §2, we introduce the notation and definitions. Our results for the special case $A_\lambda u = g + \lambda A u$ are presented in §3, mostly without proofs. The theorems and proofs for the general case are given in §4. We conclude in §5 with the applica-

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tion to the existence of multiple solutions. In [9], we will apply these results to nonlinear integral equations.

Related work on asymptotic branching and multiple solutions can be found in Krasnosel'skii [6, Chap. 5], [7, §IV.3], Amann [1], [2], Bazley and McLeod [3] and the author [10], [12].

2. Notation. Let $\mathscr{E}$ be a partially ordered Banach space with a closed positive cone $\mathscr{K} = \{x \in \mathscr{E}: x \geqq 0\}$. We assume that $\mathscr{K}$ is normal [6, p. 20]; we can then, without loss of generality, assume that the norm on $\mathscr{E}$ is such that $0 \leqq u \leqq v$ implies $\|u\| \leqq \|v\|$.

We define $\mathscr{B}^r = \{u \in \mathscr{E}: \|u\| \leqq r\}$ and $\mathscr{K}^r = \mathscr{K} \cap \mathscr{B}^r$.

Let $\mathscr{D}$ be a subset of $\mathscr{E}$, with $\mathscr{D} \cap \mathscr{K} \neq \emptyset$. An operator $A:\mathscr{D} \to \mathscr{E}$ is *positive on* $\mathscr{D}$ if $A(\mathscr{D} \cap \mathscr{K}) \subseteq \mathscr{K}$; *isotone on* $\mathscr{D}$ if $u, v \in \mathscr{D}$ and $u \leqq v$ imply $Au \leqq Av$; *forced* if $0 \in \mathscr{D}$ and $A0 > 0$; $\mathscr{K}$*-compact on* $\mathscr{D}$ if, whenever $\mathscr{D}_1$ is a set whose closure is a bounded subset of $\mathscr{D} \cap \mathscr{K}$, then $A(\mathscr{D}_1)$ has compact closure; $\mathscr{K}$*-bounded on* $\mathscr{D}$ if $A(\mathscr{D}_1)$ is bounded for any $\mathscr{D}_1$ as above.

The $\mathscr{K}$*-spectral radius* of a $\mathscr{K}$-bounded positive linear operator T on $\mathscr{E}$ is defined as $\lim_{n\to\infty} \|T^n\|_{\mathscr{K}}^{1/n}$, where $\|T\|_{\mathscr{K}} = \sup\{\|Tu\|: u \in \mathscr{K}, \|u\| \leqq 1\}$; for such T, we define $\mu_0[T] = \|T\|_{\mathscr{K}}^{-1}$, so that $0 < \mu_0[T] \leqq +\infty$. According to a generalization of the Krein–Rutman theorem due to Bonsall [4], if $\mu_0(T) < +\infty$, then a $\mathscr{K}$-bounded $\mathscr{K}$-compact linear operator T has a positive eigenvector ϕ corresponding to the characteristic value $\mu_0(T)$:

$$0 \neq \mu_0(T)T\phi = \phi \in \mathscr{K}.$$

(A characteristic value of a linear operator T is a number μ such that there exists $h \in \mathscr{E}$ with $\mu Th = h \neq 0$.)

If A is an operator defined on a set $\mathscr{D} \subseteq \mathscr{K}$ containing elements of arbitrarily large norm, then A is $\mathscr{K}$*-asymptotically linear* [6] if there exists a $\mathscr{K}$-bounded linear operator $A'(\infty):\mathscr{E} \to \mathscr{E}$ such that

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathscr{D}}} \frac{\|Au - A'(\infty)u\|}{\|u\|} = 0;$$

$A'(\infty)$ is the $\mathscr{K}$*-asymptotic derivative* of A. If A is positive, then $A'(\infty)$ is a positive operator [6, p. 109].

3. The results for a special case. Let $g \in \mathscr{K}$, and let A be a positive operator on $\mathscr{K}$. In this section we consider the equation

$$u = g + \lambda Au, \tag{3.1}$$

where λ is a positive real parameter. We denote by Λ the set of positive numbers λ for which (3.1) has a positive solution $u \in \mathscr{K}$, and set $\lambda^* = \sup(\Lambda)$.

Most of the results of this section can be obtained as corollaries of the more complicated theorems of §4, where we state and prove the results for the general case of a family $\{A_\lambda: \lambda \in J\}$.

The standard result on the value of an asymptotic branch point is the following (cf. [6, p. 159], [7, p. 207]).

THEOREM 3.1. *Let A be a continuous operator on $\mathscr{K}$ with a $\mathscr{K}$-compact $\mathscr{K}$-asymptotic derivative $A'(\infty)$. Suppose that there is a convergent sequence $\{\lambda_n\}$ in Λ corresponding to positive solutions $\{u_n\}$ of* (3.1) *such that* $\lim_{n\to\infty} \|u_n\| = \infty$. *Then* $\lim_{n\to\infty} \lambda_n$ *is a characteristic value of $A'(\infty)$ corresponding to a positive eigenvector ϕ; there exists a subsequence of $\{u_n/\|u_n\|\}$ converging to ϕ, and every convergent subsequence of $\{u_n/\|u_n\|\}$ converges to such a positive eigenvector.*

The theorem asserts, in particular, that under the stated conditions, we must have $\lim_{n\to\infty} \lambda_n > 0$. Furthermore, $A'(\infty)$ has a positive spectral radius—i.e., $\mu_0[A'(\infty)] < +\infty$—and therefore $A'(\infty)$ has a positive eigenvector corresponding to the characteristic value $\mu_0[A'(\infty)]$. Thus, if every positive eigenvector of $A'(\infty)$ corresponds to the same characteristic value, then we have $\lim_{n\to\infty} \lambda_n = \mu_0[A'(\infty)]$.

Under certain circumstances, it is possible to say that the points of the sequence $\{\lambda_n\}$ in Theorem 3.1 must all lie on one side of $\mu \equiv \lim_{n\to\infty} \lambda_n$ when the corresponding solutions $\{u_n\}$ have sufficiently large norm, and this information can be helpful in predicting the number of positive solutions of (3.1).

THEOREM 3.2. *Let A be a positive, continuous, $\mathscr{K}$-asymptotically linear operator on $\mathscr{K}$ which has the form*

$$Au = A'(\infty)u + Cu + \omega u$$

for $u > 0$, where $A'(\infty)$ is a $\mathscr{K}$-compact, bounded linear operator which has a unique positive eigenvector ϕ of norm 1, *the corresponding characteristic value is simple, and C is a continuous operator on $\mathscr{K}\backslash\{0\}$ which is homogeneous of degree $s < 1$.* (*If $g = 0$ in* (3.1), *then we assume that C is not the zero operator; if $g > 0$, then we allow C to be zero, in which case the following conditions hold for some $s < 0$.*) *The operator ω satisfies:*

(a) *for some positive number ρ_0, the set $\{\|\omega u\|/\|u\|^s : u \in \mathscr{K}, \|u\| \geqq \rho_0\}$ is bounded, and*

(b) *for all positive numbers $\rho \geqq \rho_0$,*

$$\lim_{\substack{\beta\to+\infty \\ \beta\phi+\beta^s h>0}} \beta^{-s}\|\omega(\beta\phi + \beta^s h)\| = 0 \tag{3.2}$$

uniformly on the set

$$\{h : \|h\| \leqq \rho, \beta\phi + \beta^s h > 0 \text{ for all sufficiently large } \beta\}.$$

Let $\mu(\infty) = \mu_0[A'(\infty)]$, and let ξ be a positive eigenvector of the adjoint of $A'(\infty)$ corresponding to the characteristic value $\mu(\infty)$, normalized so that $\xi(\phi) = 1$. Let $\{\lambda_n\}$ be a bounded sequence in Λ corresponding to solutions $\{u_n\}$ of (3.1) *with* $\lim_{n\to\infty} \|u_n\| = \infty$.

Then,

$$\lim_{n\to\infty} \lambda_n = \mu(\infty),$$

$$\lim_{n\to\infty} (\mu(\infty) - \lambda_n)\|u_n\|^{1-s} = \begin{cases} \mu(\infty)^2\xi(C\phi) & \text{if } \; sg > 0, \\ \mu(\infty)^2\xi(C\phi + \mu(\infty)^{-1}g) & \text{if } \; sg = 0, \\ \mu(\infty)\xi(g) & \text{if } \; sg < 0 \end{cases} \tag{3.3}$$

$$\equiv \gamma_\infty$$

and

$$u_n = \left[\frac{\gamma_\infty}{\mu(\infty) - \lambda_n}\right]^{s'} \phi + o(|\mu(\infty) - \lambda_n|^{-s'}) \tag{3.4}$$

as $n \to \infty$, *where* $s' = (1 - s)^{-1}$.

COROLLARY 3.3. *Let the conditions of Theorem* 3.2 *be satisfied. Then any of the following conditions imply that, for any number* $\mu > \mu(\infty)$, *there exists a positive number* r *such that* (3.1) *has no positive solution* u *with* $\|u\| > r$ *for* $\lambda \in [\mu(\infty), \mu]$, *and for any number* $\delta > 0$ *there exists* $\rho > 0$ *such that* (3.1) *has a positive solution* u *with* $\|u\| > \rho$ *for some* $\lambda \in [\mu(\infty) - \delta, \mu(\infty))$:

(i) $0 < s < 1$ *and* $\xi(C\phi) > 0$;
(ii) $s = 0$ *and* $\xi(C\phi + \mu(\infty)^{-1}g) > 0$;
(iii) $s < 0, g = 0$ *and* $\xi(C\phi) > 0$;
(iv) $s < 0, g > 0$ *and* $\xi(g) > 0$.

On the other hand, if any of the conditions (i)–(iii) *holds with the last inequality in that condition reversed, then there exists a positive number* r *such that* (3.1) *has no positive solution* u *with* $\|u\| > r$ *for* $\lambda \in [0, \mu(\infty)]$, *and for any number* $\delta > 0$ *there exists a positive number* ρ *such that* (3.1) *has a positive solution* u *with* $\|u\| > \rho$ *for some* $\lambda \in (\mu(\infty), \mu(\infty) + \delta)$.

The conditions (a) and (b) on ω of Theorem 3.2 are implied by the following seemingly more natural condition on the remainder ωu:

$$\lim_{\|u\| \to \infty} \|u\|^{-s}\|\omega u\| = 0; \tag{3.5}$$

however, for the integral equations considered in [9] there are cases, particularly when $s = 0$, when (3.5) is not satisfied, while (a) and (b) are. The results of Corollary 3.3 for the case $g = 0$ are illustrated in Fig. 1.

There is a partial converse to Theorem 3.1 and Corollary 3.3 when $Cu = b$ is a constant: Under appropriate conditions on the remainder ω, if $A'(\infty)$ has a simple characteristic value μ to which there corresponds a positive eigenvector, then for each λ sufficiently close to μ with

$$0 \neq \operatorname{sgn}(\mu - \lambda) = \operatorname{sgn}(\mu b + g),$$

equation (3.1) has a positive fixed point u, and for any sequence $\{\lambda_n\}$ of such numbers λ, these corresponding solutions u_n satisfy

$$\|u_n\| \sim \frac{\mu\xi(\mu b + g)}{\mu - \lambda_n} \quad \text{as } n \to \infty.$$

For details, see Theorem 4.4 below and [7, pp. 207–208].

4. Behavior of fixed points of large norm. Our first theorem gives a necessary condition for the existence of fixed points $u(\lambda)$ such that $\|u(\lambda)\| \to \infty$ as λ approaches a number μ. The proof is straightforward.

THEOREM 4.1 *Let* $\{A_\lambda : \lambda \in J\}$ *be a family of continuous operators on* $\mathscr{K}$. *Suppose that there is a sequence* $\{\lambda_n\}$ *in* J *converging to* $\mu \in J$ *such that the operators* A_{λ_n} *have fixed points* u_n *in* $\mathscr{K}$ *with* $\lim_{n\to\infty} \|u_n\| = \infty$, *and* A_μ *has a* $\mathscr{K}$*-compact* $\mathscr{K}$*-asymptotic derivative* $A'_\mu(\infty)$ *such that*

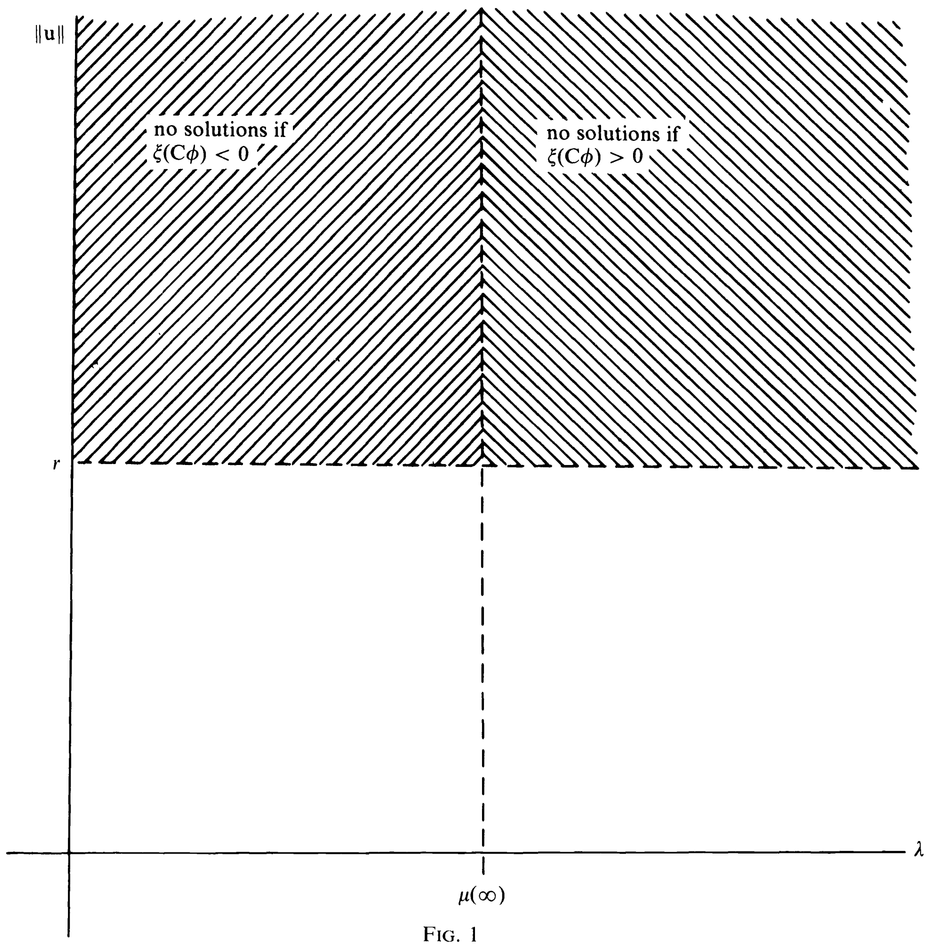

FIG. 1

$$\lim_{\substack{\|u\|\to\infty \\ u\in\mathscr{K} \\ \lambda\to\mu}} \frac{A_\lambda u - A'_\mu(\infty)u}{\|u\|} = 0. \tag{4.1}$$

Then 1 *is an eigenvalue of* $A'_\mu(\infty)$ *corresponding to a positive eigenvector* ϕ *of unit norm. There exists a subsequence of* $\{u_n/\|u_n\|\}$ *converging to* ϕ, *and every convergent subsequence of* $\{u_n/\|u_n\|\}$ *converges to such an eigenvector.*

The next theorem describes in more detail the behavior of a sequence $\{u_n\}$ of fixed points with $\|u_n\| \to \infty$.

We shall use the following definition. Let ϕ be a positive eigenvector of a bounded linear operator T corresponding to a characteristic value μ. We say that $(\phi\xi, P)$ *completely reduces* T if ξ is a bounded, positive, linear functional on $\mathscr{E}$ with $\xi(\phi) = 1$, and P is a bounded projection on $\mathscr{E}$ such that every $u \in \mathscr{E}$ may be written in the form

$$u = \xi(u)\phi + Pu,$$

where $TP = PT$, and $(I - \mu T)$ restricted to $P(\mathscr{E})$ has a bounded inverse on $P(\mathscr{E})$.

It follows that $P\phi = 0$ and $\xi(Pu) = 0$ for all $u \in \mathscr{E}$, and the functional ξ is a positive eigenvector of the adjoint of T corresponding to the characteristic value μ.

It is well known that if T is compact on $\mathscr{E}$ and μ is a simple characteristic value, then there exist (ξ, P) such that $(\phi\xi, P)$ completely reduces T. More generally, there exist (ξ, P) such that $(\phi\xi, P)$ completely reduces T if and only if μ is an isolated, simple characteristic value of T (cf. [5, §III–6]). (We call a characteristic value simple if the corresponding spectral projection has a one-dimensional range.)

If, in the preceding definition, we replace "bounded" by "$\mathscr{K}$-bounded" everywhere, and require that $u = \xi(u)\phi + Pu$ hold for all $u \in \mathscr{K}$, then we say that $(\phi\xi, P)$ *$\mathscr{K}$-completely reduces* T. We do not know whether there exists a $\mathscr{K}$-complete reduction $(\phi\xi, P)$ of a $\mathscr{K}$-compact linear operator T whenever T has a characteristic value μ corresponding to a positive eigenvector ϕ.

THEOREM 4.2. *Let $\{A_\lambda : \lambda \in J\}$ be a family of positive continuous operators on $\mathscr{K}$. Suppose there is a $\mu \in J$ and a sequence $\{\lambda_n\}$ in J converging to μ such that the operators A_{λ_n} have fixed points u_n in $\mathscr{K}$ with $\lim_{n\to\infty} \|u_n\| = \infty$. Let the operators A_λ have the form*

$$A_\lambda u = A'_\mu u + (\lambda - \mu)B_\mu u + C_\lambda u + D_\lambda u + \omega_\lambda u, \qquad u > 0, \tag{4.2}$$

where the $\mathscr{K}$-asymptotic derivative $A'_\mu = A'_\mu(\infty)$ of A_μ is $\mathscr{K}$-bounded, $\mathscr{K}$-compact, and has a unique positive eigenvector ϕ of unit norm. Also, the characteristic value $\mu_0[A'_\mu]$ is simple, B_μ is a continuous linear operator, $\{C_\lambda\}$ is a family of operators on $\mathscr{K}\backslash\{0\}$ homogeneous of degree s (i.e., for $h > 0$, $\lambda \in J$, $\alpha > 0$, $C_\lambda(\alpha h) = \alpha^s C_\lambda h$) for some number $s < 1$, the mapping $(\lambda, u) \to C_\lambda u$ of $J \times \mathscr{K}$ into $\mathscr{K}$ is continuous. In addition, the family $\{D_\lambda\}$ of continuous linear operators satisfies

$$\lim_{\lambda\to\mu} \frac{\|D_\lambda\|}{\lambda - \mu} = 0, \tag{4.3}$$

and the family $\{\omega_\lambda\}$ satisfies: For all λ in a neighborhood N (relative to J) of μ in J,

(a) *the set $\{\|\omega_\lambda u\|/\|u\|^s : u > 0, \lambda \in N\}$ is bounded, and*

(b) *for any positive number r, the limit*

$$\lim_{\substack{\beta\to+\infty \\ \beta\phi+\beta^s h > 0}} \beta^{-s}\|\omega_\lambda(\beta\phi + \beta^s h)\| = 0 \tag{4.4}$$

exists uniformly for all $\lambda \in N$ and all h such that $\beta\phi + \beta^s h > 0$ for all sufficiently large $\beta > 0$ and $\|h\| \leqq r$.

Let ξ be a positive linear functional and P be a projection such that $(\phi\xi, P)$ $\mathscr{K}$-completely reduces $A'_\mu(\infty)$; suppose $\xi(B_\mu\phi) \neq 0$. Let $s' = (1 - s)^{-1}$.

Then $1 = \mu_0[A'_\mu]$,

$$\lim_{n\to\infty} (\lambda_n - \mu)\|u_n\|^{1-s} = -\frac{\xi(C_\mu\phi)}{\xi(B_\mu\phi)}, \tag{4.5}$$

$$\begin{aligned} \lim_{n\to\infty} |\lambda_n - \mu|^{s'} u_n &= \lim_{n\to\infty} [|\lambda_n - \mu|^{s'}\xi(u_n)]\phi \\ &= \left|\frac{\xi(C_\mu\phi)}{\xi(B_\mu\phi)}\right|^{s'}\phi \end{aligned} \tag{4.6}$$

and, with $R = [1 - A'_\mu]^{-1}_{P\mathscr{E}} P$,

$$\lim_{n\to\infty} \frac{Pu_n}{\|u_n\|^s} = R\left[-\frac{\xi(C_\mu\phi)}{\xi(B_\mu\phi)} B_\mu\phi + C_\mu\phi\right]. \tag{4.7}$$

Note that for λ near μ, equation (4.2) implies that

$$A'_\lambda(\infty) = A'_\mu(\infty) + (\lambda - \mu)B_\mu + D_\lambda,$$

so that B_μ is the derivative with respect to λ of $A'_\lambda(\infty)$ at $\lambda = \mu$.

COROLLARY 4.3. *Let* $\{A_\lambda : \lambda \in J\}$ *and* $\mu \in J$ *satisfy the conditions of Theorem* 4.2. *If* $\xi(B_\mu\phi)\xi(C_\mu\phi) \neq 0$, *then there are positive numbers r and* δ *such that for all* λ *with*

$$0 \leqq (\lambda - \mu)\operatorname{sgn}[\xi(B_\mu\phi)\xi(C_\mu\phi)] \leqq \delta,$$

the operator A_λ *has no fixed points in* $\mathscr{K}$ *with norm greater than r.*

Proof. Since equation (4.1) of Theorem 4.1 is satisfied by the operators A_λ and $A'_\mu = A'_\mu(\infty)$, 1 is an eigenvalue of A'_μ corresponding to the positive eigenvector $\phi = \lim_{n\to\infty} u_n/\|u_n\|$. From (4.2) we obtain

$$0 = (\lambda_n - \mu)\xi(B_\mu u_n) + \xi(C_{\lambda_n}u_n) + \xi(D_{\lambda_n}u_n) + \xi(\omega_{\lambda_n}u_n).$$

Dividing by $(\lambda_n - \mu)\|u_n\|$ and letting $n \to \infty$, we have

$$0 = \xi(B_\mu\phi) + \lim_{n\to\infty} \frac{1}{(\lambda_n - \mu)\|u_n\|^{1-s}}\left[\xi\left(C_{\lambda_n}\frac{u_n}{\|u_n\|}\right) + \frac{1}{\|u_n\|^s}\xi(\omega_{\lambda_n}u_n)\right].$$

If $\xi(B_\mu\phi) \neq 0$, it follows from hypothesis (a) that $\limsup_{n\to\infty} |\lambda_n - \mu|\,\|u_n\|^{1-s} < \infty$. Similarly, if we divide

$$Pu_n = R\{(\lambda_n - \mu)B_\mu u_n + C_{\lambda_n}u_n + D_{\lambda_n}u_n + \omega_{\lambda_n}u_n\}$$

by $\|u_n\|^s$, we obtain

$$\frac{Pu_n}{\|u_n\|^s} = R\left\{(\lambda_n - \mu)\|u_n\|^{1-s}\frac{B_\mu u_n}{\|u_n\|} + C_{\lambda_n}\left(\frac{u_n}{\|u_n\|}\right)\right.$$
$$\left. + \frac{1}{\lambda_n - \mu} D_{\lambda_n}\left(\frac{u_n}{\|u_n\|}\right)(\lambda_n - \mu)\|u_n\|^{1-s} + \frac{\omega_{\lambda_n}u_n}{\|u_n\|^s}\right\},$$

so

$$\limsup_{n\to\infty} \frac{\|Pu_n\|}{\|u_n\|^s} < \infty.$$

Since $\lim_{n\to\infty} \xi(u_n)/\|u_n\| = 1$, we also have

$$\limsup_{n\to\infty} \frac{\|Pu_n\|}{\xi(u_n)^s} < \infty.$$

Let $\beta_n = \xi(u_n)$, $h_n = Pu_n/\beta_n^s$. Then there is a positive number r such that $\|h_n\| \leqq r$ for all n. In view of the uniformity of the limit in (4.4), we have

$$\lim_{n\to\infty} \|u_n\|^{-s}\omega_{\lambda_n}u_n = \lim_{n\to\infty} \beta_n^{-s}\omega_{\lambda_n}(\beta_n\phi + \beta_n^s h_n) = 0.$$

Equations (4.5), (4.6) and (4.7), and Corollary 4.3 follow from the equations above.

We next show that if $C_\lambda u$ is a constant b_λ for $u \in \mathscr{K}$ and if the operators ω_λ satisfy the asymptotic condition (4.4) with $s = 0$, then we can establish a converse of the preceding results for operators of the form of equation (4.4): If A'_μ has the simple characteristic value 1 to which there corresponds a positive eigenvector, then the operators A_λ have fixed points of arbitrarily large norm for λ near μ.

THEOREM 4.4. *Let $\{A_\lambda : \lambda \in J\}$ be a family of continuous operators on a cone $\mathscr{K}_1$ containing $\mathscr{K}$, with $A_\lambda \mathscr{K}_1 \subseteq \mathscr{K}$ for $\lambda \in J$, and let the operators A_λ have the form of* (4.2) *for some $\mu \in J$, where A'_μ and B_μ are continuous linear operators, $C_\lambda u = b_\lambda \in \mathscr{E}$ for all u and all $\lambda \in J$, the mapping $\lambda \to b_\lambda$ of J into $\mathscr{E}$ is continuous, D_λ is a continuous linear operator which satisfies* (4.3), *and the operators ω_λ satisfy conditions to be specified later. Let A'_μ have the simple eigenvalue* 1 *corresponding to the positive eigenvector ϕ interior to $\mathscr{K}_1$; let (ξ, P) be such that $(\phi\xi, P)$ completely reduces A'_μ.*

Let ω_λ satisfy the following conditions for any sufficiently small number $r > 0$ and for λ in an open subset N (relative to J) of J containing μ:

$$\lim_{\substack{\beta \to +\infty \\ \beta\phi + h \in \mathscr{K}_1}} \|\omega_\lambda(\beta\phi + h)\| = 0 \tag{4.8}$$

uniformly for $h \in \mathscr{B}^r = \{u \in \mathscr{E} : \|u\| < r\}$ and $\lambda \in N$; and

$$\begin{aligned} &\|\omega_\lambda(\beta_1\phi + h_1) - \omega_\lambda(\beta_2\phi + h_2)\| \\ &\qquad \leqq q_\omega(\beta_1, \beta_2; h_1, h_2; \lambda)\|(\beta_1\phi + h_1) - (\beta_2\phi + h_2)\|, \end{aligned} \tag{4.9}$$

where $q_\omega(\beta_1, \beta_2; h_1, h_2; \lambda)$ is a real-valued positive function of the numbers β_1, β_2, the vectors h_1, h_2, and the number λ, such that

$$\lim_{\substack{\beta_2 \to +\infty \\ \beta_1 > \beta_2}} q_\omega(\beta_1, \beta_2 : h_1, h_2 : \lambda) = 0 \tag{4.10}$$

uniformly for $h_1, h_2 \in \mathscr{B}^r$ and $\lambda \in N$.

Then there exists a number $\delta > 0$ such that for each $\lambda \in J$ with

$$0 < (\mu - \lambda)\,\mathrm{sgn}\,[\xi(b_\mu)\xi(B_\mu\phi)] < \infty, \tag{4.11}$$

A_λ has a fixed point $u(\lambda) \in \mathscr{K}$, and for any sequence $\{\lambda_n\}$, the elements of which satisfy relation (4.11), *the corresponding fixed points $u(\lambda_n) \equiv u_n$ of A_{λ_n} satisfy* (4.5), (4.6) *and* (4.7), *with $s = 0$.*

Proof. The proof consists of a standard Lyapunov–Schmidt technique. We seek a solution u of the equation $A_\lambda u - u = 0$ in the form $u = y + \beta\phi$ by first attempting to solve

$$P(A_\lambda u - u) = P[A_\lambda(y + \beta\phi) - (y + \beta\phi)] = 0 \tag{4.12}$$

for y in terms of β, and then choosing β so that the equation

$$\xi(A_\lambda u - u) = \xi[A_\lambda(y + \beta\phi) - (y + \beta\phi)] = 0 \tag{4.13}$$

is satisfied. Then $u = y + \beta\phi$ will satisfy $A_\lambda u - u = 0$.

Equation (4.12) is equivalent to

$$z = T(\alpha, \lambda)z, \tag{4.14}$$

where

$$\alpha = (\lambda - \mu)\beta, \qquad z = Pu - k_{\alpha,\lambda},$$
$$k_{\alpha,\lambda} = \alpha(\lambda - \mu)^{-1}R[(\lambda - \mu)B_\mu + D_\lambda]\phi + Rb_\lambda,$$
$$R = \{[I - A'_\mu]_{P\mathscr{E}}\}^{-1}P$$

and

$$(4.15) \qquad \begin{aligned} T(\alpha, \lambda)z = R[(\lambda - \mu)B_\mu(z + k_{\alpha,\lambda}) + D_\lambda(z + k_{\alpha,\lambda}) \\ + \omega_\lambda(z + k_{\alpha,\lambda} + \alpha(\lambda - \mu)^{-1}\phi)]. \end{aligned}$$

For any positive number θ and sufficiently small positive η, it is possible to find a neighborhood N_1 of μ in J such that for $|\alpha| \leqq \theta$, $\lambda \in N_1$, and $\alpha(\lambda - \mu) > 0$, $T(\alpha, \lambda)$ is a contraction mapping of $\overline{\mathscr{B}}^\eta$ into itself. Thus, (4.14) has a solution $z_{\alpha,\lambda}$, which depends continuously on α for each $\lambda \in N_1$.

Setting $y_{\alpha,\lambda} = z_{\alpha,\lambda} + k_{\alpha,\lambda}$ and substituting for y in (4.13), we find that for N_1 sufficiently small, $|\alpha| \leqq \theta$, $\beta = \alpha(\lambda - \mu)^{-1} > 0$, equation (4.13) has a solution $\alpha(\lambda)$ for $\lambda \in N$ with $|\alpha(\lambda)| \leqq \theta$ and

$$\operatorname{sgn} \alpha(\lambda) = \operatorname{sgn} (\lambda - \mu) = -\operatorname{sgn} [\xi(b_\mu)\xi(B_\mu\phi)],$$

for an appropriate choice of θ. Thus, there is a positive number δ such that for any λ satisfying inequality (4.11), the operator A_λ has a fixed point

$$u(\lambda) = y_{\alpha(\lambda),\lambda} + \frac{\alpha(\lambda)}{\lambda - \mu}\phi.$$

From (4.8), (4.9), (4.10) and (4.3), the solution $\alpha(\lambda)$ of (4.13) may be taken arbitrarily close to $-\xi(b_\mu)/\xi(B_\mu\phi)$ for λ sufficiently close to μ. Then

$$\lim_{\lambda\to\mu} \xi[u(\lambda)](\lambda - \mu) = \lim_{\lambda\to\mu} \alpha(\lambda) = -\xi(b_\mu)/\xi(B_\mu\phi).$$

Similarly, (4.7), and thence (4.5) and (4.6), may be obtained from (4.13). Let $\lambda_n \to \mu$.

Since ϕ is interior to $\mathscr{K}_1$ and $Pu_n/\xi(u_n) \to 0$ as $n \to +\infty$, $u_n/\xi(u_n) = \phi + Pu_n/\xi(u_n)$ is in $\mathscr{K}_1$ for all sufficiently large n; since $\mathscr{K}_1$ is a cone, $u_n \in \mathscr{K}_1$ for all large n. Thus $u_n = A_{\lambda_n}u_n \in \mathscr{K}$ for all large n. This completes the proof.

5. Multiple positive solutions. The preceding theorems will now be applied to obtain results on the existence of more than one positive solution of equation (3.1) or the more general equation $u = A_\lambda u$.

Our results on multiple positive solutions are based on the following result concerning the existence of minimum positive solutions of $u = A_\lambda u$ [11].

LEMMA 5.1. *Let J be an interval in* $[0, +\infty)$ *with* $\lambda_- = \inf(J)$, *and let* $\mathfrak{A}$ *be an isotone, compact operator on* $J \times \mathscr{K}$ *such that the operators* A_λ $(A_\lambda u = \mathfrak{A}(\lambda, u))$ *are forced. Suppose the set* Λ *is nonempty. Then* Λ *is an interval with* $\inf(\Lambda) = \lambda_-$; *for every* $\lambda \in \Lambda$, *there is a minimum positive solution* $u^0(\lambda)$. *The map* $\lambda \to u^0(\lambda)$ *is a nondecreasing, left-continuous function on* Λ, *and if we set* $\lambda^* = \sup(\Lambda)$, *then either* $\lambda^* = \sup(J)$ *or exactly one of the following conditions holds*:

(i) $\lim_{\lambda\to\lambda^*} \|u^0(\lambda)\| = +\infty$;

(ii) $\lambda^* \in \Lambda$.

This result is actually valid for a much broader class of operators than compact operators; cf. [11].

The next theorem is an example of the results on multiple solutions which follow from the preceding theorems.

THEOREM 5.2. *Let $g \in \mathscr{K}$ and let A be a forced, isotone, compact, $\mathscr{K}$-asymptotically linear operator on $\mathscr{E}$ which satisfies the conditions of Theorem 3.2 and $Ag > 0$. Let Λ be bounded. Suppose that either $s > 0$ and $\xi(C\phi) < 0$, or $s = 0$ and $\xi(b + \mu(\infty)^{-1}g) < 0$. Then $\mu(\infty) \in \Lambda$ and there exists $\delta > 0$ such that* (3.1) *has at least two positive solutions for each $\lambda \in (\mu(\infty), \mu(\infty) + \delta)$.*

Proof. The assertions follow from Theorem 3.2, Lemma 5.1 and the fact that under the stated conditions, (3.1) has solutions of arbitrarily large norm [6, p. 161], [13], since (3.1) is equivalent to $v = \lambda A(v + g)$, with $v = u - g$.

Remark. If A is Fréchet differentiable, then it can be shown (by using the implicit function theorem) that under the hypotheses of Theorem 5.2, equation (3.1) has at least two positive solutions for all $\lambda \in (\mu(\infty), \lambda^*)$.

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ASYMPTOTIC BRANCH POINTS AND MULTIPLE POSITIVE SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS*

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Abstract. The values of λ for which there exist large positive solutions of a nonlinear Hammerstein integral equation with an eigenvalue parameter λ are determined from the asymptotic behavior of the nonlinearity $f(w)$. With the assumption that the nonlinearity has the asymptotic form $f(w) = mw + cw^s + o(w^s)$ as $w \to +\infty$, with $m > 0$ and $0 \leqq s < 1$, the asymptotic branch point μ for positive solutions is determined by m, and the sign of $\lambda - \mu$ for values of λ corresponding to large solutions is determined by the sign of c: If $c > 0$, then $\lambda < \mu$, and if $c < 0$, then $\lambda > \mu$. The latter condition, $c < 0$, implies that if $f(0) > 0$, then the Hammerstein equation has at least two positive solutions for some values of $\lambda > \mu$. If the nonlinearity $f(w)$ is convex, then the last result stated is sharpened by assuming only that $f(w) - mw$ is negative for large w.

1. Introduction. We consider the behavior of the large solutions of the nonlinear integral equation

$$u(x) = \lambda \int_{\Omega} K(x, y) f(u(y), y)\, dy + g(x), \tag{1.1}$$

where $K(x, y)$ is a positive weakly singular kernel, the nonlinearity $f(w, x)$ is nonnegative, and $g \geqq 0$. Knowing the asymptotic behavior of $f(w, x)$ as $w \to +\infty$, we are able to predict the values of λ for which large solutions do or do not exist. When $f(0, x) > 0$, and $f(w, x)$ is nondecreasing in w, these results give sufficient conditions for the existence of at least two positive solutions of (1.1) for certain values of λ. Our results are based on the results for abstract operator equations obtained in [10].

If, in addition, $f(w, x)$ is convex in w, we are able to state conditions which are "almost" necessary and sufficient for the existence of multiple solutions of (1.1); in particular, when $f(w, x)$ is independent of x, our conditions are precisely necessary and sufficient.

In §2, we give some preliminary lemmas which enable the results of [10] to be applied to (1.1). The results for the nonconvex case are given in §3; here it is indicated how the asymptotic behavior of $f(w, x)$ determines the values of λ for which (1.1) has large solutions, and we give the asymptotic form of these solutions. We also indicate how, under certain circumstances, these results imply the existence of multiple positive solutions of (1.1). Section 4 contains the stronger result on multiple solutions for the convex case. Finally, the Appendices contain a summary of the useful properties of linear integral equations with weakly singular, positive kernels and a general lemma used in the proof of Theorem 4.2.

It is to be expected that our results apply also to the nonlinear elliptic boundary value problems (cf. [6], [15])

$$\begin{aligned} Lu(x) &= f(u(x), x), \quad && x \in \Omega, \\ Bu(x) &= g(x), && x \in \partial\Omega, \end{aligned}$$

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for a sufficiently smooth domain Ω, uniformly elliptic operator L, and boundary operator B.

The existence of multiple positive solutions of such equations with L and B self-adjoint has been discussed by Keener and Keller [5], where a result similar to, but weaker than, our Theorems 3.7 and 4.2 is proved in a different way.

2. Lemmas. The following elementary proposition, whose proof is omitted, lists some relations between the asymptotic properties of $f(w, x)$ which are useful in applying our results.

PROPOSITION 2.1. *Let Ω be a subset of R^n. Let $f(w, x)$ have a continuous partial derivative $D_1 f(w, x)$ with respect to w for each pair $(w, x) \in (\rho, +\infty) \times \Omega$, where $\rho \geqq 0$. If $\lim_{w \to +\infty} D_1 f(w, x) = m(x) \leqq +\infty$ exists (uniformly for $x \in \Omega$), then $\lim_{w \to +\infty} f(w, x)/w$ exists (uniformly for $x \in \Omega$) and equals $m(x)$. If, for some number $s \in [0, 1)$, $\lim_{w \to +\infty} w^{-s}[f(w, x) - wD_1 f(w, x)] \equiv (1 - s)b(x)$ exists (uniformly for $x \in \Omega$) and is finite, then $\lim_{w \to +\infty} D_1 f(w, x) \equiv m(x)$ exists (uniformly for $x \in \Omega$) and is finite, and $\lim_{w \to +\infty} w^{-s}[f(w, x) - m(x)w] = b(x)$ exists (uniformly for $x \in \Omega$) and is finite.*

The functions

$$f_1(w) = w + \tfrac{1}{2} \sin(w),$$

$$f_2(w) = w^2 + w[1 + \sin(2w)] + \tfrac{1}{2} \cos(2w),$$

$$f_3(w) = w + \frac{1}{2} w^s \int_0^w \frac{\sin(v)}{v} \, dv, \qquad 0 \leqq s < 1,$$

are continuously differentiable increasing functions which show that the converses of the assertions of Proposition 2.1 are not valid: $\lim_{w \to +\infty} f_i'(w)$ does not exist for $i = 1$ and 2, but $\lim_{w \to +\infty} f_1(w)/w = 1$ and $\lim_{w \to +\infty} f_2(w)/w = +\infty$;

$$\lim\nolimits_{w \to +\infty} f_3(w)/w = \lim\nolimits_{w \to +\infty} f_3'(w) = 1 \equiv m,$$

$\lim_{w \to +\infty} w^{-s}[f_3(w) - mw] = \pi/4$, but $\lim_{w \to +\infty} w^{-s}[f_3(w) - wf_3'(w)]$ does not exist.

Throughout the rest of this paper, Ω denotes a bounded open set in n-dimensional space R^n whose boundary $\partial\Omega$ has zero n-dimensional measure, and $\bar{\Omega} = \Omega \cup \partial\Omega$. We denote by $K(x, y)$ a weakly singular kernel (see Appendix A) defined for $(x, y) \in \bar{\Omega} \times \bar{\Omega}$. All integrations are over Ω unless otherwise noted. The Banach space of continuous functions on $\bar{\Omega}$ is denoted by $C(\bar{\Omega})$.

LEMMA 2.2. *Let σ be a nonpositive number, and let f be a continuous function on $[\sigma, +\infty) \times \bar{\Omega}$. Define the operator A on a subset of $C(\bar{\Omega})$ by*

$$Au(x) = \int K(x, y) f(u(y), y) \, dy. \tag{2.1}$$

Suppose that for some number $s \in [0, 1)$,

$$\lim_{w \to +\infty} w^{-s} f(w, x) = 0,$$

uniformly for $x \in \Omega$. Then for any continuous function ϕ which is positive almost everywhere on Ω, and for every positive number ρ, we have

$$\lim_{\substack{\beta\to+\infty\\ \beta\phi+\beta^s h>\sigma}} \beta^{-s}A(\beta\phi+\beta^s h)=0 \tag{2.2}$$

(*in the sense of the usual maximum norm* $\|\cdot\|$ *on* $C(\bar{\Omega})$), *uniformly for h in the set*

$$\{h\in C(\bar{\Omega}):\|h\|\leqq\rho \text{ and there exists } v_h \text{ such that } v_h\phi+v_h^s h>\sigma\}.$$

Proof. Let

$$f_m(r)=\max\{|f(w,x)|:x\in\bar{\Omega},\sigma\leqq w\leqq r\},$$
$$|K(x,y)|\leqq\kappa/|x-y|^\alpha$$

for some $\alpha\in[0,n)$ and $\kappa>0$, and let

$$\gamma=\sup\left\{\int_\Omega|K(x,y)|\,dy:x\in\bar{\Omega}\right\}.$$

Assume, without loss of generality, that $\|\phi\|=1$.

Let ε and r be given positive numbers and choose $r'>0$ such that $|f(\rho,x)|\leqq\rho^s\varepsilon[\gamma(1+r^s)]^{-1}$ for $\rho\geqq r'$ and $x\in\Omega$. Choose $\delta>0$ such that

$$\int_{B(x;\delta)}|K(x,y)|\,dy\leqq\varepsilon[f_m(r')]^{-1}$$

for all $x\in\Omega$, where $B(x;\delta)=\{y\in\Omega:|y-x|\leqq\delta\}$, and choose $\beta_0>0$ such that

$$\operatorname{meas}\left\{x\in\Omega:\phi(x)\leqq\frac{r'+\beta_0^s r}{\beta_0}\right\}\leqq\frac{\varepsilon}{\kappa}\frac{\delta^\alpha}{f_m(r')}.$$

If $\beta\geqq\beta_0$ and $\|h\|\leqq r$, then $\beta\phi(y)+\beta^s h(y)\leqq r'$ implies

$$\phi(y)\leqq\beta^{-1}[r'-\beta^s h(y)]\leqq\beta_0^{-1}r'+\beta_0^{-1+s}r.$$

Thus, setting $u=\beta\phi+\beta^s h$ and

$$\Omega_1=\{y\in\Omega:u(y)\leqq r',|x-y|>\delta\},$$
$$\Omega_2=\{y\in\Omega:u(y)\leqq r',|x-y|\leqq\delta\},$$
$$\Omega_3=\{y\in\Omega:u(y)>r'\},$$

we have

$$\begin{aligned}|A(\beta\phi+\beta^s h)(x)|&\leqq\int_{\Omega_1}+\int_{\Omega_2}+\int_{\Omega_3}|K(x,y)f(u(y),y)|\,dy\\&\leqq\left[\frac{\kappa}{\delta^\alpha}f_m(r')\right]\left[\frac{\varepsilon}{\kappa}\frac{\delta^\alpha}{f_m(r')}\right]\\&\quad+[f_m(r')]\frac{\varepsilon}{f_m(r')}+\frac{\varepsilon}{\gamma(1+r^s)}\|\beta\phi+\beta^s h\|^s\gamma\\&\leqq 2\varepsilon+\varepsilon\beta^s+\varepsilon\beta^{s^2},\end{aligned}$$

since $\|\beta\phi+\beta^s h\|^s\leqq\beta^s+r^s\beta^{s^2}$. Thus (2.2) holds uniformly for $\|h\|\leqq r$ with $\beta\phi+\beta^s h>\sigma$.

LEMMA 2.3. *Let the kernel K, the function, f, and the operator A have the properties described in the first sentence of Lemma 2.2. Suppose that there exists a number $\rho > 0$ such that for each $x \in \bar{\Omega}$, $f(w, x)$ is a continuously differentiable function of w for $w \in (\rho, +\infty)$, the partial derivative $D_1 f(w, x)$ is measurable and bounded on $(\rho, +\infty) \times \bar{\Omega}$, and there exist functions m and b on $\bar{\Omega}$ such that*

$$\lim_{w \to +\infty} D_1 f(w, x) = m(x)$$

and

$$\lim_{w \to +\infty} [f(w, x) - m(x)w] = b(x)$$

uniformly for $x \in \bar{\Omega}$. Define the operator ω by

$$\omega u(x) = \int K(x, y)[f(u(y), y) - m(y)u(y) - b(y)]\, dy.$$

Then for any sufficiently small number $r > 0$ and for every function $\phi \in C(\bar{\Omega})$ such that $\phi(x) > 0$ almost everywhere on $\bar{\Omega}$, we have

$$\lim_{\substack{\beta \to +\infty \\ \beta\phi + h > \sigma}} \|\omega(\beta\phi + h)\| = 0 \tag{2.3}$$

uniformly for $h \in \mathscr{B}^r = \{h \in C(\bar{\Omega}): \|h\| \leqq r\}$, and

$$\|\omega(\beta_1\phi + h_1) - \omega(\beta_2\phi + h_2)\| \leqq q(\beta_1, \beta_2; h_1, h_2)\|\beta_1\phi + h_1 - \beta_2\phi - h_2\| \tag{2.4}$$

if $h_i \in \mathscr{B}^r$, and $\beta_i\phi + h_i > \sigma$ for $i = 1, 2$, where

$$\lim_{\substack{\beta_2 \to +\infty \\ \beta_1 > \beta_2}} q(\beta_1, \beta_2; h_1, h_2) = 0 \tag{2.5}$$

uniformly for $h_1 \in \mathscr{B}^r$ and $h_2 \in \mathscr{B}^r$.

Proof. The proof of the preceding proposition, with $f(w, x)$ replaced by $f(w, x) - m(x)w - b(x)$, shows that ω satisfies (2.3). Condition (2.4)–(2.5) is proved in a similar way: Let

$$M(w_1, w_2; x) = \int_0^1 D_1 f(w_1 + \alpha(w_2 - w_1), x)\, d\alpha.$$

Then for any functions $u_1, u_2 \in C(\bar{\Omega})$ and any $y \in \bar{\Omega}$, we have

$$\begin{aligned} &|f(u_1(y), y) - m(y)u_1(y) - f(u_2(y), y) + m(y)u_2(y)| \\ &\qquad \leqq |M(u_1(y), u_2(y); y) - m(y)||u_1(y) - u_2(y)|. \end{aligned}$$

Using this inequality, we can show, as in the proof of the preceding proposition, that given $\varepsilon > 0$ and a sufficiently small $r > 0$, there exists β_0 such that whenever $\beta_1 \geqq \beta_2 \geqq \beta_0$, $\|h_1\| \leqq r$, and $\|h_2\| \leqq r$, we have

$$\|\omega(\beta_1\phi + h_1) - \omega(\beta_2\phi + h_2)\| \leqq \varepsilon\|(\beta_1\phi + h_1) - (\beta_2\phi + h_2)\|.$$

Thus (2.4) and (2.5) are satisfied.

3. Asymptotic branch points. We now begin our study of the integral equation (1.1). Let Ω be a bounded open set in n-dimensional space, whose boundary $\partial\Omega$ has n-dimensional measure zero, and let K be a weakly singular kernel defined on $\bar{\Omega} \times \bar{\Omega}$. We make the following assumptions on f and g:

(CP1) g is a continuous, never negative function on $\bar{\Omega}$.

(CP2) f is a continuous, never negative function on $(-\infty, +\infty) \times \bar{\Omega}$.

(AL) f is asymptotically linear; i.e., there exists a function m on $\bar{\Omega}$ such that

$$\lim_{w \to +\infty} w^{-1} f(w, x) = m(x) \quad \text{uniformly for } x \in \bar{\Omega}. \tag{3.1}$$

Then, because of (CP2), the operator A defined by (2.1) maps $C(\bar{\Omega})$ into the positive cone $\mathscr{K}$ of never negative functions in $C(\bar{\Omega})$, and A is compact (completely continuous). The operators $u \to g + \lambda Au$, for $\lambda \geqq 0$, are compact, positive operators on $C(\bar{\Omega})$, and all solutions of (1.1) for $\lambda \geqq 0$ are in $\mathscr{K}$.

In addition to the assumptions above, we may impose one or more of the following conditions on f:

(I) For each $x \in \Omega$, $f(w, x)$ is an increasing (i.e., never decreasing) function of w.

(F) For some $x \in \Omega$, $f(g(x), x) > 0$ (the *forced* case).

(UF) For all $x \in \Omega$, $f(g(x), x) = 0$ (the *unforced* case).

(L) There exist positive numbers ρ and M such that for all $x \in \Omega$ and all $w_2 \geqq w_1 \geqq \rho$, we have

$$|f(w_2, x) - f(w_1, x)| \leqq M|w_1 - w_2|.$$

We distinguish the forced and unforced cases because in the latter (1.1) has the trivial solution $u = g$ for all $\lambda > 0$, whereas in the forced case $u = g$ is a solution only for $\lambda = 0$.

In connection with (1.1) and assumption (AL), we also consider the linear eigenvalue problems

$$\phi(x) = \mu \int K(x, y) m(y) \phi(y)\, dy \tag{3.2}$$

and

$$\psi(x) = \mu \int K(y, x) m(y) \psi(y)\, dy. \tag{3.3}$$

These equations each have a unique normalized positive eigenfunction, which we denote by ϕ_∞ and ψ_∞, respectively, corresponding to the same eigenvalue $\mu_1[\infty] > 0$ (Appendix, §§A.3, A.4).

Because of (AL), the linear operator

$$A'(\infty)h(x) = \int K(x, y) m(y) h(y)\, dy \tag{3.4}$$

is the $\mathscr{K}$-asymptotic derivative [10] of the operator A, ϕ_∞ is a positive eigenvector of $A'(\infty)$, and the positive linear functional ξ on $C(\bar{\Omega})$ defined by

$$\xi(u) = \int \psi_\infty(x) m(x) u(x)\, dx \tag{3.5}$$

is a positive eigenvector of the adjoint of $A'(\infty)$.

An *asymptotic branch point* for (1.1) is a number μ for which there exists a sequence $\{u_n\}$ of positive solutions of (1.1) with $\lim_{n\to+\infty} \|u_n\| = +\infty$ such that the corresponding sequence $\{\lambda_n\}$ converges to μ.

Our first theorem is valid in both the forced and unforced case; however, we shall obtain a stronger result for the forced case below (Theorems 3.3 and 3.6). This theorem is a restatement of the general Theorem 4.4 of [10] for the case of equation (1.1).

THEOREM 3.1. *Assume* (CP1), (CP2), (AL) *and* (L). *Suppose there exists a function* $b:\bar{\Omega}\to(-\infty,+\infty)$ *such that*

$$\lim_{w\to+\infty}[f(w,x)-m(x)w]=b(x)$$

exists uniformly for $x\in\bar{\Omega}$, *and* $m(x)>0$ *for some* $x\in\Omega$. *Let* ϕ_∞, ψ_∞ *be, respectively, the positive eigenfunctions of the linear equations* (3.2) *and* (3.3), *corresponding to the eigenvalue* $\mu_1[\infty]$, *normalized so that* $\|\phi_\infty\|=1$ *and* $\int\psi_\infty(x)\phi_\infty(x)m(x)\,dx=1$. *If*

$$\gamma_\infty\equiv\int\psi_\infty(x)[\mu_1[\infty]b(x)+g(x)]\,dx\neq 0,$$

then there exists a number δ *having the same sign as* γ_∞ *such that for each* λ *between* $\mu_1[\infty]$ *and* $\mu_1[\infty]-\delta$, *there is a solution* $u(\lambda)$ *of* (1.1) *such that* $\lim_{\lambda\to\mu_1[\infty]}\|u(\lambda)\|=\infty$, *and there is a positive number r such that for each* λ *between* $\mu_1[\infty]$ *and* $\mu_1[\infty]+\delta$, *there are no solutions of* (1.1) *with norm greater than r. The solutions* $u(\lambda)$ *satisfy*

$$u(\lambda;x)=\frac{1}{\mu_1[\infty]-\lambda}\gamma_\infty\phi_\infty(x)+o(|\mu_1(\infty)-\lambda|^{-1})$$

as $\lambda\to\mu_1[\infty]$, *uniformly for* $x\in\bar{\Omega}$.

Proof. Because of the assumed uniform Lipschitz continuity in w of $f(w,x)$ for large w, we can apply Lemma 2.3 above and Theorem 4.4 of [10] to obtain the desired result.

We now consider the case that $u=g$ is not a solution of (1.1).

Suppose that $f[g(x),x]>0$ for some $x\in\Omega$. The equation $u=g+\lambda Au$ is equivalent to $v=\lambda A(v+g)$, with $v=u-g$, and $v=0$ is not a solution for $\lambda>0$; it follows from [18] that there exists an unbounded continuum of solutions (λ,v) in $[0,+\infty)\times\mathscr{K}$ containing $(0,0)$. Thus there is an unbounded continuum of solutions (λ,u) of (1.1) in $[0,+\infty)\times\mathscr{K}$ containing $(0,g)$.

Let

$$\Lambda=\{\lambda>0:(1.1)\text{ has a positive solution } u\}.$$

If Λ is bounded, then, in the forced case under consideration, there must be solutions of arbitrarily large norm; conditions guaranteeing that Λ is bounded are given in Theorems 3.4 and 3.5 below.

LEMMA 3.2. *Suppose that f satisfies* (CP2), (AL) *and* (F). *If* $m(x)=0$ *for all* $x\in\bar{\Omega}$, *then* $\Lambda=(0,+\infty)$.

Proof. For any $w_0\geqq 0$, define

$$f_{\max}(w_0)=\max\{f(x,w):x\in\bar{\Omega},0\leqq w\leqq w_0\}.$$

If $m(x)=0$ for all $x\in\bar{\Omega}$, then for any $\varepsilon>0$, there exists $\rho>0$ such that

$$f(w,x)\leqq\varepsilon w+f_{\max}(\rho).$$

Define

$$\Gamma u(x) = \int K(x, y)u(y)\, dy.$$

Then we have $Au \leqq \varepsilon \Gamma u + \Gamma f_{\max}(\rho)$. For any $\lambda > 0$, choose $\varepsilon < (\lambda \|\Gamma\|)^{-1}$, and then choose $\sigma = (\|g\| + \lambda \|\Gamma f_{\max}(\rho)\|)(1 - \varepsilon\lambda\|\Gamma\|)^{-1}$. If u satisfies $u \geqq 0$, $\|u\| \leqq \sigma$, then so does $g + \lambda Au$. By the Schauder fixed-point theorem, (1.1) has a solution. Thus, when $m(x) = 0$, $\Lambda = (0, +\infty)$.

In Theorem 3.5 below, we show that the converse of this lemma holds if f also satisfies condition (I).

THEOREM 3.3. *Suppose that f satisfies* (CP2), (AL) *and* (F). *Suppose that Λ is bounded. Then $m(x) > 0$ for some $x \in \Omega$, and there exists exactly one asymptotic branch point $\mu_1[\infty]$; $\mu_1[\infty]$ is a positive finite number, the smallest eigenvalue of the linear equation* (3.2). *If $\{u_n\}$ is any sequence of solutions of* (1.1) *such that* $\lim_{n\to+\infty} \|u_n\| = +\infty$, *then the corresponding sequence $\{\lambda_n\}$ converges to $\mu_1[\infty]$, and*

$$\lim_{n\to+\infty} \frac{u_n}{\|u_n\|} = \phi_\infty,$$

where ϕ_∞ is the positive eigenfunction of (3.2) *corresponding to the eigenvalue $\mu_1[\infty]$, with $\phi_\infty(x) > 0$ for all $x \in \Omega$.*

Proof. This is Theorem 3.1 of [10], which is applicable because of Lemmas 2.2 and 3.2 above (cf. [8, p. 209]).

We now give two conditions for Λ to be bounded.

THEOREM 3.4 (cf. [6, Cor. 3.3.4]). *Suppose there exists a never negative, not identically zero, continuous function p on $\bar{\Omega}$ such that $f(w, x) \geqq p(x)w$ for all $(w, x) \in \bar{\Omega} \times [0, +\infty)$. Then Λ is bounded above by the smallest eigenvalue of the linear integral equation* (A.1).

Proof. Any solution u of (1.1) is strictly positive on Ω and satisfies

$$u(x) \geqq \lambda \int K(x, y)p(y)u(y)\, dy \equiv \lambda Tu(x)$$

for all $x \in \bar{\Omega}$. It follows from §A.5 in the Appendix that $\lambda \leqq \mu_0[T]$.

THEOREM 3.5. *Suppose that, in addition to the usual assumptions, f satisfies* (I) *and* (F). *Then there is one and only one number $\mu_1[\infty]$ for which there is a sequence $\{\lambda_n\}$ converging to $\mu_1[\infty]$ such that* (1.1) *has positive solutions $\{u_n\}$ satisfying* $\lim_{n\to+\infty} \|u_n\| = +\infty$. *If $m(x) = 0$ for all $x \in \bar{\Omega}$, then $\mu_1[\infty] = +\infty$ and $\Lambda = (0, +\infty)$. If $m(x) > 0$ for some $x \in \bar{\Omega}$, then Λ is bounded, $\mu_1[\infty]$ is positive and finite, and Theorem* 3.3 *applies.*

Proof. Under the stated conditions, the set Λ is an interval, and for each $\lambda \in \Lambda$, there is a smallest positive solution $u^0(\lambda)$, which is an increasing function of λ (see [10, Lemma 5.1]). From $u = g + \lambda Au \geqq \lambda Ag$, it follows that if $\Lambda = (0, +\infty)$, then as $\lambda \to +\infty$, we have $u(\lambda; x) \to +\infty$ for all corresponding solutions $u(\lambda)$ of (1.1), uniformly on any closed subset of Ω.

Suppose that Λ were unbounded and $m(x) > 0$ for some $x \in \Omega$. There exists a closed ball $\bar{\Omega}_0 \subset \Omega$ such that m has a positive lower bound, say 2μ, on $\bar{\Omega}_0$, and there exists a positive number ρ such that $f(w, x) \geqq \mu w$ on $\bar{\Omega}_0$, since $f(w, x)/w \to m(x)$ uniformly on Ω. For some λ_1, the corresponding $u^0(\lambda_1) = u_1$ satisfies

$u_1(x) \geqq \rho$ for all $x \in \bar{\Omega}_0$. Therefore, for every $\lambda > \lambda_1$, any corresponding solution $u(\lambda)$ satisfies

$$u(\lambda; x) \geqq \lambda \int_{\bar{\Omega}_0} K(x, y)\mu u(y)\, dy = \lambda T u(\lambda; x),$$

where T is the compact positive linear operator given by

$$Th(x) = \int_{\Omega} K(x, y)p(y)h(y)\, dy,$$

with $p(y) = 0$ for $x \in \Omega \backslash \bar{\Omega}_0$, $p(y) = \mu$ for $x \in \bar{\Omega}_0$. The operator T has a positive spectral radius $\mu_0[T]$, and $u \geqq \lambda T u$ implies that $\lambda \leqq \mu_0[T]$, which contradicts our assumption that Λ is unbounded. Thus Λ is bounded and we can apply Theorem 3.3.

THEOREM 3.6. *Suppose, in addition to the conditions of Theorem* 3.3, *that there exists a continuous real function b on* $\bar{\Omega}$ *and a number* $s \in [0, 1)$ *such that*

$$f(w, x) = m(x)w + b(x)w^s + o(w^s) \quad as \quad w \to +\infty$$

uniformly for $x \in \Omega$. *Let* ψ_∞ *be the positive eigenfunction of* (3.3) *normalized so that* $\int \psi_\infty(x)\phi_\infty(x)m(x)\, dx = 1$. *Then*

$$\lim_{n \to \infty} (\lambda_n - \mu_1[\infty])\|u_n\|^{1-s} = \gamma_\infty,$$

where

$$\gamma_\infty = \begin{cases} \mu_1[\infty] \int \psi_\infty(x)b(x)\phi_\infty^s(x)\, dx, & s > 0, \\ \int \psi_\infty(x)[\mu_1[\infty]b(x) + g(x)]\, dx, & s = 0. \end{cases} \tag{3.6}$$

If $\gamma_\infty \neq 0$ (*in particular, if* $s = 0$ *and* $g(x)$ *is not identically zero, or if* $b(x)$ *is not identically zero and does not change sign on* $\bar{\Omega}$), *then*

$$u_n(x) = \left(\frac{\gamma_\infty}{\mu_1[\infty] - \lambda_n}\right)^{s'} \phi_\infty(x) + o(|\mu_1[\infty] - \lambda_n|^{-s'})$$

as $n \to \infty$, *uniformly for* $x \in \bar{\Omega}$, *where* $s' = (1 - s)^{-1}$. *If* $\gamma_\infty > 0$ (*or* $\gamma_\infty < 0$), *then there is a number* $r > 0$ *such that* (1.1) *has no positive eigenfunctions with norm greater than r corresponding to eigenvalues* λ *in* $[\mu_1[\infty], \infty)$ (*or* $[0, \mu_1[\infty]]$, *respectively*).

Proof. If $m(x) > 0$ for some $x \in \Omega$, then we have

$$Au = A'(\infty)u + Cu + \omega u,$$

where

$$Cu(x) = \int K(x, y)b(y)[u(y)]^s\, dy$$

and

$$\omega(u) = Au - A'(\infty)u - Cu.$$

By Lemma 2.2, ω satisfies the hypotheses of Theorem 3.2 of [10]. The conclusions of the present theorem are then exactly those of Theorem 3.2 and Corollary 3.3 of [10].

We continue the discussion of the forced case under the additional assumption (I). A complete description of the smallest positive solutions $u^0(\lambda)$ in this case is given in Lemma 5.1 of [10]. From Theorem 5.2 of [10] or directly from Theorem 3.6 above, we obtain the following condition for the existence of a second positive solution for some values of λ.

THEOREM 3.7. *Suppose that f satisfies* (CP2), (AL), (I) *and* (F). *Suppose that m is not identically zero, that for some $s \in [0, 1)$,*

$$\lim_{w \to +\infty} w^{-s}[f(w, x) - m(x)w] \equiv b(x)$$

exists uniformly for $x \in \bar{\Omega}$, and that $\gamma_\infty < 0$ (where γ_∞ is defined in (3.6)). *Then for every $\delta > 0$, there exists $\lambda \in (\mu_1[\infty], \mu_1[\infty] + \delta)$ such that* (1.1) *has at least two positive solutions.*

4. Convex nonlinearities. When the nonlinearity $f(w, x)$ in the integral equation (1.1) is convex in w, Theorem 3.7 can be improved considerably. We recall that, in general (whether or not the nonlinearity is convex), if $f(w, x)$ is forced and increasing in w and $f(w, x)/w$ is decreasing in w in a sufficiently strict sense, then the solutions of (1.1) are unique for each λ, and thus $\lim_{\lambda \to \lambda^* -} \|u^0(\lambda)\| = +\infty$ (for details, see [7, Chap. 6], [11]; for the corresponding results for partial differential equations, see [15]).

For second order ordinary differential equations in which $f(w, x)$ is independent of x, the condition that $f(w)/w$ is decreasing in w is not only sufficient but also necessary for the uniqueness of solutions [13] in the convex case.

We now show that this is very nearly true also for the general integral equation (1.1) when $f(w, x)$ satisfies (AL).

Our discussion will use the following fact about convex functions [2, I.4.4, Ex. 7b]: If $F(w)$ is a convex function of w on an interval I, with $\lim_{w \to \inf(I)} F(w) > 0$, then either $F(w)/w$ is strictly decreasing for all $w \in I$, or there exists $w_0 \in I$ such that $F(w)/w$ is strictly decreasing for $w < w_0$ and constant for $w > w_0$, or there exist $w_0, w_1 \in I$ such that $F(w)/w$ is strictly decreasing for $w < w_0$, constant for $w_0 \leqq w \leqq w_1$, and strictly increasing for $w > w_1$.

Thus the limit $\lim_{w \to +\infty} f(w, x)/w = m(x)$ exists for each $x \in \bar{\Omega}$; if this function m is bounded on $\bar{\Omega}$, then the operator defined by (3.4) is a bounded linear operator on $C(\bar{\Omega})$, and we again denote by $\mu_1[\infty]$ the reciprocal of its spectral radius.

From the convexity of $f(w, x)$ in w, it follows that

$$f(w_2, x) - f(w_1, x) \leqq m(x)(w_2 - w_1) \tag{4.1}$$

whenever $0 \leqq w_1 \leqq w_2$. Thus from [11, Chap. I.4], [1, Thm. B], [12, Thm. 3-6] we obtain the following Theorem.

THEOREM 4.1. *Assume conditions* (CP1), (CP2), (AL), (F) *and* (I), *and that $f(w, x)$ is convex in w. Then the integral equation* (1.1) *has a unique positive solution for each $\lambda \in (0, \mu_1[\infty])$.*

If strict inequality holds in (4.1) for all $x \in \Omega$, then the solution of (1.1) for $\lambda = \mu_1[\infty]$—if there is one—can be shown to be unique. The situation is more complicated if we merely assume (4.1); the results in this case are the same as the corresponding results for partial differential equations presented in [14], and we omit them here, except for the special case described in Theorem 4.4 below.

It follows from (4.1) that if $m(x) = 0$ for some $x \in \bar{\Omega}$, then $f(w, x) = f(0, x)$ for this x and all $w \geqq 0$. Thus if $m(x) = 0$ for all $x \in \bar{\Omega}$, then we have essentially a linear problem; under assumptions (F) and (I), it is in this case, and only this case, that Λ is an unbounded interval.

Our major result in this section is the next theorem and Corollary 4.3.

THEOREM 4.2. *Assume that conditions* (CP1), (CP2), (AL), (I) *and* (F) *are satisfied, and that* $f(w, x)$ *is convex in* w *for* $(w, x) \in [0, +\infty) \times \bar{\Omega}$. *Let* ψ_∞ *be a positive eigenfunction of* (3.3) *corresponding to the characteristic value* $\mu_1[\infty]$. *Define*

$$\Psi(\Omega; w) = \int_\Omega \psi_\infty(y)[m(y)w - f(w, y) - m(y)g(y)]\, dy.$$

If either of the following two equivalent conditions is satisfied:

(a) *There exists* $w_0 > 0$ *such that* $\Psi(\Omega; w_0) > 0$, *or*

(b)

$$\lim_{w \to +\infty} \Psi(\Omega; w) > 0, \tag{4.2}$$

then:

(i) *There exists a positive number r such that* (1.1) *has no solutions with* $\|u\| > r$ *corresponding to* $\lambda \in (0, \mu_1[\infty])$.

(ii) *For all positive numbers* ρ *and* δ, *there exists* $\lambda \in (\mu_1[\infty], \mu_1[\infty] + \delta)$ *for which* (1.1) *has at least two positive solutions, one of which has norm greater than* ρ.

Proof. Hypotheses (a) and (b) are equivalent since the convexity of f implies that $\Psi(\Omega; w)$ is an increasing function of w. These hypotheses imply that m is not identically zero on Ω, so by Theorem 3.5, Λ is bounded. Hence there exist solutions of arbitrarily large norm, and by Theorem 3.3, $\mu_1[\infty]$ is the asymptotic branch point. Thus, according to Lemma B.1 of Appendix B, it suffices to show that

$$\xi[A'(\infty)(u - g) - Au] > 0$$

for sufficiently large solutions $u = \lambda Au + g$, where ξ is the linear functional (3.4).

Let ϕ_∞ be the normalized positive eigenfunction of (3.2). Since $\partial\Omega$ has measure zero, there is an open subset $\Omega_0 \subseteq \bar{\Omega}_0 \subseteq \Omega$ and a positive number β such that (cf. hypothesis (a)) $\Psi(\Omega_1; w_0) \geqq \beta$ for all open subsets Ω_1 with $\bar{\Omega}_0 \subseteq \Omega_1 \subseteq \bar{\Omega}$. Since ϕ_∞ is strictly positive on $\bar{\Omega}_0$, we can choose $\varepsilon > 0$ so small that

$$\Omega_\varepsilon = \{x \in \bar{\Omega}: \phi_\infty(x) > 2\varepsilon\} \supseteq \bar{\Omega}_0$$

and

$$\int_{\Omega/\Omega_\varepsilon} \psi_\infty(y)\{m(y)g(y) + f(0, y)\}\, dy < \tfrac{1}{2}\beta.$$

By Theorem 3.3, we can find $\rho > 0$ so that if $u = \lambda Au + g$ and $\|u\| > \rho$, then $u \geqq \|u\|(\phi_\infty - \varepsilon)$ on $\bar{\Omega}$; thus on $\Omega_\varepsilon, u \geqq \|u\|\varepsilon$. Let $f_1(w, y) = m(y)g(y) + f(w, y)$; then if $u = \lambda Au + g$ and $\|u\| > \rho$,

$$
\begin{aligned}
\mu_1[\infty]\xi[A'(\infty)(u-g)-Au] &= \int \psi_\infty(y)[m(y)u(y)-f_1(u(y),y)]\,dy \\
&\geqq \int_{\Omega_\varepsilon} \psi_\infty(y)\{m(y)\|u\|\varepsilon - f_1(\|u\|\varepsilon, y)\}\,dy \\
&\quad - \int_{\Omega/\Omega_\varepsilon} \psi_\infty(y) f_1(0,y)\,dy,
\end{aligned}
\tag{4.3}
$$

where we have used the fact that $m(y)w - f_1(w, y)$ is an increasing function of w, since $f(w, y)$ is convex in w. Then $\|u\| \geqq \max\{\rho, w_0/\varepsilon\}$ in (4.3) implies

$$\mu_1[\infty]\xi[A'(\infty)(u-g)-Au] \geqq \tfrac{1}{2}\beta > 0,$$

as desired.

For sufficiently small w, $f(0, x) > 0$ implies $f_1(w, x) \geqq m(x)w$; the condition (4.2) means that for sufficiently many x and sufficiently large w, $f_1(w, x) < m(x)w$. Now the relation $f_1(w, x) < m(x)w$ for some x is true if and only if $f_1(w, x)/w$ is strictly increasing for all sufficiently large w, since otherwise we would have $f_1(w, x)/w \geqq \lim_{\rho\to+\infty} f_1(\rho, x)/\rho = m(x)$ for all $w > 0$. In particular, we have the following corollary.

COROLLARY 4.3. *The conclusions of Theorem* 4.2 *hold if we replace assumption* (4.2) *by the following: For all* $x \in \Omega$, $[f(w, x) + m(x)g(x)]/w$ *is eventually increasing in* w, *and there exists a subset* Ω_0 *of* Ω *of positive measure and a number* $\rho > 0$ *such that* $[f(w, x) + m(x)g(x)]/w$ *is strictly increasing in* w *for all* $x \in \Omega_0$ *and all* $w > \rho$.

If the nonlinearity $f(w, x)$ is independent of x, the preceding results can be expressed more elegantly as follows.

THEOREM 4.4. *Let* f *be convex, nonnegative, and increasing on* $[0, +\infty)$, *with* $f(0) + g > 0$, *where* g *is a nonnegative constant. Let* $m = \lim_{w\to+\infty} f(w)/w$, *with* $0 < m < +\infty$. *Let* $K(x, y)$ *be a weakly singular kernel on* $\bar\Omega \times \bar\Omega$, *and consider the equation*

$$u(x) = g + \lambda \int_\Omega K(x, y) f(u(y))\,dy. \tag{4.4}$$

Let $\tilde\mu$ *be the eigenvalue of the linear equation*

$$h(x) = \mu \int K(x, y)h(y)\,dy$$

corresponding to a positive eigenfunction.

Then (4.4) *has a unique positive solution* $u^0(\lambda)$ *for each* $\lambda \in (0, \tilde\mu/m)$. *If* $[mg + f(w)]/w$ *is eventually strictly increasing in* w, *then* (4.4) *has at least two solutions for some values of* $\lambda > \tilde\mu/m$. *Otherwise,* $\lambda^* \equiv \sup(\Lambda) = \tilde\mu/m$, *and either*

(a) *there is no solution for* $\lambda = \lambda^*$ *and* $\lim_{\lambda\to\lambda^*-} \|u^0(\lambda)\| = +\infty$, *or*

(b) *there are infinitely many solutions for* $\lambda = \lambda^*$, *and there is an* $\alpha_0 > 0$ *such that all solutions for* $\lambda = \lambda^*$ *have the form* $u^0(\lambda^*) + \alpha\phi_\infty$ *for* $\alpha \geqq \alpha_0$.

Case (a) *occurs if* $[mg + f(w)]/w$ *is strictly decreasing in* w *for all* $w > 0$, *or if there exists* $x \in \partial\Omega$ *such that* $K(x, y) = 0$ *for all* $y \in \Omega$.

Proof. If $[mg + f(w)]/w$ is eventually strictly increasing in w, the result follows from Corollary 4.3. In the other case, the assertions are proved in the same way as the corresponding assertions for partial differential equations in [14], [15].

COROLLARY 4.5. *Let f be convex and increasing on $(0, +\infty)$, with $f(0) > 0$ and $m = \lim_{w \to +\infty} f(w)/w < +\infty$. Suppose there exists $x \in \partial\Omega$ such that $K(x, y) = 0$ for all $y \in \Omega$. Then the following statements are equivalent*:

(i) *$f(w)/w$ is eventually strictly increasing in w.*

(ii) *$f(w) - wf'_-(w)$ is eventually negative.*

(iii) *Equation* (4.4) *has more than one positive solution for some $\lambda > \tilde{\mu}/m$.*

(iv) $\lambda^* > \tilde{\mu}/m$.

Another way of describing the nonuniqueness result of the last theorem is to say that if $f(w)/w$ is eventually strictly increasing in w, then the integral equation (without the parameter λ)

$$u(x) = \int_\Omega K(x, y) f(u(y))\, dy$$

has at least two solutions if $\tilde{\mu} < m$.

Appendix A.

A.1. We consider the linear integral equation with a weakly singular kernel

$$h(x) = \lambda \int_\Omega K(x, y) p(y) h(y)\, dy, \tag{A.1}$$

where Ω is a bounded, open, connected subset of R^n, $K(x, y)$ is continuous in (x, y) on $\bar{\Omega} \times \bar{\Omega}$ except possibly when $x = y$, there exists a constant $\kappa > 0$ and a number $\alpha \in [0, n)$ such that

$$|K(x, y)| \leqq \kappa/|x - y|^\alpha$$

for all $(x, y) \in \bar{\Omega} \times \bar{\Omega}$, $x \neq y$, and p is a bounded, measurable function on $\bar{\Omega}$. For a general description of the analysis and properties of this equation, see [3], [16], [17].

A.2. The operator $T: C(\bar{\Omega}) \to C(\bar{\Omega})$ given by

$$Tu(x) = \int_\Omega K(x, y) p(y) u(y)\, dy$$

is a compact linear operator on $C(\bar{\Omega})$ by Arzela's theorem and is positive if $N(x, y) \equiv K(x, y)p(y) \geqq 0$. We define $\Gamma u(x) = \int K(x, y) u(y)\, dy$, so $Tu = \Gamma pu$. The mth iterated kernel N_m is defined in the usual way; N_m is the kernel of T^m. Each kernel N_m is bounded on $\bar{\Omega} \times \bar{\Omega}$ for $m > n/(n - \alpha)$ [17, Chap. III]. It is possible to choose $m > n/(n - \alpha)$ so that the eigenfunctions of N corresponding to any eigenvalue λ of N are precisely the same as the eigenfunctions of N_m corresponding to the eigenvalue λ^m (the *eigenvalues* λ of N, that is, of equation (A.1), are the *characteristic values* of $T: h = \lambda Th \neq 0$). For such m, μ is an eigenvalue of N_m if and only if one of the mth roots of μ is an eigenvalue of N [17, §III.3]. In general, there may be no eigenvalues.

A.3. We assume henceforth that K is strictly positive on $\Omega \times \Omega$, that $p \geqq 0$ on Ω, and that there exists an open subset Ω_1 of Ω on which p is strictly positive.

Then there is an eigenvalue: We choose $m > n/(n - \alpha)$ as described in the preceding paragraph. The iterates of N have the form $N_m(x, y) = \tilde{K}_m(x, y)p(y)$, where $\tilde{K}_m$ is strictly positive on $\Omega \times \Omega$, and therefore $\int_\Omega N_{3m}(x, x)\,dx > 0$. Since N_m is bounded, it follows from the Fredholm theory for integral equations with bounded kernels that N_m has an eigenvalue [17, p. 178] and hence (by choice of m) N does also. The Krein–Rutman theorem [9] implies that N has a nonnegative eigenfunction ϕ corresponding to the eigenvalue $\mu_0[T]$ (the largest positive eigenvalue of N and the reciprocal of the spectral radius of T); it is easily verified that ϕ is strictly positive on Ω.

The arguments of Jentzsch ([4], cf. [3, §17.5]) applied to the bounded kernel N_m imply that $\mu_0[T]$ is a simple eigenvalue of N (the simplicity of $(\mu_0[T])^m$ as an eigenvalue of N_m implies the simplicity of $\mu_0[T]$ as an eigenvalue of N); $\mu_0[T]$ is larger than the absolute value of all other eigenvalues of N, and the positive multiples of ϕ are the only positive eigenfunctions of N.

A.4. Similarly, the integral equation

$$h(x) = \lambda \int K(y, x)p(y)h(y)\,dy$$

has a "unique" positive eigenfunction ψ, and the corresponding eigenvalue is easily seen to be $\mu_0[T]$. The linear functional $\xi : C(\bar{\Omega}) \to R$ defined by

$$\xi(h) = \int \psi(x)p(x)h(x)\,dx$$

is a positive eigenvector of the adjoint of T corresponding to the characteristic value $\mu_0[T]$; i.e., $\mu_0[T]\xi[Th] = h$ for all $h \in C(\bar{\Omega})$. This follows from the easily verified fact that, for any $v \in C(\bar{\Omega})$,

$$\mu_0[T]\xi(\Gamma v) = \int \psi(x)v(x)\,dx.$$

Note that $\xi(h) > 0$ if $h \geqq 0$ and there exists $x \in \Omega_1$ such that $h(x) > 0$.

A.5. The operator T has the following property, as can be verified using the functional ξ. Suppose that for some $h \in C(\bar{\Omega})$, $h \geqq 0$, we have $h - \lambda Th \geqq 0$ on $\bar{\Omega}$. If $h(x) > 0$ for some $x \in \Omega_1$, then $\lambda \leqq \mu_0[T]$; if $h(x) - \lambda Th(x) > 0$ for some $x \in \Omega_1$, then $\lambda < \mu_0[T]$ (cf. the property (PA) in [11] and "regularly solvable" in [1]).

Appendix B. In this appendix we state and prove the lemma used in the proof of Theorem 4.2. We use the notation and terminology of [10, §3].

LEMMA B.1. *Let $\mathscr{E}$ be a partially ordered Banach space with a positive cone $\mathscr{K}$. Let A be a positive operator on $\mathscr{K}$ which has a continuous $\mathscr{K}$-asymptotic derivative $A'(\infty)$. Suppose that there exists a solution (λ, u) of $u = g + \lambda Au$ with $u \in \mathscr{K}$, $\lambda > 0$. Let μ be a positive characteristic value of the adjoint of $A'(\infty)$ to which there corresponds a positive eigenvector ξ (that is, ξ is a positive linear functional such that $\mu\xi[A'(\infty)h] = h$ for all $h \in \mathscr{E}$) such that $\xi(u - g) \neq 0$. Then*

$$\begin{aligned}\operatorname{sgn}(\mu - \lambda) &= \operatorname{sgn}(\xi[Au - A'(\infty)u + \mu^{-1}g]) \\ &= \operatorname{sgn}(\xi[Au - A'(\infty)(u - g)]).\end{aligned}$$

Proof. We have

$$0 < \mu\xi(u-g) = \mu\lambda\xi[Au - A'(\infty)(u-g)] + \lambda\xi(u-g),$$

so that

$$\mu - \lambda = \frac{\mu\lambda}{\xi(u-g)}\,\xi[Au - A'(\infty)u + \mu^{-1}g].$$

The desired result follows immediately.

Lemma B.1 is of particular interest when we combine it with Theorem 3.1 of [10] and take μ to be an asymptotic branch point.

We assumed the existence of the positive linear functional ξ in Lemma B.1; by theorems of Krein and Rutman [9], such a ξ exists if $A'(\infty)$ is compact on $\mathscr{K}$, if $\mu = \mu_0[A'(\infty)]$, the reciprocal of the spectral radius of $A'(\infty)$, and if either $\mathscr{K} - \mathscr{K}$ is dense in $\mathscr{E}$ or $\mathscr{K}$ has interior points.

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SOLUTIONS TO A PROBLEM IN POWER SERIES REVERSION*

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Abstract. This paper presents the general solution of the following problem in two forms. Let $f(x, y)$ be defined by the formal power series $f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}x^m y^n$ with $f_{00} \neq 0$. If v satisfies $v(x, y) = f(xv^a, yv^b)$, where a and b are constants, then find the formal power series expansion of $v^c(x, y)$, where c is also a constant.

A special case of this problem, which occurs in a paper by R. A. Handelsman and J. S. Lew [1], has been proposed as a problem to be solved by computer using a symbolic algebra system [2].

1. Introduction and summary. In this paper we give two formulations of the answer to the following problem.

Let $f(x, y)$ be defined by

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}x^m y^n$$

with $f_{00} \neq 0$. If v satisfies

$$v(x, y) = f(xv^a, yv^b), \tag{1}$$

then find the formal power series expansion of $v^c(x, y)$ for arbitrary c.

First, we show that

$$(v^c)_{mn} = \begin{cases} (1 + c \cdot \ln f)_{mn}, & am + bn + c = 0, \\ \dfrac{c}{am + bn + c}(f^{am+bn+c})_{mn}, & am + bn + c \neq 0, \end{cases} \tag{2}$$

where the notation $(g)_{mn}$ denotes the coefficient of $x^m y^n$ in the series expansion of $g(x, y)$.

Equation (2) is conceptually simple but computationally difficult. To provide a formula more amenable to computation we show that (2) can be rewritten as

$$(v^c)_{mn} = \begin{cases} f_{00}^c, & m = n = 0, \\ cf_{00}^{am+bn+c} \sum_{k=1}^{m+n} F_k(m, n)(am + bn + c - 1)_{k-1} f_{00}^{-k}, & m + n > 0, \end{cases} \tag{3}$$

where $(w)_k$ is the falling factorial defined by

$$(w)_0 = 1,$$
$$(w)_k = w(w - 1) \cdots (w - k + 1), \quad k > 0,$$

and F_k is defined by the generating function

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n)x^m y^n z^k = e^{(f(x,y) - f(0,0))z} \tag{4}$$

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We then derive the following recursive formula for the computation of $F_k(m, n)$ for all k, m and n:

$$
(5) \qquad
\begin{aligned}
F_0(0,0) &= 1, & & \\
F_0(m,n) &= 0, & & m+n>0, \\
F_k(m,n) &= 0, & & m+n<k, \\
F_{k+1}(m,n) &= \frac{1}{m+n}\sum_{i=0}^{m}\sum_{j=0}^{n}(i+j)f_{ij}F_k(m-i,n-j), & & m+n>0.
\end{aligned}
$$

2. Derivation of formula (2). In order to illustrate a technique that may be applicable to other similar problems, we give a derivation based on the residue operator for formal power series[1] given in [3]. Proofs based on complex variable theory can be obtained for the one-variable case from Lagrange's theorem [4, §7.32], and for the two-variable case from the generalization of that theorem by I. J. Good in [5].

We shall briefly summarize the relevant results of [3]. Let $h(x_1, \cdots, x_r)$ be a formal power series in r variables (i.e., no convergence restrictions) of the form

$$h = \sum_{n_1=k_1}^{\infty} \cdots \sum_{n_r=k_r}^{\infty} h_{n_1\cdots n_r} x_1^{n_1} \cdots x_r^{n_r},$$

where $k_1, \cdots, k_r$ are finite but may be negative. The sum, difference, product and partial derivatives of formal power series are defined in the usual way and have the usual properties. For exponentiation, let h have a nonzero constant term h_0 and no negative exponents. Thus

$$h = h_0(1 + H(x_1, \cdots, x_n)), \tag{6}$$

where H has a zero constant term and no negative exponents. Then h^α can be defined by

$$h^\alpha = h_0^\alpha\left(\sum_{k=0}^{\infty} \frac{(\alpha)_k H^k}{k!}\right)$$

and this exponentiation has all the usual properties. For h as in (6), we define

$$
\begin{aligned}
\ln h &= \ln h_0 + \ln(1+H) \\
&= \ln h_0 + \sum_{k=1}^{\infty}(-1)^{k+1}H^k/k
\end{aligned}
$$

and

$$
\begin{aligned}
e^h &= e^{h_0}e^{h_0 H} \\
&= e^{h_0}\sum_{k=0}^{\infty}(h_0 H)^k/k!.
\end{aligned}
$$

These have the usual inverse and differentiation properties.

The basic requirement in the manipulation of formal power series is a finiteness condition: If the manipulation of the operands $g, h, \cdots$ (which are formal power series) results in a formal power series f, then the coefficient of any term in f may

[1] I. Niven gives an excellent survey of formal power series in [6].

not involve more than a finite (but possibly unbounded) number of coefficients of the operands $g, h, \cdots$.

The residue operator applied to any h is defined by

$$R(h) = \text{coefficient of } x_1^{-1} \cdots x_r^{-1} \text{ in } h.$$

As a consequence,

$$R(h/x_1^{n_1+1} \cdots x_r^{n_r+1}) = \text{coefficient of } x_1^{n_1} \cdots x_r^{n_r} \text{ in } h. \tag{7}$$

The main result for this operator deals with substitutions [3]. Let

$$g_i = x_1^{m_{i1}} \cdots x_r^{m_{ir}} G_i(x_1, \cdots, x_r), \qquad i = 1, \cdots, r,$$

where G_i has no negative exponents and a nonzero constant term and $m_{ij} \geqq 0$ with $\sum_{j=1}^r m_{ij} > 0$. Then

$$\det(m_{ij})\, R(h) = R\left(h(g_1, \cdots, g_r) \frac{\partial(g_1, \cdots, g_r)}{\partial(x_1, \cdots, x_r)}\right), \tag{8}$$

where the Jacobian is defined by

$$\frac{\partial(g_1, \cdots, g_r)}{\partial(x_1, \cdots, x_r)} = \det\left(\frac{\partial g_i}{\partial x_j}\right).$$

The fact that $h(g_1, \cdots, g_r)$ is well-defined can be shown by observing that each coefficient in its formal series will involve only a finite number of the coefficients from the g_i (see [7] for a detailed proof).

For our problem, let us first make a change of variable from x, y to s, t, so that (1) becomes

$$v(s, t) = f(sv^a(s, t), tv^b(s, t)),$$

where $f_{00} \neq 0$ and therefore $v_{00} = f_{00}$. If

$$x = sv^a(s, t),$$

$$y = tv^b(s, t),$$

then

$$v(s, t) = f(x, y)$$

giving

$$s = xf^{-a}(x, y),$$

$$t = yf^{-b}(x, y),$$

$$v(xf^{-a}(x, y), yf^{-b}(x, y)) = f(x, y),$$

which is the dual of equation (1).

Now by (7),

$$(v^c)_{mn} = R(v^c(s, t)/s^{m+1}t^{n+1}),$$

and so the substitution theorem (8) yields

$$(v^c)_{mn} = R\left(\frac{v^c(xf^{-a}, yf^{-b})}{x^{m+1}y^{n+1}} f^{am+a+bn+b} \frac{\partial(xf^{-a}, yf^{-b})}{\partial(x, y)}\right)$$
$$= R\left(\frac{f^{am+bn+a+b+c}}{x^{m+1}y^{n+1}} \cdot \frac{\partial(xf^{-a}, yf^{-b})}{\partial(x, y)}\right).$$

The Jacobian has the value

$$f^{-a-b}\left(1 - axf^{-1}\frac{\partial f}{\partial x} - byf^{-1}\frac{\partial f}{\partial y}\right)$$

giving (with $d = am + bn + c$)

$$(v^c)_{mn} = R\left(f^d\left(1 - axf^{-1}\frac{\partial f}{\partial x} - byf^{-1}\frac{\partial f}{\partial y}\right)\Big/x^{m+1}y^{n+1}\right)$$
$$= R(f^d/x^{m+1}y^{n+1}) - aR\left(f^{d-1}\frac{\partial f}{\partial x}\Big/x^m y^{n+1}\right) - bR\left(f^{d-1}\frac{\partial f}{\partial y}\Big/x^{m+1}y^n\right).$$

Now for any h,

$$R\left(\frac{\partial h}{\partial x}\Big/x^m y^{n+1}\right) = mR(h/x^{m+1}y^{n+1}). \tag{9}$$

Thus if $d \neq 0$, the second term is

$$\frac{-a}{d} R\left(\frac{\partial f^d}{\partial x}\Big/x^m y^{n+1}\right) = \frac{-am}{d} R(f^d/x^{m+1}y^{n+1})$$

and similarly the third term is $(-bn/d)R(f^d/x^{m+1}y^{n+1})$. Replacing d by $am + bn + c$ and adding, we have

$$(v^c)_{mn} = \frac{c}{am + bn + c}(f^{am+bn+c})_{mn},$$

which is the second part of (2).

If $d = 0$, the second term is

$$-aR\left(f^{-1}\frac{\partial f}{\partial x}\Big/x^m y^{n+1}\right) = -aR\left(\frac{\partial \ln f}{\partial x}\Big/x^m y^{n+1}\right)$$
$$= -amR(\ln f/x^{m+1}y^{n+1}),$$

and similarly the third is

$$-bnR(\ln f/x^{m+1}y^{n+1}).$$

Therefore, since $d = 0$ we have $c = -am - bn$ and

$$(v^c)_{mn} = (1 + c \cdot \ln f)_{mn},$$

which is the first part of (2).

3. Derivation of formulas (3) and (5). Formula (3) is a special case of the following more general result: If $h(x, y) = g(u)$ and $u = f(x, y)$ where f is a given formal power series with no negative exponents and $g(u)$ has a series expansion around the point $u_0 = f(0, 0) = f_{00}$, then the coefficients of the formal series for h are given by:

$$h_{00} = g(f(0,0)),$$

$$h_{mn} = \sum_{k=1}^{m+n} F_k(m, n) \left.\frac{d^k g}{du^k}\right|_{u=f(0,0)}, \qquad m + n > 0, \tag{10}$$

where, as before, the $F_k(m, n)$ are defined by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n) x^m y^n z^k = e^{(f(x,y) - f(0,0))z}. \tag{11}$$

Using (10) we obtain (3) in the case $am + bn + c \neq 0$ by just specializing $g(u)$ to $(c/\gamma)u^\gamma$ so that $h(x, y) = (c/\gamma)f^\gamma$. Then (10) gives

$$\left(\frac{c}{\gamma} f^\gamma\right)_{00} = \frac{c}{\gamma} f_{00}^\gamma,$$

$$\left(\frac{c}{\gamma} f^\gamma\right)_{mn} = c \sum_{k=1}^{m+n} F_k(m, n)(\gamma)_{k-1} f_{00}^{\gamma - k}, \qquad m + n > 0.$$

Setting $\gamma = am + bn + c$ and substituting these into the lower part of (2), we arrive at (3).

To obtain (3) for the case $am + bn + c = 0$, we specialize $g(u)$ to $1 + \gamma \cdot \ln u$, so that $h(x, y) = 1 + \gamma \cdot \ln f$. Then (10) gives

$$(1 + \gamma \cdot \ln f)_{00} = 1 + \gamma \cdot \ln f_{00},$$

$$(1 + \gamma \cdot \ln f)_{mn} = \gamma \sum_{k=1}^{m+n} F_k(m, n)(-1)_{k-1} f_{00}^{-k}, \qquad m + n > 0.$$

Setting $\gamma = c = -am - bn$ and substituting these into the upper part of (2), we obtain

$$(v^c)_{mn} = \begin{cases} 1, & m = n = c = 0, \\ c \displaystyle\sum_{k=1}^{m+n} F_k(m, n)(-1)_{k-1} f_{00}^{-k}, & m + n > 0, \end{cases}$$

which agrees with (3) when $am + bn + c = 0$.

Now to prove (5) and (10), we first equate the coefficients of z^k in (11) to obtain

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_k(m, n) x^m y^n = \begin{cases} 1, & k = 0, \\ \dfrac{(f(x, y) - f(0, 0))^k}{k!}, & k > 0. \end{cases} \tag{12}$$

From this, we see immediately that

$$F_0(0, 0) = 1,$$
$$F_0(m, n) = 0, \quad m + n > 0,$$
$$F_k(m, n) = 0, \quad m + n < k,$$

which are the first three parts of (5).

If we expand $h(x, y)$ in a series around the point u_0, we have

$$h(x, y) = \sum_{k=0}^{\infty} \left.\frac{d^k g}{du^k}\right|_{u=u_0} \frac{(u - u_0)^k}{k!}.$$

If $u = f(x, y)$ and $u_0 = f(0, 0)$, then

$$h(x, y) = \sum_{k=0}^{\infty} \left.\frac{d^k g}{du^k}\right|_{u=f(0,0)} \frac{(f(x, y) - f(0, 0))^k}{k!},$$

whence by (12),

$$h(x, y) = \sum_{k=0}^{\infty} \left.\frac{d^k g}{du^k}\right|_{u=f(0,0)} \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_k(m, n) x^m y^n\right).$$

Interchanging the order of summation and equating coefficients of x and y we obtain (10).

It remains to prove the last formula in (5). First differentiate (11) with respect to x to obtain

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n) m x^{m-1} y^n z^k = e^{(f(x,y) - f(0,0))z}\left(z \frac{\partial}{\partial x} f(x, y)\right)$$
$$= \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} F_k(m, n) x^m y^n z^{k+1}\right)\left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} m x^{m-1} y^n\right).$$

Equating coefficients of z^{k+1} we have

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_{k+1}(m, n) m x^{m-1} y^n = \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} F_k(m, n) x^m y^n\right)\left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn} m x^{m-1} y^n\right).$$

Applying the convolution formula for the product of two power series and equating the coefficients of $x^{m-1} y^n$, we obtain

$$m F_{k+1}(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} i f_{ij} F_k(m - i, n - j).$$

Similarly, differentiating (11) with respect to y, we obtain

$$n F_{k+1}(m, n) = \sum_{i=0}^{m} \sum_{j=0}^{n} j f_{ij} F_k(m - i, n - j).$$

Finally, adding these two equations and dividing by $m + n$ we obtain the last formula in (5).

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ON AN EXPANSION PROBLEM OCCURRING IN THE THEORY OF DIFFRACTION*

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Abstract. This paper considers an eigenfunction expansion arising in certain problems in diffraction theory involving an impedance-type boundary condition. The expansion in question is usually convergent only in part of the domain of interest and involves an infinite set of orthogonal functions which do not form a complete set, in the sense that it is not in general possible to expand a given function in terms of them no matter how well-behaved the function may be. A theorem is established which asserts the validity of the expansion whenever it converges. Also a more general expansion formula is obtained which is valid for all sufficiently general functions and which reduces to the basic expansion whenever the latter converges.

1. Introduction. This paper is devoted to an investigation of the validity of an expansion involving the orthogonal functions $H_{u_n}^{(1)}(kr)$, where $u_1, u_2, \cdots$ are the zeros of the function

$$g(u) = H_u^{(1)\prime}(ka) + iZH_u^{(1)}(ka). \tag{1}$$

Here $k, a > 0$ and Z is a given complex constant.

The expansion in question can be expressed in the form

$$f(r) = -i\pi \sum_{u=u_n} \frac{uH_u^{(1)}(kr)g_1(u)F_1(u)}{g'(u)}, \tag{2}$$

where

$$g_1(u) = J_u'(ka) + iZJ_u(ka), \tag{3}$$

$$F_1(u) = \int_a^\infty f(r)H_u^{(1)}(kr)\frac{dr}{r}. \tag{4}$$

The zeros u_n, which are discussed in [1], [2], are neither real nor purely imaginary and are located in the first and third quadrants of the complex u-plane. The summation in (2) includes only those zeros which lie in the first quadrant. Series of this type are not valid in general [6] no matter how smooth the function $f(r)$ may be or how well-behaved it is at infinity. In practice such series arise as the solutions of certain problems in diffraction theory and are frequently convergent only in part of the domain of interest, being divergent elsewhere, a phenomenon which has not been satisfactorily explained. Since many functions exist which can be represented in the form (2) it is not sufficient to argue that the above eigenfunctions are generated by a singular non-self-adjoint problem and therefore do not form a complete set. Under what conditions will the expansion (2) converge and equal the function $f(r)$? This paper provides a partial answer to this question by constructing an expansion which is valid for all sufficiently general functions and which reduces to (2) whenever the latter converges.

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In an earlier paper [4], the authors discussed the similar series involving the functions $H_{u_n}^{(1)}(kr)$, where $u_1, u_2, \cdots$ are the zeros of $H_u^{(1)}(ka)$. The series in this case takes the simpler form

$$f(r) = -i\pi \sum_{u=u_n} \frac{uH_u^{(1)}(kr)J_u(ka)F_1(u)}{(\partial/\partial u)H_u^{(1)}(ka)}. \tag{5}$$

The principal result obtained in [4] established the validity of (5) whenever the series appearing on the right-hand side is convergent. Hitherto there had been no guarantee that a formal series like (5), even if convergent, would represent the function $f(r)$ since a set of sufficient conditions to justify (5) had not been discovered.

The object of the present paper is to extend the earlier results to cover the more general expansion (2) and to obtain suitable conditions under which the series appearing in (2) does in fact represent the function $f(r)$. These conditions are stated in the following theorem.

THEOREM. *Suppose that* $f(r)$ *is twice continuously differentiable for* $r \geqq a$, $r^{-1/2}(rf_{rr} + f_r + k^2rf) \in L(a, \infty)$, $f(r)$ *and* $f'(r)$ *are* $O(r^{-1/2})$ *as* $r \to \infty$ *and* $r^{1/2}(f_r - ikf) \to 0$ *as* $r \to \infty$, *where* k *is real and positive. Then, if the parameter* λ *tends to zero through positive values,*

(i) *for* $r \geqq a$,

$$f(r) = \lim_{\lambda \to 0} \int_W \frac{e^{\lambda u^2}\Psi_u(k, r)F_1(u)u\, du}{2g(u)}, \tag{6}$$

where

$$\Psi_u(k, r) = J_u(kr)g(u) - H_u^{(1)}(kr)g_1(u); \tag{7}$$

(ii) *for* $r \geqq a$,

$$f(r) = -i\pi \lim_{\lambda \to 0} \sum_{u=u_n} \frac{ue^{\lambda u^2}H_u^{(1)}(kr)g_1(u)F_1(u)}{g'(u)}; \tag{8}$$

(iii) *equation* (2) *is valid whenever the series appearing therein is convergent.*

The path W appearing in (6) is illustrated in Fig. 1. It lies to the right of the zeros u_n of (1) and is asymptotic to the lines $\arg u = \pm \psi$, where ψ is a fixed angle in the interval $\pi/4 < \psi < \pi/2$. This choice of W is possible since it is known that the zeros u_n situated in the first quadrant are such that $\arg(u_n) \to \pi/2$ as $n \to \infty$.

2. The integral theorem. This section describes the method of derivation of the integral formula (6), the expansion (8) being readily deduced from (6) by deforming the path W onto the imaginary axis and taking the residues at the poles. The formula (6) is constructed by following the procedure developed in [5]. Let $f(r)$, the function to be expanded, satisfy the conditions of the theorem and define

$$r^2 f_{rr} + rf_r + (k^2r^2 - v^2)f = h(r), \qquad r \geqq a, \tag{9}$$

where $v > 0$. The expansion problem of interest here is that associated with (9) when the quantity $f'(a) + ikZf(a)$ is prescribed and the function $f(r)$ satisfies the radiation condition

$$\lim_{r \to \infty} r^{1/2}[f'(r) - ikf(r)] = 0. \tag{10}$$

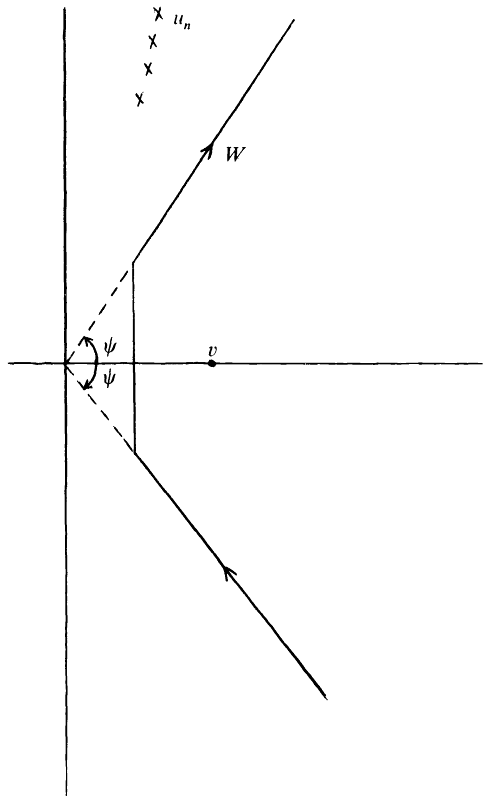

FIG. 1

To obtain the formula (6), equation (9) is inverted by means of the relation

$$f(r) = \int_a^\infty h(\rho)G(r, \rho)\frac{d\rho}{\rho} + [f'(a) + ikZf(a)]\frac{H_v^{(1)}(kr)}{kg(v)}, \tag{11}$$

where $G(r, \rho)$ denotes the Green's function defined by the equations

$$G(r, \rho) = \begin{cases} \dfrac{\pi H_v^{(1)}(k\rho)\Psi_v(k, r)}{2ig(v)}, & a \leqq r \leqq \rho, \\ \dfrac{\pi H_v^{(1)}(kr)\Psi_v(k, \rho)}{2ig(v)}, & a \leqq \rho \leqq r. \end{cases} \tag{12}$$

The Green's function as defined by the above composite expression must now be represented by means of a single formula, which will be inserted in (11). In this paper the following representation will be adopted:

$$G(r, \rho) = \lim_{\lambda \to 0} \int_W \frac{e^{\lambda u^2}\Psi_u(k, r)H_u^{(1)}(k\rho)u\, du}{2(u^2 - v^2)g(u)}. \tag{13}$$

In this formula, which is proved in the Appendix to the paper, the term $e^{\lambda u^2}$ is a summability factor, the parameter λ tending to zero through positive values.

The path W is chosen so that the point v lies to the right of it. The insertion of (13) into (11) yields the formula

$$
\begin{aligned}
f(r) = &\int_a^\infty h(\rho)\frac{d\rho}{\rho}\lim_{\lambda\to 0}\int_W \frac{e^{\lambda u^2}H_u^{(1)}(k\rho)\Psi_u(k, r)u\,du}{2(u^2 - v^2)g(u)} \\
&+ [f'(a) + ikZf(a)]\frac{H_v^{(1)}(kr)}{kg(v)}.
\end{aligned}
\tag{14}
$$

The limit $\lambda \to 0$ is now taken outside of the repeated integral. This step can be justified by verifying that the ρ-integral is uniformly convergent for $\lambda \geqq 0$, and for this purpose use will be made of the following bound:

$$
|\Gamma(u + \tfrac{1}{2})H_u^{(1)}(k\rho)| \leqq [2/(\pi k\rho)]^{1/2}\, e^{\pi|s|}\Gamma(t + \tfrac{1}{2})[2^{t-1/2} + \sqrt{\pi}\Gamma(t)(4/k\rho)^{t-1/2}].
\tag{15}
$$

In this result, which was derived in [5, p. 118], $u = t + is$ and $t > \frac{1}{2}$. The asymptotic behavior on W of the functions Ψ_u and $g(u)$ may be estimated with the aid of the relations

$$
J_u(x) \sim \frac{1}{\Gamma(u + 1)}(x/2)^u, \qquad H_u^{(1)}(x) \sim \frac{\Gamma(u)}{i\pi}(2/x)^u
\tag{16}
$$

giving

$$
\begin{aligned}
\Psi_u(k, r) &\sim \frac{i}{\pi ka}[(r/a)^u + (a/r)^u], \\
g(u) &\sim -\frac{\Gamma(u + 1)}{2i\pi}(2/ka)^{u+1},
\end{aligned}
\tag{17}
$$

where, by Stirling's formula,

$$
|\Gamma(u)| \sim (2\pi/R)^{1/2}\exp[R\cos\theta\log(R/e) - R\theta\sin\theta],
\tag{18}
$$

where $u = Re^{i\theta}$, and $|\theta| < \pi$. It is then found that as $R \to \infty$ the integrand in (14) is

$$
\begin{aligned}
&O\left\{\rho^{-1/2}R^{-2}\exp\left[\lambda R^2\cos 2\psi + R(\pi + 2\psi)\sin\psi - R\cos\psi\log\left(\frac{\rho}{2r}\right)\sec^2\psi\right]\right\} \\
&= O\{\rho^{-1/2}R^{-2}\exp[\lambda R^2\cos 2\psi - R\cos\psi\log(\rho/\rho_1)]\},
\end{aligned}
\tag{19}
$$

where $\rho_1 = 2r\cos^2\psi\exp[(\pi + 2\psi)\tan\psi]$. The bound (19) holds as $R \to \infty$ on W, uniformly for $\lambda \geqq 0$, $\rho \geqq a$. Since $\cos 2\psi < 0$, the expression in (19) is $O(\rho^{-1/2}R^{-2})$ for $\rho \geqq \rho_1$ and the contour integral in (14) is $O(\rho^{-1/2})$ uniformly for $\lambda \geqq 0$. The ρ-integral in (14) is uniformly convergent for $\lambda \geqq 0$ since by hypothesis $\rho^{-3/2}h(\rho) \in L(a, \infty)$. The limiting operation in (14) can therefore be placed outside the repeated integral. Finally the order of integration can be changed since it is evident from the above bounds that for $\lambda > 0$ the repeated integral is absolutely convergent, the dominant term in the exponentials being the $O(R^2)$ term. The formula (14) now takes the form

$$
\begin{aligned}
f(r) = &\lim_{\lambda\to 0}\int_W \frac{e^{\lambda u^2}\Psi_u(k, r)u\,du}{2(u^2 - v^2)g(u)}\int_a^\infty h(\rho)H_u^{(1)}(k\rho)\frac{d\rho}{\rho} \\
&+ [f'(a) + ikZf(a)]\frac{H_v^{(1)}(kr)}{kg(v)}.
\end{aligned}
\tag{20}
$$

The formula (6) may now be obtained from (20) by substituting the expression

$$\int_a^\infty h(\rho)H_u^{(1)}(k\rho)\frac{d\rho}{\rho} = (u^2 - v^2)F_1(u) + kaf(a)H_u^{(1)\prime}(ka) - af'(a)H_u^{(1)}(ka). \tag{21}$$

This relation follows from (9) by multiplying by $r^{-1}H_u^{(1)}(kr)$ and integrating by parts. The insertion of (21) into (20) gives the equation

$$f(r) = \lim_{\lambda\to 0}\int_W \frac{e^{\lambda u^2}\Psi_u(k, r)F_1(u)u\,du}{2g(u)} + \lambda_1 f(a) + \lambda_2 af'(a). \tag{22}$$

The definitions of the quantities λ_1, λ_2 appear below. It will be shown immediately that $\lambda_1 = \lambda_2 = 0$ so that equation (22) reduces to formula (6) of the theorem.

$$\lambda_1 = \frac{iZH_v^{(1)}(kr)}{g(v)} + \lim_{\lambda\to 0}\int_W \frac{ka\, e^{\lambda u^2}\Psi_u(k, r)H_u^{(1)\prime}(ka)u\,du}{2g(u)(u^2 - v^2)},$$

$$\lambda_2 = \frac{H_v^{(1)}(kr)}{kag(v)} - \lim_{\lambda\to 0}\int_W \frac{e^{\lambda u^2}\Psi_u(k, r)H_u^{(1)}(ka)u\,du}{2g(u)(u^2 - v^2)}.$$

The quantity λ_1 defined by the first of the above equations can be shown to be zero by the following device, which starts with the identity

$$\begin{aligned}&\int_W \frac{e^{\lambda u^2}[k\Psi_u(k, r)H_u^{(1)\prime}(ka) - H_u^{(1)}(kr)\Psi_u'(k, a)]u\,du}{(u^2 - v^2)g(u)}\\ &\qquad = \int_W \frac{k\, e^{\lambda u^2}[J_u(kr)H_u^{(1)\prime}(ka) - J_u'(ka)H_u^{(1)}(kr)]u\,du}{u^2 - v^2}.\end{aligned} \tag{23}$$

This equation follows immediately from the definition (7) of Ψ_u. The path W appearing in the integral on the right-hand side of (23) may be deformed onto the imaginary axis since it can be shown with the aid of the definition

$$H_u^{(1)}(x) = -i[J_{-u}(x) - e^{-iu\pi}J_u(x)] \operatorname{cosec} u\pi$$

and the first of the relations (16) that the integrand is $O\{u^{-1}e^{\lambda u^2}[(r/a)^u + (a/r)^u]\}$ as $u \to \infty$. Since the integrand on the right-hand side of (23) is also an odd function of u the value of the integral along the entire imaginary axis is zero. Furthermore it follows from the definition (7) in conjunction with the Wronskian identity $W(J_u, H_u) = 2i/(\pi ka)$ that $\Psi_a'(k, a) = 2Z/\pi a$. When this expression is inserted into the integral on the left-hand side of (23) and the resulting expression equated to zero we find the equation

$$\int_W \frac{k\, e^{\lambda u^2}\Psi_u(k, r)H_u^{(1)\prime}(ka)u\,du}{(u^2 - v^2)g(u)} = \frac{2Z}{\pi a}\int_W \frac{e^{\lambda u^2}H_u^{(1)}(kr)u\,du}{(u^2 - v^2)g(u)}. \tag{24}$$

As $u \to \infty$ on or to the right of W the integrand appearing on the right-hand side of (24) is $O[u^{-2}\, e^{\lambda u^2}(a/r)^u]$ and on W itself this is $O(u^{-2})$ uniformly in λ. The limiting value of the expression on the right-hand side of (24) as $\lambda \to 0$ can therefore be obtained by setting $\lambda = 0$ in the integrand and evaluating the resulting integral by closing the contour on the right and taking the residue at the pole $u = v$. It is then found that the limit equals $-2iZH_v^{(1)}(kr)/ag(v)$ so that $\lambda_1 = 0$ as required.

To show that $\lambda_2 = 0$ it is only necessary to note that the integral appearing in the expression for λ_2 is the same as that appearing in the formula (13) for the Green's function $G(r, \rho)$ with ρ set equal to a therein. Since $G(r, a)$ can be obtained from the second formula in (12) it follows that

$$\lambda_2 = \frac{H_v^{(1)}(kr)}{kag(v)} - \frac{\pi\Psi_v(k, a)H_v^{(1)}(kr)}{2ig(v)}.$$

Finally from the definition (7) of Ψ_u and the Wronskian $W(J_u, H_u) = 2i/(\pi ka)$ it is seen that $\Psi_u(k, a) = 2i/(\pi ka)$ and hence that $\lambda_2 = 0$.

3. The expansion theorem. In this section the expansion (8) of the theorem is obtained from the formula (6) already established by deforming the path W onto the imaginary axis and taking the residues at the poles that are crossed. Since $H_{-u}^{(1)}(x) = e^{iu\pi} H_u^{(1)}(x)$ it follows from the definitions (1), (4) that $g(-u) = e^{iu\pi} g(u)$ and $F(-u) = e^{iu\pi} F(u)$ while on expressing the Hankel functions appearing in (7) in terms of J_u and J_{-u} it can be shown that $\Psi_{-u} = \Psi_u$. The integrand appearing in (6) is therefore an odd function of u and the value of the integral along the imaginary axis is zero. The expansion (8) follows on evaluating the residues at the poles u_n situated in the first quadrant.

To justify the above procedure it is necessary to form a suitable sequence of paths which recede to infinity and which avoid the zeros of $g(u)$. The asymptotic behavior of this function can be obtained from (1) by expressing the Hankel functions in terms of Bessel functions and using the first relation in (16) in conjunction with Stirling's formula (18). This gives the relation

$$g(u) \sim \left(\frac{2}{ka}\right) e^{i\pi/4}(2u/\pi)^{1/2} \cosh\left\{u \log\,[2u/(kae)] + \frac{i\pi}{4}\right\}\left\{1 + O\left(\frac{1}{u}\right)\right\}$$

so that

$$u_n \log\,[2u_n/(kae)] = (n + \tfrac{1}{4})i\pi + O(u_n^{-1}). \tag{25}$$

If we set $u = Re^{i\theta}$ and define the functions

$$g_2 = R\{\cos\theta \log\,[2R/(kae)] - \theta \sin\theta\}, \tag{26}$$

$$g_0 = R\{\sin\theta \log\,[2R/(kae)] + \theta \cos\theta\}, \tag{27}$$

then the large zeros occur when $g_2 = 0$ and $g_0 = (n + \frac{1}{4})\pi$, where n is a sufficiently large integer. The equation $g_0 = (n - \frac{1}{4})\pi$ defines a path C_n which avoids the zeros. On C_n it can be shown by analogy with [5, p. 120] that $H_u^{(1)}(kr)/g(u)$ is $O[R^{-1}(r/a)^{R\cos\theta}]$ and that

$$\begin{aligned} \frac{u\Psi_u(k, r)e^{\lambda u^2}}{g(u)} = O[R^{1/2} \exp\{&\lambda R^2 \cos 2\theta + R\theta \sin\theta \\ &- R\cos\theta \log\,[2R/(ker)]\}]. \end{aligned} \tag{28}$$

This bound applies in the sector $\psi \leqq \theta \leqq \pi/2$, a similar bound holds in the sector $-\pi/2 \leqq \theta \leqq -\psi$. Suitable asymptotic bounds on the function $F_1(u)$ can be obtained from equations (22), (28) of [4] where it is shown that

$$(29)\quad F_1(u) = \begin{cases} O[\exp\{(\pi + |\theta|)|R\sin\theta| + R\cos\theta\log[(4R\cos^2\theta)/(kae)]\}], \\ \hfill \operatorname{Re}(u) \geqq \frac{1}{2}, \\ O\{\exp[(\pi + |\theta|)|R\sin\theta| - R\cos\theta\log(R/e)]\}, \quad 0 \leqq \operatorname{Re}(u) \leqq \frac{1}{2}, \end{cases}$$

as $R \to \infty$ in their respective domains.

An inspection of the bounds (28), (29) shows that the deformation of W onto the imaginary axis is permissible, the dominant term in the exponential being $\lambda R^2 \cos 2\theta$ which is negative in the region crossed.

4. The convergence theorem. With the aid of formula (8) just established it is now possible to proceed to a proof of the main result of the paper, to the effect that the function $f(r)$ can be represented by equation (2) whenever the series appearing there is convergent. The method used depends on an Abelian-type argument to show that the series in (8) will be uniformly convergent for $\lambda \geqq 0$ whenever the series appearing in (2) is convergent. In these circumstances the validity of equation (2) itself will follow from (8) on taking the term by term limit as $\lambda \to 0$.

The uniform convergence property of the series in (8) can be proved by writing this equation in the form

$$f(r) = \lim_{\lambda \to 0} \sum a_n v_n,$$

where $v_n = \exp(\lambda u_n^2)$, and verifying that the series $\sum |v_{n+1} - v_n|$ is convergent and that its sum is bounded uniformly in λ. With this aim in view we note the identity

$$(30)\qquad v_{n+1} - v_n = 2\exp\left[\frac{\lambda}{2}(u_{n+1}^2 + u_n^2)\right]\sinh\left[\frac{\lambda}{2}(u_{n+1}^2 - u_n^2)\right].$$

The asymptotic behavior of $u_n = R_n e^{i\theta_n}$ can be obtained from (25). Equations of this type have been investigated in [2], [3], [1]. The sequences R_n, $R_n\cos\theta_n$, $R_n\sin\theta_n$ tend steadily to infinity as $n \to \infty$. The angle θ_n is also steadily increasing and tends to $\pi/2$. With this notation (30) becomes

$$(31)\qquad \begin{aligned} |v_{n+1} - v_n| &= 2\exp\left[\frac{\lambda}{2}(R_{n+1}^2\cos 2\theta_{n+1} + R_n^2\cos 2\theta_n)\right] \\ &\quad \cdot[\sinh^2(\tfrac{1}{2}\lambda x) + \sin^2(\tfrac{1}{2}\lambda y)]^{1/2}, \end{aligned}$$

where

$$x = R_n^2\cos 2\theta_n - R_{n+1}^2\cos 2\theta_{n+1},$$
$$y = R_{n+1}^2\sin 2\theta_{n+1} - R_n^2\sin 2\theta_n.$$

It follows from the identities $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = 2\cos^2\theta - 1$ together with the monotone properties of the sequences $R_n\cos\theta_n$, $R_n\sin\theta_n$, that $y > 0$ and $x < R_{n+1}^2 - R_n^2$. Also since $2\theta_n \to \pi$ then

$$0 < \sin 2\theta_{n+1} \leqq \sin 2\theta_n \leqq -\cos 2\theta_n \leqq -\cos 2\theta_{n+1}$$

so that

$$y \leqq (R_{n+1}^2 - R_n^2)\sin 2\theta_n \leqq (R_n^2 - R_{n+1}^2)\cos 2\theta_n \leqq x.$$

Hence

$$0 < y \leqq x \leqq R_{n+1}^2 - R_n^2 \leqq 2(R_{n+1} - R_n)R_{n+1};$$

therefore

$$\sin(\lambda y/2) \leqq \sinh(\lambda y/2) \leqq \sinh(\lambda x/2) \leqq \sinh[\lambda(R_{n+1} - R_n)R_{n+1}].$$

In addition since $\cos 2\theta_n \to -1$ the argument in the exponential function appearing in (31) is evidently less than $-\frac{1}{4}\lambda R_{n+1}^2$ for n large enough. Equation (31) may then be written as

$$|v_{n+1} - v_n| \leqq 2\sqrt{2}\exp\left(-\frac{\lambda}{4}R_{n+1}^2\right)\sinh[\lambda(R_{n+1} - R_n)R_{n+1}] \tag{32}$$

for $n \geqq N$ say. This expression can be simplified further by using the fact that the difference $(R_{n+1} - R_n)$ tends to zero as $n \to \infty$. To obtain this property we take the modulus of (25) which gives the equation

$$R_n\sqrt{\theta_n^2 + \log\left(\frac{2R_n}{kae}\right)^2} = (n + \tfrac{1}{4})\pi + O(R_n^{-1}). \tag{33}$$

On replacing n by $(n + 1)$ and using the fact that $\theta_n \leqq \theta_{n+1}, R_n \leqq R_{n+1}$, we find that

$$R_{n+1}\sqrt{\theta_n^2 + \log\left(\frac{2R_n}{kae}\right)^2} \leqq (n + \tfrac{5}{4})\pi + O(R_n^{-1}). \tag{34}$$

From (33) and (34) by subtraction we deduce that

$$(R_{n+1} - R_n)\sqrt{\theta_n^2 + \log\left(\frac{2R_n}{kae}\right)^2} \leqq \pi + O(R_n^{-1})$$

so that $(R_{n+1} - R_n)$ tends to zero as $n \to \infty$. If this property is used together with the inequality $\sinh(bx) \leqq b\sinh x$ (which applies whenever $x \geqq 0$ and $0 \leqq b \leqq 1$) to modify the argument appearing in the sinh function in (32), we find that

$$|v_{n+1} - v_n| \leqq 2\sqrt{2}\sqrt{\lambda}(R_{n+1} - R_n)\exp\left(-\frac{\lambda}{4}R_{n+1}^2\right)\sinh(\sqrt{\lambda}R_{n+1})$$

or

$$|v_{n+1} - v_n| \leqq 2\sqrt{2}(s_{n+1} - s_n)h(s_{n+1}),$$

where $s_n = \sqrt{\lambda}R_n$ and $h(s) = \exp(-\frac{1}{4}s^2)\sinh s$. By differentiation it can be shown that the function $h(s)$ is increasing for $0 \leqq s \leqq s_0$ and decreasing for $s \geqq s_0$, where s_0 is the positive root of the equation $s = 2\coth s$.

Hence

$$\begin{aligned}\sum |v_{n+1} - v_n| &\leqq 2\sqrt{2}\sum h(s_{n+1})(s_{n+1} - s_n)\\ &\leqq 2\sqrt{2}\,s_0 h(s_0) + 2\sqrt{2}\int_{s_0}^{\infty} h(s)\,ds.\end{aligned}$$

This inequality shows that the series $\sum |v_{n+1} - v_n|$ is convergent and that its sum is uniformly bounded for $0 \leqq \lambda \leqq 1$. It follows from an Abel type of criterion that the series $\sum a_n v_n$, where a_n is independent of λ, will be uniformly convergent for $0 \leqq \lambda \leqq 1$ whenever the series $\sum a_n$ is convergent. In particular, the series appearing in (8) will be uniformly convergent whenever that appearing in (2) is convergent and in this event the validity of the representation (2) follows from (8) on letting $\lambda \to 0$.

Appendix. It remains to establish the formula (13) for the Green's function. To this end we note from (16) and (17) that

$$\frac{u\Psi_u(k, r)H_u^{(1)}(k\rho)}{(u^2 - v^2)g(u)} \sim \frac{-i}{\pi u^2}\left[\left(\frac{r}{\rho}\right)^u + \left(\frac{\rho r}{a^2}\right)^{-u}\right]$$

as $u \to \infty$ in $|\arg u| \leqq \psi \leqq \pi/2$. If $r \leqq \rho$, the above expression is $O(u^{-2})$ uniformly in λ as $u \to \infty$ on W, since $|e^{\lambda u^2}| \leqq 1$ thereon. For such values of r, ρ the integral (13) is uniformly convergent for $\lambda \geqq 0$ and the value of the limit may be obtained by setting $\lambda = 0$, closing the contour on the right and taking the residue at $u = v$. The expression (13) then reduces, for $\rho \geqq r$, to the first of the expressions appearing in (12) as required.

The above procedure cannot be used directly when $\rho < r$ for then the integral in (13) is not convergent without the summability factor. The validity of (13) for such values of r, ρ may be established as follows. On using the definition (7) it follows that

$$G(r, \rho) - G(\rho, r) = \lim_{\lambda \to 0} \int_W \frac{e^{\lambda u^2}[J_u(kr)H_u^{(1)}(k\rho) - J_u(k\rho)H_u^{(1)}(kr)]u\, du}{2(u^2 - v^2)} .$$

The integrand here is an odd function of u and is

$$O\left\{R^{-2} \exp\left[\lambda R^2 \cos 2\theta + R \cos\theta \left|\log \frac{r}{\rho}\right|\right]\right\}$$

as $R \to \infty$. The path W may be deformed onto the imaginary axis whereupon it is seen that $G(r, \rho) = G(\rho, r)$ for all values of r, ρ since the integral along the entire imaginary axis is zero. The value of the integral in (13) when $r > \rho$ can therefore be obtained by interchanging r, ρ. The resulting integral is uniformly convergent and may be evaluated as before by setting $\lambda = 0$ and taking the residue at $u = v$. This procedure shows that the value of the expression (13) when $r > \rho$ is equal to the second of the expressions in (12).

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ON THE SOLUTION OF AN INTEGRAL EQUATION OF CONVOLUTION TYPE*

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Abstract. The solution Q_λ^μ of the integral equation

$$xQ_\lambda^\mu(x) = [\{(\lambda - \mu) + \mu f\} * Q_\lambda^\mu](x)$$

is considered as a transform of the function f; f is a function of the real variable x, and λ and μ are real-valued parameters with λ positive.

The functions $Q_\lambda^\mu[f]$ with f fixed form a group under the convolution operator.

The functions $F_\lambda^\mu(x) = \Gamma(\lambda)x^{1-\lambda}Q_\lambda^\mu[f](x)$, occurring in elementary particle physics, exhibit properties, such as their analyticity and behavior at the origin and at infinity, similar to those of the input function $f(x)$.

Besides several applications concerning hypergeometric functions, a table is presented giving various input functions f and their transforms F_λ^μ.

1. Introduction. In this paper we consider the convolution equation

$$xq(x) = (k * q)(x), \qquad 0 < x < \infty, \tag{1.1}$$

where k is a given function of the real variable x, analytic in $x = 0$ with $k(0) > 0$. Moreover, we assume that $k(x)$ is the restriction of a function $k(z) = k(x + iy)$, which is holomorphic in a sector S_θ in the complex z-plane with S_θ defined as

$$-\frac{\pi}{2} + \theta < \arg z < +\frac{\pi}{2} - \theta, \qquad 0 < \theta < \frac{\pi}{2}.$$

The function $k(z) - k(0) = O(e^{\omega z})$, uniformly in arg z, for some complex number ω, as $z \to \infty$ in S_θ.

The symbol $*$ stands for the convolution operator

$$(k * q)(x) = \int_0^x k(x - y)q(y)\,dy = \int_0^x k(y)q(x - y)\,dy. \tag{1.2}$$

Because the solution of (1.1) is uniquely determined apart from a multiplicative constant, we use a suitable boundary condition. As will be shown in § 2, the function q behaves near the origin as

$$q(x) \sim C\frac{x^{k(0)-1}}{\Gamma(k(0))},$$

and we shall specify our solution of (1.1) by demanding that $C = 1$.

The boundary condition then reads

$$\lim_{x \downarrow 0} \Gamma(k(0))x^{1-k(0)}q(x) = 1. \tag{1.3}$$

The most interesting aspect of (1.1) is not so much its explicit solution as the relation between solutions for different functions k, which differ only by a real linear

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transformation, viz., $\alpha k + \beta$ with α and β real parameters. Therefore, without loss of generality, we put

$$k(x) = \lambda - \mu(1 - f(x)), \tag{1.4}$$

with $f(0) = 1$, λ, μ real numbers and $\lambda > 0$.

We consider the set of solutions $Q_\lambda^\mu[f]$ of (1.1) for different values of λ and μ and the same function f. In this way one generates from each function f a class of functions $Q_\lambda^\mu[f]$. It will be shown that in a certain sense these $Q_\lambda^\mu[f]$ form a group with respect to the index μ and that the $Q_\lambda^{\alpha\lambda}[f]$, $-\infty < \alpha < +\infty$, form a half-group with respect to the index λ; the group operation is again the convolution. Furthermore, we consider the set of functions

$$F_\lambda^\mu(x) = \Gamma(\lambda)x^{1-\lambda}Q_\lambda^\mu[f](x). \tag{1.5}$$

Owing to the boundary condition (1.3), the functions $F_\lambda^\mu(x)$ are bounded at $x = 0$ with $F_\lambda^\mu(0) = 1$.

These functions occur in the theory of many particle production phenomena in elementary particle physics (see [1], [2]). It will be proved that the functions $F_\lambda^\mu(x)$ are also restrictions of functions $F_\lambda^\mu(z)$, holomorphic in S_θ, and that they have in a neighborhood of $x = 0$ and in a neighborhood of infinity a behavior similar to that of the input function f.

In the following, we call F_λ^μ the (μ, λ) *shadow transform* of the function f, a name we have chosen to suggest the similarity between f and the F_λ^μ. Writing $\mu = \alpha\lambda$, we denote by the symbol $S[f]$ the set of all $F_\lambda^{\alpha\lambda}$ with α, λ real and $\lambda > 0$. If we take the "closure" of this set by taking the limits $\lambda \downarrow 0$ and $\lambda \to \infty$ we obtain the following interesting results:

$$\lim_{\lambda \downarrow 0} F_\lambda^{\alpha\lambda}(x) = (1 - \alpha) + \alpha f(x) \tag{1.6}$$

and

$$\lim_{\lambda \to \infty} F_\lambda^{\alpha\lambda}(x) = \exp[\alpha f'(0)x]. \tag{1.7}$$

The set $S[f]$ together with the functions $\lim_{\lambda \downarrow 0} F_\lambda^{\alpha\lambda}$ and $\lim_{\lambda \to \infty} F_\lambda^{\alpha\lambda}$ will be called the *shadow class* $\bar{S}[f]$.

Hence the function f itself as well as its linear transforms $(1 - \alpha) + \alpha f$ belong to $\bar{S}[f]$; the same is true for the exponential functions $e^{\gamma x}$ as long as $f'(0) \neq 0$.

The solution of (1.1) is constructed in §2.1, while in §2.2 the common properties of the input function and the transform F_λ^μ are deduced. Moreover, some formulas are given which express the transforms of related input functions, such as $(1 - \beta) + \beta f(\gamma x)$ and $e^{\beta x} f(x)$, in terms of the transform of the function f. In §3 we prove the above-stated group properties of the functions Q_λ^μ and $Q_\lambda^{\alpha\lambda}$; the closure $\bar{S}[f]$ of the class $S[f]$ is constructed in § 4, and in § 5 we present several examples of shadow classes $\bar{S}[f]$. Finally, in §6 we give applications of the group properties of the functions Q_λ^μ; in particular, convolution integrals involving shadow transforms can easily be calculated. Another application is the solution of a generalization of Abel's integral equation.

The motivation for the present investigation is mainly the occurrence of the shadow transforms F^{μ}_{λ} in the context of calculations concerning many particle production phenomena in elementary particle physics as dealt with by the first author [1], [2].

2. The solution of the convolution equation and some properties of the shadow transformation.

2.1. The transform $Q^{\mu}_{\lambda}[f]$. In this section we look for solutions, locally integrable and exponentially bounded at infinity, of the convolution equation

$$xq(x) = (k * q)(x), \qquad 0 < x < \infty, \tag{1.1}$$

where k is a given function of the real variable x, analytic in $x = 0$ with $k(0) > 0$.

Defining the sector S_θ in the complex z-plane as

$$-\frac{\pi}{2} + \theta < \arg z < \frac{\pi}{2} - \theta, \qquad 0 < \theta < \frac{\pi}{2}, \tag{2.1}$$

we assume also that $k(x)$ is the restriction of a function $k(z)$, holomorphic in S_θ, with $k(z) - k(0) = O(e^{\omega z})$ uniformly in arg z, for some complex number ω, as $z \to \infty$ in S_θ.

Under the hypothesis that (1.1) indeed possesses solutions which are locally integrable and exponentially bounded at infinity, we can solve (1.1) by using the Laplace transformation. By means of the convolution property of this transformation, we obtain for the transform

$$\mathscr{L}[q](s) = \tilde{q}(s) = \int_0^\infty q(x)e^{-xs}\,dx \tag{2.2}$$

the differential equation

$$-\frac{d\tilde{q}(s)}{ds} = \tilde{k}(s)\tilde{q}(s), \tag{2.3}$$

with

$$\begin{aligned}\tilde{k}(s) = \mathscr{L}[k](s) &= \int_0^\infty k(x)e^{-xs}\,dx \\ &= \int_0^\infty \{k(x) - k(0)\}e^{-xs}\,dx + \frac{k(0)}{s}.\end{aligned} \tag{2.4}$$

Because of the assumptions for $k(x)$, the transform $\tilde{k}(s)$ is holomorphic in the sector

$$-\pi + \theta < \arg(s - \omega) < +\pi - \theta, \tag{2.5}$$

with the possible exception of the point $s = 0$, (see, for instance, [3, p. 33 ff.]), and (2.3) can be solved in this sector. Hence, we have

$$\tilde{q}(s) = A \exp\left\{-\int_{s_1}^{s} \tilde{k}(s')\,ds'\right\}, \tag{2.6}$$

with s a fixed complex number and A the constant of integration; the path of integration lies inside the domain (2.5), but the point $s = 0$ is avoided.

In order to reduce (2.6) we write

$$\tilde{q}(s) = As_1^{k(0)}\left\{s^{-k(0)} \exp\left[\int_{s_1}^{s} \left\{\frac{k(0)}{s'} - \tilde{k}(s')\right\} ds'\right]\right\},$$

where $s^{-k(0)}$ is uniquely defined by $\exp[-k(0)\log s]$ for $s > 0$ and by introducing a cut along the negative real axis in the complex s-plane; the path of integration lies again inside the domain (2.5) but outside the cut. Since $k(x)$ is analytic at $x = 0$, we may apply Watson's lemma (see, for example, [3, p. 34]), and we have

$$\tilde{k}(s') = \frac{k(0)}{s'} + O\left(\frac{1}{s'^2}\right),$$

uniformly with respect to $\arg s'$ as $s' \to \infty$ in the sector (2.5).

Hence $\tilde{q}(s)$ may also be written in the form

$$\tilde{q}(s) = Cs^{-k(0)} \exp\left[-\int_{s}^{\infty} \left\{\frac{k(0)}{s'} - \tilde{k}(s')\right\} ds'\right],$$

where C is still an arbitrary constant and where the path of integration is situated inside the domain (2.5) but outside the cut.

With the aid of a well-known formula from the theory of Laplace transformation, the latter expression can be reduced to

$$\tilde{q}(s) = Cs^{-k(0)} \exp\left[-\int_{0}^{\infty} e^{-st}\left\{\frac{k(0) - k(t)}{t}\right\} dt\right]. \tag{2.7}$$

The integral

$$\int_{0}^{\infty} e^{-st}\left\{\frac{k(0) - k(t)}{t}\right\} dt$$

is uniformly bounded for $\operatorname{Re} s = \sigma \geqq \sigma_0 > \operatorname{Re} \omega$ with σ_0 any arbitrary real number larger than $\operatorname{Re} \omega$. Using again the abovementioned lemma of Watson, we see it is clear that

$$-\int_{0}^{\infty} e^{-st}\left\{\frac{k(0) - k(t)}{t}\right\} dt \sim \sum_{n=1}^{\infty} \frac{k^{(n)}(0)}{n} s^{-n} \tag{2.8}$$

uniformly in $\arg s$ as $s \to \infty$ in the sector (2.5).

Therefore, we have in this sector for s sufficiently large the uniform estimate

$$\exp\left[-\int_{0}^{\infty} e^{-st}\left\{\frac{k(0) - k(t)}{t}\right\} dt\right] = 1 + \frac{k'(0)}{s} + O\left(\frac{1}{s^2}\right). \tag{2.9}$$

Since $k(0) > 0$, the function $\tilde{q}(s)$ is the Laplace transform of the locally integrable function

$$q(x) = \frac{C}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}s^{-k(0)} \exp\left[-\int_{0}^{\infty} e^{-st}\left\{\frac{k(0) - k(t)}{t}\right\} dt\right] ds, \qquad x > 0, \tag{2.10}$$

the path of integration lies in the domain $\operatorname{Re} s \geqq \sigma_0 > \operatorname{Re} \omega$, and c may be any positive real number with $c > \operatorname{Re} \omega$.

Equation (2.10) may be written in the form

$$
\begin{aligned}
q(x) &= \frac{C}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} s^{-k(0)}\, ds \\
&\quad + \frac{C}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} s^{-k(0)} \left\{ \exp\left[- \int_0^\infty e^{-st} \left\{ \frac{k(0)-k(t)}{t} \right\} dt \right] - 1 \right\} ds \\
&= \frac{C x^{k(0)-1}}{\Gamma(k(0))} + \frac{C}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{sx} s^{-k(0)} \left\{ \exp\left[- \int_0^\infty e^{-st} \left\{ \frac{k(0)-k(t)}{t} \right\} dt \right] - 1 \right\} ds,
\end{aligned}
\tag{2.11}
$$

valid for $x > 0$; because the path of integration lies in the domain (2.5) and because in this domain the estimate (2.9) is valid, the integral in the right-hand side of (2.11) converges uniformly with respect to x.

Using the estimate (2.9), we see immediately that

$$
q(x) = O(e^{cx}) \tag{2.12}
$$

for $x \to \infty$.

The function $q(x)$ given by (2.10) is locally integrable and exponentially bounded at infinity, and therefore it is indeed the solution of the integral equation (1.1) for $0 < x < \infty$.

Alternative forms for the solution $q(x)$ are

$$
\begin{aligned}
q(x) &= C \frac{x^{k(0)-1}}{\Gamma(k(0))} \left[1 + \frac{\Gamma(k(0))}{2\pi i} \right. \\
&\qquad \left. \cdot \int_{cx-i\infty}^{cx+i\infty} e^{s} s^{-k(0)} \left\{ \exp\left[- \int_0^\infty e^{-st/x} \left\{ \frac{k(0)-k(t)}{t} \right\} dt \right] - 1 \right\} ds \right] \\
&= C \frac{x^{k(0)-1}}{\Gamma(k(0))} \left[1 + \frac{\Gamma(k(0))}{2\pi i} \right. \\
&\qquad \left. \cdot \int_{cx-i\infty}^{cx+i\infty} e^{s} s^{-k(0)} \left\{ \exp\left[- \int_0^\infty e^{-st} \left\{ \frac{k(0)-k(xt)}{t} \right\} dt \right] - 1 \right\} ds \right],
\end{aligned}
\tag{2.13}
$$

with $x > 0$ and $c > \max(\operatorname{Re} \omega, 0)$. We remark once more that the integrals in the s-plane converge uniformly with respect to x.

Because of this convergence, one verifies easily that $\Gamma(k(0))x^{1-k(0)}q(x)$ is continuous for $x \geqq 0$; taking the constant C equal to 1, we find that

$$
\lim_{x \downarrow 0} \Gamma(k(0)) x^{1-k(0)} q(x) = 1. \tag{2.14}
$$

Introducing

$$
k(x) = \lambda - \mu\{1 - f(x)\} \tag{1.4}
$$

with $f(0) = 1$ and λ, μ real numbers and $\lambda > 0$, we obtain for the image $Q^{\mu}_{\lambda}[f]$ under the transformation induced by (1.1) the formula

$$Q^{\mu}_{\lambda}[f](x) = \frac{x^{\lambda-1}}{\Gamma(\lambda)}\left[1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{cx-i\infty}^{cx+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\mu\int_0^{\infty} e^{-st}\frac{1-f(xt)}{t}\,dt\right]-1\right\}ds\right], \tag{2.15}$$

where c is any positive number larger than $\operatorname{Re}\omega$; $x > 0$.

For practical purposes we sometimes prefer the equivalent expression

$$Q^{\mu}_{\lambda}[f](x) = \mathscr{L}^{-1}\left\{s^{-\lambda}\exp\left[-\mu\mathscr{L}\left\{\frac{1-f(y)}{y}\right\}(s)\right]\right\}(x), \qquad x > 0. \tag{2.16}$$

2.2. The shadow transform F^{μ}_{λ}. The (μ, λ) *shadow transform* of the function f is defined as

$$F^{\mu}_{\lambda}(x) = \Gamma(\lambda)x^{1-\lambda}Q^{\mu}_{\lambda}[f](x), \qquad x \geqq 0, \tag{1.5}$$

with $x^{1-\lambda}$ real for $x > 0$.

According to (2.15) we have the formulas

$$F^{\mu}_{\lambda}(x) = 1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{cx-i\infty}^{cx+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\mu\int_0^{\infty} e^{-st}\frac{1-f(xt)}{t}\,dt\right]-1\right\}ds$$

$$= 1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{cx-i\infty}^{cx+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\mu\int_0^{\infty} e^{-st/x}\frac{1-f(t)}{t}\,dt\right]-1\right\}ds, \qquad x > 0, \tag{2.17}$$

with c any positive number larger than $\operatorname{Re}\omega$.

The function $F^{\mu}_{\lambda}(x)$ is continuous for $0 \leqq x < \infty$ with $F^{\mu}_{\lambda}(0) = 1$ (see (2.14)). At this stage it is not difficult to show that $F^{\mu}_{\lambda}(x)$ is the restriction of a function $F^{\mu}_{\lambda}(z)$, holomorphic in the segment S_θ. For this purpose we define

$$F^{\mu}_{\lambda}(x) = 1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\mu\int_0^{\infty} e^{-st/z}\frac{1-f(t)}{t}\,dt\right]-1\right\}ds, \qquad z \in S_\theta, \tag{2.18}$$

with $d = \max\{\operatorname{Re}(z(\omega+\varepsilon)), \operatorname{Re}\varepsilon z\}$ and ε arbitrarily small and positive.

The expression

$$\exp\left[-\mu\int_0^{\infty} e^{-st/z}\frac{1-f(t)}{t}\,dt\right]-1$$

is holomorphic in (s/z) in the sector

$$-\pi + \theta < \arg\left(\frac{s}{z} - \omega\right) < +\pi - \theta, \tag{2.19}$$

and it is uniformly asymptotically equal to $\mu f'(0)z/s + O((z/s)^2)$ for $(s/z) \to \infty$ in this sector.

The relation (2.19) is fulfilled whenever $\operatorname{Re} s = d$ and

$$-\frac{\pi}{2} + \theta < \arg z < +\frac{\pi}{2} - \theta.$$

Hence the expression in the right-hand side of (2.18) exists and the outer integral converges uniformly with respect to z for all z within a closed subdomain of S_θ; it follows that $F_\lambda^\mu(z)$ is holomorphic in S_θ and the derivative of $F_\lambda^\mu(z)$ may be obtained by differentiation under the sign of integration.

Because (2.17) is the restriction of (2.18) for $z = x$, the function $F_\lambda^\mu(x)$ is the restriction of $F_\lambda^\mu(z)$, which is holomorphic in the segment S_θ.

Using again the uniform asymptotic behavior of the exponential function in (2.18) and taking the limit under the sign of integration, we find that

$$\lim_{\substack{z\to 0\\ z\in S_\theta}} F_\lambda^\mu(z) = 1. \tag{2.20}$$

Before investigating the behavior of $F_\lambda^\mu(z)$ in S_θ for $z \to \infty$, we remark that the right-hand side of (2.18), after substitution of $s = zs'$, can be written in the form

$$F_\lambda^\mu(z) = 1 + \frac{\Gamma(\lambda)}{2\pi i} z^{-\lambda+1} \int e^{zs'}(s')^{-\lambda}\left\{\exp\left[-\mu\int_0^\infty e^{-s't}\frac{1-f(t)}{t}\,dt\right]-1\right\}ds', \tag{2.21}$$

where the integration variable s' satisfies the relations

$$\operatorname{Re}(zs') = \max\{\operatorname{Re}(z(\omega+\varepsilon)), \operatorname{Re}(\varepsilon z)\} \quad\text{and}\quad -\infty < \operatorname{Im}(zs') < +\infty, \qquad \varepsilon > 0.$$

Because s satisfies (2.19), the integration variable s' must lie in the sector $-\pi + \theta < \arg(s' - \omega) < +\pi - \theta$, and hence

$$\exp\left[-\mu\int_0^\infty e^{-s't}\frac{1-f(t)}{t}\,dt\right]-1 = \mu\frac{f'(0)}{s'} + O\left(\left(\frac{1}{s'}\right)^2\right)$$

for $s' \to \infty$ in this sector.

It follows that the outer integral in (2.21) converges uniformly with respect to z and

$$F_\lambda^\mu(z) = O(e^{\chi(z)}) \tag{2.22}$$

uniformly for $z \to \infty$ in S_θ, with

$$\chi(z) = \max\{\operatorname{Re}(z(\omega+\varepsilon)), \operatorname{Re}(\varepsilon z)\} \tag{2.23}$$

and ε arbitrarily small and positive.

Summarizing these results, we have now the following theorem.

THEOREM 1. *If $f(z)$ is regular in the segment S_θ, $f(z) - 1 = O(e^{\omega z})$, uniformly in* arg z *for some ω, as $z \to \infty$ in S_θ, and if $f(z)$ is analytic in $z = 0$ with $f(0) = 1$, then the shadow transform $F_\lambda^\mu(x)$ exists for all real $\lambda > 0$ and all real μ; moreover, $F_\lambda^\mu(x)$ is the restriction of a function $F_\lambda^\mu(z)$, also regular in the segment S_θ, and $F_\lambda^\mu(z) = O(e^{\chi(z)})$, uniformly in* arg z *as $z \to \infty$ in S_θ; $\chi(z) = \max\{\operatorname{Re}(z(\omega + \varepsilon z)), \operatorname{Re}(\varepsilon z)\}$ with ε arbitrarily small and positive. Further, $\lim_{z\to 0, z\in S_\theta} F_\lambda^\mu(z) = 1$.*

It follows from (2.17) that $F_\lambda^\mu(x)$ for $0 \leqq x < \infty$ may be written as

$$F_\lambda^\mu(x) = 1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\mu\int_0^\infty e^{-st}\frac{1-f(xt)}{t}\,dt\right]-1\right\}ds.$$

Because the integral converges uniformly in the complex s-plane, $F_\lambda^\mu(x)$ may be differentiated with respect to x by taking the derivative under the sign of integration, and so we have

$$\frac{dF_\lambda^\mu(x)}{dx} = \frac{\Gamma(\lambda)}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^s s^{-\lambda} \exp\left[-\mu\int_0^\infty e^{-st}\frac{1-f(xt)}{t}\,dt\right]\left(\mu\int_0^\infty e^{-st}f'(xt)\,dt\right)ds.$$

Using Cauchy's theorem, it is not difficult to show also that $f'(z)$ is $O(e^{\omega z})$, uniformly in any segment S^* with closure in S.

Applying Watson's lemma once again we can see that the limit for $x \to 0$ may be taken under the sign of integration, and so we obtain the interesting result that

$$\lim_{x\downarrow 0}\frac{dF_\lambda^\mu}{dx}(x) = \frac{dF_\lambda^\mu}{dx}(0) = \frac{\mu}{\lambda}f'(0). \tag{2.24}$$

The shadow transform $F_\lambda^\mu(x)$ has some more remarkable properties, which are stated in the next two theorems.

THEOREM 2. *If $f(z)$ satisfies the conditions of Theorem* 1 *and if $f(x)$ has the shadow transform $F_\lambda^\mu(x)$, then $g(x) = (1-\alpha) + \alpha f(\beta x)$ has the shadow transform*

$$G_\lambda^\mu(x) = F_\lambda^{\alpha\mu}(\beta x),$$

with α arbitrarily real and β arbitrarily positive.

Proof. Substituting the function $g(x) = (1-\alpha) + \alpha f(\beta x)$ in the first formula of (2.17), we obtain

$$G_\lambda^\mu(x) = 1 + \frac{\Gamma(\lambda)}{2\pi i}\int_{dx-i\infty}^{dx+i\infty} e^s s^{-\lambda}\left\{\exp\left[-\alpha\mu\int_0^\infty e^{-st}\frac{1-f(\beta xt)}{t}\,dt\right]-1\right\}ds,$$

with d any positive number larger than β Re ω.

It follows immediately that

$$G_\lambda^\mu(x) = F_\lambda^{\alpha\mu}(\beta x). \tag{2.25}$$

Remark. If $f(z)$ is an entire function of z and such that a neighborhood of infinity can be covered by a finite number of segments S_k and $f(z) = O(e^{\omega_k z})$, uniformly in arg z, for some ω_k, as $z \to \infty$ in S_k, then β may be taken arbitrarily complex. This remark will be used frequently later on in §5 and §6, where applications of shadow transforms will be given.

THEOREM 3. *If $f(x)$ has for $\mu = \lambda$ the shadow transform $F_\lambda^\lambda(x)$, then $g(x) = e^{\beta x}f(x)$ has for $\mu = \lambda$ the transform*

$$G_\lambda^\lambda(x) = e^{\beta x}F_\lambda^\lambda(x), \tag{2.26}$$

with β arbitrarily complex.

Proof. For the function $Q_\lambda^\lambda[f](x)$, we have the relation

$$xQ_\lambda^\lambda[f](x) = \lambda\{f * Q_\lambda^\lambda[f]\}(x),$$

and therefore,

$$xe^{\beta x}Q_\lambda^\lambda[f](x) = \lambda\{e^{\beta x}f(x) * e^{\beta x}Q_\lambda^\lambda[f](x)\}.$$

It follows that the Q_λ^λ function corresponding with $e^{\beta x}f(x)$ is given by $e^{\beta x}Q_\lambda^\lambda[f](x)$, and hence the result of the theorem follows.

3. The group properties of Q^{μ}_{λ} and $Q^{\alpha\lambda}_{\lambda}$. From the formula (2.16) and the convolution property of the Laplace transform, we obtain, for $x > 0$,

$$(Q^{\mu_1}_{\lambda_1}[f] * Q^{\mu_2}_{\lambda_2}[f])(x) = \mathscr{L}^{-1}\left\{s^{-\lambda_1-\lambda_2}\exp\left[-(\mu_1+\mu_2)\mathscr{L}\left\{\frac{1-f(y)}{y}\right\}(s)\right]\right\}(x)$$

or

$$Q^{\mu_1}_{\lambda_1}[f] * Q^{\mu_2}_{\lambda_2}[f] = Q^{\mu_1+\mu_2}_{\lambda_1+\lambda_2}[f]. \tag{3.1}$$

In particular, applying (2.16), we find that, for $\mu_1 = -\mu_2 = \mu$,

$$(Q^{\mu}_{\lambda_1}[f] * Q^{-\mu}_{\lambda_2}[f])(x) = Q^{0}_{\lambda_1+\lambda_2}[f](x) = \frac{x^{\lambda_1+\lambda_2-1}}{\Gamma(\lambda_1+\lambda_2)}, \tag{3.2}$$

valid for $x > 0$.

We introduce now the set $A[f]$ consisting of elements

$$A_{\mu}[f] \equiv \{Q^{\mu}_{\lambda}[f]; \lambda > 0\}, \tag{3.3}$$

which are in their turn the sets of all functions $Q^{\mu}_{\lambda}[f]$ with f and μ fixed and with λ ranging from zero to infinity and $\lambda > 0$.

In the set $A[f]$ we define a convolution operator as follows:

$$A_{\mu_1}[f] * A_{\mu_2}[f] = \{Q^{\mu_1}_{\lambda_1}[f] * Q^{\mu_2}_{\lambda_2}[f]; \lambda_1 > 0, \lambda_2 > 0\}. \tag{3.4}$$

Using (3.1) we obtain

$$\begin{aligned} A_{\mu_1}[f] * A_{\mu_2}[f] &= \{Q^{\mu_1+\mu_2}_{\lambda_1+\lambda_2}[f], \lambda_1 > 0, \lambda_2 > 0\} \\ &= \{Q^{\mu_1+\mu_2}_{\lambda}[f], \lambda > 0\} = A_{\mu_1+\mu_2}[f]. \end{aligned} \tag{3.5}$$

In particular, according to (3.2) we have

$$A_{\mu}[f] * A_{-\mu}[f] = A_0[f] = \{Q^{0}_{\lambda}[f]; \lambda > 0\} = \left\{\frac{x^{\lambda-1}}{\Gamma(\lambda)}; \lambda > 0\right\}. \tag{3.6}$$

From the results (3.5) and (3.6) we have obtained the following theorem.

THEOREM 4. *The set $A[f]$ is a commutative group under the convolution operation; the unit element is the element*

$$A_0[f] = \left\{\frac{x^{\lambda-1}}{\Gamma(\lambda)}; \lambda > 0\right\}.$$

Besides the set $A[f]$, we introduce also a set $B[f]$, consisting of elements

$$B_{\lambda}[f] \equiv \{Q^{\alpha\lambda}_{\lambda}[f]; -\infty < \alpha < +\infty\}, \tag{3.7}$$

which are the sets of all functions $Q^{\alpha\lambda}_{\lambda}[f]$ with f and $\lambda > 0$ fixed and α ranging from $-\infty$ to $+\infty$.

A convolution operator may be defined, similarly as in (3.4), by

$$\begin{aligned} B_{\lambda_1}[f] * B_{\lambda_2}[f] &= \{Q^{\alpha_1\lambda_1}_{\lambda_1}[f] * Q^{\alpha_2\lambda_2}_{\lambda_2}[f]; -\infty < \alpha_1, \alpha_2 < +\infty\} \\ &= \{Q^{\alpha_1\lambda_1+\alpha_2\lambda_2}_{\lambda_1+\lambda_2}[f]; -\infty < \alpha_1, \alpha_2 < +\infty\}. \end{aligned} \tag{3.8}$$

Taking $\beta = (\alpha_1\lambda_1 + \alpha_2\lambda_2)/\lambda_1\lambda_2$, we see that β ranges from $-\infty$ to $+\infty$ when α_1 and α_2 range from $-\infty$ to $+\infty$. Hence

$$(3.9)\qquad B_{\lambda_1}[f] * B_{\lambda_2}[f] = \{Q_{\lambda_1+\lambda_2}^{\beta(\lambda_1+\lambda_2)}[f];\ -\infty < \beta < +\infty\} = B_{\lambda_1+\lambda_2}[f].$$

A unit element is not so easily constructed as in the set $A[f]$ because λ is not allowed to be nonpositive. In order to do so, we add to the set $B[f]$ the element $B_0[f]$ defined as

$$(3.10)\qquad \lim_{\lambda\downarrow 0} B_\lambda[f],$$

where the limit is taken in the *weak* sense:

$$(3.11)\qquad \begin{aligned} B_0[f] * B_\lambda[f] &= \left\{\lim_{\lambda_1\downarrow 0} (Q_{\lambda_1}^{\alpha_1\lambda_1}[f] * Q_\lambda^{\alpha_2\lambda}[f]);\ -\infty < \alpha_1, \alpha_2 < +\infty\right\} \\ &= \left\{\lim_{\lambda_1\downarrow 0} Q_{\lambda+\lambda_1}^{\beta(\lambda+\lambda_1)}[f];\ -\infty < \beta < +\infty\right\} = B_\lambda[f], \end{aligned}$$

valid for all $B_\lambda[f]$ with $\lambda > 0$.

The set $B[f]$ together with the element $B_0[f]$ is denoted by $\bar{B}[f]$. The element $B_0[f]$ can be given a specified meaning as follows. In order to remove the singular factor $x^{\lambda-1}$ in $Q_\lambda^{\alpha\lambda}[f]$, we consider the limit of $F_\lambda^{\alpha\lambda}(x)$ for $\lambda \downarrow 0$.

In the expression (2.17) for $F_\lambda^\mu(x)$, we develop

$$\exp\left[-\mu\int_0^\infty e^{-st}\frac{1-f(xt)}{t}\,dt\right]$$

in a Taylor series with respect to μ. Putting $\mu = \alpha\lambda$ and taking the limit for $\lambda \downarrow 0$, we obtain, owing to the uniform convergence of the outer integral, for $x > 0$:

$$(3.12)\qquad \begin{aligned} \lim_{\lambda\downarrow 0} F_\lambda^{\alpha\lambda}(x) &= 1 - \lim_{\lambda\downarrow 0}\frac{\alpha\lambda\Gamma(\lambda)}{2\pi i}\int_{cx-i\infty}^{cx+i\infty} e^s s^{-\lambda}\left\{\int_0^\infty e^{-st}\frac{1-f(xt)}{t}\,ds\right\}ds \\ &= 1 - \frac{\alpha}{2\pi i}\int_{cx-i\infty}^{cx+i\infty} e^s\left\{\int_0^\infty e^{-st}\frac{1-f(xt)}{t}\,dt\right\}ds \\ &= 1 - \frac{\alpha x}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\left\{\int_0^\infty e^{-st}\frac{1-f(t)}{t}\,dt\right\}ds \\ &= 1 - \alpha(1 - f(x)) = (1-\alpha) + \alpha f(x). \end{aligned}$$

It is now quite natural to define $B_0[f]$ as

$$(3.13)\qquad \begin{aligned} B_0[f] &= \lim_{\lambda\downarrow 0} Q_\lambda^{\alpha\lambda}[f] = \lim_{\lambda\downarrow 0}\frac{x^{\lambda-1}}{\Gamma(\lambda)}F_\lambda^{\alpha\lambda}(x) \\ &= \lim_{\lambda\downarrow 0}\frac{x^{\lambda-1}}{\Gamma(\lambda)} + \lim_{\lambda\downarrow 0}\frac{x^{\lambda-1}}{\Gamma(\lambda)}\{\alpha(f(x)-1)\} \\ &= \lim_{\lambda\downarrow 0}\frac{x^{\lambda-1}}{\Gamma(\lambda)}, \end{aligned}$$

where the latter limit has a meaning only in the weak sense. This means that

$$B_0[f] * B_\lambda[f] = \left\{ \lim_{\nu \downarrow 0} \left(\frac{x^{\nu-1}}{\Gamma(\nu)} * Q_\lambda^{\alpha\lambda}[f] \right); -\infty < \alpha < +\infty \right\}. \tag{3.14}$$

Finally, using the well-known weak limit (see, for example, [4, pp. 64–65])

$$\lim_{\nu \downarrow 0} \frac{x_+^{\nu-1}}{\Gamma(\nu)} = \delta(x)$$

with $x_+^{\nu-1} = \theta(x)(x)^{\nu-1}$, $\theta(x)$ denoting Heaviside's unit step function, and $\delta(x)$ the Dirac distribution, we indeed obtain

$$B_0[f] * B_\lambda[f] = \{Q_\lambda^{\alpha\lambda}[f]; -\infty < \alpha < +\infty\} = B_\lambda[f].$$

Summarizing the last results, we obtain the following theorem.

THEOREM 5. *The set $\bar{B}[f]$ is a commutative half-group under the convolution operation; the unit element is*

$$B_0[f] = \lim_{\lambda \downarrow 0} \frac{x^{\lambda-1}}{\Gamma(\lambda)},$$

where the limit should be taken in the weak sense.

4. The shadow class $\bar{S}[f]$. The shadow class $\bar{S}[f]$ is the set of all shadow transforms $F_\lambda^{\alpha\lambda}$, $\lambda > 0$, $-\infty < \alpha < +\infty$, together with the functions

$$\lim_{\lambda \downarrow 0} F_\lambda^{\alpha\lambda} \quad \text{and} \quad \lim_{\lambda \to \infty} F_\lambda^{\alpha\lambda}.$$

The first limit has been calculated in the foregoing section, and the result is:

$$\lim_{\lambda \downarrow 0} F_\lambda^{\alpha\lambda}(x) = (1 - \alpha) + \alpha f(x). \tag{3.12}$$

In order to calculate $\lim_{\lambda \to +\infty} F_\lambda^{\alpha\lambda}$ we use (2.17), which may be written as

$$F_\lambda^{\alpha\lambda}(x) = \frac{\Gamma(\lambda)}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^s s^{-\lambda} \exp\left[-\alpha\lambda \int_0^\infty e^{-st} \frac{1 - f(xt)}{t} dt \right] ds,$$

valid for $0 \leqq cx < \lambda$. Expanding $(1 - f(xt))/t$ in a Taylor series and again using Watson's lemma, we obtain for all x in a bounded segment,

$$F_\lambda^{\alpha\lambda}(x) = \frac{\Gamma(\lambda)}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} e^s s^{-\lambda} \exp\left[\alpha f'(0) x \frac{\lambda}{s} \right] ds + O\left(\frac{1}{\lambda}\right).$$

Hence

$$\begin{aligned} \lim_{\lambda \to \infty} F_\lambda^{\alpha\lambda}(x) &= \lim_{\lambda \to \infty} \frac{\Gamma(\lambda)}{2\pi i} \sum_{n=0}^\infty \frac{(\alpha f'(0)x)^n}{n!} \lambda^n \int_{\lambda - i\infty}^{\lambda + i\infty} e^s s^{-\lambda - n} ds \\ &= \lim_{\lambda \to \infty} \sum_{n=0}^\infty \frac{(\alpha f'(0)x)^n}{n!} \frac{\lambda^n \Gamma(\lambda)}{\Gamma(\lambda + n)} \end{aligned}$$

or

$$\lim_{\lambda \to \infty} F_\lambda^{\alpha\lambda}(x) = \exp[\alpha f'(0)x]. \tag{4.1}$$

From the formulas (3.12) and (4.1), we have the following result.

THEOREM 6. *The shadow class* $\bar{S}[f]$ *contains the function* f *as well as all its linear transformations* $(1-\alpha)+\alpha f$, *with* $-\infty<\alpha<+\infty$. *The exponential function* $e^{\gamma x}$ *is an element of all shadow classes* $\bar{S}[f]$ *with* $f'(0)\neq 0$.

Remark. From the result (4.1) it is obvious that $\lim_{\lambda\to\infty} F_\lambda^{\alpha\lambda}(x)$ is not a shadow transform in itself, because it does not exhibit the exponential behavior with the right constant ω as is required by Theorem 1. It is clear that the limits $x\to\infty$ and $\lambda\to\infty$ should not be interchanged.

We conclude this section with another remark which will be applied frequently in the next section.

If the class $\bar{S}[f]$ is given by

$$\bar{S}[f]=\{F_\lambda^{\alpha\lambda}(x)\}, \tag{4.2}$$

then it follows from Theorem 2 that

$$\bar{S}[(1-\beta)+\beta f(\gamma x)]=\{F_\lambda^{\alpha\beta\lambda}(\gamma x)\} \tag{4.3}$$

for all real values of β and for all positive values of γ; γ may be taken arbitrarily complex if f satisfies the conditions stated in the remark following Theorem 2.

5. Examples of shadow classes.

5.1. Shadow classes of hypergeometric functions of one variable. In this section it will appear that hypergeometric functions of the types ${}_0F_k$, ${}_1F_k$, ${}_2F_k$ may be considered as representatives for certain shadow classes.

The hypergeometric function ${}_mF_n$ is defined as

$${}_mF_n(a_1,a_2,\cdots,a_m;\ b_1,b_2,\cdots,b_n;\ x)=\sum_{l=0}^{\infty}\frac{(a_1)_l(a_2)_l\cdots(a_m)_l}{(b_1)_l(b_2)_l\cdots(b_n)_l}\frac{x^l}{l!}, \tag{5.1}$$

with $(a_p)_l=\Gamma(a_p+l)/\Gamma(a_p)$ and $(b_p)_l=\Gamma(b_p+l)/\Gamma(b_p)$ (see, for example, [5, p. 182]). Hypergeometric functions of the types ${}_mF_n$ and ${}_mF_{n+k}$ are connected with each other by the formula

$$\begin{aligned}&\Gamma(\lambda)x^{1-\lambda}\mathscr{L}^{-1}[s^{-\lambda}{}_mF_n(a_1,a_2,\cdots,a_m;\ b_1,b_2,\cdots,b_n;\ \alpha s^{-k}](x)\\&\quad={}_mF_{n+k}\left\{a_1,a_2,\cdots,a_m;b_1,b_2,\cdots,b_n,\frac{\lambda}{k},\frac{\lambda+1}{k},\cdots,\frac{\lambda+k-1}{k};\ \alpha\left(\frac{x}{k}\right)^k\right\},\end{aligned} \tag{5.2}$$

valid for $m\leqq n+1$, $\lambda>0$, $-\infty<\alpha<+\infty$, $k=1,2,3,\cdots$, and $x>0$ (see [6, p. 297].

It follows now immediately from (1.5) and (2.16) that ${}_mF_{n+k}\{a_1,a_2,\cdots,a_m;$ $b_1,b_2,\cdots,b_n,\lambda/k,(\lambda+1)/k,\cdots,(\lambda+k-1)/k;\ \alpha(x/k)^k\}$ is a shadow transform $F_\lambda^\mu(x)$ whenever the hypergeometric function ${}_mF_n(a_1,a_2,\cdots,a_m;b_1,b_2,\cdots,$ $b_n;\alpha s^{-k})$ can be written in the form $\{\rho(s)\}^\mu$ with $\rho(s)$ independent of λ and μ. There are three hypergeometric functions which have this property, viz.,

$$\text{(a)}\qquad {}_0F_0(-\mu s^{-k})=\exp[-\mu s^{-k}], \tag{5.3}$$

$$\text{(b)}\qquad {}_1F_0\left(\frac{\mu}{k};-s^{-k}\right)=(1+s^{-k})^{-\mu/k}, \tag{5.4}$$

$$(5.5)\qquad \text{(c)}\quad {}_2F_1\left(\frac{\mu}{k}, \frac{\mu}{k} + \frac{1}{2}; 2\frac{\mu}{k} + 1; -s^{-k}\right) = 2^{2\mu/k}(1 + (1 + s^{-k})^{1/2})^{-2\mu/k}.$$

We now investigate each of these three cases, and the corresponding shadow classes will be constructed.

(a) From (5.2) we obtain

$$(5.6)\qquad F_\lambda^\mu(x) = {}_0F_k\left(\frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\mu\left(\frac{x}{k}\right)^k\right).$$

According to (3.12), the input function $f(x)$ is

$$f(x) = \lim_{\lambda\downarrow 0} F_\lambda^\lambda(x) = 1 + \lim_{\lambda\downarrow 0} \frac{k^k}{\lambda(\lambda+1)\cdots(\lambda+k-1)}(-\lambda)\left(\frac{x}{k}\right)^k = 1 - \frac{x^k}{(k-1)!}.$$
$$(5.7)$$

Taking the more general input function

$$(5.8)\qquad g(x) = f(\gamma x) = 1 - \frac{(\gamma x)^k}{(k-1)!},$$

where γ is an arbitrary complex number, we obtain from Theorem 2 the shadow transform

$$(5.9)\qquad G_\lambda^\mu(x) = F_\lambda^\mu(\gamma x) = {}_0F_k\left(\frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\mu\left(\frac{\gamma x}{k}\right)^k\right).$$

The shadow class $\bar{S}[g]$ consists of all functions

$$\bar{S}[g] = \left\{{}_0F_k\left(\frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\alpha\lambda\left(\frac{\gamma x}{k}\right)^k\right); \lambda \geqq 0, -\infty < \alpha < +\infty\right\},$$
$$(5.10)$$

and in particular, for $k = 1$, we obtain

$$\bar{S}[1 - \gamma x] = \{{}_0F_1(\lambda; -\alpha\gamma\lambda x); \lambda \geqq 0, -\infty < \alpha < +\infty\}$$
$$(5.11)$$
$$= \{\Gamma(\lambda)(\alpha\gamma\lambda x)^{-(\lambda-1)/2} J_{\lambda-1}(2\sqrt{\alpha\gamma\lambda x}); \lambda \geqq 0, -\infty < \alpha < +\infty\},$$

where $J_{\lambda-1}$ denotes the Bessel function of the first kind.

(b) From (5.2) we obtain

$$(5.12)\qquad F_\lambda^\mu(x) = {}_1F_k\left\{\frac{\mu}{k}; \frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\left(\frac{x}{k}\right)^k\right\}.$$

According to (3.12), the input function is

$$f(x) = \lim_{\lambda\downarrow 0} F_\lambda^\lambda(x) = {}_0F_{k-1}\left(\frac{1}{k}, \frac{2}{k}, \ldots, \frac{k-1}{k}; -\left(\frac{x}{k}\right)^k\right)$$
$$(5.13)\qquad = \sum_{m=0}^{\infty} \frac{(-1)^m}{(km)!} x^{km} = \frac{1}{k}\sum_{l=1}^{k} \exp\left[x e^{i\pi(2l-1)/k}\right].$$

Taking the more general input function

$$g(x) = (1 - \beta) + \beta f(\gamma x), \tag{5.14}$$

we obtain with the aid of Theorem 2 the shadow transform

$$G_\lambda^\mu(x) = F_\lambda^{\beta\mu}(\gamma x) = {}_1F_k\left(\frac{\beta\mu}{k}; \frac{\lambda}{k}, \frac{\lambda+1}{k}, \dots, \frac{\lambda+k-1}{k}; -\left(\frac{\gamma x}{k}\right)^k\right), \tag{5.15}$$

with β an arbitrary real and γ an arbitrary complex number.

The shadow class $\bar{S}[g]$ consists of all functions

$$\bar{S}[g] = \left\{{}_1F_k\left(\frac{\alpha\beta\lambda}{k}; \frac{\lambda}{k}, \frac{\lambda+1}{k}, \dots, \frac{\lambda+k-1}{k}; -\left(\frac{\gamma x}{k}\right)^k\right); \lambda \geqq 0, -\infty < \alpha < +\infty\right\}. \tag{5.16}$$

In particular, for $\beta = 1, \gamma = -1$ and $k = +1$, we obtain

$$\bar{S}[e^x] = \{{}_1F_1(\alpha\lambda; \lambda; x); \lambda \geqq 0, -\infty < \alpha < +\infty\}, \tag{5.17}$$

and the shadow class of e^x appears to be a set of confluent hypergeometric functions.

For $\beta = 1, \gamma = 1$ and $k = 2$ we have the result

$$\bar{S}[\cos x] = \left\{{}_1F_2\left(\frac{\alpha\lambda}{2}; \frac{\lambda}{2}, \frac{\lambda+1}{2}; -\frac{x^2}{4}\right); \lambda \geqq 0, -\infty < \alpha < +\infty\right\}. \tag{5.18}$$

A subset of this class ($\alpha = 1$) consists of the functions

$${}_0F_1\left(\frac{\lambda+1}{2}; -\frac{x^2}{4}\right) = \Gamma\left(\frac{\lambda+1}{2}\right)(\tfrac{1}{2}x)^{-(\lambda-1)/2} J_{(\lambda-1)/2}(x). \tag{5.19}$$

(c) From (5.2) we obtain

$$F_\lambda^\mu(x) = {}_2F_{1+k}\left(\frac{\mu}{k}, \frac{\mu}{k} + \frac{1}{2}; 2\frac{\mu}{k} + 1, \frac{\lambda}{k}, \frac{\lambda+1}{k}, \dots, \frac{\lambda+k-1}{k}; -\left(\frac{x}{k}\right)^k\right). \tag{5.20}$$

According to (3.12), the input function is

$$f(x) = \lim_{\lambda \downarrow 0} F_\lambda^\lambda(x) = {}_1F_k\left(\frac{1}{2}; \frac{1}{k}, \frac{2}{k}, \dots, \frac{k-1}{k}, 1; -\left(\frac{x}{k}\right)^k\right). \tag{5.21}$$

The more general input function

$$g(x) = (1 - \beta) + \beta f(\gamma x), \tag{5.22}$$

with β real and γ complex, similarly yields as above the shadow transform

$$\begin{aligned} G_\lambda^\mu(x) &= F_\lambda^{\beta\mu}(\gamma x) \\ &= {}_2F_{1+k}\left(\frac{\beta\mu}{k}, \frac{\beta\mu}{k} + \frac{1}{2}; 2\frac{\beta\mu}{k} + 1, \frac{\lambda}{k}, \frac{\lambda+1}{k}, \dots, \frac{\lambda+k-1}{k}; -\left(\frac{\gamma x}{k}\right)^k\right). \end{aligned} \tag{5.23}$$

For $\beta = 1, \gamma = -2i$ and $k = 1$ we obtain

$$g(x) = f(-2ix) = {}_1F_1(\tfrac{1}{2}; 1; 2ix) = e^{ix}J_0(x), \tag{5.24}$$

with

$$\bar{S}[e^{ix}J_0(x)] = \{{}_2F_2(\alpha\lambda, \alpha\lambda + \tfrac{1}{2}; 2\alpha\lambda + 1, \lambda; 2ix); \lambda \geqq 0, -\infty < \alpha < +\infty\}. \tag{5.25}$$

For $\beta = 1, \gamma = 1$ and $k = 2$ we obtain

$$f(x) = {}_0F_1\left(1; -\frac{x^2}{4}\right) = J_0(x), \tag{5.26}$$

with

$$\bar{S}[J_0(x)] = \left\{{}_2F_3\left(\frac{\alpha\lambda}{2}, \frac{\alpha\lambda}{2} + \frac{1}{2}; \alpha\lambda + 1, \frac{\lambda}{2}, \frac{\lambda + 1}{2}; -\frac{1}{4}x^2\right); \lambda \geqq 0, -\infty < \alpha < +\infty\right\}. \tag{5.27}$$

A subset of this class ($\alpha = 1$) consists of the functions

$${}_0F_1(\lambda + 1; -\tfrac{1}{4}x^2) = \Gamma(\lambda + 1)(\tfrac{1}{2}x)^{-\lambda}J_\lambda(x). \tag{5.28}$$

5.2. Shadow classes of hypergeometric functions of several variables. The result (5.18) may be generalized as follows.

We introduce the hypergeometric function of several variables

$$\Phi_2(a_1, a_2, \cdots, a_n; b; x_1, x_2, \cdots, x_n) = \sum_{\substack{m_i = 0\\ i = 1,2,\cdots,n}}^{\infty} \frac{(a_1)_{m_1}(a_2)_{m_2} \cdots (a_n)_{m_n}}{(b)_{m_1 + m_2 + \cdots + m_n}} \frac{x_1^{m_1}x_2^{m_2} \cdots x_n^{m_n}}{(m_1)!(m_2)! \cdots (m_n)!}. \tag{5.29}$$

We consider now

$$\mathscr{L}\left[\frac{1 - \sum_{i=1}^{n} \beta_i e^{\gamma_i x}}{x}\right] = \sum_{i=1}^{n} \beta_i \log \frac{s - \gamma_i}{s}, \tag{5.30}$$

with β_i real, γ_i complex and $\sum_{i=1}^n \beta_i = 1$. Equation (5.30) is valid for $\operatorname{Re} s > 0$ and $\operatorname{Re} s > \operatorname{Re} \gamma_i, i = 1, 2, \cdots, n$. Putting

$$f(x) = \sum_{i=1}^{n} \beta_i e^{\gamma_i x} \tag{5.31}$$

and using (2.16), we obtain

$$Q_\lambda^\mu[f](x) = \mathscr{L}^{-1}\left\{s^{-\lambda} \prod_{i=1}^{n} \left(\frac{s - \gamma_i}{s}\right)^{-\mu\beta_i}\right\}(x), \qquad x > 0,$$

or with the aid of [6, p. 222],

$$Q_\lambda^\mu[f](x) = \frac{x^{\lambda - 1}}{\Gamma(\lambda)}\Phi_2(\mu\beta_1, \mu\beta_2, \cdots, \mu\beta_n; \lambda; \gamma_1 x, \gamma_2 x, \cdots, \gamma_n x),$$

valid for $\lambda > 0$. Hence we have obtained for the shadow transform of f the formula

$$F_\lambda^\mu(x) = \Phi_2(\mu\beta_1, \mu\beta_2, \cdots, \mu\beta_n; \lambda; \gamma_1 x, \gamma_2 x, \cdots, \gamma_n x) \tag{5.32}$$

or, equivalently,

$$\bar{S}\left[\sum_{i=1}^{n}\beta_i e^{\gamma_i x}\right] = \{\Phi_2(\alpha\beta_1\lambda, \alpha\beta_2\lambda, \cdots, \alpha\beta_n\lambda; \lambda; \gamma_1 x, \gamma_2 x, \cdots, \gamma_n x); \quad \lambda \geqq 0, -\infty < \alpha < +\infty\}. \tag{5.33}$$

Finally we give two other classes of shadow transforms consisting of hypergeometric functions of the type:

$$\Phi_3(a, b; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m}{(b)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{5.34}$$

For

$$f(x) = \beta(x - 1) + (1 + \beta)e^{\gamma x}, \tag{5.35}$$

with β arbitrarily real and γ arbitrarily complex, we have

$$\mathscr{L}\left[\frac{1 - f(x)}{x}\right] = \mathscr{L}\left[-\beta + (1 + \beta)\frac{1 - e^{\gamma x}}{x}\right](s) = -\frac{\beta}{s} + (1 + \beta)\log\frac{s - \gamma}{s},$$

valid for $\operatorname{Re} s > 0$ and $\operatorname{Re} s > \operatorname{Re} \gamma$.

Using (2.16) we obtain

$$Q_\lambda^\mu[f](x) = \mathscr{L}^{-1}\left\{s^{-\lambda}\left(\frac{s - \gamma}{s}\right)^{-\mu(1+\beta)} e^{+\mu\beta/s}\right\}(x), \qquad x > 0,$$

or with the aid of [6, p. 223],

$$Q_\lambda^\mu[f](x) = \frac{x^{\lambda - 1}}{\Gamma(\lambda)}\Phi_3(\mu(1 + \beta), \lambda; \gamma x, \mu\beta x). \tag{5.36}$$

Hence we have obtained for the shadow transform of f the formula

$$F_\lambda^\mu(x) = \Phi_3(\mu(1 + \beta), \lambda; \gamma x, \mu\beta x) \tag{5.37}$$

or, equivalently,

$$\bar{S}[\beta(x - 1) + (1 + \beta)e^{\gamma x}] = \{\Phi_3((1 + \beta)\alpha\lambda, \lambda; \gamma x, \alpha\beta\lambda x), \lambda \geqq 0, -\infty < \alpha < +\infty\}. \tag{5.38}$$

For

$$f(x) = (1 + \beta x)e^{\gamma x}, \tag{5.39}$$

with β arbitrarily real and γ arbitrarily complex, we have

$$\mathscr{L}\left[\frac{1 - f(x)}{x}\right](s) = \mathscr{L}\left[-\beta e^{\gamma x} + \frac{1 - e^{\gamma x}}{x}\right](s) = -\frac{\beta}{s - \gamma} + \log\frac{s - \gamma}{s},$$

valid for $\operatorname{Re} s > 0$ and $\operatorname{Re} s > \operatorname{Re} \gamma$.

Using (2.16) again, we obtain

$$Q_\lambda^\mu[f](x) = \mathscr{L}^{-1}\left\{s^{-\lambda}\left(\frac{s - \gamma}{s}\right)^{-\mu} e^{+\beta\mu/(s-\gamma)}\right\}(x),$$

which may be reduced, with the aid of [6, p. 223], to

$$
\begin{aligned}
Q^{\mu}_{\lambda}[f](x) &= e^{\gamma x}\mathscr{L}^{-1}\{s^{-\mu}(s+\gamma)^{\mu-\lambda}e^{+\beta\mu/s}\}(x) \\
&= e^{\gamma x}\mathscr{L}^{-1}\left\{s^{-\lambda}\left(\frac{s+\gamma}{s}\right)^{\mu-\lambda}e^{+\beta\mu/s}\right\}(x) \\
&= \frac{x^{\lambda-1}}{\Gamma(\lambda)}e^{\gamma x}\Phi_3(\lambda-\mu,\lambda;\ -\gamma x.\ \beta\mu x),
\end{aligned}
\tag{5.40}
$$

valid for $\lambda > 0$. Hence we have for the shadow transform of f the formula

$$F^{\mu}_{\lambda}(x) = e^{\gamma x}\Phi_3(\lambda-\mu,\lambda;\ -\gamma x,\ \beta\mu x) \tag{5.41}$$

or, equivalently,

$$\bar{S}[(1+\beta x)e^{\gamma x}] = \{e^{\gamma x}\Phi_3((1-\alpha)\lambda,\lambda;\ -\gamma x,\ \alpha\beta\lambda x);\ \lambda \geqq 0,\ -\infty < \alpha < +\infty\}. \tag{5.42}$$

6. Applications of shadow transforms.

6.1. Table of shadow transforms. For applications concerning calculations in elementary particle physics (see [1]) and for applications to be dealt with in the next section, it is useful to summarize all the results of §5 in a table (see below) in which we use the following notation:

β real, γ complex, k positive integer, λ positive and μ real number. The functions ${}_pF_q$ denote hypergeometric functions, Φ_2 and Φ_3 are hypergeometric functions

Input function $f(x)$	Shadow transform $F^{\mu}_{\lambda}(x)$
$1 - \dfrac{(\gamma x)^k}{(k-1)!}$	${}_0F_k\left(\dfrac{\lambda}{k},\dfrac{\lambda+1}{k},\cdots,\dfrac{\lambda+k-1}{k};\ -\mu\left(\dfrac{\gamma x}{k}\right)^k\right)$
$1 - \gamma x$	$\Gamma(\lambda)(\mu\gamma x)^{-(\lambda-1/2)}J_{\lambda-1}(2\sqrt{\mu\gamma x})$
$(1-\beta) + \dfrac{\beta}{k}\sum_{l=1}^{k} e^{\gamma x \exp\{i\pi(2l-1/k)\}}$	${}_1F_k\left(\dfrac{\beta\mu}{k};\dfrac{\lambda}{k},\dfrac{\lambda+1}{k},\cdots,\dfrac{\lambda+k-1}{k};\ -\left(\dfrac{\gamma x}{k}\right)^k\right)$
e^x	${}_1F_1(\mu;\lambda;x)$
$\cos x$	${}_1F_2\left(\tfrac{1}{2}\mu;\tfrac{1}{2}\lambda,\tfrac{1}{2}(\lambda+1);\ -\dfrac{x^2}{4}\right)$
$(1-\beta) + \beta\,{}_1F_k\left(\dfrac{1}{2};\dfrac{1}{k},\dfrac{2}{k},\cdots,1;\ -\left(\dfrac{\gamma x}{k}\right)^k\right)$	${}_2F_{1+k}\left(\dfrac{\beta\mu}{k},\dfrac{\beta\mu}{k}+\dfrac{1}{2};2\dfrac{\beta\mu}{k}+1,\dfrac{\lambda}{k},\dfrac{\lambda+1}{k},\cdots,\dfrac{\lambda+k-1}{k};\ -\left(\dfrac{\gamma x}{k}\right)^k\right)$
$e^{ix}J_0(x)$	${}_2F_2(\mu,\mu+\tfrac{1}{2};2\mu+1,\lambda;2ix)$
$J_0(x)$	${}_2F_3\left(\tfrac{1}{2}\mu,\tfrac{1}{2}\mu+\tfrac{1}{2};\mu+1,\dfrac{\lambda}{2},\dfrac{\lambda+1}{2};\ -\tfrac{1}{4}x^2\right)$
$\sum_{i=1}^{n}\beta_i e^{\gamma_i x}\left(\sum_{i=1}^{n}\beta_i = 1\right)$	$\Phi_2(\mu\beta_1,\mu\beta_2,\cdots,\mu\beta_n;\lambda;\gamma_1 x,\gamma_2 x,\cdots,\gamma_n x)$
$\beta(x-1) + (1+\beta)e^{\gamma x}$	$\Phi_3(\mu(1+\beta),\lambda;\gamma x,\mu\beta x)$
$(1+\beta x)e^{\gamma x}$	$e^{\gamma x}\Phi_3(\lambda-\mu,\lambda;\ -\gamma x,\mu\beta x)$

of several variables (see § 5, formulas (5.29) and (5.34)) and J stands for the Bessel function of the first kind.

6.2. Applications of shadow transforms. From §2 it follows that $Q_\lambda^\mu[f]$ is a solution of the integral equation

$$xQ_\lambda^\mu = [\lambda - \mu(1 - f(x))] * Q_\lambda^\mu,$$

and hence for $\mu \neq 0$,

$$f(x) * Q_\lambda^\mu = \frac{x}{\mu} Q_\lambda^\mu + \left(1 - \frac{\lambda}{\mu}\right) * Q_\lambda^\mu.$$

Taking $\lambda = \mu$ and using the definition (1.5), we obtain for shadow transforms the convolution rule

$$f(x) * (x^{\lambda-1}F_\lambda^\lambda(x)) = \frac{x^\lambda}{\lambda} F_\lambda^\lambda(x), \tag{6.1}$$

valid for all positive λ. This rule may be applied to all input functions and shadow transforms, listed in the table given. For example, one has the formula

$$\begin{aligned}\left(1 - \frac{(\gamma x)^k}{(k-1)!}\right) * \left[x^{\lambda-1}{}_0F_k\left(\frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\lambda\left(\frac{\lambda x}{k}\right)^k\right)\right]& \\ = \frac{x^\lambda}{\lambda}\,{}_0F_k\left(\frac{\lambda}{k}, \frac{\lambda+1}{k}, \ldots, \frac{\lambda+k-1}{k}; -\lambda\left(\frac{\gamma x}{k}\right)^k\right)&.\end{aligned} \tag{6.2}$$

Other results concerning convolution products may be obtained by using the convolution rule (3.1) for the function $Q_\lambda^\mu[f]$:

$$Q_{\lambda_1}^{\mu_1}[f] * Q_{\lambda_2}^{\mu_2}[f] = Q_{\lambda_1+\lambda_2}^{\mu_1+\mu_2}[f]$$

or, equivalently for shadow transforms:

$$x^{\lambda_1-1}F_{\lambda_1}^{\mu_1} * x^{\lambda_2-1}F_{\lambda_2}^{\mu_2} = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} x^{\lambda_1+\lambda_2-1}F_{\lambda_1+\lambda_2}^{\mu_1+\mu_2}. \tag{6.3}$$

This formula may be applied to every function in the right column of the table. For example,

$$\begin{aligned}x^{\lambda_1-1}{}_1F_2\left(\tfrac{1}{2}\mu_1; \tfrac{1}{2}\lambda_1, \tfrac{1}{2}(\lambda_1+1); -\frac{x^2}{4}\right) * x^{\lambda_2-1}{}_1F_2\left(\tfrac{1}{2}\mu_2; \tfrac{1}{2}\lambda_2, \tfrac{1}{2}(\lambda_2+1); -\frac{x^2}{4}\right)& \\ = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)}x^{\lambda_1+\lambda_2-1}{}_1F_2\left(\tfrac{1}{2}(\mu_1+\mu_2); \tfrac{1}{2}(\lambda_1+\lambda_2), \tfrac{1}{2}(\lambda_1+\lambda_2+1); -\frac{x^2}{4}\right)&.\end{aligned} \tag{6.4}$$

Finally we mention an application in the theory of integral equations of convolution type. We consider the following integral equation

$$g(x) = \int_0^x f(y)K(x-y)\,dy = (f * K)(x), \qquad 0 < x < \infty; \tag{6.5}$$

g is a given function that is sufficiently smooth, and f is the unknown.

Whenever $K(x)$ is of the form

$$K(x) = C_1 Q_\lambda^\mu[f](x) = C_2 x^{\lambda-1} F_\lambda^\mu(x), \tag{6.6}$$

with C_1 an arbitrary constant and $C_2 = C_1/\Gamma(\lambda)$, the integral equation (6.5) can be solved easily by using (6.3). It may be remarked that (6.5) with (6.6) is a generalization of the integral equation of Abel. The method used to solve (6.5) is completely similar to that used to solve Abel's integral equation. Taking the convolution of both sides of (6.5) with $x^{\nu-1}F_\nu^{-\mu}(x)$, we obtain

$$\begin{aligned} g(x) * x^{\nu-1}F_\nu^{-\mu}(x) &= C_2\{f(x) * x^{\lambda-1}F_\lambda^\mu(x) * x^{\nu-1}F_\nu^{-\mu}(x)\} \\ &= C_2 \frac{\Gamma(\lambda)\Gamma(\nu)}{\Gamma(\lambda+\nu)}\{f(x) * x^{\lambda+\nu-1}F_{\lambda+\nu}^0(x)\} \\ &= C_2\Gamma(\lambda)\Gamma(\nu)\{f(x) * Q_{\lambda+\nu}^0[f](x)\}, \end{aligned}$$

or with the aid of (3.2),

$$g(x) * x^{\nu-1}F_\nu^{-\mu}(x) = C_2 \frac{\Gamma(\lambda)\Gamma(\nu)}{\Gamma(\lambda+\nu)}\{f(x) * x^{\lambda+\nu-1}\}. \tag{6.7}$$

We take now ν such that

$$\lambda + \nu - 1 = [\lambda],$$

$[\lambda]$ denoting the largest integer $\leqq \lambda$, and we obtain

$$g(x) * x^{\nu-1}F_\nu^{-\mu}(x) = C_2\Gamma(\lambda)\Gamma(\nu)\left\{f(x) * \frac{x^{[\lambda]}}{\Gamma([\lambda]+1)}\right\}$$

or

$$f(x) = \frac{1}{C_2\Gamma(\lambda)\Gamma(\nu)} \frac{d^{[\lambda]}}{dx^{[\lambda]}}\{g(x) * x^{\nu-1}F_\nu^{-\mu}(x)\}. \tag{6.8}$$

From (6.8) it follows that it is sufficient to assume that $g(x)$ is $[\lambda]$ times continuously differentiable in order that (6.5) has a continuous solution.

Example. The solution of the equation

$$g(x) = \int_0^x f(\xi)(x-\xi)^{\lambda-1}{}_1F_1(\mu;\lambda;x-\xi)\,d\xi,$$

with $\lambda > 0$, becomes

$$f(x) = \frac{1}{\Gamma(\lambda)\Gamma(\nu)} \frac{d^{[\lambda]}}{dx^{[\lambda]}} \int_0^x g(\xi)(x-\xi)^{\nu-1}{}_1F_1(-\mu;\nu;x-\xi)\,d\xi,$$

with $\lambda + \nu - 1 = [\lambda]$.

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AN OPERATOR RESIDUE THEOREM WITH APPLICATIONS TO BRANCHING PROCESSES AND RENEWAL TYPE INTEGRAL EQUATIONS*

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Abstract. The paper is concerned with an analytic operator-valued function, $K(\sigma)$, of a complex variable σ. Conditions are given which insure that when one is an eigenvalue of $K(\sigma_0)$, the resolvent operator $R(\sigma) = (I - K(\sigma))^{-1}$ is meromorphic in a neighborhood of σ_0 with a simple pole at $\sigma = \sigma_0$, and the residue of R at σ_0 is an operator with one-dimensional range. The residue is computed. A conjecture of T. E. Harris concerning the asymptotic behavior of the mean number of objects of a branching population is proved, and analogous formulas arising in infinite systems of renewal type integral equations are a consequence of the formalism.

1. Introduction. In *The Theory of Branching Processes* by T. E. Harris [2], a formal derivation of the asymptotic behavior for the mean number of objects of a branching process is given. This asymptotic analysis is closely allied with the behavior of a solution of a Fredholm integral equation analytically dependent on a parameter, as the parameter nears a singular point. Harris heuristically calculates a formula involving the residue at the singular point, and indicates [2, p. 90] that no rigorous verification of this formula existed at the time of writing.

Recently, Mode [5] deduced the Harris formula using the machinery of Fredholm determinants with the inherent assumption of compactness on the integral operator involved. The Harris operator is not compact, but does have a compact iterate, and presumably the Mode technique can be extended to cover this case.

In the same direction, Mode [6], [7] has obtained similar results in renewal theory. The basic system of integral equations generated are of renewal type, and (via the Laplace transform) the asymptotic behavior of the infinite system is dependent upon the operator residue at a critical point.

Our approach to these problems is similar to the one we used in [9]. We are concerned with the resolvent of an analytic operator-valued function and its meromorphic behavior near a singularity. Our main contribution is the characterization of the simplicity of its pole and the calculation of the residue. We derive the Harris formula as a consequence of our general result. Compactness never enters our argument and the machinery used is no more complicated than that of complex function theory.

Although our focus is on the behavior of the resolvent near its singularity, as a consequence of our theory, we derive a necessary and sufficient condition for a meromorphic resolvent to have a simple pole. (More precisely, when A1 and A2 hold, then A3 characterizes the simplicity of a pole with rank 1 residue.) This may be of independent interest (cf. Steinberg [10] and Howland [3]).

A brief outline of the paper follows: §2 contains our hypotheses and the statement of our results. We direct the reader's attention to assumption A3 which plays

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a crucial role in our theory. In §3, the application to branching processes is given. Proofs of the main theorems are given in §4.

2. Main results. We adopt the following notation: X is a complex Banach space with dual space X^*; if $x \in X$, $x^* \in X^*$, then $(x, x^*) = x^*(x)$; $B(X)$ is the space of bounded linear operators acting on X.

We use the standard definitions for analytic and meromorphic functions with values in a Banach space. Thus, for all σ in a complex disc D, let $f(\sigma)$ be defined with values in a Banach space B. We say f is meromorphic in D with pole at σ_0 (= center of D) if f has the representation

$$f(\sigma) = \sum_{n=-N}^{\infty} A_n(\sigma - \sigma_0)^n, \qquad A_n \in B, \tag{2.1}$$

where the series converges (for $\sigma \neq \sigma_0$) in the norm of B. The term A_{-1} is called the residue of f at σ_0. If $N = 0$ in expansion (2.1), we say f is analytic in D.

An eigenvalue is said to be geometrically simple if its geometric multiplicity is equal to 1.

We shall be concerned with a family $K(\sigma) \in B(X)$ indexed by σ in D. We consider the following three assumptions about this family:

A1 $K(\sigma)$ is analytic in D; $K(\sigma) = K_0 + K_1(\sigma - \sigma_0) + \cdots$.

A2 (i) $\lambda = 1$ is an isolated eigenvalue of K_0 and K_0^* of finite algebraic multiplicity,

(ii) $\lambda = 1$ is not an eigenvalue of $K(\sigma)$ for all σ in a deleted neighborhood of σ_0.

A3 $(K_1 e, f^*) \neq 0$ for every pair of eigenvectors e, f^* of K_0, K_0^* respectively, corresponding to eigenvalue $\lambda = 1$.

Our first result is an operator residue theorem.

THEOREM 1. *Let $K(\sigma)$ satisfy assumptions* A1, A2, *and* A3. *Then*:

(i) *$\lambda = 1$ is a geometrically simple eigenvalue of K_0 and K_0^*.*

(ii) *The resolvent operator $R(\sigma) = (I - K(\sigma))^{-1}$ is meromorphic in D and has a simple pole at σ_0.*

(iii) *The residue of R at σ_0 is the operator $A_{-1} \in B(X)$ defined by $A_{-1}x = -(x, e_0^*)e_0/(K_1 e_0, e_0^*)$ where the pair e_0, e_0^* are eigenvectors of K_0, K_0^* respectively, corresponding to $\lambda = 1$.*

THEOREM 2. *Let $K(\sigma)$ satisfy assumptions* A1 *and* A2. *Then $\lambda = 1$ is a geometrically simple eigenvalue of K_0 and K_0^*, and $\sigma = \sigma_0$ is a simple pole for $R(\sigma)$ if and only if condition* A3 *holds.*

Remark 1. The following simple example shows the necessity of condition A3 for a simple pole. (See also the example in Schumitzky and Wenska [9] for the same phenomena in a slightly different setting.) Example:

$$X = C^1 \quad \text{and} \quad K(\sigma) = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right).$$

For $\sigma_0 = 1$, $\lambda = 1$ is a simple eigenvalue of K_0 and K_0^*; but $R(\sigma) = -2\sigma/(\sigma - 1)^2$ has a double pole. Note: $K_1 = 0$.

Remark 2. Combining the conclusion of Theorem 1 with classical results in analytic perturbation theory, one can show that there exists a function $\lambda(\sigma)$ defined for σ near σ_0 with $\lambda(\sigma_0) = 1$ and $\lambda(\sigma)$ a simple eigenvalue of $K(\sigma)$ and $K^*(\sigma)$. Under suitable additional hypotheses, it can be further shown that $\lambda(\sigma)$ is analytic near σ_0 and

$$\lambda'(\sigma_0) = (K_1 e_0, e_0^*)/(e_0, e_0^*). \tag{2.2}$$

For example, it is sufficient to assume σ_0 is real and $K(\sigma)$ is self-adjoint for σ in a real interval containing σ_0 (cf. Kato [4, pp. 385–391]). Formula (2.2) will be useful later. However, we note that assumptions A1, A2 and A3 do not imply that $\lambda(\sigma)$ is differentiable at σ_0. The following example illustrates this point. Example:

$$X = C^2, \quad K(\sigma) = \begin{pmatrix} 1 & -1 \\ -\sigma & 1 \end{pmatrix} \quad \text{and} \quad \sigma_0 = 0.$$

It follows that $\lambda(\sigma) = 1 + \sqrt{\sigma}$ is a simple eigenvalue for $K(\sigma)$ and $K^*(\sigma)$, which is not differentiable at $\sigma = 0$. It can be further verified that assumptions A1, A2 and A3 are in fact satisfied.

The proof of Theorems 1 and 2 are given in §4. An immediate corollary of Theorem 1 which will be important for our application follows.

Corollary 1. *Let $K(\sigma)$ satisfy assumptions* A1, A2 *and* A3; *let $g: D \to X$ be analytic; and let $u = u(\sigma)$ be the unique solution to the equation*

$$u = g(\sigma) + K(\sigma)u,$$

where σ is sufficiently close, but not equal, to σ_0. Then the X-valued function u is meromorphic in D and has the representation

$$u(\sigma) = \frac{-(g(\sigma_0), e_0^*)e_0}{(K_1 e_0, e_0^*)(\sigma - \sigma_0)} + O(1). \tag{2.3}$$

As a concluding remark, in connection with the notion of algebraic multiplicity of operator bundles as developed by Keldys (cf. Gohberg and Krein [1, § V.9]) we note the following: Let ϕ_0 be an eigenvector of the bundle $I - K(\sigma)$ at $\sigma = \sigma_0$, and $\phi_1, \phi_2, \cdots, \phi_k$ be nonzero vectors. Then the $(k + 1)$-tuple $(\phi_0, \phi_1, \cdots, \phi_k)$ will be called a *chain* of *length* $(k + 1)$ if the entries satisfy

$$(I - K_0)\phi_p - K_1\phi_{p-1} - \cdots - K_p\phi_0 = 0, \qquad p = 0, 1, \cdots, k. \tag{2.4}$$

The eigenvector ϕ_0 is said to be of *rank r* if the largest chain associated with ϕ_0 has length r. The bundle $(I - K(\sigma))$ is of *rank s at $\sigma = \sigma_0$* if the eigenspace $N(I - K_0)$ is finite-dimensional and the rank of every eigenvector is less than or equal s with equality holding for at least one particular eigenvector. *Let $I - K(\sigma)$ be an operator bundle satisfying* A1 *and* A2 *with range of $(I - K_0)$ closed; then the rank of the bundle at $\sigma = \sigma_0$ is* 1 *and the null space of $(I - K_0^*)$ is geometrically simple if and only if* A3. One direction is clear from the Fredholm alternative. For the reverse implication, the geometric simplicity follows from Proposition 4.3, and the rest is elementary.

3. Applications to branching processes and renewal theory. The type of branching process we consider can be described informally as follows: Let Ω be a measure space. A population evolves from an initial individual in a state $x \in \Omega$ who lives a random lifetime which depends on x. At the end of his life, this individual produces a random number of offspring with a random distribution of states, the number and distribution also depending on x. All of the offspring repeat this process independently of each other according to lifetime and offspring distribution depending on their respective states. The process continues as long as there are live individuals in the population.

Under suitable hypotheses on the underlying probability distributions (cf. [2], [5]), the mean number of individuals $M(x, t, A)$ at time t with states in the set $A \subset \Omega$, given the initial individual in state x at $t = 0$ can be shown to exist. One of the fundamental problems for both theoretical and practical purposes is the determination of the asymptotic behavior of $M(x, t, A)$ as $t \to \infty$. With this as a goal, the Laplace transform $M^*(x, \sigma, A)$ of M is computed and can be shown to satisfy a Fredholm integral equation of the form

$$M^*(x, \sigma, A) = g(x, \sigma, A) + \int_\Omega k(x, y, \sigma) M^*(y, \sigma, A)\, d\mu(y), \tag{3.1}$$

where the functions g and k depend on the probability distributions of the process.

In the specific Harris example [2, p. 90], the ingredients in (3.1) are

$$g(x, \sigma, A) = \frac{1}{M} \int_A k(x, y, \sigma)\, d\mu(y),$$

$$k(x, y, \sigma) = \frac{M \exp[-(\sigma + 1)|x - y|]}{4\pi |x - y|^2}, \qquad x, y \in \Omega, \quad x \neq y,$$

where Ω is a bounded convex subset of R^3, A is a measurable subset of Ω, and M is the average number of offspring per individual.

We point out here that (3.1) occurs as the Laplace transform of a *renewal type integral equation.* We refer the reader to the development of analogous formulas by Mode [6], [7].

More generally, let $K(\sigma)$ be the integral operator defined by

$$(K(\sigma) f)(x) = \int_\Omega k(x, y, \sigma) f(y)\, d\mu(y).$$

$K(\sigma)$ will be considered as an operator on the Banach space, $B(\Omega)$, of bounded complex-valued functions on Ω. Then the examples of Harris and Mode [2], [5], [6], [7] share the following properties:

P1 $K(\sigma)$ is bounded, positive and analytic for all σ in the positive reals R^+. (An operator K on $B(\Omega)$ is positive if $(Kf)(x) > 0$ for all $x \in \Omega$ for every strictly positive function $f \in B(\Omega)$.)

P2 $K(\sigma)$ satisfies the conclusions of the Perron-Frobenius theorem, namely, for each $\sigma \in R^+$, $K(\sigma)$ and $K^*(\sigma)$ have an isolated geometrically simple eigenvalue $\lambda(\sigma)$, with corresponding eigenfunctions $e(x)$, $e^*(x)$ which are strictly positive.

(We normalize e and e^* so that $\int_\Omega e(x) e^*(x)\, d\mu(y) = 1$.)

P3 The operator $-d/d\sigma K(\sigma)$ is positive.

If we let σ_0 be the unique root of the equation $\lambda(\sigma) = 1$, then the Harris conjecture can be written in the form

$$M^*(x, \sigma, A) \sim e(x) \int_X g(y, \sigma_0, A)e^*(y)\, d\mu(y)/-\lambda'(\sigma_0)(\sigma - \sigma_0). \tag{3.2}$$

Assuming relation (3.2), a standard Laplace transform inversion theorem then shows that

$$M(x, t, A) \sim \left\{\frac{e(x)\int_X g(y, \sigma_0, A)e^*(y)\, d\mu(y)}{-\lambda'(\sigma_0)}\right\} \cdot e^{\sigma_0 t}, \qquad t \to \infty.$$

We now show that relation (3.2) is a direct consequence of Corollary 1. We first check the hypotheses of Theorem 1. It is immediately observed that P1 implies A1 and P2 + P3 imply A3. That P2 implies A2 requires further elaboration. We need the following two propositions.

PROPOSITION 3.1. *Let $T \in B(X)$; $\lambda = 1$ be an isolated geometrically simple eigenvalue of T and T^* with finite algebraic multiplicity; and e_0 and e_0^* be eigenvectors of T, T^* such that $(e, e^*) \neq 0$. Then the algebraic multiplicity of $\lambda = 1$ is equal to* 1.

Proof. We show that any vector x in the generalized eigenspace of T is in the eigenspace of T.

Thus, let x be any vector in X such that $(T - I)^2x = 0$. Set $y = (T - I)x$. Then y is in the eigenspace of T so that $y = ce_0$. It follows: $c(e_0, e_0^*) = (y, e_0^*) = ((T - I)x, e_0^*) = (x, (T^* - I^*)e_0^*) = 0$. Thus, $c = 0$ and x is an eigenvector of T. It is observed at this point that the same argument can be used to prove: *If $K(\sigma)$ satisfies* A1 *and* A3 *and K_1 commutes with K_0, then the algebraic multiplicity of K_0 corresponding to $\lambda = 1$ is* 1. It is noted that this commutativity condition is in fact satisfied in the Harris–Mode examples (although not in general). We prefer to use Proposition (3.1) for our application as the positivity assumption is quite natural in branching processes.

PROPOSITION 3.2. *If $K(\sigma)$ satisfies assumptions* A1 *and* A2(i) *and $\lambda = 1$ has algebraic multiplicity equal to* 1, *then there exist functions $\lambda(\sigma)$ and $e(\sigma)$ analytic for σ near σ_0 such that $\lambda(\sigma_0) = 1$, $e(\sigma_0) = e_0$ and $\lambda(\sigma)e(\sigma) = K(\sigma)e(\sigma)$ and*

$$\lambda'(\sigma_0) = (K_1e_0, e_0^*), \tag{3.3}$$

where e_0 and e_0^* are eigenvectors of K_0 corresponding to $\lambda = 1$.

Proposition 3.2 is a classical result in the theory of analytic perturbations. The reader is referred to Kato [4] for details of proof.

We now verify A2. Property P2 shows that $K(\sigma)$ satisfies the hypotheses of Proposition 3.1 so that the algebraic multiplicity of $\lambda = 1$ is finite (in fact equal to 1). Thus, A2(i) is satisfied. Thus $K(\sigma)$ satisfies the hypotheses of Proposition 3.2. The equation (3.3) then shows that $\lambda(\sigma_0) = 1$ is not an eigenvalue of $K(\sigma)$ for all σ near σ_0 since $(K_1e_0, e_0^*) \neq 0$. Thus, A2(ii) follows and all the hypotheses of Theorem 1 are satisfied.

In the examples of Mode and Harris, the function $g(x, \sigma, A)$ is analytic for σ near σ_0. Thus, Corollary 1 applies and the relation (2.3) along with equation (3.3) yields (3.2).

4. Proof of Theorems 1 and 2. We break up the proof into a series of propositions. Assumptions A1 and A2 will always be in force; but, A3 will be used only when explicitly stated. Our first proposition is due to Ribarič and Vidav [8, Cor. I].

PROPOSITION 4.1. *$R(\sigma)$ is meromorphic in D.*

Let $R(\sigma)$ have the following representation in D:

$$R(\sigma) = \sum_{n=-N}^{\infty} A_n(\sigma - \sigma_0)^n, \qquad A_n \in B(X). \tag{4.1}$$

PROPOSITION 4.2. *If $A_{-N}K_1A_{-N} \neq 0$, then*

$$N = 1, \tag{4.2}$$

i.e., $R(\sigma)$ has a simple pole at σ_0. Conversely, if $R(\sigma)$ has a simple pole at σ_0 then $A_{-1}K_1A_{-1} \neq 0$; in fact,

$$-A_{-1} = A_{-1}K_1A_{-1}. \tag{4.3}$$

Proof. Let $K(\sigma)$ have representation in D:

$$K(\sigma) = \sum_{n=0}^{\infty} K_n(\sigma - \sigma_0)^n. \tag{4.4}$$

Substituting (4.1) and (4.4) into the resolvent equations,

$$R(\sigma)\,(I - K(\sigma)) = (I - K(\sigma))R(\sigma) = I$$

and equating coefficients of $(\sigma - \sigma_0)^{-N}$ and $(\sigma - \sigma_0)^{-N+1}$ gives

$$A_{-N}(I - K_0) = (I - K_0)A_{-N} = 0, \tag{4.5}$$

$$A_{-N}K_1 + A_{-(N-1)}(I - K_0) = 0, \qquad N > 1, \tag{4.6}$$

$$-A_{-1}K_1 + A_0(I - K_0) = I, \qquad N = 1. \tag{4.7}$$

(Note that relation (4.5) is the same as in classical resolvent theory, cf. Yosida [11].)

Using (4.5) in (4.6) and (4.7) then implies

$$A_{-N}K_1A_{-N} = 0, \qquad N > 1,$$

and

$$A_{-1} = -A_{-1}K_1A_{-1}, \qquad N = 1. \tag{4.8}$$

The conclusion then follows from these last two relations.

PROPOSITION 4.3. *If $K(\sigma)$ satisfies assumption* A3 *then $\lambda = 1$ is a geometrically simple eigenvalue of K_0 and K_0^*, and $R(\sigma)$ has a simple pole at $\sigma = \sigma_0$.*

Proof. We first show that A3 implies the geometric simplicity of $\lambda = 1$. Assume the contrary. Specifically, let e_1 and e_2 be any two linearly independent eigenvectors of K_0, and f^* an eigenvector of K_0^* for $\lambda = 1$. Then, if either $(K_1e_1, f^*) = 0$ or $(K_1e_2, f^*) = 0$, assumption A3 is violated. Otherwise. we can choose $c \neq 0$ such that $(K_1(ce_1 + e_2),\ f^*) = c(K_1e_1, f^*) + (K_1e_2, f^*) = 0$, also contrary to A3. A similar argument proves 1 is simple for K_0^*.

We next show that $\sigma = \sigma_0$ is a simple pole of R. Let e_0 and e_0^* be eigenvectors of K_0 and K_0^* corresponding to $\lambda = 1$. (Note e_0 and e_0^* are unique up to normalization.) By virtue of (4.5) and its adjoint and the fact that $A_{-N} \neq 0$ (by definition), we have that there exist nonzero vectors $x_0 \in X$, $y_0^* \in X^*$ such that $A_{-N}x_0 = e_0$ and $A_{-N}^*y_0^* = e_0^*$. Thus, $(A_{-N}K_1A_{-N}x_0,\ y_0^*) = (K_1e_0,\ e_0^*) \neq 0$, and hence, $A_{-N}K_1A_{-N} \neq 0$. Equation (4.2) in Proposition 4.2 then shows that the pole is simple.

PROPOSITION 4.4. *If $\lambda = 1$ is a geometrically simple eigenvalue for K_0 and K_0^* and $\sigma = \sigma_0$ is a simple pole of R, then $(K_1e_0, e_0^*) \neq 0$ and A_{-1} is the operator defined by*

$$A_{-1}x = -(x, e_0^*)e_0/(K_1e_0, e_0^*), \qquad x \in X.$$

Proof. Let x be any vector in X. Then since $A_{-1} \not\equiv 0$, (4.5) implies that $A_{-1}x = c(x)e_0$ for some constant $c(x)$ (depending on x). Let $y_0^* \in X^*$ be a nonzero vector such that $A_{-1}^*y_0^* = e_0^*$. We compute the quantity $(A_{-1}x, y_0^*)$ three ways:

$$\begin{aligned}(A_{-1}x, y_0^*) &= c(x)\,(e_0, y_0^*)\\ &= (x, A_{-1}^*y_0^*) = (x, e_0^*)\\ &= -(A_{-1}K_1A_{-1}x, y_0^*) = -c(x)\,(K_1e_0, e_0^*),\end{aligned}$$

where we have used (4.3). Since $(x, e_0^*) \neq 0$ for some x, we have $(e_0, y_0^*) \neq 0$. Since $c(x) \neq 0$ for some x, we have $(e_0,\ y_0^*) = -(K_1e_0,\ e_0^*) \neq 0$ and $c(x) = (x, e_0^*)/(K_1e_0, e_0^*)$.

Propositions 4.3 and 4.4 imply Theorems 1 and 2.

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BOUNDED SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS WITH PARAMETERS*

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Abstract. Asymptotic properties of solutions of a parametrically perturbed linear ordinary differential equation are considered. Conditions are found which guarantee that a solution and its derivative with respect to the parameter are bounded. The limiting behavior of solutions as the parameter approaches zero is also exhibited.

1. Introduction. Nonlinear differential equations that include parameters have application in many areas. Many classical illustrations can be found in the monograph of N. Minorsky; in [8] for example, on page 48, the rotating pendulum problem is discussed and on page 51 the attraction of current carrying conductors is treated. In both of these formulations the mathematical model is a parametric perturbation of a linear system of differential equations. As another area of application, we note a recent contribution of H. I. Freedman and P. Waltman [3] that is concerned with the Volterra predator-prey problem; their model is again a linear system that is parametrically perturbed.

Various qualitative properties of the solutions of the

$$dx/dt = A(t)x + f(t, x, \lambda) \tag{1}$$

have been investigated in the literature (see the references). Our results discuss the asymptotic behavior for large t of solutions, $x(t, \lambda)$ of (1) and its derivative, $\partial x/\partial \lambda$, for small values of the vector parameter λ.

Our basic assumptions about (1) and the associated unperturbed linear differential equation

$$dy/dt = A(t)y \tag{2}$$

include the following. The vector field $h: R \times R^n \times R^m \to R^n$ defined by

$$h(t, x, \lambda) = A(t)x + f(t, x, \lambda)$$

and the functions $\partial f/\partial x$, $\partial f/\partial \lambda$ are continuous on $R \times R^n \times R^m$. There exists a continuous function $\omega: R \times R_+ \times R_+ \to R_+$ with the property that

$$|f(t, x, \lambda)| \leqq \omega(t, |x|, |\lambda|),$$

where $\omega(t, r, \varepsilon)$ is nondecreasing in $r[\varepsilon]$ for each $(t, \varepsilon)[(t, r)] \in R \times R_+$ and $\omega(t, r, 0) = 0$ for each $(t, r) \in R \times R_+$.

2. Conditional asymptotic stability of (2). Let $Y(t)$ denote the fundamental matrix of solutions of (2) that satisfies $Y(0) = I_n$, where I_n is the $n \times n$ identity matrix. The conditional asymptotic stability state imposed upon (2) in this section is the following.

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(H.1) Let there exist supplementary projections P_0, P_{-1}, P_1, P_∞, and constants K, q with $K > 0$ and $1 \leqq q < \infty$ such that

$$\left[\int_{-\infty}^{t} |Y(t)P_{-1}Y^{-1}(s)|^q \, ds\right]^{1/q} + \left|\int_0^t |Y(t)P_0Y^{-1}(s)|^q \, ds\right|^{1/q} + \left[\int_t^{\infty} |Y(t)P_1Y^{-1}(s)|^q \, ds\right]^{1/q} \leqq K, \qquad t \in R;$$

$$\left[\int_t^{\infty} |Y(t)P_\infty Y^{-1}(s)|^q \, ds\right]^{1/q} \leqq K, \qquad t \in R_+ \equiv [0, \infty);$$

$$\left[\int_{-\infty}^{t} |Y(t)P_\infty Y^{-1}(s)|^q \, ds\right]^{1/q} \leqq K, \qquad t \in R_- \equiv (-\infty, 0].$$

The hypothesis (H.1) effects a decomposition of the solution space in the following manner.

$$\lim_{|t| \to \infty} |Y(t)P_0| = 0, \tag{3}$$

$$\lim_{t \to \infty} |Y(t)P_{-1}| = 0 \quad \text{and} \quad \limsup_{t \to -\infty} |Y(t)P_{-1}\xi| = \infty \tag{4}$$

provided $P_{-1}\xi \neq 0$;

$$\lim_{t \to -\infty} |Y(t)P_1| = 0 \quad \text{and} \quad \limsup_{t \to \infty} |Y(t)P_1\xi| = \infty \tag{5}$$

provided $P_1\xi \neq 0$; and

$$\limsup_{|t| \to \infty} |Y(t)P_\infty\xi| = \infty \tag{6}$$

provided $P_\infty\xi \neq 0$. The above limits follow from lemmas of W. A. Coppel [2, p. 68, p. 74] in the case $q = 1$ and from work of R. Conti [1] whenever $1 < q < \infty$ (see also [5], [6]).

The monograph [2, Chap. 3] by Coppel presents an excellent introduction to the fundamentals of linear perturbation problems where conditional stability is present for the linear unperturbed system. Related problems are discussed in J. K. Hale's book [4, Chap. 4].

For q as given above, we require that the function ω that dominates f also satisfies the following condition.

(H.2) If $q = 1$, $\lim_{t\to\infty} \omega(t, r, \varepsilon) = 0$ for each fixed $(r, \varepsilon) \in R \times R_+$; if $1 < q < \infty$ and $1/p + 1/q = 1$, then $\omega(\cdot, r, \varepsilon) \in L^p(R)$ for each $(r, \varepsilon) \in R \times R_+$.

Our main results are the following theorems.

THEOREM 1. *Let* (H.1) *and* (H.2) *be satisfied for equations* (1) *and* (2). *If* $a_0 \in R^n$, *there exists a positive number* $\varepsilon_0 = \varepsilon_0(a_0)$ *such that whenever* λ *satisfies* $|\lambda| \leqq \varepsilon_0$ *then* (1) *has solutions* x_+, x *and* x_0 *with the properties:*

(i) $$x_+(\cdot, a_0, \lambda) \in L^\infty(R_+);$$

and

$$\lim_{\lambda \to 0} x_+(t, a_0, \lambda) = Y(t)\,(P_0 + P_{-1})a_0, \qquad t \in R_+.$$

(ii) $$x_-(\cdot, a_0, \lambda) \in L^\infty(R_-);$$

and

$$\lim_{\lambda \to 0} x_-(t, a_0, \lambda) = Y(t)\,(P_0 + P_1)a_0, \qquad t \in R_-.$$

If a_0 *satisfies* $(P_1 + P_{-1})a_0 = 0$, *then*

(iii) $$x_0(\cdot, a_0, \lambda) \in L^\infty(R);$$

and

$$\lim_{\lambda \to 0} x_0(t, a_0, \lambda) = Y(t)P_0 a_0, \qquad t \in R.$$

THEOREM 2. *Let the hypotheses of Theorem* 1 *be satisfied, and initial value problems for* (1) *have unique solutions.*

(iv) *If* $|f_\lambda(\cdot, Y(\cdot)(P_0 + P_{-1})a_0, 0)| \in L^p(R_+)$, *then* x_+ *of Theorem* 1 (i) *satisfies* $\lim_{\lambda\to 0} \partial x_+(\cdot, a_0, \lambda)/\partial\lambda \in L^\infty(R_+)$.

(v) *If* $|f_\lambda(\cdot, Y(\cdot)(P_0 + P_1)a_0, 0)| \in L^p(R_-)$, *then* x_- *of Theorem* 1 (ii) *satisfies* $\lim_{\lambda\to 0} \partial x_-(\cdot, a_0, \lambda)/\partial\lambda \in L^\infty(R_-)$.

(vi) *If* $|f_\lambda(\cdot, Y(\cdot)P_0 a_0, 0)| \in L^p(R)$, *then* x_0 *of Theorem* 1 (iii) *satisfies* $\lim_{\lambda\to 0} \partial x_0(\cdot, a_0, \lambda)/\partial\lambda \in L^\infty(R)$.

The conclusions of Theorem 1 discuss the behavior of solutions of (1) as functions of both t and λ. Theorem 2 reveals important information about the coefficient of the linear term λ in the Maclaurin series expansion of x about $\lambda = 0$; and hence, implicitly yields the behavior of $\partial x/\partial\lambda$ as $\lambda \to 0$.

Proof of Theorem 1. The existence of the solutions x_+, x_- and x_0 is a straightforward application of the Schauder-Tykhonov fixed-point theorem. The details relevant to the parameter λ will be stressed while the remainder of the procedure will only be indicated.

For $\alpha > 0$, $\beta > 0$ and $D = R$, R_+ or R_- define $C_{\alpha,\beta}(D) = \{x(t,\lambda) : x(\cdot,\lambda) \in L^\infty(D) \cap C(D \times R^m, R^n),\ |x(t,\lambda)| \leqq \alpha$ for each λ with $|\lambda| \leqq \beta\}$. Let $M > 0$ be chosen so that

$$|Y(t)P_{-1}| + |Y(t)P_0| \leqq M, \qquad t \in R_+,$$

and

$$|Y(t)P_1| + |Y(t)P_0| \leqq M, \qquad t \in R_-.$$

We choose $\alpha_0 = \alpha_0(a_0)$ so that

$$\alpha_0 > M_0 \equiv \max\{M|(P_0 + P_{-1})a_0|, M|(P_0 + P_1)a_0|\};$$

now, select ε_0 sufficiently small so that

$$|\omega(\cdot, \alpha_0, \varepsilon_0)|_{L^p(R)} \leqq K^{-1}(\alpha_0 - M_0). \tag{7}$$

This choice is possible since ω is continuous and $\omega(t, r, 0) = 0$. It is made in order that an operator T, as defined in the next paragraph, maps a set $C_{\alpha_0,\varepsilon_0}$ into itself.

For $D = R, R_+$ or R_-, define the operator $T: C_{\alpha_0,\varepsilon_0}(D) \to C_{\alpha_0,\varepsilon_0}(D)$ by

$$\begin{aligned} Tx(t,\lambda) &= Y(t)\,(P_0 + P_{-1})a_0 \\ &\quad + \int_0^t Y(t)\,(P_0 + P_{-1})Y^{-1}(s)f(s, x(s,\lambda),\lambda)\,ds \\ &\quad - \int_t^\infty Y(t)(P_1 + P_\infty)Y^{-1}(s)f(s, x(s,\lambda),\lambda)\,ds, \qquad t \in R_+, \end{aligned} \tag{8}$$

$$\begin{aligned} Tx(t,\lambda) &= Y(t)\,(P_0 + P_1)a_0 \\ &\quad + \int_0^t Y(t)\,(P_0 + P_1)Y^{-1}(s)f(s, x(s,\lambda),\lambda)\,ds \\ &\quad - \int_t^\infty Y(t)\,(P_{-1} + P_\infty)Y^{-1}(s)f(s, x(s,\lambda),\lambda)\,ds, \qquad t \in R_-. \end{aligned} \tag{9}$$

It follows from (8) and (9) that $TC_{\alpha_0,\varepsilon_0}(D) \subseteq C_{\alpha_0,\varepsilon_0}(D)$; in the case $D = R$, the "bifurcation" condition $(P_1 + P_{-1})a_0 = 0$ is also needed to demonstrate this inclusion. Using the compact-open topology, we see that T is a continuous operator; the details are similar to those in [5], [6], and because of this similarity are omitted. The closure of $TC_{\alpha_0,\varepsilon_0}(D)$ is compact by Ascoli's theorem: The Schauder-Tykhonov theorem implies that T has a fixed point, $x_D(t, a_0, \lambda)$, in $C_{\alpha_0,\varepsilon_0}(D)$. We shall denote these fixed points by x_0, x_+ and x_- for $D = R$, R_+ and R_- respectively.

We shall now show that these solutions have the prescribed behavior as $\lambda \to 0$. It follows from (8) that

$$|x_+(t, a_0, \lambda) - Y(t)\,(P_0 + P_{-1})a_0| \leqq K \sup_{t \in R_+} \omega(t, \alpha_0, |\lambda|), \qquad |\lambda| \leqq \varepsilon_0, \quad t \in R_+.$$

Since $\omega(t, r, 0) = 0$, we obtain

$$\lim_{\lambda \to 0} x_+(t, a_0, \lambda) = Y(t)\,(P_0 + P_{-1})a_0.$$

The corresponding limits for x_0 and x_- are verified in an analogous manner. This completes the proof of Theorem 1.

Proof of Theorem 2. We shall verify conclusion (iv). Let $x_+(t, 0, x_*, \lambda)$ denote the solution of (1) with $x_+(0, 0, x_*, \lambda) = x_*$. We wish to determine the initial position x_* with the property that

$$x_+(t, a_0, \lambda) = x_+(t, 0, x_*, \lambda), \qquad t \in R_+, \quad \lambda \in R^m.$$

To this end, let $x_* = (P_0 + P_{-1})a_0 + \xi$, where $\xi = (P_1 + P_\infty)x_*$; therefore,

$$x_+(t, a_0, \lambda) = x_+(t, 0, (P_0 + P_{-1})a_0 + \xi(a_0, \lambda), \lambda).$$

The solution $x_+(t, a_0, \lambda)$ is a fixed point of the operator T defined by (8); hence, for $t \in R_+$,

$$
\begin{aligned}
x_+(t, a_0, \lambda) = {} & Y(t)(P_0 + P_{-1})a_0 \\
& + \int_0^t Y(t)(P_0 + P_{-1})Y^{-1}(s)f(s, x_+(s, a_0, \lambda), \lambda)\, ds \\
& - \int_t^\infty Y(t)(P_1 + P_\infty)Y^{-1}(s)f(s, x_+(s, a_0, \lambda), \lambda)\, ds.
\end{aligned} \tag{10}
$$

The variation of parameters formula yields

$$
\begin{aligned}
x_+(t, 0, x_*, \lambda) = {} & Y(t)(P_0 + P_{-1})x_* + Y(t)(P_1 + P_\infty)x_* \\
& + \int_0^t Y(t)Y^{-1}(s)f(s, x_+(s, 0, x_*, \lambda), \lambda)\, ds.
\end{aligned} \tag{11}
$$

A comparison of (10) and (11) at $t = 0$ leads to

$$\xi(a_0, \lambda) = -\int_0^\infty (P_1 + P_\infty)Y^{-1}(s)f(s, x_+(s, 0, (P_0 + P_{-1})a_0 + \xi(a_0, \lambda), \lambda), \lambda)\, ds.$$

From this it follows that $\xi(a_0, 0) = 0$ and

$$\xi_\lambda(a_0, 0) = -\int_0^\infty (P_1 + P_\infty)Y^{-1}(s)f_\lambda(s, Y(s)(P_0 + P_{-1})a_0, 0)\, ds.$$

A direct computation, utilizing (1) and the fact

$$x_+(0, 0, (P_0 + P_{-1})a_0 + \xi(a_0, \lambda), \lambda) = (P_0 + P_{-1})a_0 + \xi(a_0, \lambda),$$

shows that the function $Z(t, a_0) \equiv \lim_{\lambda \to 0} \partial x_+(t, a_0, \lambda)/\partial \lambda$ is a solution of the matrix initial value problem

$$
\begin{aligned}
Z' &= A(t)Z + f_\lambda(t, Y(t)(P_0 + P_{-1})a_0, 0), \\
Z(0, a_0) &= -\int_0^\infty (P_1 + P_\infty)Y^{-1}(s)f_\lambda(s, Y(s)(P_0 + P_{-1})a_0, 0)\, ds.
\end{aligned} \tag{12}
$$

This implies that

$$
\begin{aligned}
P_1 Z(0, a_0) &= -\int_0^\infty P_1 Y^{-1}(s)f_\lambda(s, Y(s)(P_0 + P_{-1})a_0, 0)\, ds, \\
P_\infty Z(0, a_0) &= -\int_0^\infty P_\infty Y^{-1}(s)f_\lambda(s, Y(s)(P_0 + P_{-1})a_0, 0)\, ds.
\end{aligned} \tag{13}
$$

We are now in position to apply the following result (see T. G. Hallam [7]): If (H.1) is satisfied and $g(\cdot) \in L^p(R)$, then a solution $u = u(t)$ of $u' = A(t)u + g(t)$ is bounded on R_+ if and only if

$$P_1 u(0) = -\int_0^\infty P_1 Y^{-1}(s)g(s)\, ds$$

and

$$P_\infty u(0) = -\int_0^\infty P_\infty Y^{-1}(s)g(s)\,ds.$$

We obtain from (12) and (13) that $Z(\cdot, a_0) \in L^\infty(R_+)$; that is,

$$\lim_{\lambda\to 0} x_+(t, a_0, \lambda)/\partial\lambda \in L^\infty(R_+).$$

The remaining conclusions about the solutions x_0 and x_- can be obtained in a similar manner. This completes the proof of Theorem 2.

Remark 1. The function $h(t, \lambda) = \sin \lambda t$ is bounded on R but $\lim_{\lambda\to 0} \partial h(t, \lambda)/\partial\lambda = t$ is not in $L^\infty(R)$. Under the hypothesis of Theorem 2, this type of behavior is impossible for solutions of (1).

Remark 2. In the case where (2) is autonomous, the conditional asymptotic stability hypothesis (H.1) is satisfied provided no characteristic root of A has zero real part; here, we have $P_\infty = 0 = P_0$. It is interesting to observe that when A is constant, the projection P_∞ need not, in general, be zero. This can occur whenever the matrix A has a characteristic root with zero real part and multiplicity greater than one. When A is constant and (H.1) is satisfied, then it is necessary that $P_0 \equiv 0$.

A simple system to which Theorem 1 is applicable is $y = (y_1, y_2, y_3, y_4)^T$ and $A(t) = \operatorname{diag}(-2t, -1, 1, 2t)$. The fundamental matrix Y of (2) with $Y(0) = I_4$ is $Y(t) = \operatorname{diag}(e^{-t^2}, e^{-t}, e^t, e^{t^2})$. The projections P_i can be chosen as $P_{-1} = \operatorname{diag}(0, 1, 0, 0)$, $P_0 = \operatorname{diag}(1, 0, 0, 0)$, $P_1 = \operatorname{diag}(0, 0, 1, 0)$, $P_\infty = \operatorname{diag}(0, 0, 0, 1)$. A direct computation verifies hypothesis (H.1) is valid here.

3. Conditional stability of (2). Suppose that the linear system (2) satisfies the analogue of condition (H.1) for $q = \infty$. In this situation, a decomposition of the solution space into bounded solutions is not effected; however, if the additional hypotheses

$$\limsup_{t\to-\infty} |Y(t)P_{-1}\xi| = \infty \quad \text{provided} \quad P_{-1}\xi \neq 0,$$

$$\limsup_{t\to\infty} |Y(t)P_1\xi| = \infty \quad \text{provided} \quad P_1\xi \neq 0,$$

$$\limsup_{|t|\to\infty} |Y(t)P_\infty\xi| = \infty \quad \text{provided} \quad P_\infty\xi \neq 0,$$

are required, then results similar to Theorems 1 and 2 can be obtained.

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AN EXPLICIT CALCULATION OF SOME SETS OF MINIMAL CAPACITY*

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Abstract. The following extremum problem is treated: Let $\{c_1, \cdots, c_n\}$ be a finite set of complex numbers. Among the continua that contain all the c_i one with minimal capacity is to be found. The known conditions of Lavrentjev are reduced to a system of equations for the unknown critical points. For the case $n = 4$ with symmetry a detailed study is given and for the cases $n = 3$ and $n = 4$ some values are explicitly calculated.

1. Introduction. Let C be a continuum in the complex z-plane and Ω its complement. There is a conformal mapping $z = \phi(\zeta)$ which maps the exterior of the unit circle $|\zeta| > 1$ onto Ω such that $\phi(\infty) = \infty$ with the Taylor development

$$\phi(\zeta) = b\zeta + b_0 + \frac{b_1}{\zeta} + \cdots .$$

$|b|$ is an important conformal invariant called the capacity of C.

In this paper we want to treat the following extremum problem: Let $\{c_1, \cdots, c_n\}$ be a finite set of complex numbers. We want to find a continuum C which contains all the c_i and has minimal capacity.

This problem has not only applications to electrodynamics and other branches of physics but also to the theory of univalent functions. G. V. Kuz'mina [2] characterizes the solution for the case $n = 3$ by a system of equations and applies this result to the theory of univalent functions. Our method seems to be preferable for the actual calculation and applies to all n.

For the cases $n = 3$ and $n = 4$ (with symmetry) we give some explicitly calculated values of the critical points.

2. The basic system of equations. It is known that there is always a unique solution of this problem and that C consists of finitely many analytic arcs together with their limiting endpoints [2], [3]. It is also known that C solves the problem if and only if there is a polynomial $P(z) = z^{n-2} + d_1 z^{n-3} + \cdots + d_{n-2}$ of degree $n - 2$ such that the mapping function $\phi(\zeta)$ satisfies the equation

$$\log \zeta = \int \left[\frac{P(z)}{\prod_{k=1}^{n} (z - c_k)} \right]^{1/2} dz. \tag{1}$$

All the zeros of $P(z)$ must lie on C.

We denote the zeros of $P(z)$ by a_i, $i = 1, 2, \cdots, n - 2$. Each zero of order k is a limiting endpoint of $k + 2$ arcs belonging to C. The set of zeros is therefore uniquely determined by C. For proofs of these statements see [1], [3].

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It follows that there are polygonal arcs γ_i joining a_1 with a_i, $i = 2, 3, \cdots, n-2$, and δ_j joining a_1 with c_j, $j = 1, 2, \cdots, n-1$, such that

$$
(2)\qquad
\begin{aligned}
&\operatorname{Re}\int_{\gamma_i}\left[\frac{P(z)}{\prod_k (z-c_k)}\right]^{1/2} dz = 0, \qquad i = 2, \cdots, n-2,\\
&\operatorname{Re}\int_{\delta_j}\left[\frac{P(z)}{\prod_k (z-c_k)}\right]^{1/2} dz = 0, \qquad j = 1, \cdots, n-1,
\end{aligned}
$$

where one branch of the square root is followed continuously along γ_i and δ_j respectively. It is to be observed that in the above equations there is no path of integration joining a_1 and c_n. We shall see that the corresponding equation is automatically satisfied.

We want to use (2) to determine the unknowns a_i. In order to do that, we have to show that we can reconstruct the continuum C from each solution of (2). It follows then that the solution of (2) is unique.

3. A lemma. We assume now that we have a set $\{a_i\}$ and polygonal arcs γ_i and δ_j such that (2) is satisfied. For abbreviation we denote:

$$\frac{\prod_{i=1}^{n-2}(z-a_i)}{\prod_{i=1}^{n}(z-c_i)} = Q(z).$$

We prove the following lemma.

LEMMA. $|\operatorname{Re}\int_{a_1}^{z}\sqrt{Q(z)}\,dz|$ *does not depend on the path connecting* a_1 *and* z.

Proof. Let α and β be the two paths connecting a_1 and z. We have (after changing the branch of the square root along all β, if necessary):

$$\int_\alpha \sqrt{Q(z)}\,dz - \int_\beta \sqrt{Q(z)}\,dz = \int_{\alpha\beta^{-1}} \sqrt{Q(z)}\,dz.$$

According to Cauchy's integral theorem it is therefore enough to show that

$$\operatorname{Re}\int_{\sigma_l}\sqrt{Q(z)}\,dz = 0$$

for a system $\{\sigma_l : l = 1, 2, \cdots, 2n-3\}$ which generates the fundamental group of the $2n-3$ times punctured plane $\mathbb{C}\setminus[\{a_2, \cdots, a_{n-2}\} \cup \{c_1, \cdots, c_n\}]$.

Let R be a positive number such that all the c_i and a_i lie inside the circle $\{|z| < R\}$, and σ be a path connecting a_1 with the circle $\{|z| = R\}$. For σ_1 we choose the path, which we obtain by following first σ, then $\{|z| = R\}$ once around and then back to a_1 along σ^{-1}. Since $Q(z)$ has a double zero at ∞, the branch of the square root does not change when we follow the circle $\{|z| = R\}$, and the two integrals over σ and σ^{-1} therefore cancel each other.

Since

$$Q(z) = \frac{1}{z^2} + \frac{e_3}{z^3} + \cdots$$

we have

$$\sqrt{Q(z)} = \pm\left\{\frac{1}{z} + \frac{f_2}{z^2} + \cdots\right\}$$

and

$$\operatorname{Re}\int_{|z|=R}\sqrt{Q(z)}\,dz = \pm\operatorname{Re}2\pi i = 0.$$

Therefore

$$\operatorname{Re}\int_{\sigma_1}\sqrt{Q(z)}\,dz = 0.$$

For the other σ_l we follow γ_l, $l = 2, 3, \cdots, n-2$ (respectively $\delta_{l-(n-2)}$, $l = n-1, n, \cdots, 2n-3$), then once around the endpoint on a small circle and then back to a_1. The integrals over these paths do not depend on the radii of the circles around the a_i and c_j. If we let these radii tend to zero, we get from (2) that

$$\operatorname{Re}\int_{\sigma_l}\sqrt{Q(z)}\,dz = 0, \qquad l = 2, \cdots, 2n-3.$$

These $2n-3$ paths generate the fundamental group of the $2n-3$ times punctured plane. The proof of the lemma is then complete.

4. Sufficiency of the basic equations. We denote the function

$$\left|\operatorname{Re}\int_{\sigma'}^{z}\sqrt{Q(t)}\,dt\right| \text{ by } u(z).$$

It is harmonic except at the set $\{z|u(z) = 0\}$ which we denote by C plus possibly c_n. If c_n would not be in C, it would be an isolated singularity of a harmonic function which is bounded in a neighborhood of c_n. This is not possible and therefore $c_n \in C$.

Furthermore $u(z) - \log|z|$ is bounded close to ∞, i.e., $u(z)$ is the Green's function in the complement Ω of C. It has no stationary values in Ω. Therefore Ω has to be simply connected and C has to be connected (see [4, pp. 31–32]). Therefore there is a conformal mapping $z = \phi(\zeta)$ from the exterior of the unit circle onto Ω. We may assume that $\phi(\infty) = \infty$. Since the Green's function is conformally invariant, we have

$$\log|\zeta| = u(z) = \operatorname{Re}\int\sqrt{Q(z)}\,dz\,,$$

and by analytic completion we get (1). Therefore C is the minimal continuum.

It follows from the lemma that we could have begun with any paths γ_i joining a_i with a_i (respectively, δ_j joining a_1 with c_j) and we would have got the solution. We sum up our results in the following theorem.

THEOREM 1. *The system* (2) *has for any choice of paths γ_i connecting a_1 with a_i and δ_j connecting a_1 with c_j one and only one solution. The minimal continuum is the set*

$$\operatorname{Re}\int_{a_1}^{z}\left[\frac{P(z)}{\prod_k (z-c_k)}\right]^{1/2} dz = 0.$$

5. Further information on the position of the a_i. It is known that C and therefore also the a_i must lie in the convex hull of the c_j. This gives us a rough estimate for the range of the a_i.

In case the c_j are vertices of a rectangle we can say more about the a_i because of symmetry. Since $\operatorname{cap}(aC + b) = |a| \operatorname{cap} C$ (a, b complex numbers) it is no loss of generality to assume that

$$c_j = \pm 1 \pm it, \qquad 0 < t \leqq 1. \tag{3}$$

Since the solution is unique the a_1, a_2 must be symmetric with respect to both axes, i.e., they must lie on one of the axes and $a_1 = -a_2$. Also by symmetry or by direct calculation we conclude that $\operatorname{Re}\sqrt{Q(z)}\,dz = 0$ along the line segment $\overline{a_1 a_2}$. It is therefore sufficient to solve the equation

$$\operatorname{Re}\int_0^{c_1}\left[\frac{z^2 - a_1^2}{(z^2 - c_1^2)(z^2 - \bar{c}_1^2)}\right]^{1/2} dz = 0. \tag{4}$$

All the other equations of (2) then follow by symmetry.

This equation depends real-analytically on t and has for all t precisely one solution. Therefore a_1 must depend continuously on t.

The question arises whether a_1 is real or imaginary. We prove first the following.

LEMMA. *$a_1 = a_2 = 0$ solves (4) if and only if $t = 1$, i.e., $\arg c_1 = \pi/4$.*

Proof. We have for $a_1 = 0$ on the line segment $z = sc_1$, $0 \leqq s \leqq 1$,

$$\arg Q(z) = \arg \frac{s^2 c_1^2}{c_1^2(s^2 - 1)([sc_1]^2 - \bar{c}_1^2)} = \pi - \arg([sc_1]^2 - \bar{c}_1^2).$$

For geometric reasons we have on this line segment

$$\pi - \arg c_1^2 \geqq \arg([sc_1]^2 - \bar{c}_1^2) \geqq \pi/2.$$

We conclude that

$$\pi/2 \geqq \arg Q(z) \geqq \arg c_1^2,$$

and we can determine the square root such that

$$\pi/4 \geqq \arg\sqrt{Q(z)} \geqq \arg c_1;$$

this branch is continuous on our line segment. But then since $\arg dz = \arg c_1$ we have

$$2\arg c_1 \leqq \arg\{\sqrt{Q(z)}\,dz\} \leqq \pi/4 + \arg c_1 \leqq \pi/2.$$

This means that

$$\operatorname{Re}\{\sqrt{Q(z)}\,dz\} \geqq 0,$$

where equality holds on the whole line segment if and only if $\arg c_1 = \pi/4$. In all other cases we get

$$\operatorname{Re}\int_0^{c_1}\sqrt{Q(z)}\,dz > 0,$$

i.e., (4) is not satisfied, and $a_1 = a_2 = 0$ cannot be a solution.

Since a_1 depends continuously on t we conclude that either a_1 is real for all t or imaginary for all t, $0 < t \leqq 1$.

THEOREM 2. *a_1 is real for all t, $0 < t \leqq 1$.*

We prove this fact indirectly. If a_1 is imaginary, we have

$$\lim_{t \to 0} a_1 = 0$$

because a_1 must lie in the convex hull of the $\{c_i\}$. Therefore $Q(z)$ and $(d/dy) \arg Q(z)$ depend continuously on t also in the limiting case $t \to 0$ (except for $z = a_i, z = c_j$). Therefore $(d/dy) \arg Q(z)$ is bounded in the rectangle

$$R = \{1/4 \leqq \operatorname{Re} z \leqq 3/4\} \times \{-t \leqq \operatorname{Im} z \leqq +t\}.$$

One can verify directly that

$$Q(z) > 0 \quad \text{or} \quad \arg Q(z) = 0 \tag{5}$$

on the real axis. On the other hand, we have along the continuum C:

$$\operatorname{Re} \sqrt{Q(z)}\, dz = 0 \quad \text{or} \quad Q(z)\, dz^2 \leqq 0 \quad \text{or} \quad \arg Q(z) = \pi - 2 \arg dz. \tag{6}$$

C must cross both vertical sides of R and according to the mean value theorem there must be a point in R where along C

$$|\arg dz| \leqq |\arctan 4t|$$

holds or according to (6)

$$|\arg Q(z) - \pi| \leqq 2|\arctan 4t|. \tag{7}$$

This point has at most distance t from the real axis. Conditions (5) and (7) cannot both hold for all t unless $(d/dy)Q(z)$ is unbounded which cannot be the case. Therefore a_1 cannot be imaginary for all t and has to be real for all t.

6. Numerical calculations for the rectangle. We assume that the rectangle has been rotated so that its longer side is parallel to the real axis and that it is symmetric with respect to the origin. From Theorem 2 we then know that the solution lies on the real axis. We also know that the solution is unique in the rectangle. Denoting the real part of the solution by a, we use the method of binary search using the values of the integral

$$f(a) = \operatorname{Re}\left[\int_0^{c_1} \left[\frac{z^2 - a^2}{(z^2 - c_1^2)(z^2 - \bar{c}_1^2)}\right]^{1/2} dz\right] \tag{8}$$

to determine what action to take in the binary search. The starting values were chosen to be the value of (8) at the origin and at $\operatorname{Re}(c_1)$. Obviously a value a for which $f(a) = 0$ is the solution of the problem.

In order to apply a numerical quadrature to the integral (8) we subtract out the singularity at c_1 in the following manner:

$$\begin{aligned} f(a) &= \operatorname{Re}\left[\int_0^{c_1} \left[\frac{z^2 - a^2}{(z^2 - c_1^2)(z^2 - \bar{c}_1^2)}\right]^{1/2} dz\right] = \operatorname{Re}\left[\int_0^{c_1} \frac{1}{(z - c_1)^{1/2}} g(z)\, dz\right] \\ &= \operatorname{Re}\left[\int_0^{c_1} \frac{g(z) - g(c_1)}{(z - c_1)^{1/2}}\, dz - 2g(c_1)(-c_1)^{1/2}\right], \end{aligned}$$

where

$$g(z) = \left[\frac{z^2 - a^2}{(z + c_1)(z^2 - \bar{c}_1^2)}\right]^{1/2}.$$

The integral

$$\int_0^{c_1} \frac{g(z) - g(c_1)}{(z - c_1)^{1/2}}\,dz$$

is now evaluated by a suitable quadrature rule (in our case Simpson's rule). The path of integration is from 0 to c_1 in each case, i.e., the length is

$$[(\operatorname{Re} c_1)^2 + (\operatorname{Im} c_1)^2]^{1/2}.$$

We integrate over 20 subintervals and the error then is of the order $[1/20]^5 \approx 10^{-6}$, an accuracy that satisfies our needs since we also execute the binary search procedure 20 times giving an accuracy of the binary search procedure of the order $1/2^{20} \approx 10^{-6}$ as well. The results are given in Table 1.

TABLE 1
Numerical results for the rectangle

c_1	a_1
1. + i.05	.95779
1. + i.1	.91675
1. + i.15	.87692
1. + i.2	.83811
1. + i.25	.80013
1. + i.3	.76277
1. + i.35	.72582
1. + i.4	.68902
1. + i.45	.65212
1. + i.5	.61484
1. + i.55	.57683
1. + i.6	.53774
1. + i.65	.49714
1. + i.7	.45446
1. + i.75	.40900
1. + i.8	.35972
1. + i.85	.30500
1. + i.9	.24185
1. + i.95	.16290
1. + i1.	.38146E − 05

The values in Table 1 are correct to about five figures, the remaining loss of accuracy to be attributed to numerical errors.

7. Numerical calculations for the triangle. We assume for these calculations that we fix c_1 and c_2 to $c_1 = -1 - i$ and $c_2 = -1 + i$. We then let c_3 vary. By doing this we are able to find the solution for any triangle since if a triangle is rotated, the solution remains fixed with respect to the vertices. If a triangle is magnified, the solution is magnified in the same manner.

Our aim now is to find solutions of

$$f_1(a) = \operatorname{Re}\left[\int_a^{c_1}\left[\frac{z-a}{(z-c_1)(z-c_2)(z-c_3)}\right]^{1/2} dz\right],$$

$$f_2(a) = \operatorname{Re}\left[\int_a^{c_2}\left[\frac{z-a}{(z-c_1)(z-c_2)(z-c_3)}\right]^{1/2} dz\right].$$

A complex a for which $f_i(a) = 0$, $i = 1, 2$, is the required solution.

We evaluate each integral as was done in the previous section. The idea now is that $f_1(z) = 0$ determines a path from c_1 through the solution point. Similarly $f_2(z) = 0$ determines a path from c_2 through the solution point. We therefore start with a line parallel to the line through c_1 and c_2 some distance to the right of this line. We find the solution of $f_1(z) = 0$, say z_1, and $f_2(z) = 0$, say z_2. According to $\operatorname{Im}(z_1 - z_2)$ being positive or negative we move the line by a positive or negative amount. We have found our solution $a = z_1 = z_2$ when $\operatorname{Im}(z_1 - z_2) = 0$.

As in the previous section we may expect our results to be of the order of 10^{-5} in accuracy.

Some results for the triangle are given in Table 2.

TABLE 2

Numerical results for the triangle

c_3	a_2
$1. + i.0$	$-.39174533 + i.00000000$
$1. + i.00003$	$-.39174533 + i.00001525$
$1. + i.0005$	$-.39174771 + i.00024366$
$1. + i.05$	$-.39188838 + i.02414035$
$1. + i.10$	$-.39231992 + i.04824876$
$1. + i.15$	$-.39303517 + i.07229423$
$1. + i.20$	$-.39403415 + i.09624528$
$1. + i.25$	$-.39530253 + i.12005615$
$1. + i.30$	$-.39687133 + i.14374256$
$1. + i.35$	$-.39870000 + i.16722870$
$1. + i.40$	$-.40079808 + i.19050026$
$1. + i.45$	$-.40315843 + i.21353078$
$1. + i.50$	$-.40577626 + i.23629165$
$1. + i.55$	$-.40863967 + i.25875854$
$1. + i.60$	$-.41174627 + i.28090787$
$1. + i.65$	$-.41508412 + i.30271244$
$1. + i.70$	$-.41864610 + i.32415295$
$1. + i.75$	$-.42242265 + i.34521222$
$1. + i.80$	$-.42640185 + i.36586380$
$1. + i.85$	$-.43057656 + i.38610053$
$1. + i.90$	$-.43493009 + i.40590024$
$1. + i.95$	$-.43946242 + i.42525792$
$1. + i1.$	$-.44414496 + i.44414568$
$1. + i1.05$	$-.44898248 + i.46256990$

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NOTE ON ORTHOGONAL POLYNOMIALS IN v-VARIABLES*

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Abstract. We consider the polynomials in v-variables orthogonal with respect to a weight $w(\mathbf{x})$ that need not be decomposable into a product of functions of one variable and obtained by applying the orthogonalization process to an ordered sequence of monomials in v-variables. We derive recursion relations, an expression for the Christoffel–Darboux kernel and a system of partial differential equations for these polynomials.

1. Introduction. The extension to an arbitrary number of variables of the classical orthogonal polynomials in one variable presents some difficulty. One way to obtain orthogonal polynomials in several variables is by orthogonalizing an ordered sequence of monomials. In general the family of polynomials obtained in this fashion lacks symmetry. One then tries to obtain other families of polynomials having some orthogonality property. The second approach goes along these lines. One constructs two families of polynomials having the biorthogonality property [4], [10]. Once this has been done the coefficients of the expansion of an arbitrary function in terms of one of the families are obtained by taking the inner products of two polynomials, one in each family. The procedure has many similarities with the expansion of a function of one variable in terms of an orthogonal set. Finally a third approach consists in defining a single family of polynomials having some orthogonality property which in general will be that two polynomials of different degrees be orthogonal but not necessarily two polynomials with the same degrees. Examples of this approach are found in [1], [2] and in [3], [4], [5] where one bases the analysis on generating functions in several variables which are similar to the generating functions for polynomials in one variable.

In this note we concentrate on the first approach. We derive some properties of the orthogonal polynomials defined by orthogonalizing an ordered sequence of monomials. Although, in general, the approach lacks symmetry in the variables one can derive many results closely paralleling the one-variable case [6], [7], [8]. This makes the approach attractive.

We derive a recursion formula (10), an expression for the Christoffel–Darboux kernel (16) and a system of partial differential equations (22) for the polynomials.

2. Ordering θ. We want to construct an ordering of monomials of v-variables in such a way that some simple properties of the one-variable sequence 1, x, $x^2, \cdots$ are preserved. In particular, if $n < m$, the degree of the derivative of x^n will be smaller than the degree of the derivative of x^m. We want this to hold for the ordered sequence in v-variables, that is to say if $m_n(\mathbf{x})$ and $m_m(\mathbf{x})$ are two monomials of orders n and m $(n < m)$ we desire the order of any derivative of

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$m_n(\mathbf{x})$ to be less than the order of the same derivative of $m_n(\mathbf{x})$. Similarly we require that the order of the monomial $x_1 m_n(\mathbf{x})$ be less than the order of $x_1 m_m(\mathbf{x})$ if $n < m$, x_1 being any of the v variables. This is an obvious property for monomials in one variable. These properties are used in the derivation of the recursion formulas and differential equations satisfied by the polynomials in one variable.

To construct an ordering satisfying the above properties we shall order the sequence of exponent vectors $(n_1 \quad n_2 \cdots n_v)$

$$m_n(\mathbf{x}) = x_1^{n_1} x_2^{n_2} \cdots x_v^{n_v}.$$

We have to order monomials with different degrees c,

$$c = \sum_{i=1}^{v} n_i,$$

and to effect this we will order them from smaller to higher degree first. Next, we have to order monomials having the same degree. For $c = 0$ we have $m_1(\mathbf{x}) = 1$ only. To order monomials having the same degree $c \geqq 1$ we put first $(c \quad 0 \cdots 0)$. Then the next element to vector $(k_1 \quad k_2 \cdots k_v)$, $k_1 \neq 0$, will be $(k_1 - 1 \quad k_2 + 1 \cdots k_v)$. Finally for any vector $(0 \quad 0 \cdots 0 \quad k \quad r \quad s \cdots t)$, $k \neq 0$; $r, s, \cdots, t \geqq 0$, the next element is obtained as $(k - 1 \quad 0 \cdots 0 \quad 0 \quad r + 1 \quad s \cdots t)$. Proceeding similarly, the last element will be $(0 \quad 0 \cdots 0 \quad c)$.

Let $v = 3$ and order the exponent vectors of the monomials in three variables $x_1 = x$, $x_2 = y$, $x_3 = z$. The exponents $(0 \quad 0 \quad 0)$ go first. For degree $c = 1$ first put x, then y and finally z, that is to say, $(1 \quad 0 \quad 0)$, $(0 \quad 1 \quad 0)$ and $(0 \quad 0 \quad 1)$. For $c = 2$ the ordered sequence of exponents will be: $(2 \quad 0 \quad 0)$, $(1 \quad 1 \quad 0)$, $(0 \quad 2 \quad 0)$, $(1 \quad 0 \quad 1)$, $(0 \quad 1 \quad 1)$, $(0 \quad 0 \quad 2)$ according to the rules given above. List the sequence of exponent vectors for degree $c = 3$: $(3 \quad 0 \quad 0)$, $(2 \quad 1 \quad 0)$, $(1 \quad 2 \quad 0)$, $(0 \quad 3 \quad 0)$, $(2 \quad 0 \quad 1)$, $(1 \quad 1 \quad 1)$, $(0 \quad 2 \quad 1)$, $(1 \quad 0 \quad 2)$, $(0 \quad 1 \quad 2)$, $(0 \quad 0 \quad 3)$, The ordered sequence of monomials will be $1, x, y, z, x^2, xy, y^2, xz, yz, z^2, x^3, x^2y, xy^2, y^3, \cdots$ etc.

To order polynomials we order them according to the monomials of higher order in the polynomial.

We shall define two operations on the sequence of exponent vectors:

(I) For any integer k, $l[k]$, $1 \leqq l \leqq v$, is the integer corresponding to exponent vector

$$(k_1 \quad k_2 \cdots k_{l-1} \quad k_l + 1 \quad k_{l+1} \cdots k_v);$$

that is to say, with

$$k \text{———} (k_1 \quad k_2 \cdots k_{l-1} \quad k_l \quad k_{l+1} \cdots k_v)$$

then

$$l[k] \text{———} (k_1 \cdots k_{l-1} \quad k_l + 1 \quad k_{l+1} \cdots k_v).$$

(II) $l'[k] = v$ is the smallest integer greater than or equal to zero such that $l[v] \geqq k$.

The following properties hold for ordering θ:

(a) If $\partial_{x_l} m_n(\mathbf{x}) = n_l m_p(\mathbf{x})$ and $n_l > 0$, then $l[p] = n$.

(b) If $n_l > 0$, then $l[l'[n]] = n$, if $n_l = 0$, $l[l'[n]] > n$.

(c) If $k < n$, then $l[k] < l[n]$ since it is true for any pair of consecutive integers.

(d) If k is such that $k_i = \delta_{il}$ (Kronecker delta) and $q_k(\mathbf{x})\,\partial_{x_l} q_p(\mathbf{x}) = q_r(\mathbf{x})$, where $q_k(\mathbf{x})$, $q_p(\mathbf{x})$ and $q_r(\mathbf{x})$ are polynomials of orders k, p and r, then $r \leqq p$, and equality holds if $p_l > 0$.

3. The polynomials. Consider the Hilbert space H, as defined in [9], of real functions of v variables for which the inner product

$$(f_1, f_2) = \int_R w(\mathbf{x}) f_1(\mathbf{x}) f_2(\mathbf{x})\, d\mathbf{x} \quad \text{for all } f_1, f_2 \in H \tag{1}$$

exists. The region of integration R is arbitrary. If it has infinite measure, the function $w(\mathbf{x})$ should be such that the integral over R of $w(\mathbf{x}) f^2(\mathbf{x})$, $f(\mathbf{x}) \in H$, exists. The norm of an element is taken to be

$$\|f\| = (f, f)^{1/2}.$$

To obtain the orthogonal polynomials with weight $w(\mathbf{x})$, take the ordered sequence of monomials $m_i(\mathbf{x})$, $i = 1, 2, \cdots$, and consider the determinant

$$p_i(\mathbf{x}) = \begin{vmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,i} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,i} \\ & \cdots & & \\ c_{i-1,1} & c_{i-1,2} & \cdots & c_{i-1,i} \\ m_1(\mathbf{x}) & m_2(\mathbf{x}) & \cdots & m_i(\mathbf{x}) \end{vmatrix} = \sum_{k=1}^{i} a_{ki} m_k(\mathbf{x}), \tag{2}$$

where $c_{i,j} = (m_i(\mathbf{x}), m_j(\mathbf{x}))$. $p_i(\mathbf{x})$ is a polynomial of order i, in the sense that it is a linear combination of monomials $m_1(\mathbf{x})$, $m_2(\mathbf{x})$, $\cdots$, $m_i(\mathbf{x})$ in which some monomial other than $m_i(\mathbf{x})$ might not be present. $m_i(\mathbf{x})$ will appear in $p_i(\mathbf{x})$ provided $a_{ii} \neq 0$. This is true because the determinant

$$D_i = \begin{vmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,i} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,i} \\ & \cdots & & \\ c_{i,1} & c_{i,2} & \cdots & c_{i,i} \end{vmatrix} \tag{3}$$

is greater than zero since with $r_i(\mathbf{x}) = \sum_{k=1}^{i} b_k m_k(\mathbf{x})$, b_k arbitrary constants, the quadratic form

$$\|r_i\|^2 = (r_i, r_i) = \sum_{l,k=1}^{i} b_l b_k c_{l,k} \tag{4}$$

is nonnegative and $a_{ii} = D_{i-1}$.

We observe that

$$(m_k(\mathbf{x}), p_i(\mathbf{x})) = \begin{cases} 0, & k < i, \\ D_i, & k = i, \end{cases} \tag{5}$$

and as a consequence,

$$(p_i(\mathbf{x}), p_k(\mathbf{x})) = \begin{cases} 0, & k \neq i, \\ g_k = D_{k-1}D_k, & k = i. \end{cases} \tag{6}$$

Therefore, $p_i(\mathbf{x})$ is the ith orthogonal polynomial with respect to the weight $w(\mathbf{x})$ and ordering θ. We have

$$\|p_i(\mathbf{x})\| = (D_{i-1}D_i)^{1/2} \tag{7}$$

and the orthonormal sequence of polynomials will be given as

$$\bar{p}_i(\mathbf{x}) = (p_i(\mathbf{x}))/(D_{i-1}D_i)^{1/2}, \qquad i = 1, 2, \cdots.$$

4. Recursion formulas. Consider the polynomial of order p: $q_p(\mathbf{x}) = x_l p_i(\mathbf{x})$, where $1 \leqq l \leqq v$ and $p = l[i]$. We can put

$$q_p(\mathbf{x}) = \sum_{k=1}^{p} \alpha_{lik} p_k(\mathbf{x}),$$

where

$$\alpha_{lik} = \frac{1}{g_k}(x_l p_i(\mathbf{x}), p_k(\mathbf{x})) = \frac{1}{g_k}(x_l p_k(\mathbf{x}), p_i(\mathbf{x})). \tag{8}$$

We note that with $k' = l'[i]$ (see (5)),

$$\alpha_{lik} = 0 \quad \text{for } k < k'$$

since then the order of $x_l p_k(\mathbf{x})$ will never exceed i and according to (5) the inner product in (8) will be zero. Therefore the recursion will be given as

$$x_l p_i(\mathbf{x}) = \sum_{k=k'}^{p} \alpha_{lik} p_k(\mathbf{x}) \tag{9}$$

or, writing it in another form,

$$\alpha_{lip} p_p(\mathbf{x}) = x_l p_i(\mathbf{x}) - \sum_{k=k'}^{p-1} \alpha_{lik} p_k(\mathbf{x}), \tag{10}$$

where $p > i > k'$ in ordering θ.

The α's can be put in the form

$$\begin{aligned} g_k \alpha_{lik} = (x_l p_i(\mathbf{x}), p_k(\mathbf{x})) &= \sum_{n=1}^{i} a_{ni}(m_{l[n]}(\mathbf{x}), p_k(\mathbf{x})) \\ &= \sum_{n=v}^{i} a_{ni}(m_{l[n]}(\mathbf{x}), p_k(\mathbf{x})) \end{aligned}$$

and $v = l'[k]$. In particular, for $k = p$, then $v = i$ and

$$g_p \alpha_{lip} = a_{ii}(m_p(\mathbf{x}), p_p(\mathbf{x})) = a_{ii} D_p;$$

consequently

$$\alpha_{lip} = a_{ii}/D_{p-1}.$$

To obtain a recursion satisfied by the a_{lk}'s, put (10) in the form:

$$\alpha_{lip} \sum_{n=1}^{p} a_{np} m_n(\mathbf{x}) = \sum_{n=1}^{i} a_{ni} m_{l[n]}(\mathbf{x}) - \sum_{k=k'}^{p-1} \alpha_{lik} \sum_{m=1}^{k} a_{mk} m_m(\mathbf{x}). \tag{11}$$

Equate terms in $m_s(\mathbf{x})$ to obtain

$$\alpha_{lip} a_{sp} = [a_{s_1 i}] - \sum_{k=l'}^{p-1} \alpha_{lik} a_{sk}, \qquad 1 \leqq s \leqq p, \tag{12}$$

where

$$l' = \begin{cases} s & \text{for } s \geqq k', \\ k' & \text{for } s < k', \end{cases}$$

and the term $[a_{s_1 i}]$ is present if $l[s_1] = s$ for some s_1, $1 \leqq s_1 \leqq i$.

5. An expression for the Christoffel–Darboux kernel. The kernel will be given as

$$K_n(\mathbf{u}, \mathbf{x}) = \sum_{i=1}^{n} \bar{p}_i(\mathbf{u}) \bar{p}_i(\mathbf{x}).$$

To derive another expression for it we proceed as follows: From (9) we obtain

$$x_l \bar{p}_i(\mathbf{u}) \bar{p}_i(\mathbf{x}) = \sum_{k=k'}^{l[i]} \bar{\alpha}_{lik} \bar{p}_k(\mathbf{x}) \bar{p}_i(\mathbf{u}) \tag{13}$$

and

$$u_l \bar{p}_i(\mathbf{u}) \bar{p}_i(\mathbf{x}) = \sum_{k=k'}^{l[i]} \bar{\alpha}_{lik} \bar{p}_k(\mathbf{u}) \bar{p}_i(\mathbf{x}), \tag{14}$$

$$k' = l'[i], \qquad \bar{\alpha}_{lik} = (x_l \bar{p}_i(\mathbf{x}), \bar{p}_k(\mathbf{x})).$$

Subtracting (13) from (14) and adding from $i = 1, 2, \cdots, n$ we obtain

$$(u_l - x_l) K_n(\mathbf{u}, \mathbf{x}) = \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{k=k'}^{l[i]} A_{ik}(\mathbf{u}, \mathbf{x}), \tag{15}$$

where $A_{ik}(\mathbf{u}, \mathbf{x}) = \bar{\alpha}_{lik}(\bar{p}_k(\mathbf{u}) \bar{p}_i(\mathbf{x}) - \bar{p}_k(\mathbf{x}) \bar{p}_i(\mathbf{u}))$ and $A_{jj}(\mathbf{u}, \mathbf{x}) = 0$. If we set $l = v$ and use the fact that $A_{ik}(\mathbf{u}, \mathbf{x}) = -A_{ki}(\mathbf{u}, \mathbf{x})$ we can put (15) in the form

$$K_n(\mathbf{u}, \mathbf{x}) = \frac{\sum_{i=v'[n+1]}^{n} \sum_{k=n+1}^{v[i]} A_{ik}(\mathbf{u}, \mathbf{x})}{u_v - x_v} \tag{16}$$

since for this case many cancellations occur.

For the one-variable case, $v = 1$, we will have $1'[n+1] = n$ and $1[i] = i + 1$ and (16) reduces to the Christoffel–Darboux formula

$$K_n(u, x) = \frac{a_{nn}}{a_{n+1,n+1}} \frac{p_{n+1}(u) p_n(x) - p_{n+1}(x) p_n(u)}{u - x}.$$

We note that for $v > 1$ the numerator of the right-hand side of (16) will have more terms than the left-hand side.

6. A system of partial differential equations. In order to derive a system of partial differential equations for the polynomials we have to impose two strong conditions on the weight function.

(i) $w(\mathbf{x})$ satisfies the system of partial differential equations

$$\frac{\partial_{x_l} w(\mathbf{x})}{w(\mathbf{x})} = \frac{A_l(\mathbf{x})}{G_l(\mathbf{x})},$$

where

$$A_l(\mathbf{x}) = \sum_{i=1}^{a} \alpha_i m_i(\mathbf{x}), \quad a \text{------} \delta_{il}; \qquad 1 \leqq i \leqq v,$$

$$G_l(\mathbf{x}) = \sum_{i=1}^{g} \gamma_i m_i(\mathbf{x}), \quad g \text{------} 2\delta_{il}; \qquad 1 \leqq i \leqq v,$$

and either $\alpha_a \neq 0$ or $\gamma_g \neq 0$ or both.

(ii) $G_l(\mathbf{x})w(\mathbf{x}) = 0$ on the boundary of the region R. Then with $q_m(\mathbf{x})$ an arbitrary polynomial of order less than n, n such that $n_l \geqq 1$, and with

$$f_1(\mathbf{x}) = \partial_{x_l}\{G_l(\mathbf{x})w(\mathbf{x})\partial_{x_l} p_n(\mathbf{x})\},$$

let

$$I_l = \int_R q_m(\mathbf{x}) f_1(\mathbf{x})\, d\mathbf{x}. \tag{17}$$

Integrating (17) by parts twice with respect to x_l and using assumption (ii), we obtain

$$I_l = \int_R p_n(\mathbf{x}) f_2(\mathbf{x})\, d\mathbf{x},$$

where

$$f_2(\mathbf{x}) = \partial_{x_l}\{G_l(\mathbf{x})w(\mathbf{x})\partial_{x_l} q_m(\mathbf{x})\}.$$

But using assumption (i) and property (d) of ordering θ we have that

$$\begin{aligned} f_2(\mathbf{x}) &= w(\mathbf{x})\{(\partial_{x_l} G_l(\mathbf{x}) + A_l(\mathbf{x}))\partial_{x_l} q_m(\mathbf{x}) + G_l(\mathbf{x})\partial^2_{x_l} q_m(\mathbf{x})\} \\ &= w(\mathbf{x}) r_m(\mathbf{x}), \end{aligned}$$

where $r_m(\mathbf{x})$ is a polynomial of order $\leqq m$. Therefore, using (5) and the fact that $m < n$,

$$I_l = \int_R w(\mathbf{x}) p_n(\mathbf{x}) r_m(\mathbf{x})\, d\mathbf{x} = 0, \tag{18}$$

but using again assumption (i) and property (d),

$$\begin{aligned} f_1(\mathbf{x}) &= w(\mathbf{x})\{(\partial_{x_l} G_l(\mathbf{x}) + A_l(\mathbf{x}))\partial_{x_l} p_n(\mathbf{x}) + G_l(\mathbf{x})\partial^2_{x_l} p_n(\mathbf{x})\} \\ &= w(\mathbf{x}) f_n(\mathbf{x}), \end{aligned} \tag{19}$$

where $f_n(\mathbf{x})$ is a polynomial of order n.

Substituting (19) into (17) and using (18), we can write

$$(20) \qquad (q_m(\mathbf{x}), f_n(\mathbf{x})) = 0, \qquad q_m(\mathbf{x}) \text{ arbitrary polynomial of order } < n.$$

Since for a given n we can write (20) for $m = 1, 2, \cdots, n-1$ and $q_m(\mathbf{x})$ can be written as a linear combination of the first m $m_i(\mathbf{x})$'s we deduce that condition (20) implies

$$(21) \qquad (m_i(\mathbf{x}), f_n(\mathbf{x})) = 0, \qquad i = 1, 2, \cdots, n-1.$$

Since the orthogonal polynomials are uniquely determined except for a multiplicative constant, from (21) and (5) we can say that

$$f_n(\mathbf{x}) = k_n p_n(\mathbf{x})$$

which written in another form

$$(22) \qquad G_l(\mathbf{x})\partial_{x_l}^2 p_n(\mathbf{x}) + (\partial_{x_l} G_l(\mathbf{x}) + A_l(\mathbf{x}))\partial_{x_l} p_n(\mathbf{x}) - k_n p_n(\mathbf{x}) = 0$$

gives a differential equation satisfied by the orthogonal polynomials (2) under assumptions (i) and (ii).

When n is such that $n_l = 0$ the polynomial $f_n(\mathbf{x})$ in (19) will be of order $n' < n$. This follows from property (d) of ordering θ. We can also write (20) but now $f_n(\mathbf{x})$ is of order less than n. $q_m(\mathbf{x})$ being an arbitrary polynomial of order less than n, it follows that $f_n(\mathbf{x}) \equiv 0$, n such that $n_l = 0$, and we can write (22) with $k_n = 0$.

Example 1. For the generalized Laguerre polynomials over the positive v-space [3],

$$w(\mathbf{x}) = \prod_{i=1}^{v} x_i^c \exp(-x_i a_i)$$

and

$$\partial_{x_l} w(\mathbf{x}) = \frac{c - a_l x_l}{x_l} w(\mathbf{x}).$$

Therefore $A_l(\mathbf{x}) = c - a_l x_l$ and $G_l(\mathbf{x}) = x_l$. Assumptions (i) and (ii) are satisfied for $l = 1, 2, \cdots, v$. A system of partial differential equations will be

$$x_l \partial_{x_l}^2 p_n(\mathbf{x}) + (1 + c - a_l x_l)\partial_{x_l} p_n(\mathbf{x}) - k_n p_n(\mathbf{x}) = 0.$$

To determine k_n we equate to zero the coefficients of the higher order term, obtaining $-a_l n_l = k_n$.

Example 2. Take the orthogonal polynomials defined over the unit sphere with weight $w(\mathbf{x}) = 1$. We can put

$$\partial_{x_l} w(\mathbf{x}) = \frac{0}{1 - \sum_{i=1}^{v} x_i^2}.$$

Then $A_l(\mathbf{x}) = 0$ and $G_l(\mathbf{x}) = 1 - \sum_{i=1}^{v} x_i^2$. Assumptions (i) and (ii) are satisfied for $l = v$. A differential equation will be

$$\left(1 - \sum_{i=1}^{v} x_i^2\right)\partial_{x_v}^2 p_n(\mathbf{x}) - 2x_v \partial_{x_v} p_n(\mathbf{x}) - k_n p_n(\mathbf{x}) = 0.$$

To determine k_n we equate to zero the coefficient of the higher order term obtaining $k_n = -n_v(n_v + 1)$.

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A SIMPLIFICATION OF THE SCHWARZ–CHRISTOFFEL FORMULA FOR SYMMETRIC QUADRILATERAL TRANSFORMATION*

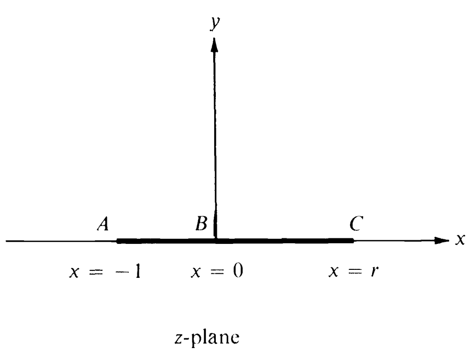
O. F. HUGHES†

Abstract. A relationship is derived which greatly simplifies the application of the Schwarz–Christoffel formula in mapping a quadrilateral having one axis of symmetry. The relationship allows all of the image points on the real axis to be evaluated explicitly, thus reducing the formula from an integral equation to an explicit integral.

1. Introduction. The well-known Schwarz–Christoffel theorem is frequently used to map the complex plane z into the region around a quadrilateral in the w-plane. In the case of quadrilaterals with one axis of symmetry, as shown in Fig. 1, the transformation is

$$w - w_0 = \int_0^z \frac{(z - x_2)^{\alpha+\beta}}{(z - x_1)^{\alpha}(z - x_3)^{\beta}}\, dz. \tag{1}$$

Because of symmetry, we need consider only the upper half-plane and in the w-plane the polygon being mapped is that portion of the upper half-plane *excluding* the triangle ABC. This polygon has four vertices, one of which is at infinity, and we therefore choose the corresponding image point in the z-plane as $x_4 = \infty$. Of



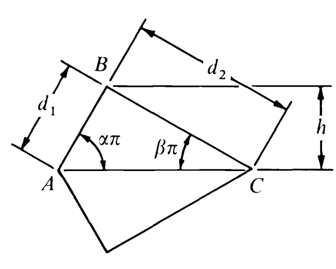


FIG. 1

the other three image points, it is well known that two of them can be chosen arbitrarily. For simplicity we choose $x_1 = -1$ and $x_2 = 0$. The location of the third image point is not arbitrary; there is some unique value, say $x_3 = r$, which it must have for the sides of the polygon to be mapped onto the real axis of z. The value of r depends on the specified geometry of the polygon, and some geometric relationship must be invoked in order for this constant to be calculated. In the present case, the appropriate geometric relationship is the condition

$$d_1 \sin \alpha\pi = d_2 \sin \beta\pi \tag{2}$$

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which is seen from Fig. 1 to be the condition that the two inclined sides of the triangle ABC have the same height at C. The quantities d_1 and d_2 are given by

$$d_1 = -\int_0^{-1} \frac{(-x)^{\alpha+\beta}}{(1+x)^\alpha(r-x)^\beta}\,dx, \tag{3}$$

$$d_2 = \int_0^{r} \frac{x^{\alpha+\beta}}{(1+x)^\alpha(r-x)^\beta}\,dx. \tag{4}$$

Substitution of these into (2) gives an implicit integral equation for r as a function of α and β. However, these integrals cannot be evaluated in terms of a finite number of elementary functions and hence numerical techniques are usually required to satisfy condition (2). It would clearly be preferable to have an explicit expression for r in terms of α and β. On the basis of some numerical results, Hughes [1] recently postulated that the relationship between r, α and β is simply

$$r = \alpha/\beta. \tag{5}$$

The present note proves this relationship in two ways, the first using properties of the hypergeometric function and the second using Cauchy's theorem.

As a preliminary step, the integrals in (3) and (4) are normalized so they have 0 and 1 as limits of integration. Substitution into (2) gives

$$\sin\alpha\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^\alpha(r+x)^\beta}\,dx = r\sin\beta\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1/r+x)^\alpha(1-x)^\beta}\,dx \tag{6}$$

and the task at hand is to prove that (6) can be reduced to (5) or, in other words, that (5) is the equivalent of (6).

2. Proof using hypergeometric functions. Our first method of proof is to substitute (5) into (6) and to show, after some manipulation with hypergeometric functions, that the resulting expression is an identity. The substitution yields, after some minor rearrangement,

$$\beta\sin\alpha\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^\alpha(\alpha/\beta+x)^\beta}\,dx = \alpha\sin\beta\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^\beta(\beta/\alpha+x)^\alpha}\,dx. \tag{7a}$$

It will be seen that the left- and right-hand sides are "images" of each other, with α and β interchanged. Therefore, let us denote the expression on the left-hand side of (7a) as $f_1(\alpha,\beta)$; then (7a) becomes

$$f_1(\alpha,\beta) = f_1(\beta,\alpha). \tag{7b}$$

The integrals in (7) may be evaluated in terms of the gamma function and the hypergeometric function, yielding

$$f_1(\alpha,\beta) = \pi\beta\left(\frac{\beta}{\alpha}\right)^\beta \frac{\Gamma(1+\alpha+\beta)}{\Gamma(\alpha)\Gamma(2+\beta)} F\left(\beta, 1+\alpha+\beta; \beta+2; -\frac{\beta}{\alpha}\right). \tag{8}$$

Substituting (8) into (7) yields, after some simplification,

$$\text{(9a)}\quad \begin{aligned}&\left(\frac{\beta}{\alpha}\right)^{\beta}\frac{\beta}{\Gamma(\alpha)\Gamma(2+\beta)}F\left(\beta,1+\alpha+\beta;\beta+2;-\frac{\beta}{\alpha}\right)\\ &=\left(\frac{\alpha}{\beta}\right)^{\alpha}\frac{\alpha}{\Gamma(\beta)\Gamma(2+\alpha)}F\left(\alpha,1+\alpha+\beta;\alpha+2;-\frac{\alpha}{\beta}\right)\end{aligned}$$

or in symbolic form

$$\text{(9b)}\qquad f_2(\alpha,\beta)=f_2(\beta,\alpha).$$

In the above expression the argument of the hypergeometric function on the right-hand side, $-\alpha/\beta$, is the inverse of the argument on the left-hand side. In order to show that the two sides are identical we apply the linear transformation

$$\begin{aligned}F(a,b,c,z)&=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{z}\right)\\ &\quad+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{z}\right)\end{aligned}$$

to the left-hand side of (9). The result is

$$\text{(10)}\quad \begin{aligned}f_2(\alpha,\beta)=\left(\frac{\beta}{\alpha}\right)^{\beta}\frac{\beta}{\Gamma(\alpha)\Gamma(2+\beta)}&\left\{\frac{\Gamma(2+\beta)\Gamma(1+\alpha)}{\Gamma(1+\alpha+\beta)\Gamma(2)}\left(\frac{\alpha}{\beta}\right)^{\beta}F\left[\beta,-1,-\alpha;-\frac{\alpha}{\beta}\right]\right.\\ &\left.+\frac{\Gamma(2+\beta)\Gamma(-1-\alpha)}{\Gamma(\beta)\Gamma(1-\alpha)}\left(\frac{\alpha}{\beta}\right)^{1+\alpha+\beta}F\left[1+\alpha+\beta,\alpha,2+\alpha;-\frac{\alpha}{\beta}\right]\right\}.\end{aligned}$$

It can easily be shown that the first of the two hypergeometric functions in (10) vanishes and that the remainder of the expression is identical to the right-hand side of (9). Equation (9) is thus reduced to an identity, and this proves the relationship which was originally postulated in (5).

3. Proof using Cauchy's theorem. Let $a_1,\cdots,a_n,\alpha_1,\cdots,\alpha_n$ be complex numbers, $a_j\neq 0$, $\arg a_j\neq\arg a_h$, and $\operatorname{Re}\alpha_j<1$, $\sum_{j=1}^n\operatorname{Re}\alpha_j>-1$, $j=1,\cdots,n$, $h=1,\cdots,n$, $h\neq j$. Define $(1-a_j/z)^{-\alpha_j}$ by its principal value outside the segment $[0,a_j]$. If C is a circle $|z|=R$ traveled counterclockwise, $R>|a_j|$, $j=1,\cdots,n$, then it may be shown that

$$2\pi i\sum_{j=1}^n a_j\alpha_j=\int_C\prod_{j=1}^n\left(1-\frac{a_j}{z}\right)^{-\alpha_j}dz.$$

The same result will be obtained by integrating around the contour shown in Fig. 2 (drawn for $n=2$) and it may be shown that the integrals around C_0 and around each of the C_h vanish as the radius ε approaches zero. Hence,

$$\begin{aligned}2\pi i\sum_{j=1}^n a_j\alpha_j&=\sum_{h=1}^n\left[\int_0^{a_h}\prod_{j=1}^n\left(1-\frac{a^j}{z}\right)^{-\alpha_j}dz+\int_{a_h}^0 e^{2\pi i\alpha_h}\prod_{j=1}^n\left(1-\frac{a_j}{z}\right)^{-\alpha_j}dz\right]\\ &=2i\sum_{h=1}^n\sin\pi\alpha_h\int_0^{a_h}\left(\frac{a_h}{z}-1\right)^{-\alpha_h}\prod_{\substack{j=1\\ j\neq h}}^n\left(1-\frac{a_j}{z}\right)^{-\alpha_j}dz.\end{aligned}$$

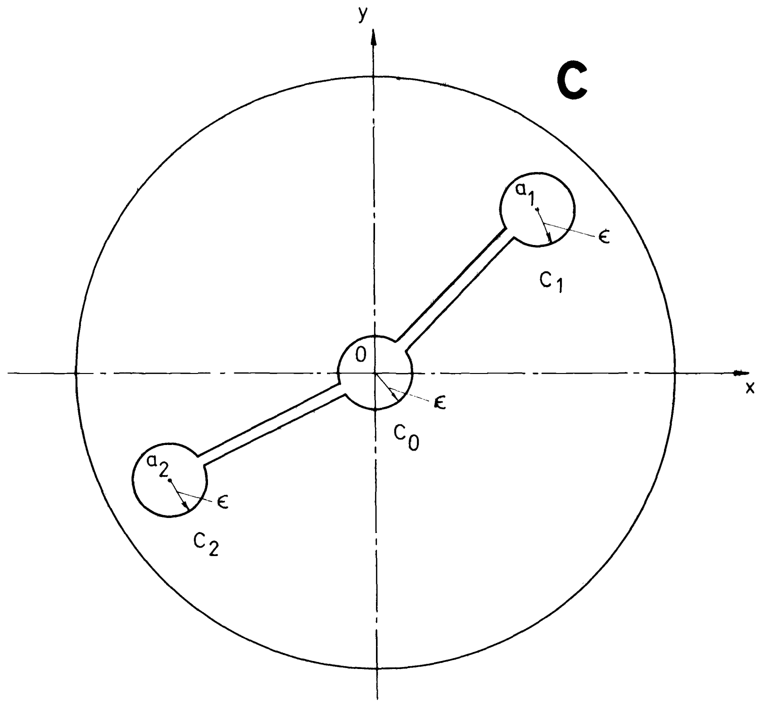

FIG. 2

In particular, if $n = 2, a_1 = 1, \alpha_1 = \alpha, a_2 = b, \alpha_2 = \beta$:

$$\pi(\alpha + b\beta) = \sin \pi\alpha \int_0^1 \left(\frac{1}{z} - 1\right)^{-\alpha}\left(1 - \frac{b}{z}\right)^{\beta} dz$$

$$+ b \sin \pi\beta \int_0^1 \left(\frac{1}{z} - 1\right)^{-\beta}\left(1 - \frac{1}{bz}\right)^{-\beta} dz.$$

Choosing $b = -\alpha/\beta$ we obtain (6) with $r = \alpha/\beta$.

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SAMPLING EXPANSIONS WITH DERIVATIVES FOR FINITE HANKEL AND OTHER TRANSFORMS*

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Abstract. A sampling expansion involving the samples of a function represented by a finite Hankel transform and the samples of the derivative of the function is derived. Also, the general procedure for obtaining sampling expansions with derivatives for functions represented by other finite integral transforms is outlined. It is shown that in parallel to the known special case of the finite Fourier transform that the advantage of sampling with N derivatives is to increase by $(N + 1)$-fold the asymptotic spacing between the sampling points. The importance of such an advantage for the Hankel transform can be realized in a time-varying or spatial-varying system.

Finally, an extension to two dimensions of the sampling theorem with N derivatives for a function having a finite double Fourier transform is stated.

Introduction. The problem of reconstructing a function by interpolating at equidistant samples of the function, using the Cardinal series, has been considered [17] and the result was introduced to communications theory [13], [14] as the well-known sampling theorem: *If a function contains no frequencies higher than* W cps, *it is completely determined by giving its ordinates at a series of points* $1/2W$ sec *apart.* A restatement of the above theorem in mathematical terms leads to the following theorem.

Let

$$f(t) = \int_{-2\pi W}^{2\pi W} F(x)\, e^{ixt}\, dx, \tag{1}$$

where $F(x) \in L_2[-2\pi W, 2\pi W]$. *Then,*

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}. \tag{2}$$

Many generalizations of this result exist. Among them are those that include showing that a sampling expansion for a function represented by (1) containing samples of the function and N derivatives simultaneously need be sampled only at every $N + 1$ sample points [4], along with an explicit result including one derivative and the function sampled simultaneously [5]. An extension of this result to an expansion containing N derivatives is given in [4], [11]. We draw the attention of the reader to the important correction of the sampling expansion of [11] which appeared in the same journal, 4 (1961), pp. 95–96. Recently, another method for obtaining a sampling series with derivatives has been given [12].

All of the sampling expansions given in [13], [14], [4], [5], [11] and [12] reconstruct a function represented by a finite Fourier transform.

It is one objective of this paper to show that sampling expansions with derivatives can be derived for functions represented by other integral transforms

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besides the finite Fourier transform. More details concerning the method, the results we present here, and other results and suggestions can be found in [10].

A generalization of the sampling theorem without derivatives for functions represented by integral transforms that includes the Fourier transform as a special case was suggested [16] and was later stated [9] as the following lemma.

LEMMA. *If*

$$f(t) = \int_I K(x, t)F(x)\, dx, \tag{3}$$

where $F(x) \in L_2(I)$, *and for each real* $t, K(x, t) \in L_2(I)$ *and there exists a countable set* $E = \{t_n\}$ *such that* $\{K(x, t_n)\}$ *is a complete orthogonal set on* $L_2(I)$, *then*

$$f(t) = \lim_{N\to\infty} \sum_{|n| \leqq N} f(t_n)S_n(t), \tag{4}$$

where

$$S_n(t) = \frac{\int_I K(x, t)\overline{K(x, t_n)}\, dx}{\int_I |K(x, t_n)|^2\, dx}. \tag{5}$$

The values of $S_n(t)$ for different cases of $K(x, t)$ can be found in [2], [8].

The following is the sampling expansion for the finite J_0-Hankel transform [9]. We state it here since we shall use it in the following sections.

The finite J_0-Hankel transform. Let the function $f(t)$ have a finite J_0-Hankel transform representation. That is,

$$f(t) = \int_0^a xJ_0(xt)F(x)\, dx, \tag{6}$$

where $F(x) \in L_2[0, a]$. Then, by (4),

$$f(t) = \sum_{k=1}^{\infty} f(t_{0,k})S_k(t), \tag{7}$$

where

$$S_k(t) = \frac{2t_{0,k}J_0(at)}{a(t_{0,k}^2 - t^2)J_1(at_{0,k})} \tag{8}$$

and $J_0(at_{0,k}) = 0$ for all k. The sampling points in (7) occur at $t_{0,k} = j_{0,k}/a$, where $j_{0,k}$ are the zeros of J_0.

1. Sampling expansions with derivatives for finite Hankel and other transforms. The method employed here will make use of the residue theorem which states that any function $h(z)$ which is meromorphic inside C_R for every R, where C_R is a circular contour of radius R centered at the origin, may be represented by an expansion of the form

$$h(z) = -\sum_j R_{z_j}\left\{\frac{h(\xi)}{z - \xi}\right\} \tag{9}$$

if the integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{h(\xi)}{\xi - z}\,d\xi \tag{10}$$

along C_R approaches zero as $R \to \infty$. In (9), R_{z_j} denotes the residue at $\{z_j\}$, and $\sum_j$ stands for the summation over the poles of $h(\xi)$. This theorem can be used to produce a variety of sampling expansions. All that we need to do is to let

$$h(z) = f(z)/g(z) \tag{11}$$

and choose the proper function for $g(z)$. This was used [5] to derive the sampling expansion (2) and the corresponding expansion with one derivative by letting $f(z)$ have a finite Fourier transform representation and letting $g(z) = \sin z$ and $\sin^2 z$ respectively. The proper choice for $g(z)$ that allows us to reproduce the sampling expansions given by (4) can be selected by comparing $S_n(z)$ in (5) with the terms in the Lagrange interpolation polynomial [5]. Each $S_n(z)$ can be expressed as a partial fraction expansion of its nontranscendental part multiplied by its transcendental part. The nonconstant transcendental portion of this expression is the proper choice for $g(z)$. Once a particular $g(z)$ has been chosen, it is a simple matter to introduce derivatives into the sampling expansion. To introduce N derivatives into the sampling expansion, we let

$$h(z) = (f(z))/g^{N+1}(z). \tag{12}$$

That this is true follows immediately from the residue theorem.

An example is $h(z) = f(z)/J_n^{N+1}(az)$ for the finite J_n-Hankel transform. In the following we shall illustrate the method for the case of $n = 0$ and $N = 1$. Other explicit sampling expansions with derivatives derived by this method can be found in [10]. This includes the sampling expansion with one derivative for a function represented by a finite Legendre transform and with N derivatives for the finite Fourier and the J_n-Hankel transforms.

The finite J_0-Hankel transform. If $g(z) = J_0(az)$, then we may use (9) to reproduce (7). That the condition for the integral in (10) is satisfied is shown in details at the end of this section. In accordance with (12), if we desire a sampling expansion for a function $f(z)$ having a finite J_0-Hankel representation which includes the samples of one derivative in addition to samples of the function, we should choose

$$g(z) = J_0^2(az). \tag{13}$$

The sampling expansion with one derivative is given by

$$f(z) = \sum_{k=1}^{\infty}\left[\frac{z^2}{t_{0,k}^2}f(t_{0,k}) + \frac{(z^2 - t_{0,k}^2)}{2t_{0,k}}f'(t_{0,k})\right]S_n^2(z), \tag{14}$$

where $S_n(z)$ is defined as in (8).

To evaluate the residues needed in (9), we evaluate the sum of the residues of the integrals:

$$\oint_{C_R}\frac{f(\xi)}{(\xi - z)J_0^2(a\xi)}\,d\xi \quad\text{and}\quad \oint_{C_R}\frac{f(\xi)}{(\xi + z)J_0^2(a\xi)}\,d\xi. \tag{15}$$

The residue for the first of these integrals is the coefficient of the term $(\xi - t_{0,k})^{-1}$ in the product of the following three expansions:

$$f(\xi) = f(t_{0,k}) + (\xi - t_{0,k})f'(t_{0,k}) + \cdots, \tag{16}$$

$$\frac{1}{\xi - z} = \frac{1}{t_{0,k} - z} - \frac{\xi - t_{0,k}}{(t_{0,k} - z)^2} + \frac{(\xi - t_{0,k})^2}{(t_{0,k} - z)^3} + \cdots, \tag{17}$$

and

$$J_0^{-2}(a\xi) = 1 + \frac{4}{a^2}\sum_{k=1}^{\infty}\left[\frac{t_{0,k}^2}{(\xi^2 - t_{0,k}^2)^2} + \frac{1}{\xi^2 - t_{0,k}^2}\right]\frac{1}{J_1^2(at_{0,k})}. \tag{18}$$

With regard to (18), see [6], [3]. Summing the results obtained from (16), (17) and (18) with similar results obtained by using the second integral of (15), we have the total residue contributions as:

$$\frac{1}{a^2}\sum_{k=1}^{\infty}\left[\frac{z^2}{t_{0,k}^2}f(t_{0,k}) + \frac{z^2 - t_{0,k}^2}{2t_{0,k}}f'(t_{0,k})\right]\left[\frac{2t_{0,k}}{(t_{0,k}^2 - z^2)J_1(at_{0,k})}\right]^2. \tag{19}$$

Thus, (19) becomes (14). It is important to point out that in considering $J_0^2(az)$ here instead of $J_0(az)$, the condition needed for $f(z)$ to satisfy the integral condition (10) is somewhat relaxed. This is in the sense that (14) can represent a finite J_0-Hankel transform with double the finite limit and hence the asymptotic spacing between the sampling points is doubled. In the following, this will become evident as we show that the integral in (10) vanishes along C_R as $R \to \infty$.

If in (9), $h(z) = f(z)/g(z)$ where $f(z)$ is defined by a finite Hankel transform and $g(z)$ is defined by $g(z) = J_n(az)$, then we shall make use of the following asymptotic relationship [1] for large values of $|z|$:

$$J_\nu(z) = (\tfrac{1}{2}\pi z)^{-1/2}\{\cos(z - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + e^{|\mathscr{I}z|}\cdot O(|z|^{-1})\}, \tag{20}$$

$|\arg z| \leqq \pi - 0 < \pi$, so that

$$\frac{1}{|J_n(az)|} = O(|z|^{1/2}e^{-a|y|}), \tag{21}$$

where $z = x + iy$. So to satisfy the condition on (10) to vanish it is necessary that $f(z)$ increase, less rapidly than $e^{a|y|}/|z|^{1/2}$ as $y \to \infty$.

To show this for the J_0-Hankel transform consider

$$f(z) = \int_0^b \omega J_0(\omega z)F(\omega)\,d\omega,$$

where again for large $|z|$ we use (20), and consider the total variation of $\omega^{1/2}F(\omega)$ over $(0, b)$ to obtain

$$|f(z)| = O(|z|^{-1/2}e^{b|y|}). \tag{22}$$

Now with the aid of (22) it becomes clear that (10) is satisfied if $b < a$. That is, the Hankel transform is band-limited and the limit should not exceed a, where a determines the asymptotic sample spacing. The case of $a = b$ can be proved by showing that $zf(z)$ is again a finite J_1-Hankel transform. Thus (6) represents $f(z)$, where $f(z)$ is defined by a Hankel transform over the interval $(0, b)$ if $b \leqq a$.

Similarly, if $g(z) = J_0(az)$ and $N = 1$ in (12), then we may again make use of (20) to show that (14) represents $f(z)$, where

$$f(z) = \int_0^b \omega J_0(\omega z)F(\omega)\,d\omega. \tag{23}$$

In this case, (21) becomes

$$\frac{1}{|J_0(az)|^2} = O(|z|\,e^{-2a|y|})$$

and the integral will vanish if $b < 2a$.

It is very important to point out that in considering $J_0^2(az)$ instead of $J_0(az)$, the condition on $f(z)$ for (10) to vanish is somewhat relaxed. This is in the sense that (14) can represent a finite Hankel transform with larger limit $b = 2a$.

Hence the sampling points for this function are $t_{0,k} = j_{0,k}/a = 2j_{0,k}/b$ which is double that of $t_{0,k} = j_{0,k}/b$ for $f(z)$ in (7).

If N derivatives are to be introduced into the sampling expansion of a function represented by (23), then let

$$h(z) = (f(z))/J_0^{N+1}(az)$$

and the same analysis will be followed to show that the sampling points can be taken at $t_{0,k} = (N + 1)j_{0,k}/b$. Hence the asymptotic sample spacing becomes $N + 1$ times that when the function samples alone are employed.

Of course this advantage is realized when (14) is used for time-variant systems in contrast to the sampling expansion with derivatives for the Fourier transform which is most useful for time-invariant systems.

The analysis for a function represented by a finite Legendre transform to show that (10) is satisfied follows very closely the analysis done for the case involving $J_0^2(az)$ except $g(z) = \sin a(z - \frac{1}{2})$ and $N = 1$ in (12). Again the doubling of the asymptotic sample spacing is achieved.

2. *R*-derivative sampling with double Fourier transforms. The sampling theorem with R derivatives for a function represented by a band-limited Fourier transform has been developed [11]. In this section, we state that we have generalized the result given in [11] to a sampling expression with partial derivatives for a function represented by a double Fourier transform which is band-limited in two dimensions. The proof uses a generalization to two dimensions of the lemma given in [11]. The result and the details of this proof are given in [10]. This extension is of particular interest to us because it is well known [12] that a Hankel transform can be represented as a double Fourier transform of a function with circular symmetry. Thus, if a function has a Hankel transform representation and has circular symmetry, we can use the sampling expression developed here to reconstruct the function.

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ON THE STABILITY OF A PERIODIC SOLUTION OF A DIFFERENTIAL DELAY EQUATION*

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Abstract. This paper considers the class of scalar, first order, differential delay equations $y'(t) = -f(y(t-1))$. It is shown that under certain restrictions there exists an annulus A in the $(y(t), y(t-1))$-plane whose boundary is a pair of slowly oscillating periodic orbits and A is asymptotically stable. These results are applied to the frequently studied equation $x'(t) = -\alpha x(t-1)[1 + x(t)]$. The techniques used are related to the Poincaré–Bendixson method, used in the $(y(t), y(t-1))$-plane.

1. Introduction. The differential delay equation

$$x'(t) = -\alpha x(t-1)[1 + x(t)] \tag{1.1}$$

is one which occurs in several applications and has been studied by many authors. It was first proposed by Cunningham [1] as a nonlinear population growth model. Later, it was mentioned by Wright [10] as arising in the application of probability methods to the theory of asymptotic prime number density. Jones, in [3], states that this equation may also describe the operation of a control system working with potentially explosive chemical reactions. In that same paper Jones demonstrated the existence of a periodic solution of (1.1) for $\alpha > \pi/2$ through the use of an asymptotic fixed-point theorem. This method has been extended and clarified by Grafton [2] and Nussbaum [8], [9]. A result which establishes the existence of a periodic solution for a class of differential delay equations which also includes equation (1.1), but is based upon unrelated techniques, has been given by Kaplan and Yorke [6]. This equation, and similar ones, also appear in the recent book by May [7] on ecology.

In numerical studies related to [3], Jones [4] suggested that this periodic orbit possessed some type of stability, although this was never proved.

In this paper we consider a class of equations of the form

$$y'(t) = -f(y(t-1)), \tag{E}$$

where $f(0) = 0, f: R \to R$ is continuously differentiable and $(d/dy)f(y) > 0$ for all $y \in R$. This class can be shown to include (1.1) by making an appropriate change of variables. (See Example 5.1.) We show, for example, that for (1.1) if $\alpha > \pi/2$ there is an annulus A in the $(y(t), y(t-1))$-plane whose boundary is a pair of orbits in R^2 of slowly oscillating periodic solutions and A is asymptotically stable. The region of attraction includes all solutions which do not oscillate too quickly, in the sense that higher harmonics do oscillate too quickly. We also show that

$$x'(t) = -\alpha x(t-1)[1 - x^2(t)], \tag{1.2}$$

which is also transformable into (E), has a slowly oscillating solution (see Example

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5.2). We have chosen to study equations of type (E) because they include the simplest oscillatory differential delay equations.

The techniques introduced are related to the Poincaré–Bendixson method, which we would like to have used in the $(y(t), y(t-1))$-plane. The difficulty that arises, however, is that trajectories of solutions of (E) cross in this plane because points in R^2 do not determine solutions uniquely. We are able to show that for equations of type (E) if two trajectories in R^2 do not cross for a sufficiently large time interval, then the curves will not cross in the future (see the trajectory crossing lemma of § 3). These lemmas permit the use of analysis similar to that of Poincaré–Bendixson.

2. Statement of main results. Consider the scalar, first order, differential delay equation

$$y'(t) = -f(y(t-1)), \tag{E}$$

where $f(0) = 0$, $f: R \to R$ is continuously differentiable and $(d/dy)f(y) > 0$ for all $y \in R$. Let $J_0 = [-1, 0)$ and $J = [-1, 0]$. Denote by $C = C(J, R)$ the set of all continuous functions mapping J into R. Let C_* denote the set of all $\phi \in C$ satisfying

$$\phi \text{ has at most one zero on } J, \tag{2.1}$$

$$\text{if } \phi \text{ has a zero in } (-1, 0), \text{ then } \phi \text{ must change sign there.} \tag{2.2}$$

We shall write $y_t(\cdot)$ to denote the function $y_t(s) = y(t+s)$ for $s \in J$, provided $y(\cdot)$ is defined on $[t-1, t]$. Thus $y_t(\cdot) \in C$. Similarly, we shall write $y'_t(\cdot)$ for $y'_t(s) = y'(t+s)$ for $s \in J$. We let $y(t; t_0, \phi)$ denote the unique solution of (E) for $t \geqq t_0$ for which $y_{t_0}(\cdot) = y_{t_0}(\cdot\,; t_0, \phi) = \phi(\cdot)$.

PROPOSITION 2.1. *Let $\phi \in C_*$, and write $y(t) = y(t; t_0, \phi)$. Then $y_t \in C_*$ for all $t \geqq t_0$ and $y'_t \in C_*$ for all $t \geqq t_0 + 1$. Moreover, for $t_1 \geqq t_0$, $y(t_1) = 0$ implies $y'(t_1) \neq 0$, and for $t_1 > t_0$, $y'(t_1) = 0$ implies t_1 is a strict local maximum of $|y(\cdot)|$.*

Proof. Since (E) is autonomous, in order to show that $y_t \in C_*$ for all $t \geqq t_0$, it will suffice to let $t_0 = 0$ and to show that $y_t(\cdot\,; 0, \phi) \in C_*$ for all $0 < t \leqq 1$.

If ϕ is nonnegative or nonpositive on J, then since $yf(y) > 0$ for all $y \in R$, and from the fact that $y'(s) = -f(y(s-1)) = -f(\phi(s-1))$ for all $s \in [0, 1]$, we see that $y_1(\cdot\,; 0, \phi)$ is monotonic on J. It follows immediately that $y_t(\cdot\,; 0, \phi) \in C_*$ for $0 < t \leqq 1$.

Suppose, therefore, that there exists $t_1 \in (-1, 0)$ such that $\phi(t_1) = 0$. For simplicity, we shall suppose that $\phi(s) < 0$ for $s \in [-1, t_1)$, while $\phi(s) > 0$ for $s \in (t_1, 0]$. These assumptions imply that $y_1(\cdot\,; 0, \phi)$ is monotonically increasing on $[-1, t_1)$, and thus $y_1(s; 0, \phi) > 0$ for all $s \in [-1, t_1)$. Also $y_1(s; 0, \phi)$ is monotonically decreasing on $(t_1, 0]$. It is thus clear that $y_1(\cdot\,; 0, \phi)$ can have at most one zero on J, at which it must change sign. Obviously, if $y_1(\cdot\,; 0, \phi)$ has no zero on J, then $y_t(\cdot\,; 0, \phi) \in C_*$ for all $0 < t \leqq 1$.

Suppose $y_1(\cdot\,; 0, \phi)$ has a zero at $t_2 \in (-1, 0]$. It will be shown that $y_s \in C_*$ for $0 < s \leqq 1$. To do so, observe that $y(t+s; 0, \phi)$ can be zero only at $t + s = t_1$ and $t + s = t_2 + 1$. It suffices to show these zeros satisfy $(t_2 + 1) - t_1 > 1$, i.e., $t_2 > t_1$. But, as observed above, $y_1(\cdot\,; 0, \phi)$ increases from the positive number

$\phi(0)$ on $-1 \leqq t \leqq t_1$, and decreases thereafter, hence $t_2 > t_1$. Therefore, $y_s \in C_*$, $0 < s \leqq 1$.

Notice that $f(y) = 0$ if and only if $y = 0$, and f changes sign at 0. Thus for $z(t) = f(y(t-1))$ we have $z_t \in C_*$ if and only if $y_{t-1} \in C_*$. But $y'(t) = -f(y(t-1))$, so that $y_t \in C_*$ for $t \geqq t_0$ implies $y'_t \in C_*$ for $t \geqq t_0 + 1$.

Since y' changes sign at any of its zeros t_1, t_1 is a local extremum of $y(t)$, and so by (2.1) it cannot be a zero of y. Thus, for $t_1 \geqq t_0$, $y(t_1) = 0$ implies $y'(t_1) \neq 0$.

Finally, let $t_1 > t_0$, and suppose that $y'(t_1) = 0$. Then

$$t_1 - 1 = \sup\{t \leqq t_1 ; y(t) = 0\}.$$

For the sake of definiteness, we suppose that $y(t_1) > 0$. Then $y(t)$ changes from negative to positive at $t_1 - 1$. Since $-f$ is a strictly monotonically decreasing function, $y'(t) = -f(y(t-1))$ changes from positive to negative at t_1. Therefore, t_1 is a strict local maximum for $|y(t)|$.

DEFINITION 2.1. The real-valued function $y(\cdot)$ is said to be *slowly oscillating* (*on* $[t_0, \infty)$) if $[t_0, \infty) \subset$ domain $(y(\cdot))$ and $y_t \in C_*$ for all $t \geqq t_0 + 1$.

The following proposition increases the motivation for restricting our consideration to slowly oscillating solutions.

PROPOSITION 2.2. *For $\eta \geqq -1$, define the subset $C_*^\eta = \{\Psi \in C; y(\cdot\,; 0, \Psi)$ is slowly oscillating on $[\eta, \infty)\}$. Then for $\eta > 0$, C_*^η is an open neighborhood of C_* in the topology of uniform convergence on compact subsets of $[\eta, \infty)$.*

A proof of Proposition 2.2 can be easily supplied by the reader.

For the remainder of the paper we shall write $p(t) = (y(t), y(t-1))$ and $q(t) = (y(t), -y(t-1))$ whenever y is a solution of (E). The use of $q(t)$ rather than $p(t)$ is of no substantial consequence, but will allow simpler notation. Let $d[\cdot,\cdot]$ denote the usual Euclidean distance in the plane.

DEFINITION 2.2. Let $S \subset R^2$ be closed. We say that S is *C_*-stable in R^2* (for (E)) if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that when y is a slowly oscillating solution of (E) on $[t_0, \infty)$ satisfying

$$d[q(t), S] < \delta(\varepsilon) \quad \text{for } t \in [t_0, t_0 + 1]$$

we have

$$d[q(t), S] < \varepsilon \quad \text{for all } t \geqq t_0.$$

We say that S is a *C_*-global attractor* in R^2 if for each solution y of (E) such that $y_t \in C_*$ for some t, we have $d(q(t), S) \to 0$ as $t \to \infty$.

We say S is *C_*-globally asymptotically stable* if S is C_*-stable and is a C_*-global attractor.

DEFINITION 2.3. For a periodic solution y of (E) we define the *orbit of y in R^2* by $O_y = \{q(t) : t \in R\}$. We say y is a *simple periodic solution* of (E) if O_y is a simple closed curve.

For a simple periodic solution we let Int O_y and Ext O_y denote the closures of the interior and exterior of the simple closed curve O_y, in the sense of the Jordan curve theorem.

DEFINITION 2.4. We say $A \subset R^2$ is a *periodic C_* annulus for* (E) if there are simple periodic solutions x and y with $x_t, y_t \in C_*$ for all $t \in R$ such that

$A = \text{Int}\, O_x \cap \text{Ext}\, O_y$. Observe that $O_x \subset A$ and $O_y \subset A$. We allow $x = y$, in which case $A = O_x$.

THEOREM 2.1. *Let $f: R \to R$ be a continuously differentiable function such that $f(0) = 0$ and*

$$f'(y) > 0 \quad \textit{for all } y \in R. \tag{2.3}$$

Assume there exists a $B > 0$ such that

$$f(y) > -B \quad \textit{for all } y \in R. \tag{2.4}$$

If $\alpha = 2f'(0)/\pi > 1$, then there is a periodic C_* annulus $A \subset R^2$ which is C_*-globally asymptotically stable for (E).

COROLLARY 2.1. *Assume that the hypotheses of Theorem 2.1 are satisfied. Assume that there is a unique slowly oscillating periodic solution y. Then O_y is a C_*-globally asymptotically stable set.*

Remark 2.1. Corollary 2.1 follows immediately from Theorem 2.1. In Example 5.2 we give an equation for which we are unable to show that the periodic solution is unique. G. S. Jones' computer experiments made it rather clear that there was a unique periodic solution for (1.1), but this fact remains without a rigorous proof.

Remark 2.2. If $0 \leqq \alpha \leqq 1$, it is still possible for 0 to be globally asymptotically stable. This will be the case, for example, if $0 < f'(y) < \pi/2$ for all y.

Remark 2.3. Condition (2.4) can be replaced by

$$f(y) < B \quad \text{for all } y \in R, \tag{2.5}$$

since if this condition held we could make the change of variable $y \to -y$ and let $f_1(y) = -f(-y)$. Then f_1 would satisfy the hypotheses of Theorem 2.1.

Remark 2.4. Recall that for a solution y of (E) we write $p(t) = (y(t), y(t-1))$. Clearly, for $y \equiv 0$, $p(t)$ does not tend to A as $t \to \infty$. In general, by considering linear equations, we are led to expect that there is always an infinite-dimensional "surface" $M \subset C(J, R) - C_*$ such that $\phi \in M$ implies $y(t; 0, \phi) \to 0$ as $t \to \infty$. At the same time we may expect $C_{**} = \{\phi : y_t(\cdot\,; 0, \phi) \in C_* \text{ for all sufficiently large } t\}$ to be an open dense subset of C. For linear equations of the type (E), if $f(y) > \pi y/2$, then C_{**} is the complement of a proper infinite-dimensional subspace.

Example 2.1. There can be nontrivial periodic solutions of (E) which are "higher harmonics" and are not in C_*. G. S. Jones pointed out the following example.

Let $f(u) = \alpha f_0(u)$, where $f_0'(0) = \pi/2$. Suppose that y^0 is a nontrivial periodic solution with period p of (E), for α equal to some α_0. Then y^0 also satisfies

$$(y^0)' = -\alpha_0 f_0(y^0(t - np - 1))$$

for $n = 0, 1, 2, \cdots$. Define $y^n(t) = y^0((np + 1)t)$. Then $y^n(t-1)$ is $y^0((np+1)t - np - 1)$ and so y^n satisfies $y' = -\alpha f_0(y(t-1))$ for $\alpha = (np+1)\alpha_0$. Of course, for large n, y^n_t would not be in C_* since it would have at least two zeros in each unit interval.

Remark 2.5. For a simple periodic solution x of (E), the conclusion of Corollary 2.1 that "O_x is a C_*-globally asymptotically stable set" implies a result for the more usual Banach space formulation of differential delay equations. Let $O_x(C) = \{x_t : t \in R\}$, which is a compact subset of the Banach space C. Then, if we consider the "flow" $\pi : R^+ \times C \to C$ given by $\pi(t, \phi) = y_t(\cdot\,; 0, \phi)$, the set $O_x(C)$ is asymptotically stable (that is, x_t is orbitally asymptotically stable) and the region of attraction is C_{**}.

This follows from Proposition 2.2.

The main purpose of this paper is to establish this asymptotic stability result. Existence of periodic solutions under the hypotheses of Theorem 2.1 is known, and the methods of Jones and Grafton used to establish this existence extend to some equations with several lags.

3. Trajectory crossing lemma. In this section we study functions $x : I_x \to R$ and $y : I_y \to R$, where I_x and I_y are intervals. Let $G = G(v)$ denote a continuous function from R into R which is strictly monotonically increasing. We assume that

$$G(0) = 0. \tag{3.1}$$

For those points $t \in I_x$ and $\tau \in I_y$ at which $x'(t)$ and $y'(\tau)$ exist (using right-hand derivatives at the left endpoints of I_x and I_y), we shall write

$$r(t) = (x(t), G(x'(t))), \qquad s(\tau) = (y(\tau), G(y'(\tau))). \tag{3.2}$$

DEFINITION 3.1. We say a function $x : I_x \to R$ is *simply oscillatory* on I if

(i) $I \subset I_x$ is an interval, $\sup I = \infty$;

(ii) x is continuously differentiable on I;

(iii) the zeros of x and x' are strictly interlaced on the interval I, and x has infinitely many zeros on I.

Observe that these conditions imply that if $x'(t) = 0$, $t \in I$, then t is a strict local maximum of $|x(\cdot)|$.

We would like to examine the notion of the curve $r(t)$ being "outside" the curve $s(\tau)$, where x and y are simply oscillatory functions. This idea will be used to examine the crossings of trajectories, and to define the concepts of "spiraling outward" or "spiraling inward".

In order to do this, it is necessary to first define what is meant by the point $r_0 \in R^2$ being "outside" the point $s_0 \in R^2$. The most obvious definition is based upon a lexicographic ordering on the polar decompositions of r_0 and s_0; that is, we might say that r_0 is outside of s_0 if both points lie on the same ray through $(0, 0)$ and $|r_0| > |s_0| > 0$. Unfortunately, such a procedure will not produce the kind of phase plane performance we demand from $r(t)$ and $s(\tau)$, even when $r(t)$ and $s(\tau)$ are periodic. Instead, we shall use the following definition.

DEFINITION 3.2. Given $r_0 = (r_1, r_2)$ and $s_0 = (s_1, s_2)$ in R^2, we say r_0 is *outside* s_0 if $r_0 \neq (0, 0)$, $s_0 \neq (0, 0)$ and either of the following is satisfied:

$$r_1 = s_1, \quad |r_2| > |s_2| \quad \text{and} \quad r_2 s_2 > 0; \tag{3.3a}$$

$$s_2 = 0, \quad |r_1| \geqq |s_1| \quad \text{and} \quad r_1 s_1 > 0. \tag{3.3b}$$

Notice that if $s_2 \neq 0$ then the set of all r_0 outside s_0 is an open, vertical, half-line. If $s_2 = 0$, the set of all r_0 outside s_0 is a closed half-plane minus s_0 (which is on the boundary).

Let us elaborate further on Definition 3.2. Our goal is to construct a non-decreasing function $T(t)$ such that $r(t)$ and $s(T(t))$ wind around the origin of the phase plane at about the same rate. In particular, it is required that if $r(t)$ crosses the x-axis then $s(T(t))$ crosses the x-axis at $\tau = T(t)$, and when $r(t)$ crosses the y-axis, so does $s(T(t))$. Toward this end, it will be useful for us to give an alternate characterization of the notion of outside. It is, of course, equivalent to the definition just given, but it will be more convenient for some further developments.

DEFINITION 3.2′. Given $r_0 = (r_1, r_2)$ and $s_0 = (s_1, s_2)$ in R^2, we say r_0 is *outside* s_0 if $r_0 \neq (0,0)$, $s_0 \neq (0,0)$, $r_0 \neq s_0$, and one of the following is satisfied:

$$s_2 \neq 0, \quad r_1 = s_1, \quad r_2 = (1+\lambda)s_2, \quad \lambda > 0; \tag{3.3a′}$$

$$s_2 = 0, \quad r_1 = (1+\lambda)s_1, \quad \lambda > 0; \tag{3.3b′}$$

$$s_2 = 0, \quad r_1 = s_1, \quad r_2 \neq 0. \tag{3.3c′}$$

If $r_0 = r(t)$ and $s_0 = s(\tau)$, then, by (3.1), we see that (3.3) or (3.3′) is in turn equivalent to

$$y'(\tau) \neq 0, \quad x(t) = y(\tau), \quad (x'(t))/y'(\tau) > 1; \text{ or} \tag{3.4a}$$

$$y'(\tau) = 0, \quad x(t) = (1+\lambda)y(\tau), \quad \lambda > 0; \text{ or} \tag{3.4b}$$

$$y'(\tau) = 0, \quad x(t) = y(\tau), \quad x'(t) \neq 0. \tag{3.4c}$$

In particular, we find that $x(t) = (1+\lambda)y(\tau)$, where $\lambda \geqq 0$ in (3.4a–c).

PROPOSITION 3.1. *Let x and y be simply oscillating functions on $[t_1, \infty)$ and $[T_1, \infty)$, respectively, and assume that $r(t_1)$ is outside $s(T_1)$. Then there exists ω, $t_1 < \omega \leqq \infty$, and a function $T:[t_1, \omega) \to [T_1, \infty)$ such that*

$$T \text{ is continuous, nondecreasing,} \quad T(t_1) = T_1, \quad \text{and} \quad r(t) = s(T(t)) \quad \text{for } t_1 \leqq t < \omega. \tag{3.5}$$

Furthermore, T is uniquely determined by (3.5), *and*

$$\text{if } \omega < \infty, \text{ then } T \text{ can be continuously defined at } \omega \text{ and } r(\omega) = s(T(\omega)); \tag{3.6}$$

$$\text{if } \omega = \infty, \text{ then } T(t) \to \infty \text{ as } t \to \infty. \tag{3.7}$$

Proof. Let us first show that there exists an $\omega_0 > t_1$ and a function $T:[t_1, \omega_0) \to [T_1, \infty)$ satisfying (3.5) with $\omega = \omega_0$.

Suppose that (3.4) holds with $t = t_1$ and $T = T_1$. In case (3.4a) y^{-1} exists in a neighborhood $(y(T_1) - \varepsilon_1, y(T_1) + \varepsilon_2)$, $\varepsilon_1, \varepsilon_2 > 0$. Since $x(t)$ starts in this neighborhood at $t = t_1$, it remains in the neighborhood on some interval. Therefore, $T(t) = y^{-1}(x(t))$ is continuous near t_1. If $\omega_0 > t_1$ is chosen appropriately, then $r(t)$ and $s(T(t))$ will satisfy (3.4a) on $[t_1, \omega_0)$, hence (3.5) with $\omega = \omega_0$.

In case (3.4b), we have $y(T_1) \neq 0$ and $\lambda = [x(t_1)/y(T_1)] - 1 > 0$. Therefore, $x(t)/y(T_1) > 1$ on some interval $t_1 \leqq t < \omega_0$, and we may define $T(t) = T_1$ in order to have $r(t)$ and $s(T(t))$ satisfy (3.4b) on $t_1 \leqq t < \omega_0$. Thus, T satisfies (3.5) with $\omega = \omega_0$.

If (3.4c) holds, then $x(t_1) = y(T_1)$, $y'(T_1) = 0 \neq x'(t_1)$. Suppose first that $x(t_1) > 0$, $x'(t_1) > 0$. Then $x(t)$ increases from $x(t_1)$ for $t - t_1 > 0$ sufficiently small, and we see that taking $T(t) = T_1$ gives (3.4b) ($r = r(t)$, $s = s(T(t))$) for some interval $[t_1, \omega_0)$. Hence (3.5) is satisfied with $\omega = \omega_0$.

Suppose next that $x(t_1) > 0$, $x'(t_1) < 0$. Since y is simply-oscillating, we must have $y'(T) < 0$ for $T - T_1 > 0$ sufficiently small. Strict monotonicity implies that $\omega_0 > t_1$ and $T_2 > T_1$ can be selected such that $y(T_2) \leqq x(t) \leqq y(T_1)$ for $t_1 \leqq t \leqq \omega_0$, y^{-1} exists and is continuous in $[y(T_2), y(T_1)]$, and $x'(t)/y'(y^{-1}(x(t))) > 1$ on $t_1 < t \leqq \omega_0$. Therefore, we let $T(t) = y^{-1}(x(t))$ on $[t_1, \omega_0]$, and observe that $r(t)$ and $s(T(t))$ satisfy (3.4c) at $\tau = \tau_1$ and (3.4a) on $t_1 < t \leqq \omega_0$. Hence (3.5) holds with $\omega = \omega_0$.

The separate cases $x(t_1) < 0$, $x'(t_1) < 0$ and $x(t_1) < 0$, $x'(t_1) > 0$, can be obtained from the preceding by replacing x by $-x$ and y by $-y$. We conclude that if (3.4c) holds, then there exists a function T and an ω_0 such that (3.5) holds with $\omega = \omega_0$.

Let $\omega = \sup\{\omega_0\}$, where $\omega_0 > t_1$ and there exists a function T with (3.5) holding, ω replaced by ω_0. To obtain a function T as in (3.5) with this choice of ω, it suffices to show that any two functions T_1 and T_2, which satisfy (3.5) on $t_1 \leqq t < \omega_0$, $\omega_0 < \omega$, are equal.

To show $T_1 \equiv T_2$ on $[t_1, \omega_0)$, it suffices to show that $T_1(t_0) = T_2(t_0)$, $t_1 \leqq t_0 < \omega_0$, implies $T_1 \equiv T_2$ for $t - t_0 > 0$ small enough. This is done by showing that T_1 and T_2 agree with the function T, constructed as in the first part of the proof, except with t_1 replaced by t_0 and T_1 by $T_0 = T_1(t_0) = T_2(t_0)$.

Using limiting arguments and/or the invertibility of y, we show that $T \equiv T_1 \equiv T_2$ in a right neighborhood of t_0, provided either (3.4a) or (3.4b) holds at $t = t_0$, $\tau = T_0$. The verification in case (3.4c) is more involved. The idea is to observe that the equation $y'(T) = 0$ has only the solution $T = T(t_0)$ when T is near $T(t_0)$, and $y(T) - y(T(t_0)) \leqq 0$ for T near $T(t_0)$. Then one examines the possible definitions of $T(t)$, $t - t_0$ small. The details are left to the reader.

The argument supplied above shows that T is unique in (3.5). Relations (3.6) and (3.7) will now be established.

On the interval $t_1 \leqq t < \omega$ we have $x(t) = [1 + \lambda(t)]y(T(t))$ by (3.4), where $\lambda(t) \geqq 0$. Therefore, $T(t)$ maps the zeros of $x(t)$ one-to-one and onto the zeros of $y(s)$.

If $\omega < \infty$, then $x(t)$ has a finite number k of zeros in $[t_1, \omega)$. Therefore, $T(t)$ is bounded above by the $(k+1)$st zero of y. Conversely, if $T(t)$ is bounded above, then $x(t)$ has only a finite number of zeros in $[t_1, \omega)$, hence $\omega < \infty$. This establishes (3.7), and shows that in (3.6) we may define $T(\omega) = \sup\{T(t) : t < \omega\}$.

To verify that $r(\omega) = s(T(\omega))$ in (3.6), observe that (3.4) holds for $t < \omega$, $\tau = T(t)$. Suppose that $r(\omega) \neq s(T(\omega))$. Since one of (3.4a)–(3.4c) must hold in every left neighborhood of ω, we arrive at three cases: (a) $x(\omega) = y(T(\omega))$, $y'(T(\omega)) \neq 0$, $x'(\omega)/y'(T(\omega)) > 1$; (b) $x(\omega) = y(T(\omega))$, $y'(T(\omega)) = 0$, $x'(\omega) \neq 0$; (c) $y'(T(\omega)) = 0$, $y(T(\omega)) \neq 0$, $x(\omega)/y(T(\omega)) \geqq 1$. In arriving at these cases we used the fact that $y'(T) = 0$ implies $y(T) \neq 0$, and $r(\cdot) \neq s(T(\cdot))$ if and only if $(x(\omega), x'(\omega)) \neq (y(T(\omega)), y'(T(\omega)))$. In any case, $r(\omega)$ is outside $s(T(\omega))$, hence T can be extended beyond ω, as in the first part of the proof, a contradiction. Thus, (3.6) holds.

Remark 3.1. Let x and y be given as in Proposition 3.1, with $r(t_1)$ outside $s(T_1)$. The first four paragraphs of the proof of Proposition 3.1 show how to construct $T(t)$ explicitly. The reader may find it instructive to do so for $x(t) = 2 \sin t$, $y(t) = \sin t$, $t_1 = \pi/6$, $T_1 = \pi/2$, $G(x') \equiv x'$.

DEFINITION 3.3. Let x and y be simply-oscillatory functions. Let J_x and J_y be intervals, with J_x and J_y subsets of the domains of x and y, respectively. We shall say $x|J_x$ is outside $y|J_y$ if and only if there exists a function $T: J_x \to J_y$ that maps J_x onto J_y, T is continuous and nondecreasing, and $r(t)$ is outside $s(T(t))$ for all $t \in J_x$. In particular, the identity $x(t) = [1 + \lambda(t)]y(T(t))$, $\lambda(t) \geqq 0$, is valid for $t \in J_x$.

PROPOSITION 3.2. *Let x and y be simply-oscillatory functions and suppose that J_x and J_y are nonvoid, bounded open intervals. Assume $x|J_x$ is outside $y|J_y$ and y is monotonic on J_y. Then*

$$\text{length } J_x < \text{length } J_y.$$

Proof. By Definition 3.1, $y'(\tau) \neq 0$ for $\tau \in J_y$. From (3.4a) we have

$$T(t) = y^{-1}(x(t)) \quad \text{and} \quad T'(t) = x'(t)/y'(T(t)) > 1$$

for $t \in J_x$. Thus (3.9) follows from the mean value theorem.

We now wish to compare x and y as solutions of the differential delay equation

$$x'(t) = -g(x(t-1)), \tag{3.8}$$

where $g: R \to R$ is continuous, and strictly monotonically increasing. We assume that

$$g(0) = 0. \tag{3.9}$$

We also assume that $x|[t_1 - 1, \infty)$ and $y|[T_1 - 1, \infty)$ are continuous so that $x|[t_1, \infty)$ and $y|[T_1, \infty)$ are continuously differentiable. Then for g as above, satisfying (3.9), we have the following result.

LEMMA 3.1 (Trajectory crossing lemma). *Let x and y be solutions of* (3.8). *Assume that $x_{t_0}, y_{T_0} \in C_*$, and the sets of zeros of x and y are both infinite. Assume for some t_1 and T_1 satisfying*

$$t_1 \geqq t_0 + 1, \qquad T_1 \geqq T_0 + 1 \tag{3.10}$$

that either $x|[t_0, t_1)$ is outside $y|[T_0, T_1)$ or that $x|(t_0, t_1]$ is outside $y|(T_0, T_1]$. Then $x|(t_0, \infty)$ is outside $y|(T_0, \infty)$.

Proof. Define $G(\cdot)$ by $G(\cdot) = -[(-g)^{-1}(\cdot)]$, where $(-g)^{-1}$ denotes the inverse of $-g$. It follows that G is strictly monotonically increasing (because g is), and (3.9) implies that G satisfies (3.1). Hence

$$r(t) = (x(t), -x(t-1)), \qquad s(t) = (y(t), -y(t-1)), \tag{3.11}$$

since $G(x'(t)) = -[(-g)^{-1}(x'(t))] = -x(t-1)$, and similarly for y.

Let T be the function of Proposition 3.1. Then, if the conclusion of the trajectory crossing lemma is false there must exist $\omega < \infty$ such that $r(\omega) = s(T(\omega))$. In particular, using (3.11), this says that

$$x(\omega - 1) = y(T(\omega) - 1). \tag{3.12}$$

Moreover, since $\omega \geqq t_0 + 1$, $T(\omega) \geqq T_0 + 1$, we must have

$$r(\omega - 1) \quad \text{is outside} \quad s(T(\omega - 1)). \tag{3.13}$$

Let t_2 be the largest number such that $x|(t_2, \omega)$ is outside $y|(T(\omega) - 1, T(\omega))$. If

y is not monotonic on $(T(\omega) - 1, T(\omega))$, then from the definition of "outside", x is not monotonic on (t_2, ω). In this case, it follows from (3.12) that we must have $t_2 = \omega - 1$. Therefore x is not monotonic in $(\omega - 1, \omega)$ (see Fig. 1).

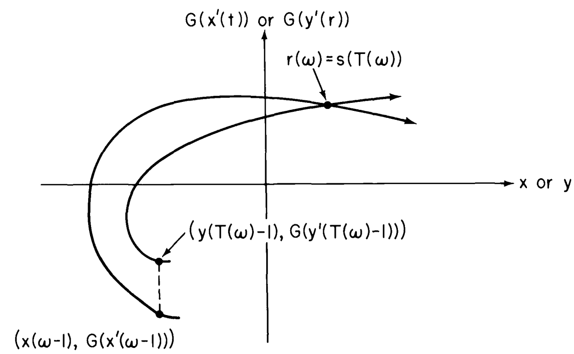

FIG. 1

If, on the other hand, y is monotonic on $(T(\omega) - 1, T(\omega))$, then since $x|(t_2, \omega)$ is outside $y|(T(\omega) - 1, T(\omega))$, Proposition 3.2 implies that $\omega - t_2 < 1$. Furthermore, $x(t) = y(T(t))$ for $t \in (t_2, \omega)$, so by continuity and (3.12) we have

$$x(t_2) = y(T(\omega) - 1) = x(\omega - 1).$$

But $\omega - t_2 < 1$, and so x is not monotonic on $(\omega - 1, \omega)$, a fact which is therefore true whether or not y is monotonic on $(T(\omega) - 1, T(\omega))$ (see Fig. 2). For definiteness we shall assume that $x'(t_2) > 0 > x'(t_2 - 1)$. The other case (in which the signs are reversed) is similar and is omitted.

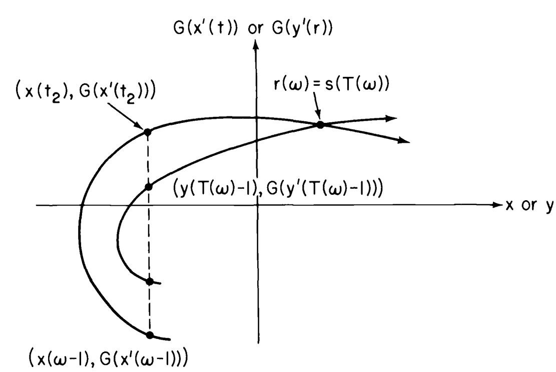

FIG. 2

Observe that, since $r(\omega) = s(T(\omega))$,

$$G(x'(\omega)) = G(y'(T(\omega)))$$

and $y'(T(\omega)) > 0$. Also, for some $\varepsilon > 0$, writing $I_\varepsilon = (\omega - \varepsilon, \omega)$, we have that for all $t \in I_\varepsilon$, $x(t) = y(T(t))$ and

$$G(x'(t)) > G(y'(T(t))).$$

For $t \in (\omega - \varepsilon, \omega]$, let

$$\delta(t) = G(x'(t)) - G(y'(T(t))).$$

Thus $\delta(t) = -x(t-1) + y(T(t)-1)$. Since $\delta(\cdot) > 0$ on I_ε and $\delta(\omega) = 0$, we must have (using left-hand derivatives), that $\delta'(\omega) \leqq 0$; that is,

$$g'(T(\omega) - 1)T'(\omega) \leqq x'(\omega - 1). \tag{3.14}$$

But $x(t) = y(T(t))$. Thus, for $t = \omega$, we have

$$T'(\omega) = \frac{x'(\omega)}{y'(T(\omega))} = \frac{-g(x(\omega - 1))}{-g(y(T(\omega) - 1))} = 1.$$

Therefore, from (3.14), we see that

$$y'(T(\omega) - 1) \leqq x'(\omega - 1). \tag{3.15}$$

Now $r(\omega - 1)$ is outside $s(T(\omega - 1))$ by (3.13), and $G(x'(\omega - 1)) < 0$, so

$$G(x'(\omega - 1)) < G(y'(T(\omega - 1))) \leqq G(y'(T(\omega) - 1)),$$

with equality holding if $T(\omega - 1) = T(\omega) - 1$. But $x(\omega - 1) = y(T(\omega) - 1)$, combined with the monotonicity of G, imply that

$$x'(\omega - 1) < y'(T(\omega) - 1),$$

contradicting (3.15).

Thus, there cannot exist any $\omega < \infty$, $\omega \geqq t_0 + 1$, for which $r(\omega) = s(T(\omega))$.

LEMMA 3.2. *Let x and y be oscillatory functions. Let C be a simple closed curve in R^2 such that $r(t) \in C$ and $s(t) \in C$ for all $t \geqq 0$. Then r, s, x and y are periodic functions and for some $c > 0$, $r(t) \equiv s(t + c)$.*

Proof. Clearly $r(R) = s(R) = C$. Let $c > 0$ be the first strictly positive time for which $r(0) = s(c)$. As in Proposition 2.1 (although the details are simpler in this case), there exists a unique $T: R \to R$ such that $T(0) = c$ and $r(t) = s(T(t))$, for $t \geqq 0$. It follows from the technique of Proposition 3.2, since $x'(t) = y'(T(t))$, that $T'(t) = 1$. Hence $T(t) = t + T(0) = t + c$. Hence

$$r(t) = s(t + c) \quad \text{for all } t \geqq 0. \tag{3.16}$$

For any oscillatory x, we may choose $y = x$, in which case $r(t) = s(t + c) = r(t + c)$, by (3.16). Since c was chosen strictly positive, r must be periodic. The periodicity of r and (3.16) imply that s is periodic.

4. Proof of main results. The reason for imposing the condition that $f'(0) > \pi/2$ in Theorem 2.1 is that if $f'(0) < \pi/2$, it is possible for all solutions of (E) to tend to zero as $t \to \infty$. For $f'(0) = \pi/2$, the following is easily verified.

Example 4.1. The functions $v_0(t) = \rho \sin(\pi/2)t$ and $v_1(t) = \rho\cos(\pi/2)t$ satisfy

$$v'(t) = -(\pi/2)v(t - 1) \tag{4.1}$$

for all t. Of course, v_0 and v_1 are slowly oscillating.

We now wish to compare solutions of the equation (3.8). Throughout the remainder of the paper we shall let

$$r(t) = (x(t), -x(t - 1)), \quad s(t) = (y(t), -y(t - 1)) \tag{4.2}$$

(as was done in (3.11)) since we can make the appropriate choice of G as we did in Lemma 3.1.

DEFINITION 4.1. We say the simply-oscillatory function x *spirals outward on* $[t_0, \infty)$ if there exists some $t_1 > t_0$ for which $x|[t_1, \infty)$ is outside $x|[t_0, \infty)$. In particular, $r(t_1)$ is outside $r(t_0)$ and there exists $T:[t_1, \infty) \to [t_0, \infty)$ with $T(t_1) = t_0$, as given in Definition 3.3.

We say x *spirals inward on* $[t_0, \infty)$ if there exists some $t_1 > t_0$ for which $x|[t_0, \infty)$ is outside $x|[t_1, \infty)$.

We shall assume (as in the paragraph preceding Lemma 3.1) that $g: R \to R$ is continuous, and strictly monotonically increasing. We shall assume that g satisfies

$$g(0) = 0. \tag{4.3}$$

We assume, further, that $g'(v)$ exists and is continuous, and that the following condition is satisfied:

(4.4) There exists some $\delta_0 > 0$ such that $g'(v) \geqq 1$ whenever $|v| < \delta_0$.

This assumption implies that any solution y of (3.8) for which $y_t \in C_*$, is simply-oscillatory. To see this, suppose that y is a nonsimply-oscillatory solution of (3.8), for example. Then $y(t)$ must tend to 0 monotonically as $t \to \infty$. Thus for some $T_1 > 0$, and all $t > T_1$, $0 < |y(t)| < \delta/2$. Assume for the sake of definiteness that $y(T_1) > 0$. For $T_2 > T_1 + 1$, write $I = [T_2, T_2 + 1]$. Then

$$\begin{aligned} y(T_2 + 1) &= y(T_2) - \int_I g(y(s-1))ds \\ &\leqq y(T_2) - \int_I y(s-1)ds < y(T_2) - \int_I y(T_2)ds = 0. \end{aligned}$$

Hence $y(T_2 + 1) < 0$, contradicting our hypothesis.

We now return our investigation to the study of (E), where we shall assume that $f: R \to R$ satisfies all the hypotheses of Theorem 2.1.

Let $B_1 = \sup \{|f(v)| : -\infty < v \leqq B\}$, where B is given in condition (2.4).

LEMMA 4.1. *Every slowly oscillating solution y on $[t_0, \infty)$ of* (E) *is oscillatory. If τ_2 denotes the second zero of y on $[t_0, \infty)$, then*

$$|y(t)| \leqq B_1 \quad \textit{for all } t \geqq \tau_2.$$

Proof. The fact that any slowly oscillating solution y of (E) is simply-oscillatory follows from the remark following hypothesis (4.4), and the fact that $f'(0) > \pi/2$.

Let τ_1 denote the first zero of y on $[t_0, \infty)$. If t_M is any local maximum of y on $[\tau_1, \infty)$, then $t_M - 1$ is a zero of y (since $y'(t_M) = 0$). Writing $I = [t_M - 1, t_M]$ and using (2.4) we have

$$y(t_M) = 0 + \int_I y'(s)\, ds < B$$

from (2.4). Hence $y(t) < B$ for all $t \leqq \tau_1$. If τ_2 is the second zero of y on $[t_0, \infty)$, then $\tau_2 > \tau_1 + 1$. Let t_m be any local minimum of y on $[\tau_2, \infty)$. Then, proceeding as above, we see that we must have $y(t) > f(B)$ for all $t \geqq \tau_2$.

Let P be the set of all slowly oscillating periodic solutions of (E).

LEMMA 4.2. *P is nonempty and there exists a "smallest" $\lambda \in P$; that is, for all $w \in P$, $\operatorname{Int} O_\lambda \subset \operatorname{Int} O_w$ and either $O_\lambda = O_w$ or $O_\lambda \cap O_w = \varnothing$. Also, λ is a simple periodic solution.*

Furthermore, $\operatorname{Ext} O_\lambda$ is C_*-globally asymptotically stable.

Proof. Let $\phi_\rho = \rho \sin(\pi t/2)$, $t \in [-1, 0]$. Consider now the linearization of (E) in a neighborhood of the origin

$$x'(t) = -f'(0)x(t-1). \tag{4.5}$$

Denote by x_ρ the solution of (4.5) on $[-1, \infty)$ with initial condition ϕ_ρ, and let y denote the corresponding solution of (E). Since $f'(0) > \pi/2$ it is well known (and in fact may be verified by direct computation) that x_ρ spirals outward on $[-1, T]$ for any fixed $T > 0$. Now $|f(x) - f'(0)x| = o(|x|)$ and solutions of (E) depend continuously upon the right-hand side of the equation. It follows that if ρ is sufficiently small, we must have that y_ρ spirals outward on $[-1, T]$. Suppose that T has been chosen sufficiently large that there exists $T_0 \in (0, T]$ such that $y_\rho|[T_0 - 1, \infty)$ is outside $y_\rho|[0, \infty)$; that is, y_ρ spirals outward on $[0, \infty)$. By Lemma 4.1, the curve $r = r_\rho(t) = (y_\rho(t), -y_\rho(t-1))$ is bounded. Clearly $r_\rho(t)$ spirals outward to some simple closed curve C.

Let c be any point of C. Choose $t_n \to \infty$ such that $r(t_n) \to c$. Let $\Psi_n(\Gamma) = y_\rho(t_n + \Gamma)$ for $\Gamma \geqq -1$. Since Ψ_n and Ψ_n' are uniformly bounded, it follows from Ascoli's lemma that there exists a function $\delta: [-1, \infty) \to R$ and a subsequence Ψ_{n_i} such that for each $T > 0$,

$$\Psi_{n_i}(t) \to \delta(t) \quad \text{uniformly for } t \in [-1, T]$$

and

$$f(\Psi_{n_i}(t-1)) \to f(\delta(t-1)) \quad \text{uniformly for } t \in [0, T].$$

Hence δ satisfies (E). Obviously $(\delta(t), -\delta(t-1)) \in C$ for all $t \geqq 0$. Moreover, since C is bounded away from $(0, 0)$, δ is oscillatory. It is easy to verify that δ is slowly oscillating. It follows from Lemma 3.2 that δ is periodic, and thus P is nonempty.

We now wish to demonstrate the existence of a "smallest" $\lambda \in P$. Choose ρ_0 sufficiently small that for all $0 < \rho \leqq \rho_0$, y_ρ spirals outward on $[0, \infty)$. Denote by λ that element of P to which y_{ρ_0} converges; that is, y_{ρ_0} spirals outward to O_λ. Define $\tilde{\rho} = \inf\{\rho_1 \leqq \rho_0 | y_\rho \text{ spirals outward to } O_\lambda \text{ for all } \rho_1 < \rho \leqq \rho_0\}$. If $\tilde{\rho} = 0$, then λ is the "smallest" element of P and we are done. Suppose, therefore, that $\tilde{\rho} > 0$.

CLAIM. *$y_{\tilde{\rho}}$ spirals outward to O_λ.*

Suppose not. Then there would exist some $\tilde{\lambda} \in P$, $\tilde{\lambda} \neq \lambda$, such that $y_{\tilde{\rho}}$ spirals outward to $O_{\tilde{\lambda}}$ and $O_{\tilde{\lambda}} \subset \operatorname{Int} O_\lambda$. Let

$$d = \sup_{t \in [0,1]} \operatorname{dist}[(y_\rho(t), -y_\rho(t-1)), O_{\tilde{\lambda}}].$$

By the continuous dependence of solutions upon initial conditions, if we choose

$\rho > \tilde{\rho}$, $|\rho - \tilde{\rho}|$ sufficiently small, then

$$\sup_{t \in [0,1]} \operatorname{dist}[(y_\rho(t), -y_\rho(t-1)), (y_{\tilde{\rho}}(t), -y_{\tilde{\rho}}(t-1))] < d/2,$$

so that $y_\rho(t)$ is inside $O_{\tilde{\lambda}}$ for $t \in [0, 1]$. It follows from the trajectory crossing lemma that $y_\rho(t)$ is inside $O_{\tilde{\lambda}}$ for all $t \in [0, \infty)$. This contradicts the fact that $\tilde{\rho} < \rho \leqq \rho_0$ implies that y_ρ spirals outward to O_λ, and establishes our claim.

Consider next a sequence ρ_i such that $0 < \rho_i < \tilde{\rho}$, $\rho_i < \rho_{i+1}$, $\rho_i \to \tilde{\rho}$. We know that the corresponding solutions y_{ρ_i} must spiral outward to some $\lambda_i \in P$. Moreover, $y_{\rho_i} \to y_{\tilde{\rho}}$ uniformly on compact t-subsets. This in turn implies that $O_{\lambda_i} \to O_\lambda$, in the sense that $O_\lambda \subset \overline{\bigcup_i \operatorname{Int} O_{\lambda_i}}$. It follows that there must exist some integer N so large that $\operatorname{dist}[O_{\lambda_N}, O_\lambda] < d/2$. Thus $y_{\tilde{\rho}}|[0, 1]$ is inside O_{λ_N}. This implies that $y_{\tilde{\rho}}$ must remain inside O_{λ_N} for all future time. This contradicts the fact that $y_{\tilde{\rho}}$ spirals outward to O_λ. Thus we must have $\tilde{\rho} = 0$, and λ is the "smallest" element of P, establishing the first part of our lemma.

To see that $\operatorname{Ext} O_\lambda$ is a C_*-global attractor, let y be any slowly oscillating solution of (E) on $[t_0, \infty)$. Let $s(t) = (y(t), -y(t-1))$ for $t \geqq t_0$. Recall that in the first paragraph of the proof of this lemma we constructed a family of curves $r_\rho(t) = (y_\rho(t), -y_\rho(t-1))$, such that y_ρ is a solution of (E) and each y_ρ spirals outward. Choose $t_1 \geqq 0$ so that $y(t_1) = y_\rho(0) = 0$. Now choose ρ so small that $y|[t_1, t_1 + 1]$ is outside $y_\rho|[0, t_2]$ for some $t_2 \geqq 1$. Then by Lemma 3.1, $y|(t_1, \infty)$ is outside $y_\rho|[0, \infty)$. Thus there exists a monotonically nondecreasing function T satisfying the requirements of Proposition 3.1, so that $s(t)$ is outside $r_\rho(T(t))$ for all $t \in [t_1, \infty)$. Since y_ρ spirals outward to some slowly oscillating periodic solution of (E), it follows that for ρ sufficiently small

$$d[r_\rho(T(t)), O_\lambda] \to 0 \quad \text{as } t \to \infty.$$

Since $y|[t_1, \infty)$ is outside $y_\rho|[0, \infty)$ it follows that $d[s(t), \operatorname{Ext} O_\lambda] \to 0$ as $t \to \infty$ (and possibly $d[s(t), \operatorname{Ext} O_\lambda] = 0$ for all t sufficiently large). Hence $\operatorname{Ext} O_\lambda$ is a C_*-global attractor.

It remains to show that $\operatorname{Ext} O_\lambda$ is C_*-stable in R^2 for (E). An equivalent formulation of C_*-stability of a set $\operatorname{Ext} O_\lambda$ is given by the following:

Let $\{z_n\}$ be any sequence of slowly oscillating solutions of (E) on $[0, \infty)$. Write $\phi_n(t) = (z_n(t), -z_n(t-1))$. Then $\operatorname{Ext} O_\lambda$ is C_*-stable if

$$d[\phi_n(t), \operatorname{Ext} O_\lambda] \to 0 \quad \text{as } n \to \infty \text{ uniformly for } t \in [0, 1] \tag{4.6}$$

implies that

$$d[\phi_n(t), \operatorname{Ext} O_\lambda] \to 0 \quad \text{as } n \to \infty \text{ uniformly for all } t \in [0, \infty). \tag{4.7}$$

Let $\{z_n\}$ be a sequence satisfying (4.6). Let x be any solution of (E) which spirals outward on $[0, \infty)$, and such that $r(t) = (x(t), -x(t-1)) \to O_\lambda$ as $t \to \infty$. For n sufficiently large, let T_n denote the largest time such that $\phi_n(0)$ is outside $r(T_n)$, and there exist numbers $t_n \geqq 1$ and $T_n' \geqq T_n + 1$ such that $z_n|[0, t_n]$ is outside $x|[T_n, T_n']$. But then $z_n|[0, \infty)$ is outside $x|[T_n, \infty)$ by the trajectory crossing lemma. Since

$$\sup\{d[r(t), \operatorname{Ext} O_\lambda] : t \geqq T_n\} \to 0 \quad \text{as } n \to \infty,$$

it follows that (4.7) is satisfied and $\operatorname{Ext} O_\lambda$ is C_*-stable. Therefore $\operatorname{Ext} O_\lambda$ is C_*-globally asymptotically stable.

LEMMA 4.3. *There exists a "largest"* $\Psi \in P$; *that is, for all* $w \in P$, $\operatorname{Ext} O_\Psi \subset \operatorname{Ext} O_w$ *and either* $O_\Psi = O_w$ *or* $O_\lambda \cap O_w = \varnothing$. *Also,* Ψ *is a simple periodic solution.*

Furthermore, $\operatorname{Int} O_x$ *is* C_**-globally asymptotically stable.*

Proof. The proof of this lemma resembles the proof of Lemma 4.2. Let us first establish the existence of a largest $\Psi \in P$. Define a partial ordering on $\theta = \overline{\{O_w : w \in P\}}$ as follows: If $w_1, w_2 \in P$, then $O_{w_1} \leqq O_{w_2}$ if $O_{w_1} \subset \operatorname{Int} O_{w_2} = \overline{\operatorname{Int} O_{w_2}}$. Let $\{O_w^\alpha\}_{\alpha \in A}$ be a chain in O. Then we may find a sequence $w_i \in P$ (where the w_i need not all be distinct), such that $w_i \to w \in P$ and O_w is the boundary of $\bigcup_{\alpha \in A} O_w^\alpha$. (Note that this union is a bounded set by Lemma 4.1.) Clearly, O_w is an upper bound for the chain. By Zorn's lemma, θ must have a maximal element O_Ψ for some $\Psi \in P$, and $\operatorname{Ext} O_\Psi \subset \operatorname{Ext} O_w$ for all $w \in P$.

As in the proof of Lemma 4.1, we shall be able to show that $\operatorname{Int} O_\Psi$ is C_*-globally asymptotically stable once we are able to establish the existence of a slowly oscillating solution x of (E) which spirals inward, and such that

$$r(t) = (x(t), -x(t-1)) \to O_w$$

as $t \to \infty$. To do this, let $A_1 > B_1$, where B_1 is given in Lemma 4.1. Define x to be the (slowly oscillating) solution of (E) on $[0, \infty)$ for which $x(t) = 2A_1$ for $t \in [-1, 0]$. It follows that

$$|r(t)| = (|x(t)|^2 + |x(t-1)|^2)^{1/2} > 2A_1 \quad \text{for } t \in [0, 1].$$

Assume (by a translation of Ψ if necessary) that $x|[0, 1]$ is outside $\Psi|[0, t_0]$ for some $t_0 \geqq 1$, and hence $x|[0, \infty)$ is outside $\Psi|[0, \infty)$, by Lemma 3.1. But by Lemma 4.1 we must have $|r(t)| < 2B_1 < 2A_1$, for $t > \tau_2$, and so x must spiral inward. Since the limit set of $r(t)$ is a simple closed curve C, which is the orbit of some solution, it follows that we must have $C = O_\Psi$, and Ψ is a simple periodic solution. The remainder of the proof proceeds as in Lemma 4.2.

Proof of Theorem 2.1. With the proof of Lemma 4.3, we have completed the proof of Theorem 2.1, since $A = \operatorname{Ext} O_\lambda \cap \operatorname{Int} O$ is a periodic C_*-annulus, and it is clear that A must be C_*-globally asymptotically stable since $\operatorname{Ext} O_\lambda$ and $\operatorname{Int} O_\Psi$ are.

5. Examples.

Example 5.1. Consider equation (1.1). If $x(t)$ satisfies (1.1) for $t \geqq t_0$, define $a(t) = -\alpha x(t-1)$ for $t \geqq t_0$. Then this solution x obviously satisfies

$$x'(t) = a(t)[1 + x(t)] \quad \text{for all } t \geqq t_0.$$

It follows immediately from inspection of this ordinary differential equation that

$$\begin{aligned} x(t_0) &= -1 \quad \text{implies} \quad x(t) \equiv 1 && \text{for all } t \geqq t_0; \\ x(t_0) &< -1 \quad \text{implies} \quad x(t) < 1 && \text{for all } t \geqq t_0; \\ x(t_0) &> -1 \quad \text{implies} \quad x(t) > -1 && \text{for all } t \geqq t_0. \end{aligned}$$

In the second case it follows that $x(t) \to -\infty$ as $t \to \infty$. Hence for the interesting case, $x(t_0) \in (-1, \infty)$. Perform the change of variable $y = \ln(1 + x)$ taking

$(-1, \infty)$ onto R. Write

$$f(v) = \alpha[e^v - 1].$$

Then $f(0) = 0, f'(0) = \alpha, f'(v) > 0$ for all $v \in R$, and (1.1) becomes

$$y'(t) = \frac{x'(t)}{1 + x(t)} = -\alpha x(t - 1) = -f(y(t - 1)).$$

Thus by Theorem 2.1 there are two (possibly equal) slowly oscillating simple periodic solutions which bound an annulus $A \subset R^2$, and A is C_*-globally asymptotically stable.

Example 5.2. Consider equation (1.2). This equation was studied by Jones in [5]. As in the above example, there is a region of particular interest, which in this case is $(-1, 1)$. Let $y(t) = \frac{1}{2} \ln [(1 + x)(1 - x)]$, taking $(-1, 1)$ into R. Define

$$f(v) = \alpha[e^{2v} - 1]/[e^{2v} + 1].$$

Then (1.2) becomes

$$y'(t) = \frac{x'(t)}{1 - x^2(t)} = -\alpha x(t - 1) = -f(y(t - 1)).$$

Again, $f(0) = 0$, $f'(0) = \alpha$, $f'(v) > 0$ for all $v \in R$. Thus there exists a periodic orbit O_y, and this orbit is C_*-globally asymptotically stable. Using the fact that $f(y) = -f(-y)$, Jones proved that there exists a slowly oscillating periodic solution y of (1.2) having period 4. (Its existence also follows from Theorem 2.2.) Furthermore, he showed that any other slowly oscillating periodic solution of period 4 was just a translate of y. However, he was unable to rule out the existence of a slowly oscillating periodic solution with another period.

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THE APPROXIMATION OF CERTAIN PARABOLIC EQUATIONS BACKWARD IN TIME BY SOBOLEV EQUATIONS*

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Abstract. For any nonnegative, self-adjoint operator A, which does not depend on time, the backward solution to the parabolic equation, $u'(t) = -Au(t)$, $t \geqq 0$, in a cylinder can be approximated by the solution to the Sobolev equation, $u'(t) = -(I + \beta A)^{-1}Au(t)$. The solution to the backward Sobolev equation can be more readily computed than the solution to the adjoint of the parabolic equation. In a Hilbert space setting, if the norm of the solution is assumed to be bounded by a positive constant E at the base $t = 0$ of the cylinder and the data error at $t = T$ is less than a prescribed $\varepsilon > 0$, then the norm of the difference in the solutions is $O([-\log(\varepsilon/E)]^{-1})$. This logarithmic continuity is essentially the best that can be obtained for this approximation.

The above result can be generalized to operators A which are sectorial with semiangle $\pi/4$ and such that $-A$ generates a contraction semigroup of operators. Simple numerical results for the heat equation in a rectangle illustrate the approximation results.

1. Introduction. Consider the region of the plane given by $0 \leqq x \leqq \pi$ and $0 \leqq t \leqq 1$. Suppose the solution $u(x, t)$ to the heat equation, $u_{xx} = u_t$, in the above region is known approximately for all x when $t = 1$. The object of this paper is to discuss in a Hilbert space setting the numerical approximation and continuous dependence on data of solutions for $t < 1$ to a fairly general class of equations containing the heat equation.

The problem

$$u_{xx} = u_t \qquad \text{for } 0 < x < \pi, \quad 0 < t < 1, \tag{1.1a}$$

$$0 = u(0, t) = u(\pi, t) \quad \text{for } 0 < t < 1, \tag{1.1b}$$

$$u(x, 1) = \chi(x) \qquad \text{for } 0 < x < \pi, \tag{1.1c}$$

is unstable and not well-posed in the sense of Hadamard [10]. However, continuous dependence of the solution on the data can often be brought about by the additional requirement of a prescribed global bound upon the class of solutions considered [11]. Therefore, we add the restriction

$$|u(x, 0)| < E \quad \text{for } 0 < x < \pi, \tag{1.2}$$

where E is some known positive constant.

Since the heat operator cannot be time-inverted to obtain a well-posed problem (irreversibility), we would like to find an operator "near" the heat operator in some sense for which the backward problem is well-posed. We then compare the solution of the backward problem for the perturbed operator with the desired solution of the original problem (1.1)–(1.2).

Many people have considered this type of problem. Among these are Cannon, Douglas, John, Lattés and Lions, Lavrentiev, Miller, Payne, Pucci, Showalter,

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Buzbee and Carasso, and others [2], [3], [4], [5], [6], [7], [11], [13], [14], [15], [16], [17], [18], [19], [20], [22].

We now consider the differential equation on a Hilbert space,

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{1.3}$$

where A is a self-adjoint operator that is not dependent on t and is nonnegative, which means the numerical range of A is contained in the right half of the complex plane. The method of *quasi-réversibilité* introduced by Lattés and Lions [13] replaces A in (1.3) by a function of the operator, $f(A) = A - \delta A^2$, with spectrum bounded above and then solves the backward problem for the new operator. Using the final value for this new backward problem as initial data for the original operator, they obtained an approximation which converged to the data in their control theory problem.

Using the quasi-reversibilité idea, Miller [17] employs the requirement of Hölder continuity to determine constraints on $f(A)$. He then shows that an $f(A)$ satisfying these constraints can be found which results in a Hölder degree of approximation. This method leads to rational functions of the operator for which the numerical computations require complex arithmetic and several complicated inversions of the operator at each time step. The purpose of this paper is to consider a perturbation of the operator A which allows much easier numerical computations and still retains logarithmic continuity.

Consider the "pseudoparabolic" [21] or Sobolev equation

$$v'(t) + \beta A v'(t) = -Av(t) \tag{1.4}$$

with $\beta > 0$. Since A is nonnegative and $\beta > 0$, we see that $I + \beta A$ is invertible and we obtain the equation

$$v'(t) = -(I + \beta A)^{-1} A v(t). \tag{1.5}$$

Thus, in the quasi-réversibilité setting we are choosing

$$f(A) = (I + \beta A)^{-1} A. \tag{1.6}$$

The idea of approximating (1.3) by (1.5) is due to Yosida. He uses this idea in his proof of the generation theorem for semigroups of operators [23]. We see that the Sobolev equation (1.4) satisfies the requirement of a bounded spectrum. Also, numerical techniques do not require complex arithmetic. For some numerical methods see [8] and Part II of the author's Ph.D. thesis [7a].

We now state the problem considered in this paper.

Problem. Suppose $u(t)$ is an unknown solution of

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{1.7a}$$

$$\|u(1) - \chi\| < \varepsilon, \tag{1.7b}$$

$$\|u(0)\| < E, \tag{1.7c}$$

where χ is a given "data" vector in a Hilbert space H, $\varepsilon > 0$ is a known small number, E is a known positive constant, and A is any nonnegative, self-adjoint operator which does not depend on t. H incorporates the side boundary conditions and has norm $\|\cdot\|$. We want to approximate $u(t)$ with $v(t)$, a solution of the approximate problem

$$v'(t) = -(I + \beta A)^{-1}Av(t), \qquad t \geqq 0, \tag{1.8a}$$

$$v(1) = \chi. \tag{1.8b}$$

We shall show that for each $t > 0$, we can choose a β in (1.8) such that

$$\|u(t) - v(t)\| = O([-\log(\varepsilon/E)]^{-1}). \tag{1.9}$$

In § 3, we consider generalizations of the results obtained in § 2 using different techniques. We describe the notion of an operator being sectorial. Then, for any operator A which is sectorial with semiangle $\pi/4$ and such that $-A$ generates a contraction semigroup of operators, we obtain the same type of logarithmic continuity as in (1.9) for the problem related to (1.7). Finally, in § 4, we describe some simple numerical results for the heat equation in the problem (1.1).

2. Continuous dependence on data. It is well known [9], [13], [17] that solutions to (1.7a) have the representation

$$u(t) = e^{-tA}u_0, \qquad t \geqq 0, \quad u_0 \in H, \tag{2.1}$$

where e^{-tA}, the strongly continuous contraction semigroup generated by $-A$, is easily defined in terms of the spectral representation of A. Now we recall a well-known result which stabilizes the problem (1.7).

THEOREM 2.1 (Stability estimate) [1]. *If $u(t)$ is a solution of equation* (1.7), *then* $\log \|u(t)\|$ *is a convex function of t. Consequently, if*

$$\|u(1)\| \leqq \varepsilon, \tag{2.2a}$$

$$\|u(0)\| \leqq E, \tag{2.2b}$$

then

$$\|u(t)\| \leqq \varepsilon^t E^{1-t} \quad \textit{for } 0 \leqq t \leqq 1. \tag{2.3}$$

This stability estimate clearly gives a backward uniqueness result for (1.7). This uniqueness result implies that for $t = 1$, e^{-1A} is a 1–1 operator. Thus the kernel of e^{-A} consists only of the zero vector. An easy computation shows that the kernel of e^{-A} is the orthogonal complement of the range of the adjoint, $(e^{-A})^*$. However, in our discussion, A is self-adjoint and since $(e^{-A})^* = e^{-(A^*)}$ [9], we have that $(e^{-A})^* = e^{-A}$. Thus since the zero vector is the orthogonal complement of the range of e^{-A}, we have the range of e^{-A} is dense in H. Therefore, given any data vector χ in H and $\varepsilon > 0$, we can write for some u_0 in the domain of A,

$$\chi = e^{-A}u_0 + \psi, \tag{2.4a}$$

with

$$\|\psi\| < \varepsilon. \tag{2.4b}$$

Thus we can write any data vector χ in the form (2.4) and be compatible with (1.7b).

Since χ is the exact data for (1.8a), the exact solution for $0 \leqq t \leqq 1$ of (1.8) is given by

$$
\begin{aligned}
v(t) &= e^{(1-t)(I+\beta A)^{-1}A}\chi \\
&= e^{(1-t)(I+\beta A)^{-1}A}(e^{-A}u_0 + \psi) \qquad (2.5)\\
&= e^{(1-t)(I+\beta A)^{-1}A-A}u_0 + e^{(1-t)(I+\beta A)^{-1}A}\psi,
\end{aligned}
$$

where the strongly continuous semigroup $e^{(1-t)(I+\beta A)^{-1}A}$ is defined in terms of its spectral representation.

THEOREM 2.2. *Let u be a solution of* (1.7) *and let v be given by* (2.5). *If we choose $\beta = 1/\log(E/\varepsilon)$, we obtain for each $t > 0$,*

$$
\begin{aligned}
\|u(t) - v(t)\| &\leqq \frac{4(1-t)E}{t^2 \log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t \qquad (2.6)\\
&= O([-\log(\varepsilon/E)]^{-1}).
\end{aligned}
$$

Proof. We compare $u(t)$ and $v(t)$ in the norm. From (2.1), (2.4), and (2.5), we have

$$
\begin{aligned}
\|u(t) - v(t)\| &= \|e^{-tA}u_0 - e^{(1-t)(I+\beta A)^{-1}A-A}u_0 + e^{(1-t)(I+\beta A)^{-1}A}\psi\| \qquad (2.7)\\
&\leqq \|e^{-tA} - e^{(1-t)(I+\beta A)^{-1}A-A}\|\,\|u_0\| + \|e^{(1-t)(I+\beta A)^{-1}A}\|\,\|\psi\|.
\end{aligned}
$$

Thus defining

$$B(t) = e^{-tA} - e^{(1-t)(I+\beta A)^{-1}A-A} \qquad (2.8a)$$

and

$$C(t) = e^{(1-t)(I+\beta A)^{-1}A}, \qquad (2.8b)$$

it follows from (1.7) and (2.7) that

$$\|u(t) - v(t)\| \leqq \|B(t)\|E + \|C(t)\|\varepsilon. \qquad (2.9)$$

Now we consider the roles that $B(t)$ and $C(t)$ play in our problem. We note that $\|B(t)\|$ just measures the amount by which the Sobolev operator differs from the parabolic operator. It is clear that as $\beta \to 0$, $\|B(t)\| \to 0$ in some sense. As in the author's thesis, one can show that $\|B(t)\|$ is at most $O(\beta)$ with the bound

$$\|B(t)\| \leqq \frac{4(1-t)}{t^2}\beta. \qquad (2.10)$$

$\|C(t)\|\,\|\psi\|$ measures the effect of the backward Sobolev equation on the error term ψ in the data. We can easily obtain the bound

$$\|C(t)\| < e^{(1-t)/\beta}. \qquad (2.11)$$

As $\beta \to 0$, the bound $e^{(1-t)/\beta}$ grows very rapidly. Thus we must balance the two terms against each other to obtain a best estimate. If we could solve for the bound for $\|B(t)\|$ in closed form in terms of β as we did for $\|C(t)\|$, we could obtain the best β in closed form. At present, we can only approximate β.

The best choice of β for the second bound is given by

$$\beta = 1/\log(E/\varepsilon). \tag{2.12}$$

With this choice of β, it follows from (2.9), (2.10), and (2.11) that for any $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{4(1-t)E}{t^2 \log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t = O([-\log(\varepsilon/E)]^{-1}). \tag{2.13}$$

Remark. The choice (2.12) is not the best possible choice of β since it gives the second term significantly better continuity properties than the first. In regard to this problem, we conducted a simple numerical experiment on the computer for the heat equation (1.1). If we take

$$\chi = e^{-1}\sin x, \tag{2.14}$$

we know a priori the exact solution to the backward heat equation. Using these data and then perturbing it we considered the differences in the Fourier series representations of $u(t)$ and $v(t)$ as we varied ε. The literature tells us we cannot expect usable results all the way back to the time $t = 0$. For $t = 0.5$, we obtained numerically

$$\|u(.5) - v(.5)\| \leqq (.130)[\log(1/\varepsilon)]^{-1.113}. \tag{2.15}$$

Thus even with this very simple problem, where an essentially best possible β was used, we don't get significantly better than logarithmic continuity.

3. Generalizations. We recall that all the results in § 2 hold for any nonnegative self-adjoint operator A which does not depend on t. The techniques that were used depended heavily on the self-adjointness of A. We now extend these results to more general operators.

First we assume that $-A$ generates a strongly continuous contraction semigroup of operators on the complex Hilbert space H. We shall add another restriction later and shall need the following theorem.

THEOREM 3.1 [23]. *The operator $-A$ is the generator of a contraction semigroup if and only if $-A$ is closed, densely defined, each $\lambda > 0$ is in the resolvent set of $-A$, and*

$$\|(\lambda + A)^{-1}\| \leqq 1/\lambda \quad \text{for all } \lambda > 0. \tag{3.1}$$

COROLLARY 3.2 [23]. *If the operator $-A$ is the generator of a contraction semigroup, then for every $\beta > 0$, the operator $J_\beta = (I + \beta A)^{-1}$ is a contraction, or*

$$\|J_\beta\| \leqq 1 \quad \text{for every } \beta > 0. \tag{3.2}$$

From the identity

$$J_\beta A = (1/\beta)(1 - J_\beta) = AJ_\beta, \tag{3.3}$$

we clearly see that A commutes with J_β. Then from (3.2) and (3.3) we see that $J_\beta A$ is a bounded linear operator:

$$\begin{aligned} \|J_\beta A\| &= \|(1/\beta)(I - J_\beta)\| \\ &\leqq (1/\beta)(1 + \|J_\beta\|) \\ &\leqq 2/\beta. \end{aligned} \tag{3.4}$$

Due to (3.4) we can define the group of linear operators by

$$T_\beta(t) \equiv \exp(-tJ_\beta A), \tag{3.5}$$

where we use the power series to define the exponential function. Yosida showed [23] that for each $\beta > 0$, $t \geqq 0$, $T_\beta(t)$ is a contraction and that the strong limit

$$T(t)x = \operatorname*{s-lim}_{\beta \to 0} T_\beta(t)x, \qquad x \in H, \tag{3.6}$$

exists and is the semigroup generated by $-A$. Yosida also showed that for $x \in D(A)$, the domain of A, then $x \in D((d/dt)T(t))$, $x \in D((d/dt)T_\beta(t))$,

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax, \qquad t \geqq 0, \tag{3.7a}$$

and

$$\frac{d}{dt}T_\beta(t)x = J_\beta AT_\beta(t)x = T_\beta(t)J_\beta Ax, \qquad t \geqq 0. \tag{3.7b}$$

From (3.7b) we see that for any $\alpha, \beta > 0$,

$$T_\alpha(s)T_\beta(t) = T_\beta(t)T_\alpha(s), \qquad s, t > 0. \tag{3.8}$$

Also from (3.7b) if $x \in D(A)$, then $T_\beta(t)x \in D(A)$ and we see that from (3.6) and (3.8), we have for $s, t > 0$,

$$T(s)T_\beta(t)x = \operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)T_\beta(t)x = \operatorname*{s-lim}_{\alpha \to 0} T_\beta(t)T_\alpha(s)x. \tag{3.9}$$

Now from (3.6),

$$\operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)x = T(s)x. \tag{3.10}$$

Then since $T_\beta(t)$ is continuous and thus closed,

$$\operatorname*{s-lim}_{\alpha \to 0} T_\beta(t)T_\alpha(s)x = T_\beta(t) \quad \operatorname*{s-lim}_{\alpha \to 0} T_\alpha(s)x = T_\beta(t)T(s)x. \tag{3.11}$$

Thus combining (3.9) and (3.11) we see that for $x \in D(A)$, $s, t > 0$,

$$T(s)T_\beta(t)x = T_\beta(t)T(s)x. \tag{3.12}$$

Now we need to consider some additional terminology. An unbounded operator A on H is called *sectorial* with semiangle θ if the numerical range of A, $\{(Ax, x) : x \in D(A)\}$, is contained in the sector $\{z : |\arg(z)| \leqq \theta\}$. We recall the following theorem.

THEOREM 3.3 [12]. *If $-A$ generates a contraction semigroup T and A is sectorial with semiangle θ, where $0 \leqq \theta < \pi/2$, then T is a holomorphic semigroup. For each $t > 0$ and $x \in H$, $T(t)x \in D(A)$ and $AT(t)$ is a bounded linear operator on H with $\|AT(t)\| \leqq M_1/t$, where M_1 is a positive constant. The identity $T(t) = T(t/m)^m$ holds for $t > 0$ and $m \geqq t$, and we also have*

$$\|A^m T(t)\| \leqq M_m/t^m, \tag{3.13}$$

where M_m are a sequence of positive constants.

Note. The last inequality in Theorem 3.1 is crucial in obtaining the last results above.

Showalter [22] introduces a collection of semigroups which he calls Q-R semigroups generated by $-(A - J_\beta A)$ and denoted by

$$E_\beta(t) \equiv \exp(-t(A - J_\beta A)), \tag{3.14}$$

and proves the following theorem.

THEOREM 3.4 [22]. *Let $-A$ generate a contraction semigroup and define the Q-R semigroups, E_β, by (3.14). Then the Q-R semigroups are each contractions if and only if A is sectorial with semiangle $\pi/4$. In this case we have $\lim_{\beta\to 0} E_\beta(t)x = x$ for each $x \in H$, uniformly on bounded intervals, and the following estimates hold for any $t \geqq 0$:*

$$\|E_\beta(t)x - x\| \leqq t\|Ax - J_\beta Ax\|, \qquad x \in D(A), \tag{3.15a}$$

$$\|E_\beta(t)x - x\| \leqq t\beta\|A^2x\|, \qquad x \in D(A^2). \tag{3.15b}$$

We can now state the main theorem of this section.

THEOREM 3.5. *Let $-A$ generate a contraction semigroup and let A be sectorial with semiangle $\pi/4$. Let u be an unknown solution to*

$$u'(t) = -Au(t), \qquad t \geqq 0, \tag{3.16a}$$

$$\|u(1) - \chi\| < \varepsilon, \tag{3.16b}$$

$$\|u(0)\| < E, \tag{3.16c}$$

where $\chi = e^{-A}u_0 + \psi$ and $\|\psi\| < \varepsilon$. Let

$$v(t) = e^{(1-t)J_\beta A}\chi \tag{3.17}$$

be a solution to

$$v'(t) = -J_\beta Av(t), \tag{3.18a}$$

$$v(1) = \chi, \tag{3.18b}$$

where $J_\beta = (I + \beta A)^{-1}$. For the choice

$$\beta = 2/\log(E/\varepsilon), \tag{3.19}$$

$$\|u(t) - v(t)\| = O([-\log(\varepsilon/E)]^{-1}) \tag{3.20}$$

holds for each $t > 0$, where the constant depends on t and is displayed below in (3.30).

Proof. In defining his Q-R semigroups, instead of the notation in (3.9), Showalter [22] defines

$$E_\beta(t) \equiv T(t)T_\beta(-t), \qquad t \geqq 0, \quad \beta > 0, \tag{3.21}$$

where $T(t)$ and $T_\beta(t)$ are defined in (3.6) and (3.5). In this form, (3.15b) becomes, for $t = 1$,

$$\lim_{\beta\to 0} T(1)T_\beta(-1)x = x, \qquad x \in D(A). \tag{3.22}$$

Since by Theorem 3.1, $-A$ is densely defined, the range of $T(1) = e^{-A}$ is dense in

H and the requirement that

$$\chi = e^{-A}u_0 + \psi, \tag{3.23a}$$

where

$$\|\psi\| < \varepsilon \tag{3.23b}$$

is not a restriction.

Consider $u(t)$ and $v(t)$. From (3.5), (3.6), (3.16), (3.17), and (3.18) we have

$$\begin{aligned} \|u(t) - v(t)\| &= \|e^{-tA}u_0 - e^{(1-t)J_\beta A}(e^{-A}u_0 + \psi)\| \\ &\leqq \|T(t)u_0 - e^{-(A-J_\beta A)}T_\beta(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\|. \end{aligned} \tag{3.24}$$

Using the fact that

$$e^{-(A-J_\beta A)} = E_\beta(1) = T(1)T_\beta(-1) \tag{3.25}$$

from (3.14) and (3.21), we have from (3.8),

$$\begin{aligned} \|u(t) - v(t)\| &\leqq \|T(t)u_0 - T(1)T_\beta(-1)T_\beta(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\| \\ &= \|T(t)u_0 - T(1-t)T_\beta(-1+t)T(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\| \\ &= \|T(t)u_0 - E_\beta(1-t)T(t)u_0\| + \|e^{(1-t)J_\beta A}\psi\|. \end{aligned} \tag{3.26}$$

Theorem 3.3 implies that $-A$ generates a holomorphic semigroup. Hence, $u_0 \in D(A)$ implies $u_0 \in D(A^2)$. Then from (3.4), (3.7), and (3.15b), we have for $t > 0$,

$$\|u(t) - v(t)\| \leqq (1-t)\beta\|A^2T(t)u_0\| + e^{2(1-t)/\beta}\varepsilon. \tag{3.27}$$

Now from (3.13) of Theorem 3.3, we have for $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{(1-t)}{t^2}M_2E\beta + e^{2(1-t)/\beta}\varepsilon. \tag{3.28}$$

Setting

$$\beta = 2/\log(E/\varepsilon) \tag{3.29}$$

in (3.28), we obtain for each $t > 0$,

$$\|u(t) - v(t)\| \leqq \frac{2(1-t)M_2E}{t^2}\frac{1}{\log(E/\varepsilon)} + E^{(1-t)}\varepsilon^t = O([-\log(\varepsilon/E)]^{-1}). \tag{3.30}$$

4. Numerical results. In this section we present some results of numerical comparison of the Fourier sine series representations of the solutions to the heat equation in a rectangle (1.1) and the corresponding Sobolev equation approximation. We also considered the Crank–Nicolson method and got very comparable numerical results, but these results are not presented here.

First we describe the numerical method for choosing the parameter β in (1.4). We know from the literature that it would be overly optimistic to expect very good numerical results all the way back to $t = 0$. Thus we built in the requirement that the choice of β would be best for results at $t = .5$, half the way back. We noted that for $t = .5$ we can obtain the bound

$$\|u(t) - v(t)\| < (1 - e^{-8\beta})E + e^{1/(2\beta)}\varepsilon. \tag{4.1}$$

To minimize this bound, we differentiated, set the result equal to zero, and used an interval halving method on the computer to obtain an approximation for the required β. Table 1 gives an idea of the optimal choice of β for different data errors with $E = 1$. Using the choices of β found in Table 1 we obtained the result in (2.15).

TABLE 1

Data error	β
10^{-2}	0.196
10^{-3}	0.113
10^{-4}	0.080
10^{-5}	0.061
10^{-6}	0.049
10^{-7}	0.041

Now we describe a few experiments consisting of various perturbations of the fundamental mode of sine curves for which the exact solution of the heat equation is known a priori. We first perturbed the fundamental mode with .01 sin $2x$ and then .01 sin $3x$ in Figs. 1 and 2 for $t = 0.5$. Then in Fig. 3 we perturbed the sine curve

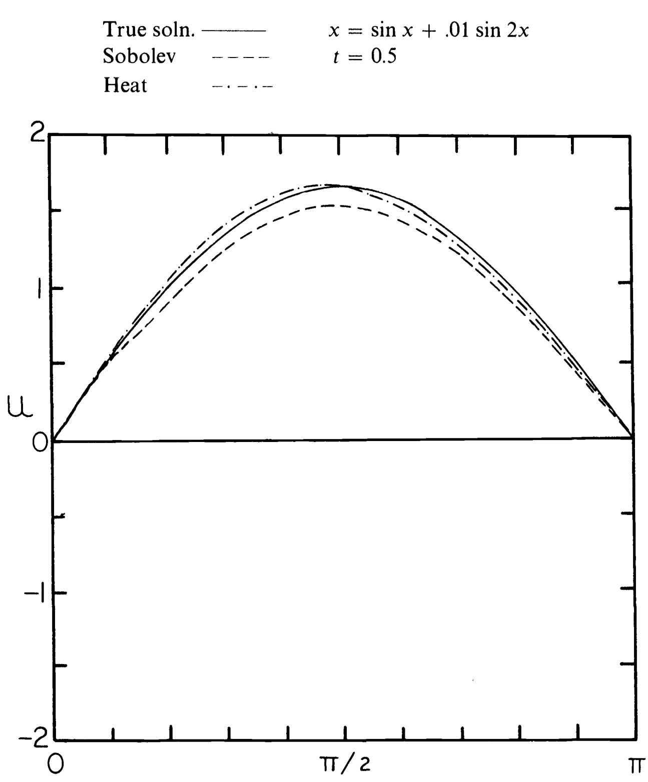

FIG. 1

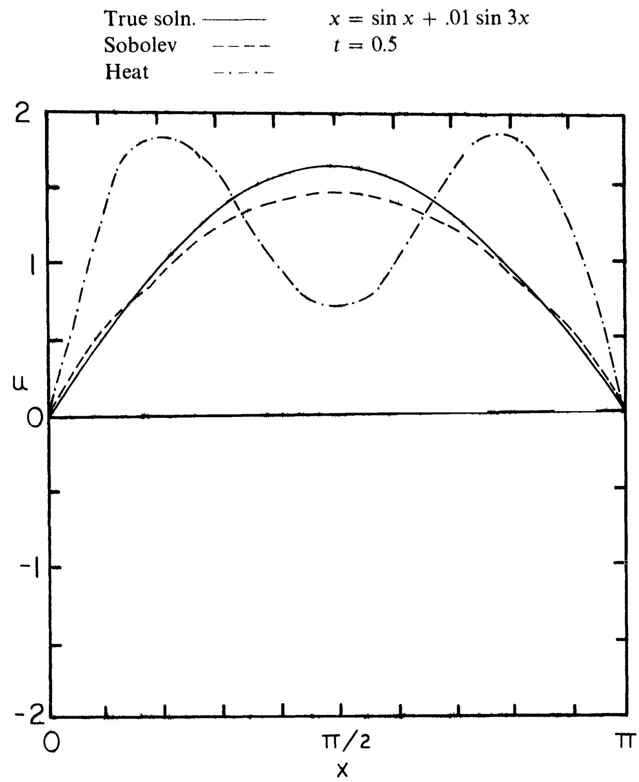

FIG. 2

with a uniform error of .01 (expanded in the first ten terms of this Fourier series) and considered the results at $t = .5$. In each case 100 intervals were used in the x-direction and 10 intervals in the y-direction. Simpson's rule was used for the numerical integration. Truncation in the Fourier series always occurred after 20 terms. In each of the figures all solutions have zero boundary data on $x = 0$ and $x = \pi$. "True solution" means the exact solution u of the heat equation with $u(x, 1) = \sin x$, "Sobolev" refers to the solution w of the Sobolev equation $w_{xx} = w_t - \beta w_{xxt}$ with $w(x, 1) = \sin x +$ perturbation, and "Heat" refers to the solution z of the heat equation (which was obtained by setting $\beta = 0$ in the above equation) with $z(x, 1) = \sin x +$ perturbation. The severe problems with the backward numerical computations on the heat equation are apparent.

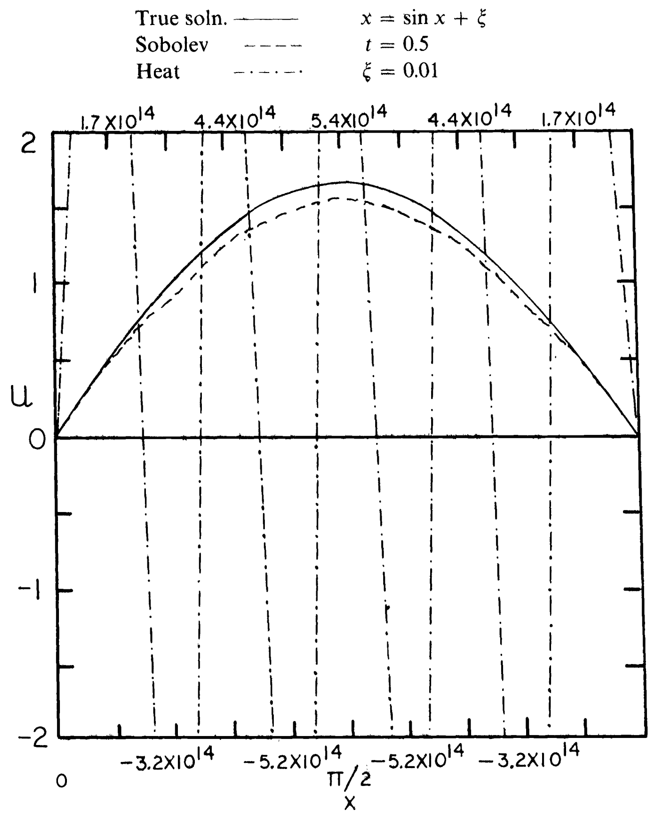

FIG. 3

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ALGEBRAIC METHOD FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS WHOSE COEFFICIENTS ARE FUNCTIONS OF ONE VARIABLE*

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Abstract. The present paper is devoted to an algebraic method for solving linear differential equations, whose coefficients are functions of one or several variables. This method is based on the composition product of tensor products of kernel distributions and permits transformation of differential equations into algebraic composition equations.

In our subsequent papers, we shall apply the same method for solving partial differential equations whose coefficients are functions of several variables.

1.1. Preliminaries. It is known that the integral transforms (Fourier, Laplace) permit transformation of linear differential and partial differential equations into algebraic equations by means of the convolution product. Likewise, by means of the algebraic operational calculus of distributions, linear differential and partial differential equations with constant coefficients are transformed into algebraic equations in a field of operators (cf. [1], [2], [3], [4]). In the present paper we present an algebraic method for solving differential equations whose coefficients are functions of one or several variables, by means of the composition product of tensor products of kernel distributions. This method permits the transformation of linear differential equations into algebraic composition equations.

1.2. The composition algebra $\mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}$. First of all, we recall the definition of the composition product of tensor products of kernel distributions, such as that given in [5, Chap. I] and [6, p. 181]. For a more general algebraic composition product of tensor products of finite families of bimodules, see [6, pp. 100–103] and [7].

Let X^n (resp. Y^n) be a topological vector space isomorphic with the Euclidean space R^n, $n \geqq 1$. Let (cf. [2, Chap. II, § 2]) $\mathscr{D}_{(+\Gamma_x)}$ (resp. $\mathscr{D}_{(-\Gamma_y)}$) be the locally convex space of indefinitely differentiable functions with support limited to the left (resp. to the right), for $x \in X^n$ (resp. $y \in Y^n$). Let (cf. [2, § 2, no. 2]) $\mathscr{D}'_{(-\Gamma_x)}$ (resp. $\mathscr{D}'_{(+\Gamma_y)}$) be the strong dual of $\mathscr{D}_{(+\Gamma_x)}$ (resp. $\mathscr{D}_{(-\Gamma_y)}$). On the other hand, let (cf. [8, § 2, pp. 4–7]) $\mathscr{D}_{(+\Gamma_x)(-\Gamma_y)}$ be the locally convex space of indefinitely differentiable functions with support limited to the left for $x \in X^n$ and with support limited to the right for $y \in Y^n$. Let (cf. [8, § 4, pp. 7–9]) $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_y)}$ be the topological dual of $\mathscr{D}_{(+\Gamma_x)(-\Gamma_y)}$.

$\mathscr{D}'_{(-\Gamma_x)(+\Gamma_y)}$ is the locally convex space of distributions with support limited to the right for $x \in X^n$ and to the left for $y \in Y^n$. We have (cf. [8, Thm. 1, § 3, no. 2])

$$\mathscr{D}'_{(-\Gamma_x)(+\Gamma_y)} = \mathscr{D}'_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)} \quad \text{(kernel theorem)}.$$

The locally convex space $\mathscr{D}_{(-\Gamma_x)}$ may be considered as a subspace of $\mathscr{D}'_{(-\Gamma_x)}$, endowed

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with the topology induced by the latter. Then,

$$E_{xy} = \mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)} \subset \mathscr{D}'_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}$$

is a composition algebra, in which the composition $(S, T) \to S \circ T$ is the bilinear operation of

$$(\mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}) \times (\mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}) \quad \text{into} \quad \mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_\xi)}.$$

defined as follows: for $S(x, \xi) \in \mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_\xi)}$ and $T(\xi, y) \in \mathscr{D}_{(-\Gamma_\xi)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}$, set

$$(S \circ T)(x, y) = \langle S(x, \xi), T(\xi, y)\rangle = \int_{\Xi^n} S(x, \xi)T(\xi, y)\, d\xi. \tag{1.1}$$

On the other hand (cf. [9, § 5, pp. 848–849]), if we take $y \leqq \xi \leqq x \Leftrightarrow y_j \leqq \xi_j \leqq x_j$ for all $j \in [1, n] \in N$, we obtain

$$(S \circ T)(x, y) = \int_y^x S(x, \xi)T(\xi, y)\, d\xi \in \mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}. \tag{1.2}$$

The relation (1.2) shows that the operation of composition in $\mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(+\Gamma_y)}$ is an extension of the composition product of the *first kind*, as given in the theory of Volterra's integral equations. More precisely, (1.2) is a composition product *with variable limits of integration.*

In another connection, the composition product in $\mathscr{D}_{-x} \hat{\otimes} \mathscr{D}'_y$, such as defined in [6, p. 181], is an extension of the composition product of the *second kind*, such as defined in the theory of Fredholm's integral equations (operation of composition with *fixed limits of integration*).

On the general theory of topological composition algebras, see [10].

Remark. For $x \in X^n$, $y \in Y^n$, let $\delta(x - y)$ be the Dirac kernel. On the properties of $\delta(x - y)$, see Schwartz [5, Chap. I, § 4, pp. 102–105]. In particular, it is known that $\delta(x - y)$ belongs to $\mathscr{D}_x \hat{\otimes} \mathscr{D}'_y$, $\mathscr{D}'_x \hat{\otimes} \mathscr{D}_y$, $\mathscr{S}_x \hat{\otimes} \mathscr{S}'_y$, $\mathscr{S}'_x \hat{\otimes} \mathscr{S}_y$, $\mathscr{E}'_x \hat{\otimes} \mathscr{E}_y$, $\mathscr{E}_x \hat{\otimes} \mathscr{E}'_y$, $\mathscr{D}_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}'_{(-\Gamma_y)}$ and $\mathscr{D}'_{(-\Gamma_x)} \hat{\otimes} \mathscr{D}_{(+\Gamma_y)}$. Moreover, $\delta(x - y)$ is the unit element for the composition product, i.e., $\delta(x - y) \circ S = S \circ \delta(x - y) = S$ for all $S \in \mathscr{D}'_{xy}$. Indeed, $S \in \mathscr{D}'_{xy} = \mathscr{D}'_x \hat{\otimes} \mathscr{D}'_y$ and $\delta(x - y)\mathscr{D}'_x \hat{\otimes} \mathscr{D}_y \Rightarrow \delta(x - y) \circ S(x, y) = \langle \delta(x - \xi), S(\xi, y)_\xi \rangle$ has meaning.

Indeed, $\delta(x - \xi)$ is the identity distribution kernel $\mathscr{I}(\hat{x}, \xi) \in \mathscr{D}'_x \hat{\otimes} \mathscr{D}_\xi$ (cf. [5, Chap. I, pp. 102–103]), with $\mathscr{I} \in \mathscr{L}(\mathscr{D}', \mathscr{D}') = \mathscr{D} \in \mathscr{D}'$, such that

$$\mathscr{I} \circ S = \int_{X^n} \mathscr{I}(x, \xi)S(\xi, y)\, d\xi = S(x, y).$$

Likewise, $\mathscr{I}(\xi, y) \in \mathscr{D}_\xi \hat{\otimes} \mathscr{D}'_y$ implies $S \circ \delta(x - y) = S$.

2. The topological composition algebras $L^p_n(L^q_y)$, $1 \leqq p < \infty$, $1 \leqq q < \infty$, and $\mathscr{C}^{l,m}_{xy}$.

2.1. The spaces $(L^p_{\text{loc}})_x$ and $(L^p_{\text{loc}})_y$. Let X^n (resp. Y^n) be a topological space isomorphic with the Euclidean space $\mathbb{R}^n$, $n \geqq 1$. For $1 \leqq p < \infty$, $1 \leqq q < \infty$, $1/p + 1/q = 1$, let $(L^p_{\text{loc}})_x$ (resp. $(L^q_{\text{loc}})_y$) be the vector space of classes of functions whose pth (resp. qth) powers are locally integrable with respect to the Lebesgue measure on X^n (resp. Y^n). Provided with the topology defined by the sequence of norms

$$P_\nu(f) = \left(\int_{K_\nu} |f(x)|^p\,dx\right)^{1/p} \tag{2.1}$$

$$\left(\text{resp. } Q_\nu(g) = \left(\int_{L_\nu} |g(y)|^q\,dy\right)^{1/q}\right),$$

in which $(K_\nu)_{\nu\in\mathbb{N}}$ (resp. $(L_\nu)_{\nu\in\mathbb{N}}$) is a covering of X^n (resp. Y^n) with compact subsets, such that

$$K_\nu \subset K^0_{\nu+1} \text{ (resp. } L_\nu \subset L^0_{\nu+1}) \quad \text{for all } \nu \in \mathbb{N},$$

it is easy to show that $(L^p_{\text{loc}})_x$ (resp. $(L^p_{\text{loc}})_y$) is a Fréchet space.

Consider, on the other hand, the vector space of (classes of) functions $f(x, y)$ with pth locally integrable powers with respect to Lebesgue measure on X^n, for each fixed $y \in Y^n$, and with qth locally integrable powers with respect to Lebesgue measure on Y^n, for each fixed $x \in X^n$. One can consider $f(x, y)$ as a vector-valued function of X^n with values in $(L^q_{\text{loc}})_y$, i.e., as an element of the vector space $(L^p_{\text{loc}})_x\,((L^p_{\text{loc}})_y)$.

For each compact K_ν (resp. L_ν) of X^n (resp. Y^n), we denote by $L^p(K_\nu)$ (resp. $L^q(L_\nu)$) the vector space of restrictions of elements of $(L^p_{\text{loc}})_x$ (resp. $(L^q_{\text{loc}})_y$) to K_ν (resp. L_ν). Under these conditions, the vector space $(L^p_{\text{loc}})_x\,((L^q_{\text{loc}})_y) = (L^p_{\text{loc}})_y\,((L^p_{\text{loc}})_x)$ up to isomorphism, provided with the topology defined by the sequence of norms

$$\begin{aligned} P_\nu(f) = \|f\|_\nu &= \left(\int_{K_\nu} \|f(x,y)\|^p_{L^q_y}\,dx\right)^{1/p} = \left(\int_{L_\nu} \|f(x,y)\|^{q}_{L^p_x} dy\right)^{1/q} \\ &= \left(\int_{K_\nu}\left(\int_{L_\nu} |f(x,y)|^q\,dy\right)^{p/q} dx\right)^{1/p} = \|f\|_\nu \\ &= \left(\int_{L_\nu}\left(\int_{K_\nu} |f(x,y)|^p\,dx\right)^{q/p} dy\right)^{1/q}, \end{aligned} \tag{2.2}$$

is a Fréchet space.

Remark 2.1. It is known that the Banach space $L^p(K_\nu) \hat{\otimes}_\pi L^q(K_\nu)$ is a vector subspace of the Banach space $L^p(K_\nu)(L^q(L_\nu))$, and that the topology induced by $L^p(K_\nu)(L^q(L_\nu))$ is weaker than the projective topology. For $p = 1$, one has

$$L^1(K_\nu) \hat{\otimes} L^\infty(L_\nu) = L^1(K_\nu)(L^\infty(L_\nu)),$$

algebraically and topologically, and in this case, for each $f \in L^1(K_\nu) \hat{\otimes}_\pi L^\infty(L_\nu)$ we have

$$\|f\| = \int_{K_\nu} \|f(x,y)\|_\infty\,dx. \tag{2.3}$$

2.2. The topological composition algebras $L^p(K_\nu)(L^q(K_\nu))$, $\nu \in \mathbb{N}$. For each pair f, g of elements of $(L^p_{\text{loc}})_x((L^q_{\text{loc}})_y)$ and each $\nu \in \mathbb{N}$, let $f \circ g$ be the composition product in $L^p(K_\nu)(L^q(K_\nu))$ defined by

$$f \circ g = \int_{K_\nu} f(x,\xi)g(\xi,y)\,d\xi.$$

PROPOSITION 2.1. *The Banach space $L^p(K_\nu)(L^q(K_\nu))$ is a Banach algebra of composition, for each compact K_ν of the covering of $\mathbb{R}^n$.*

Proof. First of all, we shall prove that $f \circ g$ belongs also to $L^p(K_\nu)(L^q(K_\nu))$. Indeed, from (2.2) we get, for each fixed $(x, y) \in X^n \times Y^n$,

$$\begin{aligned}|f \circ g| &\leqq \int_{K_\nu} |f(x,\xi)||g(\xi,y)|\,d\xi \\ &\leqq \left(\int_{K_\nu} |f(x,\xi)|^q\,d\xi\right)^{1/q}\left(\int_{K_\nu} |g(\xi,y)|^p\,d\xi\right)^{1/p}\end{aligned} \tag{2.4}$$

by virtue of Hölder's inequality, whence

$$|f \circ g|^q \leqq \left(\int_{K_\nu} |f(x,\eta)|^q\,d\eta\right)\left(\int_{K_\nu} |g(\xi,y)|^p\,d\xi\right)^{q/p}$$

and

$$\left(\int_{K_\nu} |f \circ g|^q\,dy\right)^{1/q} \leqq \left(\int_{K_\nu} |f(x,\xi)|^q\,d\xi\right)^{1/q}\left(\int_{K_\nu}\left(\int_{K_\nu} |g(\xi,y)|^p\,d\xi\right)^{q/p} dy\right)^{1/q}. \tag{2.5}$$

But

$$\left(\int_{K_\nu}\left(\int_{K_\nu} |g(\xi,y)|^p\,d\xi\right)^{q/p} dy\right)^{1/q} = \|g\|_\nu$$

by virtue of (2.2). Therefore,

$$\left(\int_{K_\nu}\left(\int_{K_\nu} |f \circ g|^q\,dy\right)^{p/q} dx\right)^{1/p} \leqq \left(\int_{K_\nu}\left(\int_{K_\nu} |f(x,\eta)|^q\,d\eta\right)^{p/q} dx\right)^{1/p} \|g\|_\nu. \tag{2.6}$$

But by virtue of (2.2) we have $(\int_{K_\nu} (\int_{K_\nu} |f(x,\eta)|^q\,d\eta)^{p/q})^{1/p} = \|f\|_\nu$, whence

$$\|f \circ g\|_\nu \leqq \|f\|_\nu \cdot \|g\|_\nu. \tag{2.7}$$

Inequality (2.5) shows that $(f \circ g)(x, y)$ belongs to $L^q(K_\nu)$ for each fixed $x \in X^n$. Likewise, one can prove that $(f \circ g)(x, y)$ belongs to $L^p(K_\nu)$ for each fixed $y \in Y^n$. Therefore $f \circ g$ belongs to $L^p(K_\nu)(L^q(K_\nu))$.

On the other hand, $L^p(K_\nu)((L^q(K_\nu))$ is a Banach composition algebra by virtue of (2.7).

2.3. The topological composition algebra $\mathscr{C}_{xy}^{(l,m)}$. Suppose $n = 2$, X and Y are topological spaces isomorphic with $\mathbb{R}$ and $(K_\nu)_{\nu\in\mathbb{N}}$ (resp. $(L_\nu)_{\nu\in\mathbb{N}}$) is a covering of X (resp. Y) with compact subsets such that $K_\nu \subset \mathring{K}_{\nu+1}$) resp. $L_\nu \subset \mathring{L}_{\nu+1}$) for any $\nu \in N$. Let $\mathscr{C}_{xy}^{(l,m)}$ be the vector space of real functions continuously differentiable of order $\leqq l$ with respect to $x \in X$, and of order $\leqq m$ with respect to $y \in Y$.

$\mathscr{C}_{xy}^{(l,m)}$ provided with the topology defined by the sequence of norms

$$P_\nu(f) = \sup_{\substack{s\leqq l\\ t\leqq m}}\left|\frac{\partial^{s+t}f(x,y)}{\partial X^s\,\partial y^t}\right| = \|f\|_\nu, \qquad \nu \in \mathbb{N}, \qquad (x,y) \in (K_\nu) \times (L_\nu), \tag{2.8}$$

is a Fréchet space.

PROPOSITION 2.2. *The locally convex space $\mathscr{C}_{xy}^{(l,m)}$ provided with a second internal operation defined by the composition product*

$$(f \circ g)(x, y) = \int_y^X f(x, \xi) g(\xi, y)\, d\xi \tag{2.9}$$

is a topological composition algebra.

Proof. We have

$$\begin{aligned} \frac{\partial^{s+t}(f \circ g)}{\partial x^s\, \partial y^t} = {} & \int_y^x f_{x^s}^{(s)}(x, \xi) g_{y^t}^{(t)}(\xi, y)\, d\xi + \frac{\partial^{s-1}}{\partial x^{s-1}}[f(x, x) g_{y^t}^{(t)}(x, y)] \\ & + \frac{\partial^{s-2}}{\partial x^{s-2}}[f'_x(x, \xi)|_{\xi = x} g_{y^t}^{(t)}(x, y)] + \cdots \\ & + \frac{\partial}{\partial x}[f_{x^{(s-2)}}^{(s-2)}(x, \xi)|_{\xi = x} g_{y^t}^{(t)}(x, y)] + f_{x^{s-1}}^{(s-1)}(x, \xi)|_{\xi = x} g_{y^t}^{(t)}(x, y) \\ & - \frac{\partial^{t-1}}{\partial y^{t-1}}[f_{x^s}^{(s)}(x, \xi)|_{\xi = y} g(y, y)] - \frac{\partial^{t-2}}{\partial y^{t-2}}[f_{x^s}^{(s)}(x, \xi)|_{\xi = y} g_y'^{(\xi - y)}|_{\xi = y}] \\ & - \cdots - \frac{\partial^2}{\partial y^2}[f_{x^s}^{(s)}(x, s)|_{\xi = y} g_{y^{t-3}}^{(t-3)}(\xi, y)|_{\xi = y}] \\ & - \frac{\partial}{\partial y} f_{x^s}^{(s)}(x, \xi)|_{\xi = y} g_{y^{t-2}}^{(t-2)}(\xi, y)|_{\xi = y} - f_{x^s}^{(s)}(x, \xi)|_{\xi = y} g_{y^{t-1}}^{(t-1)}(\xi, y)|_{\xi = y}, \end{aligned}$$

whence

$$\|f \circ g\|_v = \sup_{\substack{s \leqq l \\ t \leqq m}} \left| \frac{\partial^{s+t}(f \circ g)}{\partial x^s\, \partial y^t} \right| \leqq M \|f\|_v \cdot \|g\|_v, \qquad (x, y) \in K_v \times L_v,$$

where M depends on l and m.

Therefore the bilinear mapping $(f, g) \mapsto f \circ g$ of $\mathscr{C}_{xy}^{(l,m)} \times \mathscr{C}_{y}^{(l,m)}$ into $\mathscr{C}_{xy}^{(l,m)}$ defined by the composition product (2.9) is continuous (cf. [11, Chap. II, p. 1, no. 4, Prop. 4]).

3. Composition product of kernel functions.

3.1. Preliminaries. Let $\mathscr{H}_{xy} = (L_{\text{loc}}^0)_x ((L_{\text{loc}}^\infty)_y)$ be the vector space of real functions $f(x, y)$ in $X \times Y$, such that for each fixed $y \in Y$, f is locally integrable with respect to Lebesgue measure on X, and for each fixed $x \in X$, f is a locally bounded and measurable function of $y \in Y$, where X and Y are one-dimensional. Then, for f, g arbitrary elements of $\mathscr{H}_{xy}$, the composition product

$$f \circ g = \int_y^x f(x, \xi) g(\xi, y)\, d\xi$$

has meaning for $-\infty < a \leqq y \leqq \xi \leqq x \leqq b < \infty$ (cf. § 2).

3.2. Heaviside kernel.

DEFINITION 3.1. We call Heaviside's kernel in $X \times Y$ the function $Y(x-y)$ given by

$$Y(x-y) = \begin{cases} 1 & \text{for } x \geqq y, \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

3.3. Kernel functions in $\mathscr{D}'_{(+\Gamma_x)(-\Gamma_y)}$. For f an arbitrary element of $\mathscr{H}_{xy}$, set

$$\{f\} = Y(x-y)f(x,y) = \begin{cases} f(x,y) & \text{for } x \geqq y, \\ 0 & \text{elsewhere.} \end{cases}$$

Under this hypothesis, for each fixed $x \in X$, $\{f\}$ is a distribution in y, an element of $\mathscr{D}'_{(-\Gamma_y)}$, and for each fixed $y \in Y$, $\{f\}$ is an element of $\mathscr{D}'_{(+\Gamma_x)}$. In the sequel the elements $\{f\}$ will be called *kernel functions.*

Now, we shall prove that for $x \in X$, $y \in Y$ arbitrary elements such that $x \geqq y$, $\{f\}$ is a distribution of $\mathscr{D}'_{(+\Gamma_x)(-\Gamma_y)}$. First of all, let us prove that $\{f\}$ is an element of $\mathscr{D}'_{(+\Gamma_x)}(\mathscr{D}'_{(-\Gamma_y)})$, i.e., a vector-valued function with values in $\mathscr{D}'_{(-\Gamma_y)}$ (cf. [8, § 3, no. 1, Def. 1]). Indeed, by virtue of kernel distribution theory (cf. [8, § 3]), for $\phi = \mathscr{D}_{(-\Gamma_x)}$ and $\Psi \in \mathscr{D}_{(+\Gamma_y)}$, such that $\operatorname{supp}\phi \subset\,]-\infty, b]$ and $\operatorname{supp}\Psi \subset [a, \infty[$, we have, using Fubini's theorem:

$$\begin{aligned} \langle \{f\}, \phi(x) \otimes \Psi(y)\rangle &= \iint Y(x-y)f(x,y)\phi(x)\Psi(y)\,dx\,dy \\ &= \int \Psi(y)\,dy \int Y(x-y)f(x,y)\phi(x)\,dx \\ &= \int \phi(x)\,dx \int Y(x-y)f(x,y)\Psi(y)\,dy, \end{aligned}$$

i.e.,

$$\begin{aligned} \langle \{f\}, \phi(x) \otimes \psi(y)\rangle &= \int_a^b \psi(y) \int_y^b f(x,y)\phi(x)\,dx \\ &= \int_a^b \phi(x)\,dx \int_a^x f(x,y)\psi(y)\,dy \end{aligned}$$

since the support conditions give us the inequalities $a \leqq y \leqq x \leqq b$. Therefore $\{f\} \in \mathscr{D}'_{(+\Gamma_x)}(\mathscr{D}'_{(-\Gamma_y)})$ (cf. [8, § 3, no. 3, p. 13]).

On the other hand, we have

$$\mathscr{D}'_{(+\Gamma_x)}(\mathscr{D}'_{(-\Gamma_y)}) = \mathscr{D}'_{(+\Gamma_x)} \hat{\otimes}\, \mathscr{D}'_{(-\Gamma_y)} = \mathscr{D}'_{(+\Gamma_x)(-\Gamma_y)}$$

by virtue of the kernel theorem (cf. [8, § 3, no. 4, Thm. 1]). Hence,

$$\{f\} \in \mathscr{D}'_{(+\Gamma_x)} \hat{\otimes}\, \mathscr{D}'_{(-\Gamma_y)} = \mathscr{D}'_{\Gamma(+x)(-\Gamma_y)}.$$

3.4. Composition product of kernel functions. For each pair (f, g) of arbitrary elements of $\mathscr{H}_{xy}$, the composition product of the kernel distributions $\{f\}$, $\{g\}$ is given by

$$\begin{aligned}\{f\}\circ\{g\} &= \langle Y(x-\xi)f(x,\xi),\, Y(\xi-y)g(\xi,y)_\xi\rangle \\ &= \left\{\int_{\mathbb{R}} Y(x-\xi)f(x,\xi)Y(\xi-y)g(\xi,y)\,d\xi\right\} \\ &= \left\{\int_y^x f(x,\xi)g(\xi,y)\,d\xi\right\} \\ &= \begin{cases}\int_y^x f(x,\xi)g(\xi,y)\,d\xi & \text{for } y\leqq\xi\leqq x,\\ 0 & \text{elsewhere},\end{cases}\end{aligned}$$

where $\int_y^x f(x,\xi)g(\xi,y)\,d\xi = f\circ g$ is the composition product in $\mathscr{H}_{xy}$ (cf. no. 1).

Remark 3.1. In the same manner one can define the kernel functions of the elements of $(L^p_{\text{loc}})_x((L^p_{\text{loc}})_y)$ and $\mathscr{C}^{(l,m)}_{xy}$, and also the composition product of these elements.

3.5. Derivatives of kernel functions in $\mathscr{C}^{(l,m)}_{x\xi}$. For all $f\in\mathscr{C}^{l,m}_{x\xi}$, the kernel functions $\{f\} = Y(x-\xi)f(x,\xi)$ give us

$$\frac{\partial\{f\}}{\partial x} = \delta(x-\xi)f(\xi,\xi) + Y(x-\xi)\frac{\partial f}{\partial x} = \left\{\frac{\partial f}{\partial x}\right\} + \delta(x-\xi)f(\xi,\xi), \tag{3.2}$$

where

$$\left\{\frac{\partial f}{\partial x}\right\} = Y(x-\xi)\frac{\partial f}{\partial x} \tag{3.3}$$

is the kernel function of $\partial f/\partial x$, considered as a distribution in $x\in X$, for $x\geqq\xi$. Likewise,

$$\frac{\partial\{f\}}{\partial\xi} = \left\{\frac{\partial f}{\partial\xi}\right\} - \delta(x-\xi)f(x,x), \tag{3.4}$$

in which $\{\partial f/\partial\xi)$ is the kernel function of $\partial f/\partial\xi$, considered as a distribution in $\xi\in\Xi$, for $\xi\leqq x$. Then, from (3.2) we obtain for the derivative of order $\nu\leqq 1$:

$$\frac{\partial^\nu\{f\}}{\partial x^\nu} = \left\{\frac{\partial^\nu f}{\partial x^\nu}\right\} + \sum_{k=0}^{\nu-1}\delta^{(k)}(x-\xi)\left.\frac{\partial^{\nu-k-1}f(x,\xi)}{\partial x^{\nu-k-1}}\right|_{x=\xi}, \tag{3.5}$$

for $\{f\}$ considered as a distribution in $x\in X$, for $x\geqq\xi$.

3.6. Composition powers of $Y(x-\xi)$. We have

$$[Y(x-\xi)^2]^\circ = \int Y(x-y)Y(y-\xi)\,dy = \left\{\frac{x-\xi}{1!}\right\} = Y(x-\xi)\circ Y(x-\xi), \tag{3.6}$$

whence, by recurrence on the powers of composition,

$$[Y(x-\xi)^\nu]^\circ = \frac{(x-\xi)^{\nu-1}}{(\nu-1)!}\quad\text{for all } \nu\in\mathbb{N}^* = \{1,2,3,\cdots\}. \tag{3.7}$$

3.7. Primitives of $\{f(x, \xi)\}$. We have

$$\begin{aligned}(3.8)\qquad Y(x-\xi)\circ\{f\} &= \left\{\int_\xi^x f(\eta,\xi)\,d\eta\right\}\\ &= \text{primitive of } \{f\} \text{ with respect to } x\end{aligned}$$

and

$$\begin{aligned}(3.9)\qquad \{f\}\circ Y(x-\xi) &= \left\{\int_\xi^x f(x,\eta)\,d\eta\right\}\\ &= \text{primitive of } \{f\} \text{ with respect to } \xi.\end{aligned}$$

3.8. Unit element of composition of kernel functions. We have

$$(3.10)\qquad \delta(x-\xi)\circ\{f\} = \left\{\int_\xi^x \delta(x-\eta)f(\eta,\xi)\,d\eta\right\} = \{f(x,\xi)\}$$

and

$$(3.11)\qquad \{f\}\circ\delta(x-\xi) = \left\{\int_\xi^x f(x,\eta)\delta(\eta-\xi)\,d\eta\right\} = \{f(x,\xi)\}.$$

Therefore, $\delta(x-\xi)$ is the unit element of the composition for elements of $\mathscr{C}_{x\xi}^{(l,m)}$ (resp. $\mathscr{H}_{xy}$) considered as kernel functions.

PROPOSITION 3.1. *We have*

$$(3.12)\qquad \delta_x'(x-\xi)\circ\{Y(x-\xi)\} = \{Y(x-\xi)\}\circ\delta_x'(x-\xi) = \{\delta(x-\xi)\}.$$

Indeed,

$$\begin{aligned}\delta_x'(x-\xi)\circ\{Y(x-\xi)\} &= \left\{\int_\xi^\infty \delta_x'(x-\eta)\,d\eta\right\}\\ &= \left\{\int_\xi^\infty -\frac{\partial\delta(x-\eta)}{\partial\eta}\,d\eta\right\} = \{\delta(x-\xi)\}.\end{aligned}$$

Likewise

$$\begin{aligned}\{Y(x-\xi)\}\circ\{\delta_x'(x-\xi)\} &= \left\{\int Y(x-\eta)\delta_\eta'(\eta-\xi)\,d\eta\right\}\\ &= \left\{\int_{-\infty}^x \delta_\eta'(\eta-\xi)\,d\eta\right\} = \{\delta(x-\xi)\}.\end{aligned}$$

Therefore, we can write by definition

$$(3.13)\qquad [\delta'(x-\xi)^{-1}]^\circ = \{Y(x-\xi)\} \quad\text{and}\quad \{\delta_x'(x-\xi)\} = [\{Y(x-\xi)\}^{-1}]^\circ.$$

Thus, by definition, the general formulas

$$(3.14)\qquad \begin{aligned}\{\delta_x'(x-\xi)\}^{-\nu} = [\{Y(x-\xi)\}^{-1}]^\circ = \{\delta^{(-\nu)}(x-\xi)\} \quad\text{and}\\ [\{\delta_x'(x-\xi)\}^\nu]^\circ = [\{Y(x-\xi)\}^{(-\nu)}]^\circ.\end{aligned}$$

3.9. Derivatives of the composition product. We have

$$\delta'_x(x-\xi)\circ\{f\} = \frac{\partial\{f\}}{\partial x} \quad\text{and}\quad \{f\}\circ\delta'_x(x-\xi) = -\frac{\partial\{f\}}{\partial\xi}. \tag{3.15}$$

Indeed,

$$\begin{aligned}\delta'_x(x-\xi)\circ\{f\} &= \left\{\int_\xi^\infty \delta'_x(x-\eta)f(\eta,\xi)\,d\eta\right\}\\ &= \left\{\int_\xi^\infty -\frac{\partial}{\partial\eta}\delta(x-\eta)f(\eta,\xi)\,d\eta\right\}\\ &= \{\delta(x-\xi)f(\xi,\xi)\} + \left\{\frac{\partial f}{\partial x}\right\},\end{aligned}$$

whence, by virtue of (3.2),

$$\delta'_x(x-\xi)\circ\{f\} = \frac{\partial\{f\}}{\partial x}.$$

Likewise,

$$\begin{aligned}\{f\}\circ\delta'_x(x-\xi) &= \left\{\int f(x,\eta)\delta'_\eta(\eta-\xi)\,d\eta\right\}\\ &= \left\{\delta(x-\xi)f(x,x) - \frac{\partial f(x,\xi)}{\partial\xi}\right\} = -\frac{\partial\{f\}}{\partial\xi}\end{aligned}$$

by virtue of (3.4).

From (3.15), we get

$$\begin{aligned}&\delta_x^{(k)}(x-\xi)\circ\{f\} = \frac{\partial^k}{\partial x^k}\{f\},\\ &\{f\}\circ\delta_{x^k}^{(k)}(x-\xi) = (-1)^k\frac{\partial^k\{f\}}{\partial\xi^k},\\ &\delta_{x^k}^{(k)}(x-\xi)\circ\{f\}\circ\delta_{x^k}^{(l)}(x-\xi) = (-1)^l\frac{\partial^{k+l}\{f\}}{\partial x^k\,\partial\xi^l}.\end{aligned} \tag{3.16}$$

Remark. If f is a function of the single variable x, we set

$$\{f\} = Y(x-\xi)f(x)\otimes 1_\xi = \{f(x)\otimes 1_\xi\}, \tag{3.17}$$

in which 1_ξ is the constant function of $\xi\in\Xi$ equal to 1. Then

$$\{f\}\circ\delta'_x(x-\xi) = -\frac{\partial}{\partial\xi}\{f\otimes 1_\xi\} = \{0\}.$$

Likewise, if f is a function of the single variable ξ, we set

$$\{f\} = \{1_x\otimes f(\xi)\}, \quad\text{where } 1_x(x) = 1 \quad\text{for each } x, \tag{3.18}$$

whence

$$\delta'_x(x-\xi)\circ\{f\} = \frac{\partial}{\partial x}\{1_x \times f(\xi)\} = \{0\}.$$

PROPOSITION 3.2. *We have*

$$\delta'_x(x-\xi)\circ\{f\}\circ\{g\} = \frac{\partial\{f\}}{\partial x}\circ\{g\}, \tag{3.19}$$

$$\{f\}\circ\delta'_x(x-\xi)\circ\{g\} = \{f\}\circ\frac{\partial\{g\}}{\partial x} = -\frac{\partial\{f\}}{\partial\xi}\circ\{g\} \tag{3.20}$$

and

$$\{f\}\circ\{g\}\circ\delta'_x(x-\xi) = -\{f\}\circ\frac{\partial\{g\}}{\partial\xi}. \tag{3.21}$$

These formulas are an immediate consequence of the formulas (3.15) and of the associativity of the composition product. More generally, we have

$$\begin{aligned}
&\delta^{(k)}_{x^k}(x-\xi)\circ\{f\}\circ\{g\} = \frac{\partial^{(k)}\{f\}}{\partial x^k}\circ\{g\},\\
&\{f\}\circ\delta^{(k)}_{x^k}(x-\xi)\circ\{g\} = \{f\}\circ\frac{\partial^k\{g\}}{\partial x^k} = (-1)^k\frac{\partial^k\{f\}}{\partial\xi^k}\circ\{g\},\\
&\{f\}\circ\{g\}\circ\delta^{(k)}_{x^k}(x-\xi) = (-1)^k\{f\}\circ\frac{\partial^k\{g\}}{\partial\xi^k}.
\end{aligned} \tag{3.22}$$

3.10. Operators of multiplication in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$.

DEFINITION. Let $a(x)$ be an indefinitely differentiable function of $x\in X$. For each $S(x,\xi)\in\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$ we say that $a(x)$ is a multiplication operator, if we have

$$a(x)\{S(x,\xi)\} = \{a(x)S(x,\xi)\}\in\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}.$$

PROPOSITION 3.3. *Let $a(x)$ be a multiplication operator in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$. Then*

$$\{Y(\mathring{x}-\xi)\}^{(k)}\circ\{a(x)(\delta^{(j)}_{x^j}(x-\xi))\} = (-1)^j\frac{\partial^j}{\partial\xi^j}\,\frac{(x-\xi)^{k-1}}{(k-1)!}a(\xi)\ . \tag{3.23}$$

Proof. We have

$$\begin{aligned}
[Y(\mathring{x}-\xi)^k]^\circ\{a(x)\delta^{(j)}_{x^j}(x-\xi)\} &= \left\{\int_\xi^x\frac{(x-z)^{k-1}}{(k-1)!}a(z)(\delta^{(j)}_{x^j}(z-\xi))\,dz\right\}\\
&= \left\{\frac{(x-\xi)^{k-1}}{(k-1)!}a(\xi)\right\}\circ\delta^{(j)}_{x^j}(x-\xi).
\end{aligned}$$

On the other hand, keeping in mind the formulas (3.22), we obtain

$$\left\{\frac{(x-\xi)^{k-1}}{(k-1)!}a(\xi)\right\}\circ\delta^{(j)}_{x^j}(x-\xi) = (-1)^j\frac{\partial^j}{\partial\xi^j}\left\{\frac{(x-\xi)^{k-1}}{(k-1)!}a(\xi)\right\},$$

whence (3.23). The formula (3.23) is fundamental, because it leads to the following important theorem.

THEOREM 3.1. *Each linear differential operator, whose coefficients are functions of a variable, may be transformed in a composition product in* $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$.

Proof. Let Ω be a linear differential operator of order $n \geqq 1$, acting on an element $S(x, \xi) \in \mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$, defined by

$$\Omega(S) = \sum_{j=0}^{n} a_{n-j}(x) \frac{\partial^{(j)} S(x, \xi)}{\partial x^j}, \tag{3.24}$$

where the $a_j(x)$, $j \in [0, n] \in N$, are multiplication operators in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$. By virtue of (3.22), we can write

$$\Omega(S) = \sum_{j=0}^{n} a_{n-j}(x)(\delta^{(j)}_{x^j}(x - \xi) \circ S(x, \xi)). \tag{3.25}$$

On the other hand, we have

$$\{a_{n-j}(x)\delta^{(j)}_{x^j}(x - \xi)\} \circ S(x, \xi) \in \mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)} \quad \text{for all } j \in [0, n] \subset N.$$

Suppose $a_0(x) \equiv 1$. Then, we can write

$$\Omega(S) = \{\delta^{(n)}_{x^n}(x - \xi)\} \circ S(x, \xi) + \left\{ \sum_{j=0}^{n-1} a_{n-j}(x)(\delta^{(j)}_{x^j}(x - \xi)) \right\} \circ S(x, \xi). \tag{3.26}$$

But $\delta^{(n)}_{x^n}(x - \xi)$ possesses an inverse in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$ given by (cf. Prop. 1, 3, no. 8, formulas (14)):

$$[\{\delta^{(n)}_{x^n}(x - \xi)\}^{-1}]^\circ = [\{\delta'_{x'}(x - \xi)\}^{(-n)}]^\circ = [\{Y(x - \xi)\}^n]^\circ. \tag{3.27}$$

Therefore, by composition to the left in (3.26) with (3.27), we get

$$\begin{aligned} &[\{Y(x - \xi)\}^n]^\circ \circ \{\Omega(S)\} \\ &\quad = S(x, \xi) + [\{Y(x - \xi)\}^n]^\circ \circ \left\{ \sum_{j=0}^{n-1} a_{n-j}(x)(\delta^{(j)}_{x^j}(x - \xi)) \right\} \circ S(x, \xi) \end{aligned} \tag{3.28}$$

whence, by virtue of (3.23), the following fundamental formula is obtained:

$$\begin{aligned} &[\{Y(x - \xi)\}^n]^\circ \circ \{\Omega(S)\} \\ &\quad = S(x, \xi) + \left\{ \sum_{j=0}^{n-1} (-1)^j \frac{\partial^j}{\partial \xi^j} \frac{(x - \xi)^{n-1}}{(n-1)!} a_{n-j}(\xi) \right\} \circ S(x, \xi) \\ &\quad = \left\{ \delta(x - \xi) + \sum_{j=0}^{n-1} (-1)^j \frac{\partial^j}{\partial \xi^j} \frac{(x - \xi)^{n-1}}{(n-1)!} a_{n-j}(\xi) \right\} \circ S(x, \xi) \\ &\quad = \{\delta(x - \xi) + H(x, \xi)\} \circ S(x, \xi), \end{aligned} \tag{3.29}$$

where

$$\{H(x, \xi)\} = \sum_{j=0}^{n-1} \frac{\partial^j}{\partial \xi^j} \left\{ \frac{(x - \xi)^{n-1}}{(n-1)!} a_{n-j}(\xi) \right\}. \tag{3.30}$$

4. Algebraic operational methods for solving differential equations with variable coefficients.

4.1. Fundamental kernels. Consider in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$ the equation

$$\frac{d^n\{E\}}{dx^n} + a_1(x)\frac{d^{n-1}\{E\}}{dx^{n-1}} + \cdots + a_n(x)\{E\} = \delta(x - \xi) \tag{4.1}$$

in which $(a_j(x))_{1 \leqq j \leqq n}$ are indefinitely differentiable functions of $x \in R$. Then the fundamental formula (3.29) allows us to write (4.1) as follows:

$$\{\delta(x - \xi) + H(x, \xi)\} \circ \{E\} = \left\{\frac{(x - \xi)^{n-1}}{(n - 1)!}\right\} \tag{4.2}$$

in which

$$H(x, \xi) = \sum_{k=1}^{n} (-1)^{n-k} \frac{\partial^{n-k}}{\partial \xi^{n-k}} \left\{\frac{(x - \xi)^{n-1}}{(n - 1)!} a_k(\xi)\right\}. \tag{4.3}$$

Now, let $[[\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)})^{\mathbb{N}}]]$ be the algebra of formal series,[1] the terms of which are elements of $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$. Then $\{\delta(x - \xi) + H(x, \xi)\}$ is invertible in $[[(\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)})^{\mathbb{N}}]]$; whence

$$\begin{aligned} \{E(x, \xi)\} &= \overset{\circ}{\overbrace{\{\delta(x - \xi) + H(x, \xi)\}}}^{(-1)} \circ \overset{\circ}{\overbrace{\{Y(x - \xi)\}}}^{n} \\ &= \sum_{\nu \in \mathbb{N}} (-1)^{\nu} \overset{\circ}{\overbrace{\{H(x, \xi)\}}}^{(\nu)} \circ \overset{\circ}{\overbrace{\{Y(x, \xi)\}}}^{n}, \end{aligned} \tag{4.4}$$

where the composition powers of the kernel $\{H(x, \xi)\}$ are given by

$$\begin{aligned} &\overset{\circ}{\overbrace{\{H(x, \xi)\}}}^{(0)} = \delta(x - \xi), \\ &\overset{\circ}{\overbrace{\{H(x, \xi)\}}}^{(1)} = \{H(x, \xi)\}, \\ &\overset{\circ}{\overbrace{\{H(x, \xi)\}}}^{(2)} = \left\{\int^{x} H(x, \eta)H(\eta, \xi)\, d\eta\right\} = \{H^{(2)}(x, \xi)\}, \\ &\quad\vdots \\ &\overset{\circ}{\overbrace{\{H(x, \xi)\}}}^{(\nu)} = \left\{\int_{\xi}^{x} H^{(\nu-1)}(\xi, \eta)H(\eta, \xi)\, d\eta\right\} = \{H^{(\nu)}(x, \xi)\}. \\ &\quad\vdots \end{aligned} \tag{4.5}$$

Let

$$\{\Gamma(x, \xi)\} = \sum_{\nu \in \mathbb{N}} (-1)^{\nu} \{H^{(\nu)}(x, \xi)\} \tag{4.6}$$

be the "resolvent kernel" formed with the composition powers of $\{H(x, \xi)\}$. If

[1] On the formal series and their extensions, cf. [12].

we set

$$M = \sup_{a \leqq \xi \leqq x \leqq b} |H(x, \xi)|,$$

we obtain

$$|H^{(\nu)}(x, \xi)| \leqq M^{\nu} \frac{(b-a)^{\nu-1}}{(\nu-1)!}.$$

Therefore $\{\Gamma(x, \xi)\}$ is an absolutely and uniformly convergent series on each closed finite interval of R. Then the same is true for the solution

$$\text{(4.7)} \quad \begin{aligned} \{E(x, \xi)\} &= \left\{ \sum_{\nu \in \mathbb{N}} (-1)^{\nu} \{H^{(\nu)}(x, \xi)\} \circ \{Y(x-\xi)\}^{n} \right\}^{\circ} \\ &= \left\{ \int_{\xi}^{x} \Gamma(x, y) \frac{(\eta-\xi)^{(n-1)}}{(n-1)!} \, d\eta \right\}. \end{aligned}$$

4.2. Differential equation of first order. In particular, if we consider the first order equation

$$\text{(4.8)} \quad \frac{d\{E\}}{dx} + a(x)\{E\} = \delta(x-\xi),$$

the formulas (4.5) give us: $H(x, \xi) = a(\xi) = 1_x \otimes a(\xi)$, whence

$$\begin{aligned} \{a(\mathring{\xi})\}^{(0)} &= \delta(x), \\ \{a(\mathring{\xi})\}^{(1)} &= \{a(\xi)\}, \\ \{a(\mathring{\xi})\}^{(2)} &= \left\{ \int_{\xi}^{x} a(\xi) a(\eta) \, d\eta \right\} = \left\{ a(\xi) \int_{\xi}^{x} a(\eta) \, d\eta \right\} \\ &= -\frac{1}{2!} \frac{\partial}{\partial \xi} \left\{ \left(\int_{\xi}^{x} a(\eta) \, d\eta \right)^{2} \right\} = \{a^{(2)}(x, \xi)\}, \\ \{a(\mathring{\xi})\}^{(3)} &= \left\{ \int_{\xi}^{x} a^{(2)}(x, \eta) a(\eta) \, d\eta \right\} \\ &= -\frac{1}{2!} \left\{ \int_{\xi}^{x} \frac{\partial}{\partial \eta} \left(\int_{\eta}^{x} a(\eta_1) \, d\eta_1 \right)^{2} a(\eta) \, d\eta \right\} \\ &= -\frac{1}{2!} a(\xi) \left[\left(\int_{\xi}^{\eta} a(\eta_1) \, d\eta_1 \right)^{2} \right]_{\xi}^{x} = \{a^{(3)}(x, \xi)\} \\ &= \frac{a(\xi)}{2!} \left(\int_{\xi}^{x} a(\eta) \, d\eta \right)^{2} = -\frac{1}{3!} \frac{\partial}{\partial \xi} \left(\int_{\xi}^{x} a(\eta) \, d\eta \right)^{3} \\ &\;\vdots \\ \{a(\xi)\}^{(\nu)} &= \{a^{(\nu)}(x, \xi)\} = -\frac{1}{\nu!} \frac{\partial}{\partial \xi} \left(\int_{\xi}^{x} a(\eta) \, d\eta \right)^{\nu} \quad \text{for all } \nu \in \mathbb{N}. \end{aligned}$$

On the other hand, we have

$$\left\{-\frac{\partial}{\partial\xi}\left(\int_\xi^x a(\eta)\,d\eta\right)^\nu\right\}\circ\{Y(x-\xi)\} = \left\{\left(\int_\xi^x a(\eta)\,d\eta\right)^\nu\right\}.$$

Therefore

$$\begin{aligned}\Gamma(x,\xi) &= \sum_{\nu\in\mathbb{N}}(-1)^\nu\{a(\xi)\}^{(\nu)}\\ &= \sum_{\nu\in\mathbb{N}}(-1)^\nu\frac{1}{\nu!}\left\{-\frac{\partial}{\partial\xi}\left(\int_\xi^x a(\eta)\,d\eta\right)^\nu\right\},\end{aligned}$$

whence

$$\begin{aligned}\{E(x,\xi)\} &= \{\Gamma(x,\xi)\}\circ Y(x-\xi) = \sum_{\nu\in\mathbb{N}}(-1)^\nu\frac{1}{\nu!}\left\{\left(\int_\xi^x a(\eta)\,d\eta\right)^\nu\right\}\\ &= \left\{\exp\left[-\int_\xi^x a(\eta)\,d\eta\right]\right\} = Y(x-\xi)\exp\left[-\int_\xi^x a(\eta)\,d\eta\right].\end{aligned} \tag{4.9}$$

5. Algebraic method for solving Cauchy's problem for linear differential equations with coefficients functions of an independent variable.

5.1. Statement of the problem. Consider the linear differential equation of order $n \geqq 1$:

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = f(x) \tag{5.1}$$

in which $(a_j(x))_{1\leqq j\leqq n}$, $f(x)$ are given functions satisfying some conditions of differentiability in any closed finite interval $I = [a, b]$ of the real line $\mathbb{R}$.

It is required to find a solution of (5.1) in a neighborhood of $\alpha \in I$, satisfying the following conditions (Cauchy's problem):

$$y(\alpha) = C_1,\quad y'(\alpha) = C_2,\quad \cdots,\quad y^{(n-1)}(\alpha) = C_n. \tag{5.2}$$

To do this, proceed as follows: by transfer of (5.1) into $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$, we get

$$\left\{\frac{d^n y}{dx^b}\right\} + \left\{a_1(x)\frac{d^{n-1}y}{dx^{n-1}}\right\} + \cdots + \{a_n(x)y\} = \{f\}. \tag{5.3}$$

Of course, we suppose the functions $(a_j(x))_{1\leqq j\leqq n}$ are multiplication operators in $\mathscr{D}'_{(-\Gamma_x)(+\Gamma_\xi)}$. On the other hand, the fundamental formula (3.5) gives us, for $\xi = \alpha$:

$$\left\{\frac{d^\nu y}{dx^\nu}\right\} = \frac{d^\nu\{y\}}{dx^\nu} - \sum_{k=0}^{\nu-1}\left\{\delta^{(k)}(x-\alpha)\left.\frac{d^{\nu-k-1}y}{dx^{\nu-k-1}}\right|_{x=\alpha}\right\} \tag{5.4}$$

for $0 \leqq \nu \leqq n$. By virtue of (5.2),

$$\left\{\frac{dy}{dx}\right\} = \frac{d\{y\}}{dx} - \delta(x-\alpha)C_1,$$

$$\left\{\frac{d^2y}{dx^2}\right\} = \frac{d^2\{y\}}{dx^2} - \delta(x-\alpha)C_1 - \delta'_x(x-\alpha)C_2,$$

$$\vdots$$

$$\left\{\frac{d^\nu y}{dx^\nu}\right\} = \frac{d^\nu\{y\}}{dx^\nu} - \sum_{k=0}^{\nu-1}\{\delta^{(k)}_{x^k}(x-\alpha)C_{\nu-k}\},$$

$$\left\{\frac{d^n y}{dx^n}\right\} = \frac{d^n\{y\}}{dx^n} - \sum_{k=0}^{n-1}\{\delta^{(k)}_{x^k}(x-\alpha)C_{n-k}\}.$$

Then, equation (5.3) takes the form

$$\begin{aligned}&\frac{d^n\{y\}}{dx^n} + a_1(x)\frac{d^{n-1}\{y\}}{dx^{n-1}} + \cdots + a_n(x)\{y\}\\ &\quad = \{f\} + \sum_{k=0}^{n-1}\delta^{(k)}_{x^k}(x-\alpha)C_{n-k} + a_1(x)\sum_{k=0}^{n-2}\delta^{(k)}_{x^k}(x-\alpha)C_{n-1-k}\\ &\qquad + a_2(x)\sum_{k=0}^{n-3}\delta^{(k)}_{x^k}(x-\alpha)C_{n-2-k} + \cdots + a_{n-1}(x)\delta(x-\alpha)(x-\alpha)C_1.\end{aligned} \tag{5.5}$$

For solving (5.5) it is required to find first of all the fundamental solution $\{E\}$ of the equation:

$$\frac{d^n\{E\}}{dx^n} + a_1(x)\frac{d^{n-1}\{E\}}{dx^{n-1}} + \cdots + a_n(x)\{E\} = \delta(x-\alpha) \tag{5.6}$$

whose solution, by virtue of the formula (4.7), is given by

$$\begin{aligned}\{E(x,\alpha)\} &= \sum_{\nu\in\mathbb{N}}(-1)^\nu\{H^{(\nu)}(x,\alpha)\}\circ\{Y(\mathring{x}-\alpha)\}^{(n)}\\ &= \sum_{\nu\in\mathbb{N}}(-1)^\nu\int_\alpha^x H^{(\nu)}(x,\eta)\frac{(\eta-\alpha)^{n-1}}{(\eta-1)!}\,d\eta\ .\end{aligned} \tag{5.7}$$

If we denote by $\{\phi(x,\alpha)\}$ the right-hand side of (5.5), the solution of (5.5) is given by

$$\{y\} = \{E(x,\alpha)\}\circ\{\phi(x,\alpha)\}. \tag{5.8}$$

5.2. Cauchy's problem for the first order linear differential equation. In particular, the first order linear differential equation

$$\frac{dy}{dx} + a(x)y = f(x), \tag{5.9}$$

whose solution satisfies the *Cauchy condition*

$$y(\alpha) = C_1, \tag{5.10}$$

is obtained as follows: From (5.9) and (5.10), we get the equation

$$\frac{d\{y\}}{dx} + a(x)\{y\} = \{f\} + \delta(x - \alpha)C_1. \tag{5.11}$$

The fundamental solution of (5.11) is given by (cf. formula (4.9)),

$$\{E(x,\alpha)\} = \left\{ \exp\left[-\int_\alpha^x a(\eta)\, d\eta \right] \right\} = Y(x - \alpha) \exp\left[-\int_\alpha^x a(\eta)\, d\eta \right]; \tag{5.12}$$

therefore, by (5.8) we get:

$$\begin{aligned} \{y\} &= \{E(x,\alpha)\} \circ \{f\} + \{E(x,\alpha)\} \circ \delta(x - \alpha)C_1 \\ &= \left\{ \int_\alpha^x E(x,s)f(s)\, ds \right\} + C_1\{E(x,\alpha)\} \\ &= \left\{ \int_\alpha^x \exp\left[-\int_s^x a(s_1)\, ds_1 \right] f(s)\, ds \right\} + C_1 \left\{ \exp\left[-\int_\alpha^x a(s)\, ds \right] \right\}. \end{aligned}$$

For $\alpha \leqq x$, we obtain the classical formula

$$y(x) = \int_\alpha^x \exp\left[-\int_s^x a(s_1)\, ds_1 \right] f(s)\, ds + C_1 \exp\left[-\int_\alpha^x a(s)\, ds \right], \tag{5.13}$$

in which

$$Z(x,\alpha) = \exp\left[-\int_\alpha^x a(s)\, ds \right]$$

is the solution of the homogeneous equation

$$\frac{dZ}{dx} + a(x)Z = 0$$

satisfying the Cauchy condition

$$Z(\alpha) = 1.$$

In a forthcoming paper we shall prove that the functions which multiply the coefficients $(C_j)_{1 \leqq j \leqq n}$ in (5.8) form a fundamental system of solutions of the homogeneous equation

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_n(x)y = 0, \tag{5.14}$$

whereas the function

$$Y_0(x) = E(x,\alpha) \circ f(x) = \int_\alpha^x E(x,s)f(s)\, ds \tag{5.15}$$

is a particular solution of the nonhomogeneous equation (5.1), satisfying the Cauchy conditions:

$$Y_0^{(\nu)}(\alpha) = \begin{cases} 0 & \text{for } 0 \leqq \nu \leqq n - 2, \\ 1 & \text{for } \nu \leqq n - 1. \end{cases} \tag{5.16}$$

More precisely, our algebraic methods give simultaneously a fundamental system of solution of the homogeneous equation (5.14) and a particular solution of the nonhomogeneous equation (5.1) (cf. also S. Vasilach [11], where we have obtained the same results by a different method).

Our next papers will be devoted to our algebraic method for solving partial differential equations whose coefficients are functions of several variables.

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ON A NONLINEAR INTEGRAL EQUATION FOR POPULATION GROWTH PROBLEMS*

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Abstract. We consider a nonlinear Volterra integral equation which arises in the study of population growth assuming a growth rate depending only on population size, and a probability of death depending only on age. The same type of equation also arises in the study of the spread of a disease for which recovery from the disease confers no immunity to reinfection. We obtain a result on boundedness of solutions, and a condition under which no solution can tend to zero.

1. Introduction. The study of population growth, assuming a growth rate which depends only on the size of the population and a probability of death which depends only on age, leads to a Volterra integral equation of the form

$$x(t) = f(t) + \int_0^t g(x(t - s))P(s)\, ds. \tag{1}$$

Here, $g(x)$ is the number of members added to the population in unit time when the population size is x, so that $g(x)/x$ is the rate of growth of population size in unit time per unit of population size. In many situations, for example, populations governed by a logistic equation, with $g(x) = ax - bx^2$ for $0 \leqq x \leqq a/b$, this growth rate decreases as the population increases. The function $P(t)$ represents the probability that a member of the population services to age t. Thus $g(x(t - s))P(a)\,\Delta s$ approximates the number of members added to the population between time $(t - s)$ and time $(t - s + \Delta s)$ who survive to age s at time t. The term $f(t)$ represents the number of members of the population who were already present at time $t = 0$ and who are still alive at time t.

The equation (1) may also be used to describe the number of members of a given population who are afflicted with some disease, provided that recovery from the disease confers negligible immunity (so that the growth rate depends only on the number of diseased members), and provided that the disease has a negligible incubation period (so that no delay terms appear in the integral). This model has been derived and used to study the spread of gonorrhea [2], [3].

Another problem which leads to the same equation is an economic model, where $x(t)$ represents the total value of capital of time t, where the production of new capital within the economy depends only on $x(t)$ and the rate of production is $g(x(t))$, where $P(t)$ is the value at age t of a unit of equipment (or where equipment retains its full value until breakdown and $P(t)$ is the probability of survival to age at least t), and where $f(t)$ is the value at time t of the original capital from time $t = 0$.

In studying the behavior of solutions of (1) we are interested in questions of boundedness of solutions, of whether all bounded solutions tend to limits, and what limits are possible. In this paper we give a result on boundedness of solutions under hypothesis which are appropriate for many population problems. We also

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give conditions under which solutions cannot tend to zero. This gives a global asymptotic behavior result if the growth rate $g(x)/x$ is a monotone decreasing function of x.

2. Boundedness of solutions. In studying the equation

$$x(t) = f(t) + \int_0^t g(x(t-s))P(s)\,ds, \tag{1}$$

we shall always make the following assumptions on the functions f, g, and P:

H_f. We assume that $f(t)$ is nonnegative, continuous, and of bounded variation on $0 \leqq t < \infty$, so that

$$f(\infty) = \lim_{t\to\infty} f(t) \tag{2}$$

exists.

H_P. We assume that $P(t)$ is nonnegative, monotone nonincreasing, and differentiable on $0 \leqq t < \infty$, and is normalized so that $P(0) = 1$. We also assume that

$$\int_0^\infty P(s)\,ds < \infty. \tag{3}$$

H_g. We assume that $g(0) = 0$, that $g(x)$ is continuous and nonnegative on $0 \leqq x < \infty$, and that $g'(x)$ is continuous on $0 \leqq x < \infty$.

Given $f(\infty)$, there is a set of functions $f(t)$ satisfying H_f and tending to this limit as $t \to \infty$. To each $f(t)$ in this set corresponds a unique solution $x(t)$ of (1). When we speak of the collection of all solutions of (1), we shall mean the collection of solutions $x(t)$ corresponding to some $f(t)$ in this set.

THEOREM 1. *Suppose that the hypotheses* H_f, H_P, *and* H_g *are satisfied and that*

$$\left(\int_0^\infty P(s)\,ds\right) \limsup_{x\to\infty} \frac{g(x)}{x} < 1. \tag{4}$$

Then every nonnegative solution of (1) *is bounded on* $0 \leqq t < \infty$.

Proof. Pick a sufficiently large $K > 0$, and then choose $\rho < 1$ such that

$$\left(\int_0^\infty P(s)\,ds\right) g(x) < \rho x \tag{5}$$

for $x \geqq K$, which is possible because of (4). Now define

$$I_1 = \{u \mid x(u) > K\}, \qquad I_2 = \{u \mid x(u) \leqq K\}.$$

Since $f(t)$ is of bounded variation on $0 \leqq t < \infty$, there exists $M > 0$ such that $f(t) \leqq M$ on $0 \leqq t < \infty$. Since $g(x)$ is continuous, there exists $L > 0$ such that $g(x(u)) \leqq L$ for $x(u) \leqq K$, that is, for $u \in I_2$. Thus, using (5), we obtain

$$\begin{aligned} x(t) &= f(t) + \int_0^t g(x(u))P(t-u)\,du \\ &= f(t) + \int_{I_1} g(x(u))P(t-u)\,du + \int_{I_2} g(x(u))P(t-u)\,du \end{aligned}$$

(cont.)

$$\leqq M + \frac{\rho}{\int_0^\infty P(s)\,ds}\int_{I_1} x(u)P(t-u)\,du + L\int_{I_2} P(t-u)\,du$$

$$\leqq M + \rho \sup_{u\in I_1, u\leqq t} x(u) + L\int_0^\infty P(s)\,ds$$

$$\leqq M + \rho \sup_{0\leqq u\leqq t} x(u) + L\int_0^\infty P(s)\,ds.$$

Since this is valid for every $t \geqq 0$, we have

$$\sup_{0\leqq t\leqq T} x(t) \leqq M + \rho \sup_{0\leqq u\leqq T} x(u) + L\int_0^\infty P(s)\,ds$$

for every $T > 0$, and since $\rho < 1$, we obtain

$$\sup_{0\leqq t\leqq T} x(t) \leqq \frac{M + L\int_0^\infty P(s)\,ds}{1-\rho}.$$

Since this bound is independent of T, we have

$$\sup_{0\leqq t\leqq \infty} x(t) \leqq \frac{M + L\int_0^\infty P(s)\,ds}{1-\rho},$$

and the theorem is proved.

COROLLARY. *Suppose that* H_f, H_P, *and* H_g *are satisfied and that* $\limsup_{x\to\infty} g(x)/x = 0$. *Then every nonnegative solution of* (1) *is bounded* on $0 \leqq t < \infty$.

Since the hypothesis $\limsup_{x\to\infty} g(x)/x = 0$ implies (4), the proof of the corollary is immediate.

3. Limits of solutions. It has been shown recently ([5], [6], [7], [8], [9]), that under quite general hypotheses every bounded solution of (1) tends to a limit as $t \to \infty$. For example, we quote the following result, actually established under much less stringent hypotheses.

THEOREM 2 (Londen [8]). *Suppose the conditions* H_f, H_P, *and* H_g *are satisfied. Then every bounded solution* $x(t)$ *of* (1) *satisfies*

$$\lim_{t\to\infty}\left[x(t) - g(x(t))\int_0^\infty P(s)\,ds\right] = f(\infty). \tag{6}$$

It is easy to see that if $x(t)$ is a solution of (1) with $\lim_{t\to\infty} x(t) = c$, then

$$\lim_{t\to\infty}\int_0^t g(x(t-s))P(s)\,ds = g(c)\int_0^\infty P(s)\,ds.$$

It follows that the limit c must satisfy the equation

$$c = f(\infty) + g(c)\int_0^\infty P(s)\,ds. \tag{7}$$

It follows easily from Theorem 2 that if the roots of (7) are isolated, then every bounded solution of (1) tends to a limit c which is a root of (7). In other words, the possible limits of solutions of (1) are given by the abscissae of the intersections of the curve $y = g(x)$ and the straight line $y = (x - f(\infty))/\int_0^\infty P(s)\,ds$. If $g'(c)\int_0^\infty P(s)\,ds < 1$, then the curve crosses the line from above to below (in the direction of increasing x). If $g'(c)\int_0^\infty P(s)\,ds > 1$, then the curve crosses the line from below to above in the direction of increasing x. Since by assumption $g(0) = 0$, if $f(\infty) = 0$, then $c = 0$ is a root of (7). Our main result is that if $g'(0)\int_0^\infty P(s)\,ds > 1$, then even though $c = 0$ is a root of (7), it is not a possible limit of a nontrivial solution of (1). The proof will involve linearization of (1) and examination of the solutions of the linearized equation. We make use of the following obvious generalization of a classical result on linear integral equations.

THEOREM 3 (Feller [4]). *Consider the linear integral equation*

$$z(t) = F(t) + \int_0^{t-\tau} z(t - s)a(s)\,ds \tag{8}$$

for $t \geqq \tau$, where $z(t)$, $F(t)$, and $a(t)$ are continuous and nonnegative on $0 \leqq t < \infty$ and where $\lim_{t\to\infty} F(t) = 0$. Suppose that $a(t)$ is of bounded variation and that $\int_0^\infty a(t)\,dt < \infty$.

(i) *If $\int_0^\infty a(t)\,dt = 1$, $\int_0^\infty ta(t)\,dt = m_1 < \infty$, and $\int_0^\infty t^2a(t)\,dt < \infty$, and if $\int_\tau^\infty F(t)\,dt = b < \infty$, then (8) has a unique solution $z(t)$ on $\tau \leqq t < \infty$ which is nonnegative and satisfies $\lim_{t\to\infty} z(t) = b/m_1$. In particular, if $F(t)$ is not identically zero for $\tau \leqq t < \infty$, then $\lim_{t\to\infty} z(t) \neq 0$.*

(ii) *If $\int_0^\infty a(t)\,dt > 1$, choose $\sigma > 0$ so that $\int_0^\infty e^{-\sigma t}a(t)\,dt = 1$. Then*

$$\lim_{t\to\infty} z(t)e^{-\sigma(t-\tau)} = \frac{\int_\tau^\infty e^{-\sigma(t-\tau)}F(t)\,dt}{\int_0^\infty te^{-\sigma t}a(t)\,dt}.$$

In particular, if $F(t)$ is not identically zero for $\tau \leqq t < \infty$, then $z(t)$ grows exponentially as $t \to \infty$.

To reduce this result to the case $n = 2$ of Theorem 4 of [4], we need only make the change of variable $t - \tau = u$ and define $\zeta(u) = z(u + \tau) = z(t)$, $\Phi(u) = F(u + \tau) = F(t)$ for $u \geqq 0$; then the equation (8) for $t \geqq \tau$ becomes

$$\zeta(u) = \Phi(u) + \int_0^u z(t - s)a(s)\,ds = \Phi(0) + \int_0^u \zeta(u - s)a(s)\,ds$$

for $u \geqq 0$.

We may now establish our main result.

THEOREM 4. *Suppose the hypotheses H_f, H_P, H_g are satisfied, and suppose $f(\infty) = 0$, so that $c = 0$ is a root of (7). If $g'(0)\int_0^\infty P(s)\,ds > 1$, then no solution of (1) which is not identically zero for all large t can tend to zero as $t \to \infty$.*

Proof. Choose a number α such that $g'(0) > \alpha > 1/\int_0^\infty P(s)\,ds$. Then define $\phi(x) = g(x) - \alpha x$ for $x \geqq 0$. Since $\phi'(0) = g'(0) - \alpha > 0$, $\phi(x)$ is positive for all sufficiently small $x > 0$. Suppose $x(t)$ is a solution of (1) which tends to zero as $t \to \infty$ but does not vanish identically for all large t. Then we may choose $\tau \geqq 0$ so that $x(\tau) > 0$ and $x(t)$ is small enough for $t \geqq \tau$ so that $\phi(x(t)) \geqq 0$ for $t \geqq \tau$.

We now rewrite (1) as

$$x(t) = f(t) + \int_{t-\tau}^{t} g(x(t-s))P(s)\,ds + \int_{0}^{t-\tau} g(x(t-s))P(s)\,ds$$

$$= f(t) + \int_{t-\tau}^{t} g(x(t-s))P(s)\,ds + \int_{0}^{t-\tau} \phi(x(t-s))P(s)\,ds$$

$$+ \int_{0}^{t-\tau} \alpha x(t-s)P(s)\,ds,$$

or

$$x(t) = F(t) + \int_{0}^{t-\tau} x(t-s)a(s)\,ds \tag{9}$$

with

$$F(t) = f(t) + \int_{t-\tau}^{t} g(x(t-s))P(s)\,ds + \int_{0}^{t-\tau} \phi(x(t-s))P(s)\,ds, \tag{10}$$

$$a(s) = \alpha P(s). \tag{11}$$

Since $\phi(x) \to 0$ as $x \to 0$, $x(t) \to 0$ as $t \to \infty$, $\int_0^\infty P(s)\,ds < \infty$, $f(\infty) = 0$, and $g(x(t))$ is locally bounded, it is easy to deduce from (10) that $F(t) \geqq 0$ and $\lim_{t\to\infty} F(t) = 0$. It also follows from (10) that $F(\tau) = f(\tau) + \int_0^\tau g(x(\tau - s))P(s)\,ds = x(\tau) \neq 0$, so that $F(t)$ is not identically zero for $t \geqq \tau$. From (11) and the choice of α, we see that $\int_0^\infty a(s)\,ds > 1$. By the second part of Theorem 3, $x(t)$ grows exponentially, a contradiction which establishes the result.

A similar result can be obtained if $g'(0)\int_0^\infty P(s)\,ds = 1$. In this case, we choose $\alpha = g'(0) = 1/\int_0^\infty P(s)\,ds$. To make $\phi(x) > 0$ for x positive, we must assume $g''(0) > 0$. To apply the first part of Theorem 3, we must also assume $\int_0^\infty sP(s)\,ds < \infty$ and $\int_0^\infty s^2P(s)\,ds < \infty$. Otherwise, the proof is similar to that of Theorem 4 and we have the following result.

THEOREM 5. *Suppose the hypotheses* H_f, H_P, H_g *are satisfied, and suppose* $f(\infty) = 0$. *Suppose also that* $\int_0^\infty sP(s)\,ds < \infty$, $\int_0^\infty s^2P(s)\,ds < \infty$. *If* $g'(0)\int_0^\infty P(s)\,ds = 1$ *and* $g''(0) > 0$, *then no solution of* (1) *which is not identically zero for all large* t *can tend to zero as* $t \to \infty$.

The situation covered in Theorem 5 is that in which the curve $y = g(x)$ and the line $y = x/\int_0^\infty P(s)\,ds$ are tangent at $x = 0$, but the line is above the curve for small positive x, as in the situation covered in Theorem 4.

Relevant to these results, although not overlapping with them, are those of Chover, Ney, and Wainger [1], especially Theorems 10 and 11, dealing with the rate of approach of a solution to its limit.

4. An example. A particularly interesting situation arises when the growth rate $h(x) = g(x)/x$ is a monotone nonincreasing function of x, which we have suggested as a plausible assumption for some population problems. In this case,

(7) becomes

$$c = f(\infty) + ch(c) \int_0^\infty P(s)\,ds. \tag{12}$$

If $f(\infty) = 0$, then $c = 0$ is a root of (12). If in addition,

$$g'(0) \int_0^\infty P(s)\,ds = h(0) \int_0^\infty P(s)\,ds > 1,$$

then there is a second root of (12) given by $h(c)\int_0^\infty P(s)\,ds = 1$. By Theorem 4, however, no solution of (1) can tend to zero; and every bounded solution tends to this second root of (12). On the other hand, if $h(0)\int_0^\infty P(s)\,ds < 1$, $c = 0$ is the only root of (12), and every bounded solution of (1) tends to zero. If $f(\infty) > 0$, it is easy to verify that (12) has a unique root in either case. This argument, together with Theorem 4 and Theorem 1, gives the following global asymptotic behavior result.

THEOREM 6. *Suppose that the hypotheses of Theorem* 1 *are satisfied, and in addition that* $g(x)/x$ *is a monotone nonincreasing function of* x. *Then every solution of* (1) *is bounded on* $0 \leqq t < \infty$ *and tends to the same limit as* $t \to \infty$.

It is natural to conjecture that a condition like that of Theorem 4 should be necessary for nonzero limits of solutions of (1). However, the proof of Theorem 4 depends strongly on the fact that the approach to the limit zero is necessarily from above. For a nonzero limit, the solution may oscillate about the limit and it does not appear that any general result of this nature can be true. It might be possible to give conditions on f which would imply that certain roots of (7) cannot be limits of solutions of (1), and this question certainly deserves further exploration.

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INTEGRALS OF PRODUCTS OF LAGUERRE POLYNOMIALS*

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Abstract. If $L_n(x)$ is the nth Laguerre polynomial, let $A_{rst}(\alpha) = \int_0^\infty e^{-\alpha x} L_r(x)L_s(x)L_t(x)\,dx$. It has recently been shown that $A_{rst}(\alpha) > 0$ for $\alpha \geqq 2$, $r, s, t = 0, 1, \cdots$, while $(-1)^{r+s+t}A_{rst}(\alpha) \geqq 0$ for $0 < \alpha \leqq 1$. It has been conjectured that $(-1)^t A_{rrt}(3/2) > 0$ for $r \geqq t$. The complete conjecture has not yet been proved, but it is established here for the cases $0 \leqq t \leqq 10$, $r \geqq t$, by obtaining, for each such t, an explicit expression for $A_{rrt}(3/2)$ as a function of r. The numbers $A_{rrt}(3/2)$ have been evaluated by means of recurrence relations up to $r = t = 500$ and the conjecture has been found to hold. In addition $A_{rrt}(3/2)$ has been evaluated asymptotically as $r \to \infty$ both for fixed t and for $t = r$. The asymptotic expressions verify the conjecture. The main technique used is that of recurrence relations and generating functions, with the asymptotic estimates derived from generating functions by Darboux's method.

On the basis of numerical computations it seems as though the original conjecture about $A_{rrt}(3/2)$ may in fact be true for $A_{rrt}(\alpha)$, $1 < \alpha \leqq 3/2$.

1. Introduction. We define the Laguerre polynomials by

$$L_n(x) = \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} x^\alpha/\alpha!$$

and the Laguerre functions by $\lambda_n(x) = e^{-(1/2)x}L_n(x)$, $n = 0, 1, 2, \cdots$. It is well known that the functions $\lambda_n(x)$ form an orthonormal set, complete in $L^2(0, \infty)$. If we wish to linearize the product of two such functions, say

$$\lambda_r(x)\lambda_s(x) = \sum_{t=0}^{\infty} C_{rst}\lambda_t(x), \tag{1.1}$$

then we see from the orthonormal property that

$$C_{rst} = \int_0^\infty \lambda_r(x)\lambda_s(x)\lambda_t(x)\,dx; \tag{1.2}$$

and, in particular, that C_{rst} is symmetric in r, s, t.

The coefficients C_{rst} were discussed rather extensively in an earlier paper [6], and several recurrence relations and other methods for computing them were derived. A table of numerical values of C_{rst} for $0 \leqq r \leqq s \leqq t \leqq 10$ led to the conjecture that

$$(-1)^t C_{rrt} > 0 \tag{1.3}$$

for all $0 \leqq t \leqq r$. The complete conjecture has not yet been established. However, we shall prove in this note that (1.3) holds in the following cases:

(i) For $0 \leqq t \leqq 10$ and all $r \geqq t$.
(ii) For $0 \leqq t \leqq r \leqq 500$.
(iii) For each $t \geqq 0$ and all sufficiently large r; though the "sufficiently large" has not been shown to be uniform with respect to t.
(iv) For all sufficiently large r and $t = r$; i.e., $(-1)^r C_{rrr} > 0$ for large r.

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We shall establish (i) by obtaining, for each value of t up to $t = 10$, an explicit formula for C_{rrt} as a function of r. The method could certainly have been continued to larger t, but it becomes progressively more complicated and does not seem capable of being extended indefinitely.

The verification of (1.3) for $0 \leqq t \leqq r \leqq 500$, as stated in (ii), is by direct computation, based on recurrence relations which will be derived in § 3.

To prove (iii) we shall in fact do rather more, and shall obtain an asymptotic estimate for C_{rrt}, for fixed t, as $r \to \infty$. Similarly we shall prove (iv) by evaluating C_{rrr} asymptotically for large r.

If, more generally, we define

$$A_{rst}(\alpha) = \int_0^\infty e^{-\alpha x} L_r(x) L_s(x) L_t(x)\, dx, \tag{1.4}$$

then $C_{rst} = A_{rst}(3/2)$.

Some time ago Szegö [9] and Kaluza [8] showed that $A_{rst}(3) > 0$, $r, s, t = 0, 1, \cdots$, and more recently Askey and Gasper (private communication also quoted in [1]) have proved a similar result for $\alpha = 2$. Moreover, they have pointed out that if, for any $\alpha > 0$, $A_{rst}(\alpha) > 0$ for all r, s, t, then the same will be true for any $\alpha' > \alpha$. This follows at once from the identity

$$L_n(\mu x) = \sum_{k=0}^{n} \binom{n}{k} \mu^{n-k}(1-\mu)^k L_{n-k}(x) \tag{1.5}$$

([5, p. 192]).

For suppose that $\alpha' > \alpha > 0$ and let $\mu = \alpha/\alpha'$. Then $0 < \mu < 1$, and

$$\begin{aligned} A_{rst}(\alpha') &= \int_0^\infty e^{-\alpha' x} L_r(x) L_s(x) L_t(x)\, dx \\ &= \mu \int_0^\infty e^{-\alpha y} L_r(\mu y) L_s(\mu y) L_t(\mu y)\, dy \\ &= \mu \sum_{k_1, k_2, k_3} \binom{r}{k_1}\binom{s}{k_2}\binom{t}{k_3} \mu^{r+s+t-(k_1+k_2+k_3)}(1-\mu)^{k_1+k_2+k_3} \\ &\quad \cdot A_{r-k_1, s-k_2, t-k_3}(\alpha) > 0. \end{aligned} \tag{1.6}$$

We know therefore, that $A_{rst}(\alpha) > 0$, $\alpha \geqq 2$; $r, s, t = 0, 1, 2, \cdots$.

On the other hand it is known that

$$(-1)^{r+s+t} A_{rst}(1) \geqq 0 \tag{1.7}$$

([7] and [10]) and we may deduce, by an argument similar to the above, that

$$(-1)^{r+s+t} A_{rst}(\alpha) \leqq 0 \tag{1.8}$$

for $0 < \alpha \leqq 1$; $r, s, t = 0, 1, \cdots$.

In this paper we shall limit the discussion to the case $\alpha = 3/2$, with particular reference to the special values $r = s \geqq t$. However we mention here that we have performed a large number of direct computations of $A_{rst}(\alpha)$, for different values of α, r, s, t, and these seem to suggest the possibility that, in fact, $(-1)^t A_{rrt}(\alpha) \geqq 0$ for $r \geqq t$ and $\alpha \leqq 3/2$.

2. Generating functions.

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} C_{rst} x^r y^s z^t = 2\{3 - (x + y + z) - (yz + zx + xy) + 3xyz\}^{-1} \tag{2.1}$$

and

$$\sum_{r=0}^{\infty} \sum_{t=0}^{\infty} C_{rrt} u^r z^t = 2(1 - u)^{-1/2}\{(3 - z)^2 - u(1 - 3z)^2\}^{-1/2}. \tag{2.2}$$

Proof of (2.1). It is known ([5, p. 189]) that

$$\sum_{r=0}^{\infty} L_r(\theta) x^r = (1 - x)^{-1} \exp\left\{\frac{-x\theta}{1 - x}\right\} \tag{2.3}$$

and hence

$$\begin{aligned}&\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \lambda_r(\theta)\lambda_s(\theta)\lambda_t(\theta) x^r y^s z^t \\ &\quad = [(1 - x)(1 - y)(1 - z)]^{-1} \exp\left\{-\theta\left(\frac{x}{1 - x} + \frac{y}{1 - y} + \frac{z}{1 - z} + \frac{3}{2}\right)\right\}.\end{aligned} \tag{2.4}$$

Integrating (2.4) term by term for θ from 0 to infinity gives

$$\begin{aligned}&\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} C_{rst} x^r y^s z^t \\ &\quad = \left[(1 - x)(1 - y)(1 - z)\left\{\frac{x}{1 - x} + \frac{y}{1 - y} + \frac{z}{1 - z} + \frac{3}{2}\right\}\right]^{-1},\end{aligned} \tag{2.5}$$

i.e., (2.1).

We can also write this as

$$\begin{aligned}&\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} C_{rst} x^r y^s z^t \\ &\quad = \frac{2}{\{3 - z - xy(1 - 3z)\} - (x + y)(1 + z)} \\ &\quad = 2 \sum_{n=0}^{\infty} \frac{(x + y)^n (1 + z)^n}{\{3 - z - xy(1 - 3z)\}^{n+1}}.\end{aligned} \tag{2.6}$$

Proof of (2.2). We pick out from (2.6) the terms in which x, y enter with equal powers. This can only happen for even n, and in these we need only the middle term from the expansion of each such $(x + y)^n$ in the numerators on the right-hand side. Making this selection, and writing $xy = u$,

$$\sum_{r=0}^{\infty} \sum_{t=0}^{\infty} C_{rrt} u^r z^t = 2 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(1+z)^{2n} u^n}{\{3 - z - u(1 - 3z)\}^{2n+1}}$$
$$= \frac{2}{[3 - z - u(1 - 3z)]} \left\{1 - \frac{4(1+z)^2 u}{[3 - z - u(1 - 3z)]^2}\right\}^{-1/2},$$

and this is equivalent to (2.2).

3. Recurrence relations. We shall establish the following recurrence relations, which will be used later:

$$\begin{aligned} 3C_{rst} &- (C_{r-1,s,t} + C_{r,s-1,t} + C_{r,s,t-1}) \\ &- (C_{r,s-1,t-1} + C_{r-1,s,t-1} + C_{r-1,s-1,t}) \\ &+ 3C_{r-1,s-1,t-1} = 0, \quad \text{unless } r = s = t = 0. \end{aligned} \tag{3.1}$$

Also, writing $K_{rt} = C_{rrt}$,

$$\begin{aligned} &(r+1)(9K_{r+1,t} - 6K_{r+1,t-1} + K_{r+1,t-2}) \\ &- (2r+1)(5K_{r,t} - 6K_{r,t-1} + 5K_{r,t-2}) \\ &+ r(K_{r-1,t} - 6K_{r-1,t-1} + 9K_{r-1,t-2}) = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} 9(t+1)K_{r,t+1} &- 3(2t+1)K_{r,t} + tK_{r,t-1} - (t+1)K_{r-1,t+1} \\ &+ 3(2t+1)K_{r-1,t} - 9tK_{r-1,t-1} = 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} 3(t+1)(t+2)K_{r,t+2} &= 2(t+1)(5t + 1 - 8r)K_{r,t+1} \\ &- 2t(5t - 4 - 8r)K_{r,t-1} + 3t(t-1)K_{r,t-2}. \end{aligned} \tag{3.4}$$

In all of the above relations we assume $r \geqq 0$, $s \geqq 0$, $t \geqq 0$. Moreover any term $C_{\alpha,\beta,\gamma}$ or $K_{\alpha,\beta}$, in which any of the subscripts is negative, is to be replaced by 0. The exception to (3.1) which arises when $r = s = t = 0$ is

$$C_{000} = \tfrac{2}{3}.$$

Proof of (3.1). This follows immediately from (2.1), if we multiply both sides of the latter by $\{3 - (x + y + z) - (yz + zx + xy) + 3xyz\}$ and then compare coefficients of $x^r y^s z^t$.

Proof of (3.2). Write $F(u, z) = \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} K_{rt} u^r z^t$. By (2.2),

$$F(u, z) = 2(1 - u)^{-1/2} \{(3 - z)^2 - u(1 - 3z)^2\}^{-1/2} \tag{3.5}$$

and, therefore,

$$\begin{aligned} \frac{1}{F} \frac{\partial F}{\partial u} &= \frac{1}{2} \left[\frac{1}{1 - u} + \frac{(1 - 3z)^2}{(3 - z)^2 - u(1 - 3z)^2} \right] \\ &= \frac{5 - 6z + 5z^2 - u(1 - 3z)^2}{(3 - z)^2 - 2u(5 - 6z + 5z^2) + (1 - 3z)^2 u^2}. \end{aligned} \tag{3.6}$$

It follows that

$$(3.7)\quad \begin{aligned}&[(9 - 6z + z^2) - 2u(5 - 6z + 5z^2) + u^2(1 - 6z + 9z^2)] \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} rK_{r,t}u^{r-1}z^t \\ &\qquad = [5 - 6z + 5z^2 - u(1 - 6z + 9z^2)] \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} K_{r,t}u^r z^t.\end{aligned}$$

Comparison of the coefficients of $u^r z^t$ on both sides of (3.7) immediately yields (3.2).

Proof of (3.3). This is very similar to that of (3.2) except that we consider $(\partial/\partial z)(\log F)$ instead of $(\partial/\partial u)(\log F)$. We omit the details.

Proof of (3.4). The function $y = \lambda_r(x)$ satisfies the differential equation [5]

$$(3.8)\qquad xy'' + y' + (r + \tfrac{1}{2} - \tfrac{1}{4}x)y = 0.$$

If we set $u = x^{1/2}\lambda_r$, then u satisfies

$$(3.9)\qquad u'' + Pu = 0,$$

where

$$(3.10)\qquad P = (1/4x^2)\{1 + 2(2r + 1)x - x^2\}.$$

Let $v = u^2$. Then

$$(3.11)\qquad v' = 2uu',$$

$$(3.12)\qquad \begin{aligned} v'' &= 2uu'' + 2u'^2 \\ &= -2Pu^2 + 2u'^2 \\ &= -2Pv + 2u'^2,\end{aligned}$$

and hence $(v'' + 2Pv)' = 4u'u'' = -2Pv'$, by (3.9) and (3.11). It follows that $v = x\lambda_r^2$ satisfies

$$(3.13)\qquad v''' + 4Pv' + 2P'v = 0,$$

and hence that $z(= \lambda_r^2 = x^{-1}v)$ satisfies

$$(3.14)\qquad \begin{aligned}\mathscr{D}(z) \equiv x^2z''' + 3xz'' + \{1 + 2(2r + 1)x - x^2\}z' \\ + (2r + 1 - x)z = 0.\end{aligned}$$

We now substitute $z = \sum_{t=0}^{\infty} K_{rt}\lambda_t(x)$ in (3.14). To calculate $\mathscr{D}(\lambda_t)$, note that

$$(3.15)\qquad x\lambda_t = -(t + 1)\lambda_{t+1} + (2t + 1)\lambda_t - t\lambda_{t-1},$$

$$(3.16)\qquad x\lambda_t' = \tfrac{1}{2}\{(t + 1)\lambda_{t+1} - \lambda_t - t\lambda_{t-1}\},$$

$$(3.17)\qquad \begin{aligned}x^2\lambda_t = -\tfrac{1}{2}(t + 1)(t + 2)\lambda_{t+2} + (t + 1)(t + 2)\lambda_{t+1} - (2t + 1)\lambda_t \\ - t(t - 1)\lambda_{t-1} + \tfrac{1}{2}t(t - 1)\lambda_{t-2}.\end{aligned}$$

In fact (3.15) follows immediately from the usual recurrence relation for L_t, and the other two relations are obtained from it immediately by differentiation followed by repeated use of (3.15) itself. Moreover higher derivatives of λ_t are all expressible in terms of λ_t, λ_t' by means of the differential equation for λ_t. We finally obtain

$$
\begin{aligned}
0 = x\mathscr{D}(z) = {} & \sum_{t=0}^{\infty} K_{rt}\left\{\frac{3}{8}(t+1)(t+2)\lambda_{t+2} + \frac{1}{4}(t+1)(8r-5t-1)\lambda_{t+1}\right. \\
& \left. - \frac{1}{4}t(8r-5t+4)\lambda_{t-1} - \frac{3}{8}t(t-1)\lambda_{t-2}\right\} \\
= {} & \sum_{t=0}^{\infty}\left\{\frac{3}{8}t(t-1)K_{r,t-2} + \frac{1}{4}t(8r-5t+4)K_{r,t-1}\right. \\
& \left. - \frac{1}{4}(t+1)(8r-5t-1)K_{s,t+1} - \frac{3}{8}(t+2)(t+1)K_{r,t+2}\right\}\lambda_t.
\end{aligned}
\tag{3.18}
$$

Now the functions λ_t form a complete orthonormal set on $L^2(0, \infty)$, while λ_r^2 and all of its derivatives clearly belong to this function space. It follows that the coefficients of the individual λ_t's in (3.18) must all vanish, and this establishes (3.4).

4. Application of the recurrence relations. The relation (3.1) was used to compute C_{rst} for $0 \leqq r \leqq s \leqq t \leqq 250$, using the Golem A computer at the Weizmann Institute. As a check on the stability of the computation the work was first done with single precision and then repeated using double precision. There was very close agreement up to the tenth significant figure. The coefficients K_{rt} were computed for $0 \leqq r, t \leqq 400$, using (3.3). In this range (1.3) was verified for all $r \geqq t$. Indeed the tabulated values rather led one to guess that (1.3) might be true whenever $r > t - \phi(t)$, where $\phi(t)$ is some positive slowly increasing function of t. Again we could verify the accuracy of our results by using both single and double precision computation. Moreover, for r, t up to 250, we could check with the previously computed table of C_{rst}.

Now consider (2.2). We have

$$
\begin{aligned}
F(u, z) &= \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} K_{rt}u^r z^t \\
&= \frac{2}{\{(3-z)^2 - 2u(5-6z+5z^2) + (1-3z)^2u^2\}^{1/2}} \\
&= \frac{2}{3-z}\sum_{m=0}^{\infty}\left(\frac{1-3z}{3-z}\right)^m P_m\left[\frac{5-6z+5z^2}{(1-3z)(3-z)}\right]u^m,
\end{aligned}
\tag{4.1}
$$

where P_m is the mth Legendre polynomial. Hence

$$
F(u, 0) = \sum_{r=0}^{\infty} K_{r0}u^r = \frac{2}{3}\sum 3^{-m}P_m\left(\frac{5}{3}\right)u^m,
\tag{4.2}
$$

i.e.,

$$
K_{r0} = C_{rr0} = (2/3^{r+1})p_r,
\tag{4.3}
$$

where we have written $P_r(\frac{5}{3}) = p_r$. We also seek an explicit formula for K_{r1}.

Write

$$w = \frac{5 - 6z + 5z^2}{(1 - 3z)(3 - z)} \tag{4.4}$$

and

$$q_m = P'_m(\tfrac{5}{3}). \tag{4.5}$$

Then

$$\begin{aligned} \frac{\partial F}{\partial z} &= \frac{2}{(3 - z)^2} \sum_{m=0}^{\infty} \left(\frac{1 - 3z}{3 - z}\right)^m P_m(w) u^m \\ &\quad - \frac{2}{3 - z} \sum_{m=0}^{\infty} \frac{8m(1 - 3z)^{m-1}}{(3 - z)^{m+1}} P_m(w) u^m \\ &\quad + \frac{2}{3 - z} \sum_{m=0}^{\infty} \left(\frac{1 - 3z}{3 - z}\right)^m P'_m(w) \frac{dw}{dz} u^m. \end{aligned} \tag{4.6}$$

Now, for any w, (cf. [5, p. 179])

$$(1 - w^2)P'_m(w) = -mwP_m(w) + mP_{m-1}(w), \tag{4.7}$$

and hence, setting $w = \frac{5}{3}$, we have

$$q_m = (3m/16)(5p_m - 3p_{m-1}). \tag{4.8}$$

Also $(dw/dz)_{z=0} = \frac{32}{9}$. Hence, by (4.6),

$$\begin{aligned} \sum_r K_{r1} u^r &= \left(\frac{\partial F}{\partial z}\right)_{z=0} \\ &= \frac{2}{9} \sum_{m=0}^{\infty} 3^{-m} p_m u^m - \frac{16}{3} \sum_{m=0}^{\infty} 3^{-m-1} m p_m u^m \\ &\quad + \frac{2}{3} \sum_{m=0}^{\infty} 3^{-m} \frac{3m}{16} (5p_m - 3p_{m-1}) \frac{32}{9} u^m, \end{aligned} \tag{4.9}$$

giving

$$\begin{aligned} K_{r,1} &= \frac{2}{3^{r+2}} [p_r - 8rp_r + 2r(5p_r - 3p_{r-1})] \\ &= \frac{2}{3^{r+2}} [(2r + 1)p_r - 6rp_{r-1}]. \end{aligned} \tag{4.10}$$

Again, setting $t = 0$ in (3.4) gives

$$6K_{r,2} = 2(1 - 8r)K_{r,1}, \tag{4.11}$$

i.e.,

$$K_{r,2} = -\frac{8r - 1}{3} K_{r,1};$$

and, setting $t = 1$ gives

$$18K_{r,3} = 4(6 - 8r)K_{r,2} - 2(1 - 8r)K_{r,0} \tag{4.12}$$

leading to

$$K_{r,3} = 2.3^{-r-5}(8r - 1)\{(32r^2 - 8r - 3)p_r - 24r(4r - 3)p_{r-1}\}. \tag{4.13}$$

We can calculate $K_{r,t}$, for successive values of t, by repeated application of (3.4). In this way formulas for $K_{r,t}$ as functions of r have been obtained for $0 \leqq t \leqq 10$. These are tabulated in the Appendix.

5. Verification of (1.3) for $0 \leqq t \leqq 10$, $r \geqq t$. To exploit the formulas derived in § 4 we need an estimate of p_r/p_{r-1} for large r. To begin with we note that, for large r,

$$P_r(\cosh \zeta) = \frac{e^{(r+1/2)\zeta}}{\sqrt{2r\pi \sinh \zeta}}\left\{1 + O\left(\frac{1}{r}\right)\right\} \tag{5.1}$$

provided that $\operatorname{Re} \zeta > 0$; ([3, p. 98]).

Setting $\zeta = \ln 3$, $\cosh \zeta = \frac{5}{3}$, $\sinh \zeta = \frac{4}{3}$, we get

$$p_r = \frac{3^{r+1}}{\sqrt{8r\pi}}\left\{1 + O\left(\frac{1}{r}\right)\right\}, \tag{5.2}$$

and so $p_r/p_{r-1} \to 3$ as $r \to \infty$. To obtain an asymptotic series we try an expansion of the form

$$\sigma_r \sim 3\left(1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \cdots\right), \tag{5.3}$$

where we have written σ_r for p_r/p_{r-1}. Now, for any x, ([5, p. 179])

$$rP_r(x) - (2r - 1)xP_{r-1}(x) + (r - 1)P_{r-2}(x) = 0. \tag{5.4}$$

Dividing by $P_{r-1}(x)$ and setting $x = \frac{5}{3}$, we get

$$3r\sigma_r - 5(2r - 1) + 3(r - 1)/\sigma_{r-1} = 0. \tag{5.5}$$

Let

$$\begin{aligned} \sigma_{r-1} &\sim 3\left[1 + \frac{a_1}{r-1} + \frac{a_2}{(r-1)^2} + \cdots\right] \\ &\sim 3\left[1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots\right]. \end{aligned} \tag{5.6}$$

Then it follows, by direct expansion, that $b_1 = a_1$,

$$b_2 = a_1 + a_2, \qquad b_3 = a_1 + 2a_2 + a_3, \cdots.$$

Rewriting (5.5) in the form

$$\begin{aligned} 9r\left(1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots\right)\left(1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots\right) - 5(2r - 1)\left(1 + \frac{b_1}{r} + \frac{b_2}{r^2} + \cdots\right) \\ + (r - 1) = 0 \end{aligned} \tag{5.7}$$

and comparing coefficients to obtain $a_1, a_2, \cdots$, we get

$$\tag{5.8} \begin{aligned} \sigma_r \sim 3\Big\{1 &- \frac{1}{2r} - \frac{1}{32r^2} - \frac{7}{128r^3} - \frac{865}{8192r^4} - \frac{3797}{2^{14}r^5} - \frac{617{,}153}{2^{20}r^6} \\ &- \frac{3{,}629{,}173}{2^{21}r^7} - \frac{3{,}126{,}992{,}365}{2^{29}r^8} - \frac{23{,}752{,}522{,}907}{2^{30}r^9} \\ &- \frac{6{,}434{,}124{,}927{,}143}{2^{36}r^{10}} - \frac{60{,}819{,}747{,}327{,}687}{2^{37}r^{11}} - \cdots\Big\}. \end{aligned}$$

We now apply this expansion to the formulas for K_{rt}, $0 \leqq t \leqq 10$. Let us take, for example, the case $t = 3$. We have

$$\tag{5.9} K_{r,3} = \frac{2(8r-1)}{3^{r+5}} p_{r-1}(32r^2 - 8r - 3)\left\{\sigma_r - \frac{24r(4r-3)}{32r^2 - 8r - 3}\right\}.$$

Now we can verify that

$$\tag{5.10} \frac{24r(4r-3)}{32r^2 - 8r - 3} > 3\left\{1 - \frac{1}{2r} - \frac{1}{32r^2} - \frac{7}{128r^3} - \frac{1}{32r^4}\right\}$$

for $r \geqq 5$.

On the other hand we can show, e.g., by induction using (5.5), that

$$\tag{5.11} \sigma_r < 3\left\{1 - \frac{1}{2r} - \frac{1}{32r^2} - \frac{7}{128r^3} - \frac{1}{32r^4}\right\}$$

for $r \geqq 6$.

It follows from (5.9), (5.10), (5.11) that $K_{r,3} < 0$ for $r \geqq 6$. The values of $K_{r,3}$ for $3 \leqq r \leqq 6$ can be verified, by direct numerical computation, to satisfy (1.3).

Higher values of r, up to $r = 10$, can be handled similarly, except that the expansions have to be taken to more terms and the work is correspondingly more complicated. Incidentally, we note from the table in Appendix A that it is sufficient to perform the verification for odd r.

6. Proof of (iii). In this section we shall derive an asymptotic estimate of K_{rt} for fixed t as r tends to infinity. The proof will depend mainly on a classical theorem of Darboux [4].

DARBOUX'S THEOREM. *Let $h(w) = \sum_{n=0}^{\infty} A_n w^n$ be regular for $|w| < 1$ and have a finite number of singularities $w_1, w_2, \cdots, w_l$ on the circle $|w| = 1$. In the vicinity of each $w_k, k = 1, 2, \cdots, l$, let $h(w)$ have an expansion of the form*

$$\tag{6.1} h(w) = \sum_{\nu=0}^{\infty} c_\nu^{(k)}(1 - w/w_k)^{a_k + \nu b_k},$$

where $b_k > 0$. Then the expression

$$\tag{6.2} \sum_{\nu=0}^{\infty} \sum_{k=1}^{l} c_\nu^{(k)} \binom{a_k + \nu b_k}{n} (-w_k)^n$$

furnishes an asymptotic expansion for A_n as $n \to \infty$.

Consider the expansion (2.2). We can write it in the form

$$\sum_{r=0}^{\infty}\sum_{t=0}^{\infty} K_{rt}u^r z^t = 2(1-u)^{-1/2}\{(9-u) - 6z(1-u) + z^2(1-9u)\}^{-1/2}$$

$$(6.3)\qquad = 2(1-u)^{-1/2}(9-u)^{-1/2}\left\{1 - \frac{6(1-u)}{9-u}z + \frac{(1-9u)^{t/2}}{(9-u)^{t+1/2}}z^t,\right.$$

$$= \sum_{t=0}^{\infty} \frac{2}{(1-u)^{1/2}} P_t\left[\frac{3(1-u)}{\sqrt{(9-u)(1-9u)}}\right]\frac{(1-9u)^{t/2}}{(9-u)^{t+1/2}}z^t,$$

where P_t is the Legendre polynomial of order t. It follows that

$$(6.4)\qquad \begin{aligned}\sum_{r=0}^{\infty} K_{rt}u^r &= 2(1-u)^{-1/2}(1-9u)^{(1/2)t}(9-u)^{-t-1/2}P_t\left[\frac{3(1-u)}{\sqrt{(9-u)(1-9u)}}\right]\\ &= F_t(u) \quad \text{(say)}.\end{aligned}$$

Now the only singularities of $F_t(u)$ are at $u = 1$, and $u = 9$; since the factor $(1-9u)^{1/2}$ in the denominator of the argument of $P_t(\cdot)$ is balanced by the factor $(1-9u)^{(1/2)t}$ outside. It follows, by Darboux's theorem, that it is sufficient to consider the behavior of $F_t(u)$ in the neighborhood of $u = 1$.

Write $\zeta = 1 - u$, and set $x = (3(1-u)/\sqrt{(9-u)(1-9u)})$. We find at once that

$$(6.5)\qquad x^2 \sim -\frac{9\zeta^2}{64}\left[1 + \zeta + \frac{73\zeta^2}{64} + O(\zeta^3)\right].$$

We distinguish two cases:

(a) *t even* $(= 2s)$. Then

$$P_t(x) = P_{2s}(x) \sim \binom{-\frac{1}{2}}{s}\{1 - s(2s+1)x^2 + O(x^4)\} \qquad \text{(cf. [5])}$$

$$(6.6)\qquad \sim \binom{-\frac{1}{2}}{s}\left\{1 + \frac{9s(2s+1)}{64}\zeta^2 + \frac{9s(2s+1)}{64}\zeta^3 + O(\zeta^4)\right\},$$

by (6.5).

Also,

$$(6.7)\qquad \begin{aligned}\left(\frac{1-9u}{9-u}\right)^{(1/2)t} &= (-1)^s\left(\frac{8-9\zeta}{8+\zeta}\right)^s\\ &= (-1)^s\left\{1 - \frac{5s\zeta}{4} + \frac{5s(5s-4)}{32}\zeta^2 + O(\zeta^3)\right\},\end{aligned}$$

while

$$(6.8)\qquad (9-u)^{-1/2} = (8+\zeta)^{-1/2} \sim \frac{1}{2\sqrt{2}}\left\{1 - \frac{\zeta}{16} + \frac{3\zeta^2}{512} + O(\zeta^2)\right\}.$$

From (6.4), (6.6), (6.7), (6.8) we obtain

$$F_t(u) = \frac{(-1)^s}{\sqrt{2\zeta}}\binom{-\frac{1}{2}}{s}\left\{1 + \frac{9s(2s+1)}{64}\zeta^2 + O(\zeta^3)\right\}$$

$$(6.9)\qquad \cdot\left\{1 - \frac{5s\zeta}{4} + \frac{5s(s-4)}{32}\zeta^2\right\}\left\{1 - \frac{\zeta}{16} + \frac{3\zeta^2}{512}\right\}$$

$$= \frac{(-1)^s}{\sqrt{2\zeta}}\binom{-\frac{1}{2}}{s}\left\{1 - \frac{20s+1}{16}\zeta + \frac{544s^2 - 208s + 3}{512}\zeta^2 + O(\zeta^3)\right\}.$$

This is of the form (6.1) with $a_k = -\frac{1}{2}$, $b_k = 1$, and it follows by (6.2) that

$$K_{r,2s} \sim \frac{(-1)^s}{\sqrt{2}}\binom{-\frac{1}{2}}{s}(-1)^r\left\{\binom{-\frac{1}{2}}{r} - \frac{20s+1}{16}\binom{\frac{1}{2}}{r} + \frac{544s^2 - 208s + 3}{512}\binom{\frac{3}{2}}{r}\cdots\right\}$$

$$= 2^{-1/2}(-1)^{r+s}\binom{-\frac{1}{2}}{r}\binom{-\frac{1}{2}}{s}\left\{1 + \frac{20s+1}{16}\,\frac{1}{2r-1}\right.$$

$$(6.10)\qquad \left. + \frac{544s^2 - 208s + 3}{512}\,\frac{3}{(2r-1)(2r-3)}\cdots\right\}$$

$$= 2^{-2r-t-1/2}\binom{2r}{r}\binom{t}{\frac{1}{2}t}\left\{1 + \frac{10t+1}{16}\left(\frac{1}{2r} + \frac{1}{4r^2}\right) + \frac{3(136t^2 - 104t + 3)}{2048r^2}\right.$$

$$\left. + O(r^{-3})\right\}.$$

But it follows from Stirling's formula that

$$(6.11)\qquad 2^{-2r}\binom{2r}{r} \sim (\pi r)^{-1/2}\left\{1 - \frac{1}{8r} + \frac{1}{128r^2} + O(r^{-3})\right\}.$$

Combining (6.10) with (6.11) we get that, for even t,

$$(6.12)\qquad K_{r,t} \sim 2^{-t}\binom{t}{\frac{1}{2}t}(2\pi r)^{-1/2}\left\{1 + \frac{10t-3}{32r} + \frac{408t^2 - 72t + 49}{2048r^2} + O(r^{-3})\right\},$$

and is, in particular, positive.

(b) *t odd* ($= 2s + 1$). We now have, for small x (cf. [5])

$$P_{2s+1}(x) = \binom{-\frac{3}{2}}{s} \times \left\{1 - \frac{1}{3}s(2s+3)x^2 + O(x^4)\right\}$$

$$(6.13)$$

$$\sim \binom{-\frac{3}{2}}{s} \times \left\{1 + \frac{3s(2s+3)\zeta^2}{64} + O(\zeta^3)\right\}$$

by (6.5).

Hence, by (6.4) and the definition of x, ζ,

$$
\begin{aligned}
F_{2s+1}(u) &= 2\binom{-\frac{3}{2}}{s}\{(1-u)^{-1/2}(1-9u)^{s+1/2}(9-u)^{-s-1}\} \\
&\cdot\{3(1-u)(1-9u)^{-1/2}(9-u)^{-1/2}\} \\
&\cdot\left\{1+\frac{3s(2s+3)}{64}\zeta^2+\cdots\right\} = 6\left\{\binom{-\frac{3}{2}}{s}\zeta^{1/2}\left(\frac{1-9u}{9-u}\right)^s(9-u)^{-3/2}\right\} \\
&\cdot\left\{1+\frac{3s(2s+3)}{64}\zeta^2+O(\zeta^4)\right\}.
\end{aligned}\tag{6.14}
$$

But, as in (6.7),

$$\left(\frac{1-9u}{9-u}\right)^s = (-1)^s\left\{1-\frac{5s\zeta}{4}+\frac{5s(5s-4)}{32}\zeta^2+O(\zeta^3)\right\},\tag{6.15}$$

while

$$(9-u)^{-3/2} = (8+\zeta)^{-3/2} = 2^{-9/2}\left\{1-\frac{3\zeta}{16}+\frac{15\zeta^2}{512}+O(\zeta^3)\right\}.\tag{6.16}$$

Substituting (6.15) and (6.16) into (6.14) yields

$$F_{2s+1} = \frac{3(-1)^s}{8\sqrt{2}}\binom{-\frac{3}{2}}{s}\zeta^{1/2}\left\{1-\frac{20s+3}{16}\zeta+\frac{448s^2-128s+15}{512}\zeta^2+\cdots\right\}\tag{6.17}$$

and so, again applying Darboux's theorem,

$$
\begin{aligned}
K_{r,2s+1} &\sim \frac{3(-1)^s}{8\sqrt{2}}\binom{-\frac{3}{2}}{s}(-1)^r\left\{\binom{\frac{1}{2}}{r}-\frac{20s+3}{16}\binom{\frac{3}{2}}{r}\right. \\
&\left.+\frac{448s^2-128s+15}{512}\binom{\frac{5}{2}}{r}+\cdots\right\}.
\end{aligned}\tag{6.18}
$$

But
$$\binom{\frac{1}{2}}{r} = -\frac{1}{2r-1}\binom{-\frac{1}{2}}{r},$$

$$\binom{\frac{3}{2}}{r} = +\frac{3}{(2r-1)(2r-5)}\binom{-\frac{1}{2}}{r},$$

and

$$\binom{\frac{5}{2}}{r} = -\frac{15}{(2r-1)(2r-3)(2r-5)}\binom{-\frac{1}{2}}{r}.$$

Substituting these in (6.18), and applying (6.11), we get, after some straightforward manipulation, that, for odd t,

$$
\begin{aligned}
K_{r,t} &\sim -\frac{3t}{2^{t+3}}\binom{t-1}{\frac{1}{2}(t-1)}\frac{1}{\sqrt{2\pi r^3}}\left\{1+\frac{3(10t-3)}{32r}\right. \\
&\left.+\frac{5(336t^2-144t+149)}{2048r^2}\cdots\right\}.
\end{aligned}\tag{6.19}
$$

7. Proof of (iv). It follows from (4.1) that

$$\sum_{t=0}^{\infty} K_{rt}z^t = 2(3-z)^{-r-1}(1-3z)^r P_r\{(5-6z+5z^2)/(1-3z)(3-z)\} \tag{7.1}$$

and so

$$\begin{aligned} C_{rrr} = K_{rr} = \frac{1}{\pi i}\int_{|z|=1} & z^{-r-1}(3-z)^{-r-1}(1-3z)^r \\ & \cdot P_r\{(5-6z+5z)/(1-3z)(3-z)\}\,dz. \end{aligned} \tag{7.2}$$

In the integral in (7.2) we make the change of variable

$$w = \frac{3-z}{1-3z}. \tag{7.3}$$

This transforms the unit circle into itself and we get

$$C_{rrr} = \frac{1}{\pi i}\int_{|w|=1} \frac{(1-3w)^r}{(3-w)^{r+1}} P_r\left\{\frac{1}{2}\left(w+\frac{1}{w}\right)\right\} w^{-r-1}\,dw. \tag{7.4}$$

Let $w = e^{-i\theta}$. Then

$$C_{rrr} = \frac{(-1)^r}{\pi}\int_{-\pi}^{\pi} \frac{(3-e^{+i\theta})^r}{(3-e^{-i\theta})^{r+1}} P_r(\cos\theta)\,d\theta. \tag{7.5}$$

To evaluate C_{rrr} asymptotically, consider the generating function

$$\begin{aligned} H(z) &= \sum_{r=0}^{\infty} (-1)^r C_{rrr} z^r \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{r=0}^{\infty} \frac{(3-e^{i\theta})^r}{(3-e^{-i\theta})^{r+1}} z^r P_r(\cos\theta)\,d\theta \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \{(3-e^{-i\theta})^2 - 4z\cos\theta(5-3\cos\theta) + z^2(3-e^{+i\theta})^2\}^{-1/2}\,d\theta. \end{aligned} \tag{7.6}$$

The evaluation of the sum under the integral sign is an immediate consequence of the generating function for Legendre polynomials. The change of order of summation and integration is justified by the uniform convergence of the power series in any domain $|z| \leqq \rho < 1$. The integral in (7.6) will converge unless z is such that the expression under the square root sign, regarded as a function of θ, has a zero of order higher than 1. It is an elementary exercise to show that this will happen if and only if $z = \pm 1$, and hence that $H(z)$ is regular in $|z| < 1$ with singularities at $z = \pm 1$ on its circle of convergence. To apply Darboux's theorem we have therefore to study the behavior of $H(z)$ in the neighborhoods of these two points.

We begin by writing $t = \tan\theta/2$, and get

$$\begin{aligned} H(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} & \{4(1+z)^2t^4 + 4i(1-z^2)t^3 + 3(1-z)^2t^2 + 2it(1-z^2) \\ & + (1-z)^2\}^{-1/2}\,dt. \end{aligned} \tag{7.7}$$

If we now write $t = -iu^{-1}$, $w = (1+z)/(1-z)$, then

$$(7.8)\qquad H(z) = -\frac{i}{\pi(1-z)}\int_{-i\infty}^{i\infty}\{u^4 + 2wu^3 - 3u^2 - 4wu + 4w^2\}^{-1/2}\,du.$$

We consider the two singularities separately, and shall denote by X_r, Y_r the respective contributions of $z = 1$ and $z = -1$ to the asymptotic development.

(a) *The singularity at* $z = 1$. To transform the integral in (7.8) we suppose $z = 1 - \zeta$, where ζ is real, small, and positive.

Let $\mu = (1 + z^{1/2})^{1/3}(1 - z^{1/2})^{-1/3}$, $\sigma = \mu + \mu^{-1}$. Then μ, σ are real and clearly both tend to infinity as $z \to 1$, $\zeta \to 0$. We find that

$$(7.9)\qquad H(z) = -i(\pi\zeta)^{-1}\int_{-i\infty}^{i\infty}\{(u-a)(u-b)(u-c)(u-\bar{c})\}^{-1/2}\,du.$$

Here

$$(7.10)\qquad \begin{aligned} a &= -\sigma,\\ b &= -\sigma(\sigma^2 - 3),\\ c &= -(\omega\mu + \omega^2/\mu),\\ \bar{c} &= -(\omega^2\mu + \omega/\mu), \end{aligned}$$

where $\omega = e^{2\pi i/3}$.

We now make some transformations of standard type. Define

$$(7.11)\qquad \begin{aligned} b_1 &= \tfrac{1}{2}(c + \bar{c}) = \tfrac{1}{2}\sigma,\\ a_1^2 &= -\tfrac{1}{4}(c - \bar{c})^2 = \tfrac{3}{4}(\sigma^2 - 4), \end{aligned}$$

$$(7.12)\qquad \begin{aligned} A^2 &= (a - b_1)^2 + a_1^2 = 3(\sigma^2 - 1),\\ B^2 &= (b - b_1)^2 + a_1^2 = (\sigma^2 - 1)^2(\sigma^2 - 3),\\ G^2 &= (b_1 - a)(b_1 - b) + a_1^2 = \tfrac{3}{2}(\sigma^4 - 2\sigma^2 - 2), \end{aligned}$$

$$(7.13)\qquad v = \frac{(A - B)u - (Ab - Ba)}{(A + B)u - (Ab + Ba)},$$

and we get

$$(7.14)\quad H(z) = i\sqrt{2}(\pi\zeta)^{-1}\int_C \{(1 - v^2)[(AB + G^2)v^2 + (AB - G^2)]\}^{-1/2}\,dv,$$

where C, the image of the imaginary u-axis under the transformation (7.13), is a circle with center on the real axis in the v-plane, and which intersects that axis at v_1, v_2 (say), where

$$(7.15)\qquad \begin{aligned} v_1 &= \frac{A - B}{A + B} = -1 + \frac{2\sqrt{3}}{\sigma^2} + O(\sigma^{-4})\\ &\text{and}\\ v_2 &= \frac{Ab - Ba}{Ab + Ba} = 2 - \sqrt{3} + O(\sigma^{-2}). \end{aligned}$$

The singularities of the integrand in (7.14) are at $v = \pm 1$ (both of which are outside of C) and at $v = \pm i\lambda$, where

$$\lambda^2 = \frac{AB - G^2}{AB + G^2} \sim \frac{\sqrt{3} - \frac{3}{2}}{\sqrt{3} + \frac{3}{2}} + O(\sigma^{-2}) \tag{7.16}$$
$$= (2 - \sqrt{3})^2 + O(\sigma^{-2}).$$

It is easily verified that the points $\pm i\lambda$ lie inside the circle (at least when σ is large). We now deform the path of integration from C to Γ, where Γ consists of small circles with centers at $\pm i\lambda$, connected by parallel straight lines, A, B, C, D on opposite sides of the imaginary axis (see Fig. 1). It is seen that the contributions

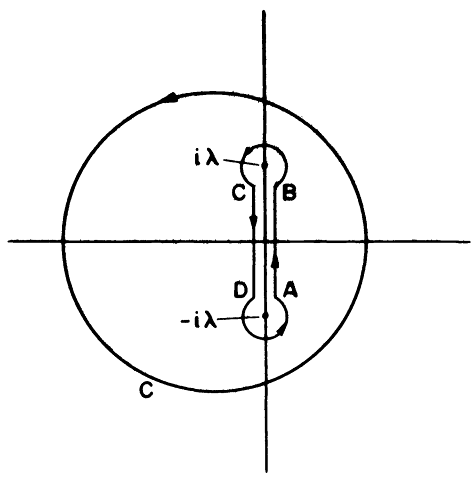

FIG. 1

of the two circles to the integral tend to zero with their radii. On the other hand the integrand clearly changes sign as we go round either circle from B to C. Letting the radii of the two circles tend to zero, as also the distance of AB, CD from the imaginary axis, we see that

$$\int_C \{(1 - v^2)[(AB + G^2)v^2 + (AB - G^2)]\}^{-1/2}\, dv$$
$$= 2\int_{-i\lambda}^{i\lambda} \{(1 - v^2)[(AB + G^2)v^2 + (AB - G^2)]\}^{-1/2}\, dv \tag{7.17}$$
$$= \frac{4}{\sqrt{AB + G^2}} \int_0^{i\lambda} [(1 - v^2)(v^2 + \lambda^2)]^{-1/2}\, dv.$$

Setting $v = i\lambda \cos\theta$ leads to

$$H(z) = 4\sqrt{2}(\pi\zeta)^{-1}(AB + G^2)^{-1/2} \int_0^{\pi/2} (1 + \lambda^2 - \lambda^2 \sin^2\theta)^{-/12}\, d\theta \tag{7.18}$$
$$= 4(\pi\zeta)^{-1}(AB)^{-1/2} K\left(\frac{\lambda}{\sqrt{1 + \lambda^2}}\right),$$

where, in accordance with the usual notation,

$$K(k) = \int_0^{1/2\pi} (1 - k^2 \sin^2 \theta)^{-1/2}\, d\theta \tag{7.19}$$

for $0 \leqq k < 1$.

We also define, for any k in $(0, 1)$,

$$k' = +\sqrt{1 - k^2} \tag{7.20}$$

and

$$E(k) = \int_0^{(1/2)\pi} (1 - k^2 \sin^2 \theta)^{1/2}\, d\theta. \tag{7.21}$$

It is known ([2, p. 282]) that

$$\frac{dK}{dk} = \frac{1}{kk'^2}\{E(k) - k'^2 K(k)\} \tag{7.22}$$

and

$$\frac{d^2K}{dk^2} = \frac{1}{k^2k'^4}\{(3k^2 - 1)E(k) + k'^2(k'^2 - k^2)K(k)\}. \tag{7.23}$$

In our case,

$$\begin{aligned} \frac{\lambda}{\sqrt{1 + \lambda^2}} &= \sqrt{\frac{1}{2}\left(1 - \frac{G^2}{AB}\right)} \\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}} - \frac{\sqrt{6}(\sqrt{3} + 1)}{8\sigma^2} - \frac{\sqrt{2}(9 + 7\sqrt{3})}{32\sigma^4} + O(\sigma^{-6}) \end{aligned} \tag{7.24}$$

by (7.12). We write

$$\begin{aligned} \frac{\sqrt{3} - 1}{2\sqrt{2}} &= k_0, \\ K(k_0) &= K_0 = 1.598147, \\ E(k_0) &= E_0 = 1.544151, \\ k_0' &= \sqrt{1 - k_0^2} = \frac{\sqrt{3} + 1}{2\sqrt{2}}. \end{aligned} \tag{7.25}$$

We note that $k_0 k_0' = \frac{1}{4}$. It follows from (7.24), (7.22), (7.23) that

$$K\left(\frac{\lambda}{\sqrt{1 + \lambda^2}}\right) = 1.59815 - \frac{0.18380}{\sigma^2} + \frac{0.13978}{\sigma^4} + O(\sigma^{-6}). \tag{7.26}$$

On the other hand we deduce from (7.12) that

$$(AB)^{-1/2} \sim 3^{-1/4}\sigma^{-2}\left(1 + \frac{3}{2\sigma^2} + \frac{21}{8\sigma^4}\right). \tag{7.27}$$

Combining (7.26), (7.27) with (7.18) yields

$$H(z) \sim \frac{1.54613}{\zeta\sigma^2}\left\{1 + \frac{1.38499}{\sigma^2} + \frac{2.53995}{\sigma^4} + \cdots\right\}. \tag{7.28}$$

If we now recall the definitions of μ, σ at the beginning of this section, we see that

$$\begin{aligned} \mu^3 &= \{1 + (1-\zeta)^{1/2}\}/\{1 - (1-\zeta)^{1/2}\} \\ &= 4\zeta^{-1}\left[1 - \frac{1}{2}\zeta - \frac{1}{16}\zeta^2 + O(\zeta^3)\right], \end{aligned} \tag{7.29}$$

and therefore,

$$\mu = 4^{1/3}\zeta^{-1/3}\left\{1 - \frac{1}{6}\zeta - \frac{7}{144}\zeta^2 + O(\zeta^3)\right\}. \tag{7.30}$$

It follows that

$$\begin{aligned} \sigma &= \mu + \mu^{-1} \\ &\sim 4^{1/3}\zeta^{-1/3}\{1 + 2^{-4/3}\zeta^{2/3} - \tfrac{1}{6}\zeta + O(\zeta^{5/3})\}. \end{aligned} \tag{7.31}$$

If we now substitute this into (7.28) we obtain, after some rather laborious work,

$$H(z) \sim 0.61358\zeta^{-1/3} - 0.14976\zeta^{1/3} + 0.20453\zeta^{2/3} + O(\zeta). \tag{7.32}$$

It follows now by Darboux's theorem that

$$X_r \sim (-1)^r\left\{0.61358\binom{-\frac{1}{3}}{r} - 0.14976\binom{\frac{1}{3}}{r} + 0.20453\binom{\frac{2}{3}}{r} + \cdots\right\}. \tag{7.33}$$

But for any α,

$$\begin{aligned} (-1)^r\binom{-\alpha}{r} &= \frac{\Gamma(r+\alpha)}{\Gamma(\alpha)\Gamma(r+1)} \\ &\sim \frac{1}{\Gamma(\alpha)}\frac{1}{r^{1-\alpha}}\left\{1 + \frac{\alpha(\alpha-1)}{2r} + O(r^{-2})\right\}, \end{aligned} \tag{7.34}$$

by Stirling's formula. Substituting from this into (7.33) we get

$$\begin{aligned} X_r &\sim \frac{0.61358}{\Gamma(\frac{1}{3})}\frac{1}{r^{2/3}}\left\{1 - \frac{1}{9r} + O(r^{-2})\right\} \\ &\quad - \frac{0.14976}{\Gamma(-\frac{1}{3})}\cdot\frac{1}{r^{4/3}} + \frac{0.20453}{\Gamma(-\frac{2}{3})}\cdot\frac{1}{r^{5/3}} \end{aligned} \tag{7.35}$$

$$\sim \frac{0.2290}{r^{2/3}} + \frac{0.0369}{r^{4/3}} - \frac{0.0763}{r^{5/3}} + O(r^{-8/3}). \tag{7.36}$$

(b) *The singularity at* $z = -1$. It remains to estimate Y_r. We consider $z = -1 + \eta$, where η is to tend to zero through positive values. We previously defined μ by

$$\mu = \{(1 + z^{1/2})/(1 - z^{1/2})\}^{1/3}. \tag{7.37}$$

This now becomes a complex number and $|\mu| = 1$. We therefore write

$$\mu = e^{2i\alpha}, \tag{7.38}$$

where α is real, and note that

$$\lim_{z\to -1} \alpha = \pi/12. \tag{7.39}$$

The roots of the quartic polynomial in the integrand in (7.8) are now all real and

$$H(z) = -\frac{i}{\pi(1-z)}\int_{-i\infty}^{i\infty} \{(u-e)(u-f)(u-g)(u-h)\}^{-1/2}\,du,$$

where

$$\begin{aligned} e &= 2\cos(\pi/3 - 2\alpha),\\ f &= 2\cos(\pi/3 + 2\alpha),\\ g &= -2\cos 6\alpha,\\ h &= -2\cos 2\alpha. \end{aligned} \tag{7.40}$$

We make the standard transformation

$$v = \frac{u-f}{u-e} \tag{7.41}$$

and obtain

$$H(z) = \frac{-i}{\pi(1-z)}\int_{C_1} \frac{dv}{\{v[(e-g)v-(f-g)][(e-h)v-(f-h)]\}^{1/2}}, \tag{7.42}$$

where C_1 is a circle with center on the real axis in the v-plane and which cuts that axis at $v_R = +1$ and at $v_L = \cos(\pi/3 + 2\alpha)/\cos(\pi/3 - 2\alpha)$, and it is easily seen that, for α close to $\pi/12$, $0 < v_L < v_R$. (See Fig. 2.) The singularities of the integrand

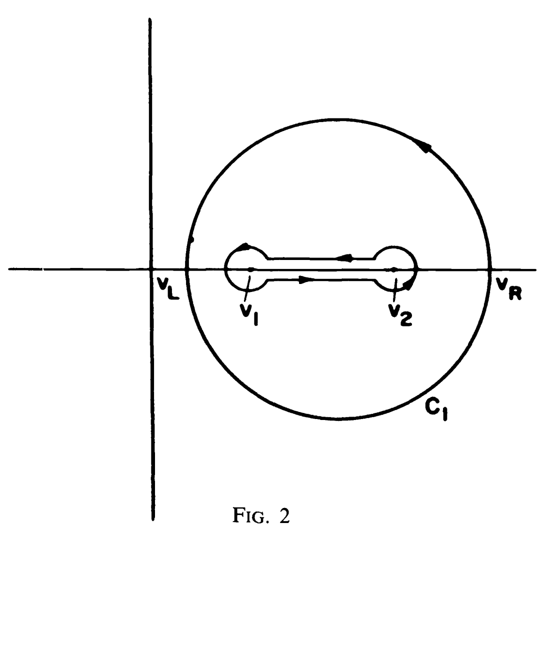

FIG. 2

(apart from the singularity at $v = 0$, which is outside of C_1) are at

$$v_1 = \frac{f-g}{e-g}$$

(7.43) and

$$v_2 = \frac{f-h}{e-h}.$$

By the same kind of argument as was used in transforming the integral in (7·14) we see that

$$H(z) = \frac{2}{\pi(1-z)\sqrt{(e-g)(e-h)}}\int_{v_1}^{v_2}\frac{dv}{\sqrt{v(v-v_1)(v_2-v)}}. \tag{7.44}$$

We now write

$$\sin^2\theta = \frac{v_2 - v}{v_2 - v_1}, \tag{7.45}$$

and obtain

$$H(z) = \frac{4}{\pi(1-z)\sqrt{(e-g)(f-h)}}K(k), \tag{7.46}$$

where

$$\begin{aligned} k^2 &= 1 - \frac{v_1}{v_2} \\ &= 1 - \frac{(f-g)(e-h)}{(f-h)(e-g)} \\ &= 1 - \gamma \quad \text{(say)}. \end{aligned} \tag{7.47}$$

If we write $\alpha = \pi/12 - \beta$ we obtain

$$\gamma = \frac{\sin 4\beta \cos^2 2\beta}{\sin(\pi/3 + 4\beta)\sin^2(\pi/6 + 2\beta)}. \tag{7.48}$$

As $z \to -1$ we have $\beta \to 0$, and we deduce from (7.37), (7.38), (7.39) that

$$\beta \sim \frac{1}{12}\left(\eta + \frac{1}{2}\eta^2\right) + O(\eta^3), \tag{7.49}$$

and therefore, by (7.48), that

$$\gamma \sim \frac{8\eta}{3\sqrt{3}}\left\{1 - \frac{8 - 3\sqrt{3}}{6\sqrt{3}}\eta\right\}. \tag{7.50}$$

But it is known that, ([12, p. 298]), as $k \to 1$,

$$K(k) \sim (1 + \tfrac{1}{4}k'^2)\ln\left(\frac{1}{k'}\right) + O(k'^4 \log k'). \tag{7.51}$$

To complete our evaluation of (7.46) we easily calculate that

$$
\begin{aligned}
\sqrt{(e-g)(f-h)} &= \sqrt{3}\{1 + 16\beta/\sqrt{3} + O(\beta^2)\}^{1/2} \\
&\sim \sqrt{3}\{1 + 8\beta/\sqrt{3} + O(\beta^2)\} \\
&= \sqrt{3}\{1 + 2\eta/(3\sqrt{3}) + O(\eta^2)\}.
\end{aligned} \tag{7.52}
$$

Substituting in (7.46) we obtain

$$
\begin{aligned}
H(z) &\sim \frac{4}{\pi(2-\eta)} 3^{-1/2}\left\{1 - \frac{2\eta}{3\sqrt{3}}\right\} \times \left\{-\frac{1}{2}\left(1 + \frac{2\eta}{3\sqrt{3}}\right)[\ln \eta + O(1)]\right\} \\
&\sim -\frac{3^{-1/2}}{\pi}\left(1 + \frac{\eta}{2}\right)\ln \eta.
\end{aligned} \tag{7.53}
$$

We use now the easily established extension of Darboux's theorem to logarithmic singularities. The term $\ln \eta$ contributes to Y_r simply the coefficient of x^r in $\ln(1 + x)$, i.e., $(-1)^{r-1}/r$, while the term $\eta \ln \eta$ similarly contributes $(-1)^{r-1}/(r(r-1))$.

We deduce that

$$
\begin{aligned}
Y_r &\sim \frac{(-1)^r}{\pi\sqrt{3}}\left\{\frac{1}{r} + \frac{1}{2r^2} + O(r^{-3})\right\} \\
&\sim (-1)^r \times 0.1838\left\{\frac{1}{r} + \frac{1}{2r^2} + O(r^{-3})\right\}.
\end{aligned} \tag{7.54}
$$

We can now combine (7.36) and (7.54) and obtain

$$
\begin{aligned}
(-1)^r C_{rrr} &\sim \frac{0.2290}{r^{2/3}} + \frac{0.0369}{r^{4/3}} - \frac{0.0763}{r^{5/3}} \\
&\quad + (-1)^r\left(\frac{0.1838}{r} + \frac{0.0919}{r^2}\right).
\end{aligned} \tag{7.55}
$$

8. Some numerical values. As a matter of interest we compare in Table 1 some values of K_{rt} as given by our asymptotic estimates with actual values computed directly by the aid of the recurrence relations.

TABLE 1

r	t	Exact value	Asymptotic estimate	Formula used
20	2	+.04588	+.04587	(6.12)
50	5	−.0004350	−.0004347	(6.17)
200	5	−.0000507	−.0000507	(6.17)
200	20	+.005134	+.005133	(6.12)
400	20	+.0035705	+.0035704	(6.12)
20	20	+.0405	+.0407	(7.55)
21	21	−.0217	−.0214	(7.55)
22	22	+.0375	+.0379	(7.55)
100	100	+.0125	+.0125	(7.55)
500	500	+.00401	+.00401	(7.55)

Appendix.

$$C_{rr0} = \frac{2}{3^{r+1}} p_r,$$

$$C_{rr1} = \frac{2}{3^{r+2}} \{(2r + 1)p_r - 6rp_{r-1}\},$$

$$C_{rr2} = -\frac{8r - 1}{3} C_{rr1},$$

$$C_{rr3} = \frac{2(8r - 1)}{3^{r+5}} \{(32r^2 - 8r - 3)p_r - 24r(4r - 3)p_{r-1}\},$$

$$C_{rr4} = -\frac{1}{3}\,\frac{32r^2 - 48r + 1}{8r - 1} C_{rr3},$$

$$C_{rr5} = \frac{2(32r^2 - 48r + 1)}{5.3^{r+7}} \{(512r^3 - 1{,}152r^2 + 46r + 15)p_r$$
$$- 6r(256r^2 - 704r + 303)p_{r-1}\},$$

$$C_{rr6} = -\frac{256r^3 - 1{,}056r^2 + 872r - 3}{9(32r^2 - 48r + 1)} C_{rr5},$$

$$C_{rr7} = \frac{2}{35}\,\frac{256r^3 - 1{,}056r^2 + 872r - 3}{3^{r+10}} \{(8{,}192r^4 - 45{,}056r^3 + 53{,}440r^2$$
$$- 352r - 105)p_r - 192r(128r^3 - 768r^2 + 1{,}183r - 408)p_{r-1}\},$$

$$C_{rr8} = -\frac{512r^4 - 4{,}096r^3 + 9{,}280r^2 - 5{,}792r + 3}{3(256r^3 - 1{,}056r^2 + 872r - 3)} C_{rr7}.$$

$$C_{rr9} = \frac{2}{5.7}\,\frac{512r^4 - 4{,}096r^3 + 9{,}280r^2 - 5{,}792r + 3}{3^{r+14}}$$
$$\cdot \{(131{,}072r^5 - 1{,}310{,}720r^4 + 3{,}873{,}280r^3 - 3{,}345{,}280r^2$$
$$+ 3{,}378r + 945)p_r - 6r(65{,}536r^4 - 688{,}128r^3 + 2{,}262{,}272r^2$$
$$- 2{,}624{,}064r + 806{,}409p_r)\}$$

$$C_{rr10} = -\frac{1}{15}\,\frac{4{,}096r^5 - 53{,}760r^4 + 232{,}960r^3 - 388{,}800r^2 + 206{,}104r - 15}{512r^4 - 4{,}096r^3 + 9{,}280r^2 - 5{,}792r + 3} C_{rr9},$$

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ON THE SEPARATION OF VARIABLES FOR THE LAPLACE EQUATION $\Delta\psi + K^2\psi = 0$ IN TWO- AND THREE-DIMENSIONAL MINKOWSKI SPACE*

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Abstract. Using well established methods we classify all coordinate systems in two-dimensional Minkowski space which allow a separation of variables of the Laplace equation $\Delta\psi + K^2\psi = 0$. With each such coordinate system we associate an operator L which determines the choice of basis functions. The connection between these operators and symmetric second order operators in the generators of the group $E(1, 1)$ is discussed. We also give a classification of all orthogonal separable coordinate systems in three-dimensional Minkowski space.

1. Introduction. The problem of the separation of variables for the Laplace equation $\Delta\psi + K^2\psi = 0$ in a two- or three-dimensional Euclidean space has been solved by Eisenhart [1]. He showed that to each of the distinct eleven types of orthogonal coordinate systems in Euclidean 3-space which have second order coordinate curves there corresponds a separation of variables of the Laplace equation. These coordinate systems were investigated by earlier workers from a purely geometrical point of view [2], [3]. The fundamental criterion that an orthogonal coordinate system in Riemannian n-space R_n with differential form

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{nn}(dx^n)^2 \tag{1.1}$$

admit a separation of variables for the Laplace equation is that the metric coefficients $g_{ii}, i = 1, \cdots, n$, be of Stäckel form. In other words, there is a Stäckel matrix S of functions

$$S = \begin{pmatrix} \Phi_{11}(x^1) \cdots \Phi_{1n}(x^1) \\ \Phi_{21}(x^2) \cdots \Phi_{2n}(x^2) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Phi_{n1}(x^n) \cdots \Phi_{nn}(x^n) \end{pmatrix} \tag{1.2}$$

such that

$$g_{ii} = \frac{\det S}{M_{i1}}, \qquad \frac{g^{1/2}}{\det S} = \prod_{i=1}^{n} f_i(x^i), \tag{1.3}$$

where M_{i1} is the cofactor of Φ_{i1} in $\det S$ and $g = |g_{11} \quad g_{22} \cdots g_{nn}|$. This form of the metric was originally investigated by Stäckel [4] and has been shown by Moon and Spencer [5] to be necessary and sufficient for the separation of variables in Riemannian n-space. It should also be mentioned at this stage that Olevski [6] has found all the coordinate systems in a four-dimensional space having constant positive or negative curvature for which the Laplace equation admits a separation of variables.

More recently there has been renewed interest in the problem of separation of variables in Riemannian spaces with constant curvature [7]–[9] (including zero curvature). This work has connected the separation of variables for the Laplace equation with various possible choices of basis for the harmonic functions of the Riemannian space in question. For the orthogonal groups $SO(p, q)$ and Euclidean

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groups $E(p, q)$ the connection of the Laplace equation with the corresponding group is clear. The Laplacian Δ is in fact one of the Casimir operators of the group in question. The Laplace equation then solves for basis functions of the most degenerate irreducible representations (IRs) of the group, i.e., IRs for which all the remaining Casimir operators introduce no new labels (*Note*: there is generally more than one Casimir operator labeling the IR of the group we are considering). Choosing a separable coordinate system for the Laplace equation amounts to choosing a basis set for degenerate unitary irreducible representation (UIR) of the corresponding invariance group. This has been shown explicitly for some of the lower dimensional orthogonal and Euclidean groups.

To each such basis (for the cases that have been evaluated) there corresponds a set of symmetric second order operators in the generators of the group whose eigenvalue equations together with the Laplacian completely specify the basis functions. The basis defining operators commute, if there is more than one, so as to be simultaneously diagonalizable. Each such basis set is inequivalent to any other set under inner automorphisms of the associated group, the addition of arbitrary real multiples of the Laplace operator Δ and the formation of arbitrary linear combinations of operators in each set. Furthermore in the cases investigated to date these bases sets exhaust all such possibilities under this equivalence relation. The known examples of such classification of second order operators are the works of Winternitz et al. [7] in which the little groups of the Poincaré group are treated, Makarov et al. [8] in which pairs of operators are found for the group $E(3)$ and Smorodinski and Tugov [9] who treat the proper Lorentz group $SO(3, 1)$.

The interesting result of this classification is that not only do these basis sets give those bases in which the defining operators are Casimir operators of a definite subgroup chain, but there are also bases in which the defining operators do not belong to a subgroup chain. This latter type of basis is what is known as a nonsubgroup basis. The simplest example of bases of this type is the case of the three-dimensional rotation group [7] $SO(3)$ in which we have one subgroup-type basis specified by making $\underline{J}^2$ and J_3 diagonal (this gives spherical harmonics) and one nonsubgroup-type basis in which $\underline{J}^2$ and $J_1^2 + rJ_2^2$ $(0 < r < 1)$ are diagonal (this gives the product of two Lamé polynomials as the harmonic functions). This nonsubgroup basis for $SO(3)$ has recently been investigated in detail by Patera and Winternitz [10]. The introduction of nonsubgroup-type bases is of considerable interest from the mathematical and physical points of view. From the mathematical point of view this is a group theoretic treatment of new special functions. (Systematic group theoretic treatments of special functions are already well developed; see, for instance, Talman [11], Vilenkin [12] and Miller [13]). More recently, Miller [14], [15] has examined parabolic-type nonsubgroup bases for the groups $E(2)$ and $E(3)$ and obtained new identities for the corresponding special functions. On the physical side Macfadyen and Winternitz [16] have used a Lamé function basis for the group $SO(2, 1)$ to obtain an explicitly crossing symmetric expansion of the spinless scattering amplitude. The corresponding Lamé basis for the rotation group examined by Patera and Winternitz [10] has its applications in the quantum mechanical treatment of the asymmetric top.

It is the purpose of this paper to investigate the problem of the separation of variables in two- and three-dimensional Minkowski space. In particular, we give a detailed treatment of the two-dimensional case and find the basis generating operators in terms of the generators of the group $E(1, 1)$. For the three-dimensional case we content ourselves with a classification of the differential forms corresponding to a separation of variables of the Laplace operator and the calculation of the corresponding special functions appearing in the solution. In doing this we obtain a further idea of the correspondence between separation of variables and quadratic operators in the group generators.

The content of the article is arranged as follows. In § 2 we discuss the group $E(1, 1)$ and some of its properties with regard to separation of variables. In § 3 we evaluate the different classes of inequivalent second order symmetric operators with respect to the group of distance preserving transformations in the pseudo-Euclidean plane (two-dimensional Minkowski space) viz. $E(1, 1)$. In § 4 we carry out the separation of variables for the two-dimensional Laplacian in the pseudo-Euclidean plane (two-dimensional Minkowski space). In § 5 we list the differential forms and special functions which are solutions of the Laplace equation for the case of three-dimensional Minkowski space. In § 6 we draw our conclusions from the results we have established.

2. The group $E(1, 1)$ and the separation of variables for the Laplace equation in the pseudo-Euclidean plane. The pseudo-Euclidean plane (i.e., two-dimensional Minkowski space) is such that the distance between two points (t_i, x_i), $i = 1,2$ (these are Cartesian or perpendicular coordinates), is

$$s^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2. \tag{2.1}$$

The distance s can thus be positive, zero or pure imaginary: The group of motions in this plane which preserve the distance s is the one-dimensional Poincaré or pseudo-Euclidean group $E(1, 1)$. A general transformation of this group on the defining t- and x-coordinates can be written in matrix form as

$$\begin{pmatrix} t' \\ x' \\ 1 \end{pmatrix} = \begin{pmatrix} \cosh a & \sinh a & s_1 \\ \sinh a & \cosh a & s_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ 1 \end{pmatrix}. \tag{2.2}$$

The Lie algebra commutation relations for the group $E(1, 1)$ are

$$[P_1, P_2] = 0, \quad [P_2, M] = P_1, \quad [M, P_1] = -P_2, \tag{2.3}$$

where P_1 and P_2 are the generators of translations along the t- and x- axes respectively and M is the generator of hyperbolic rotations. In the coordinate representation these generators are given by

$$P_1 = \frac{\partial}{\partial t}, \quad P_2 = \frac{\partial}{\partial x}, \quad M = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}. \tag{2.4}$$

There is one Casimir operator of the group $E(1, 1)$ viz. $\Delta = P_1^2 - P_2^2$, the Laplacian in the pseudo-Euclidean plane. Each complex eigenvalue $-K^2$ of this operator specifies an irreducible representation of $E(1, 1)$. These representations are also unitary if K^2 is real. The standard realization of these IRs is on the space of functions $f(b)$ over the real line with scalar product.

$$(f, h) = \int_{-\infty}^{\infty} f(b)\overline{h(b)}\, db < \infty. \tag{2.5}$$

The action of a general group element in this space is [12]

$$T_K(g)f(b) = \exp[iK(-s_1 \cosh a + s_2 \sinh a)]f(b - a) \tag{2.6}$$

with a, s_1, s_2 specifying the group element as in (2.2). For the unitary case the basis functions which span this space are the eigenfunctions $e^{i\tau b}$ (τ real) of the operator M. For an alternate realization of the IRs of $E(1, 1)$ and the properties of related matrix elements we refer the reader to the treatment of Vilenkin [12].

By way of a specific example we review here some of the pertinent facts for the corresponding problem for the group $E(2)$. This will help to illustrate some of the general remarks in the Introduction. For each of the four coordinate systems for which the two-dimensional Laplacian equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + K^2\psi = 0 \tag{2.7}$$

is separable there is a symmetric second order operator L which together with the Laplace equation (2.7) completely determines a basis for the corresponding UIR of $E(2)$ specified by K. The details are given in Table 1.

In Table 1, P_1 and P_2 are the generators of x and y translations and M generates the rotations. The two eigenvalue equations

$$\Delta\psi + K^2\psi = 0, \qquad L\psi = B\psi \tag{2.8}$$

then completely determine the corresponding coordinate system. The separation constant in the separated equations of the Laplace equation is related to the eigen-

TABLE 1
The separation of variables in the Euclidean plane and associated bases

Coordinate system	Laplace operator Δ	Additional operator L
Rectangular coordinates x, y	$\Delta = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$	$L = P_1^2 - P_2^2$
Polar coordinates $x = r\cos\phi, \quad y = r\sin\phi$	$\Delta = \dfrac{1}{r}\dfrac{\partial}{\partial r} r \dfrac{\partial}{r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\phi^2}$	$L = M^2$
Parabolic coordinates $x = \frac{1}{2}(\xi^2 - \eta^2), \quad y = \xi\eta$	$\Delta = \dfrac{1}{(\xi^2 + \eta^2)}\left(\dfrac{\partial^2}{\partial\xi^2} + \dfrac{\partial^2}{\partial\eta^2}\right)$	$L = MP_2 + P_2M$
Elliptic coordinates $x = \cosh\xi\cos\eta$, $y = \sinh\xi\sin\eta$	$\Delta = \dfrac{1}{(\sinh^2\xi + \sin^2\eta)}\left(\dfrac{\partial^2}{\partial\xi^2} + \dfrac{\partial^2}{\partial\eta^2}\right)$	$L = M^2 + P_1^2$

value B of the additional operator L. The four operators L correspond to a complete classification of all symmetric second order operators in the generators of $E(2)$ under the equivalence relation stated in the Introduction.

In the next section we shall investigate the corresponding relation between the separation of variables for the pseudo-Euclidean Laplacian $\Delta = P_1^2 - P_2^2$ and second order symmetric operators in the group generators. The corresponding problem for $E(1, 1)$ is however more complicated. For instance, it is not true that every coordinate system for which Δ separates variables covers all of the pseudo-Euclidean plane. An example of this is polar coordinates. t and x are then parametrized according to $t = r \cosh a$, $x = r \sinh a$. This parametrization only covers the quadrant $x \pm t > 0$. By similarly parametrizing the other remaining quadrants, all of the pseudo-Euclidean plane can be parametrized except for the lines $t = \pm x$. The Laplacian Δ then admits a separation of variables in each of these regions. This behavior is typical of the coordinate systems we shall introduce.

3. Classification of all inequivalent second order symmetric operators for the group $E(1, 1)$. The general problem we consider here is the classification of the operator

$$f = aM^2 + b_1(MP_1 + P_1M) + b_2(MP_2 + P_2M) + c_1P_1^2 + c_2P_1P_2 + c_3P_2^2, \tag{3.1}$$

where equivalence is defined in much the same sense as given in the Introduction, i.e., $f \sim f'$ if (i) f' can be obtained from f by an inner automorphism of the group $E(1, 1)$. Under such a group motion the generators transform according to

$$P_1 \to \cosh aP_1 - \sinh aP_2, \tag{3.2}$$

$$P_2 \to \sinh aP_1 + \cosh aP_2, \tag{3.3}$$

$$M \to M + (s_2 \cosh a - s_1 \sinh a)P_1 + (s_1 \cosh a - s_2 \sinh a)P_2 \tag{3.4}$$

and the general quadratic operator f transforms into $\bar{f}$ with the new coefficients given by

$$\begin{aligned}
\bar{a} &= a, \\
\bar{b}_1 &= as_1 + b_1 \cosh a - b_2 \sinh a, \\
\bar{b}_2 &= as_2 - b_1 \sinh a + b_2 \cosh a, \\
\bar{c}_1 &= as_1^2 + 2s_1b_1 \cosh a - 2s_1b_2 \sinh a + c_1 \cosh^2 a - c_2 \sinh a \cosh a \\
&\quad + c_3 \sinh^2 a, \\
\bar{c}_2 &= 2as_1s_2 + 2b_1(s_2 \cosh a - s_1 \sinh a) + 2b_2(s_1 \cosh a - s_2 \sinh a) \\
&\quad - 2(c_1 + c_3) \sinh a \cosh a + c_2 \cosh^2 a, \\
\bar{c}_3 &= as_2^2 - 2s_2b_1 \sinh a + 2s_2b_2 \cosh a + c_1 \sinh^2 a - c_2 \sinh a \cosh a \\
&\quad + c_3 \cosh^2 a.
\end{aligned} \tag{3.5}$$

Then from our definition $f \sim \bar{f}$, (ii) two such operators f, f' are also equivalent if they differ by a real multiple of Δ, i.e., if

$$f = \mu f' + \lambda\Delta, \quad \lambda, \mu \text{ real}, \quad \mu \neq 0. \tag{3.6}$$

In order to evaluate the different classes of operators f we consider each possibility and demonstrate mutual inequivalence by inspection.

Case 1. *Operators of the type* $f = c_1P_1^2 + c_2P_1P_2 + c_3P_2^2$ *with* $a = b_1 = b_2 = 0$. Without loss of generality we can take $c_1 = 0$. If in addition c_2 or c_3 are also zero, then we obtain the forms P_2^2 and P_1P_2 respectively. If $c_2 = \pm c_3$, then f can be made equivalent to one of the forms $(P_1 \pm P_2)^2$. Finally if $c_2 \gtrless c_3$, then by suitable choice of hyperbolic rotation angle f can be reduced to one of the forms P_2^2 or P_1P_2. The four inequivalent forms of f of this type are then $f \sim P_2^2$, P_1P_2, $(P_1 \pm P_2)^2$.

Case 2. *Operators for which* $a \neq 0$. For operators of this type we can always choose s_1, s_2 such that $b_1 = b_2 = 0$. Having ensured this condition initially it can be maintained under inner automorphism by making only hyperbolic rotations. The degrees of freedom for the coefficients c_1, c_2, c_3 then coincide with those of Case 1. The inequivalent forms of operators of this type are then

$$\begin{aligned} &f \sim M^2 + \alpha P_2^2, && f \sim M^2 + \alpha P_1P_2, \\ &f \sim M^2 + \alpha(P_1 \pm P_2)^2, && f \sim M^2, \end{aligned} \tag{3.7}$$

where $\alpha \gtrless 0$.

Case 3. The final type of operator is that corresponding to $a = 0, b_1, b_2 \neq 0$. There are a number of cases to consider here. If $b_1 > b_2$, then it is possible to choose a hyperbolic angle a such that $\bar{b}_2 = 0$. Then by appropriate choice of translations s_1 and s_2 such that $\bar{c}_1 + \bar{c}_3 = 0$ and $\bar{c}_2 = 0$, f can be reduced to the form $MP_1 + P_1M$. Similar remarks apply to the case $b_1 < b_2$ for which we obtain the form $MP_2 + P_2M$. There are two remaining cases to be considered. If $b_1 = b_2$ and (i) $c_2 = c_1 + c_3$, then $f \sim MP_1 + P_1M + MP_2 + P_2M$, (ii) $c_2 \neq c_1 + c_3$, then $f \sim MP_1 + P_1M + MP_2 + P_2M + \alpha(P_1 - P_2)^2$, $\alpha > 0$. These are the two possible cases. Similarly for the case $b_1 = -b_2$ there are two distinct inequivalent forms. We now list all the forms for the operators of type 3:

$$\begin{aligned} &f \sim MP_1 + P_1M, \qquad f \sim MP_2 + P_2M, \\ &f \sim MP_1 + P_1M + P_2M + MP_2 + \alpha(P_1 - P_2)^2, \qquad \alpha \geqq 0, \\ &f \sim MP_1 + P_1M - MP_2 - P_2M + \alpha(P_1 + P_2)^2, \qquad \alpha \geqq 0. \end{aligned} \tag{3.8}$$

This then completes the evaluation of the inequivalent operators f of the form (3.1) in the generators of $E(1, 1)$.

Subsequently it will prove convenient to consider in addition to the group motions of $E(1, 1)$ in the pseudo-Euclidean plane two further discrete (or improper) transformations which leave the Laplacian Δ invariant apart from, in one case, a change of sign. The two transformations are defined by the matrices

$$R = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.9}$$

In coordinate space these transformations correspond to R; $(t, x) \to (-t, x)$ (reflection through the x-axis) and $I:(t, x) \to (x, t)$ (permutation of coordinates). If these discrete transformations are added to the group of inner automorphisms we have used to define the equivalence of the operators f (i.e., added to the group $E(1, 1)$), we can reduce the number of inequivalent operators of this type considerably. We shall see that this is necessary in obtaining a correspondence between separable coordinates for the Laplace equation and symmetric second order operators of the type (3.1). It should also be noted here that the form of the Laplace equation in the pseudo-Euclidean plane is invariant under the scale transformation $S(t, x) \to (rt, rx)$ (r real) (for the nonunitary equation r could also be complex). The point to be made from this observation is that under S the generators of $E(1, 1)$ transform according to

$$S:M \to M, \qquad S:P_i \to \frac{1}{r}P_i, \qquad i = 1, 2. \tag{3.10}$$

This follows from the coordinate representation (2.4). Therefore a basis which is specified by the Laplacian Δ and an operator of the type (3.7), e.g., $M^2 + \alpha P_2^2$ ($\alpha > 0$), can equally well be specified by a normalized operator $M^2 + P_2^2$ having applied the scale transform S with $r = (\alpha)^{-1/2}$ to both the basis defining operators. In a similar way the operators in (3.8) with $\alpha \neq 0$ can be unit normalized (note in the case of the Euclidean plane the scale transformation has been used in Table 1 to normalize the interfocal distance). This normalization procedure although not always used by other authors does make our problem of classification more straightforward. In Table 2 we summarize the classification of operators of type (3.1) under the equivalence relation given in the beginning of this section. The effect of the discrete transformations R and I in reducing the number of inequivalent classes is also given. From the original 19 classes of such operators the discrete transformations reduce that number to 12.

TABLE 2

Quadratic symmetric operators in the generators of $E(1, 1)$ classified up to equivalence

	Operators for $E(1, 1)$	Operators for $E(1, 1) \oplus R(\oplus I)$
1	P_2^2	P_2^2
2	P_1P_2	P_1P_2
3–4	$(P_1 \pm P_2)^2$	$(P_1 + P_2)^2$
5–6	$M^2 \pm P_2^2$	$M^2 \pm P_2^2$
7–8	$M^2 \pm P_1P_2$	$M^2 + P_1P_2$
9–12	$M^2 \pm (P_1 \pm P_2)^2$	$M^2 \pm (P_1 + P_2)^2$
13	M^2	M^2
14	$MP_1 + P_1M$	$\left.\begin{matrix} MP_1 + P_1M \\ MP_2 + P_2M \end{matrix}\right\} \xrightarrow{I} MP_1 + P_1M$
15	$MP_2 + P_2M$	
16–17	$MP_1 + P_1M \pm MP_2 \pm P_2M$	$MP_1 + P_1M + MP_2 + P_2M$
18–19	$MP_1 + P_1M \pm MP_2 \pm P_2M + (P_1 \mp P_2)^2$	$MP_1 + P_1M + MP_2 + P_2M + (P_1 - P_2)^2$

4. The separation of variables for the Laplace equation in the pseudo-Euclidean plane. We now proceed to find all the different coordinate systems for which the Laplace equation

$$\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} + K^2\psi = 0 \tag{4.1}$$

allows the separation of variables. The criterion for the separability of a form of the type (1.1) has already been outlined in the Introduction. The means for making a complete classification of all separable coordinate systems in an n-dimensional Riemannian space has been given by Eisenhart [1]. He has shown that for such a space with metric diagonal as in (1.1) the conditions that the equations of the geodesics admit $n - 1$ quadratic first integrals of the form

$$a_{ij}\frac{dx_i}{ds}\frac{dx_j}{ds} = \text{const.} \tag{4.2}$$

together with some additional conditions (for which we refer to Eisenhart [1]) are necessary and sufficient for the metric to be in Stäckel form. The remaining constraints on the metric coefficients amount to the specification of the curvature of the space, i.e., the Riemann curvature tensor.

We now specialize to the case of two-dimensional Riemannian space viz. the pseudo-Euclidean plane. For the separation of the Laplacian corresponding to the metric

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 \tag{4.3}$$

(i.e., there exists a Stäckel matrix

$$S = \begin{pmatrix} \Phi_{11}(x^1) & \Phi_{12}(x^1) \\ \Phi_{21}(x^2) & \Phi_{22}(x^2) \end{pmatrix} \tag{4.4}$$

such that conditions (1.3) for $n = 2$ are satisfied), the necessary and sufficient condition that we have separation as given by Eisenhart [1] is given by the two equations

$$\begin{aligned} \frac{\partial^2 \log H_1^2}{\partial x_1\,\partial x_2} + \frac{\partial \log H_1^2}{\partial x_2}\cdot\frac{\partial \log H_2^2}{\partial x_1} &= 0, \\ \frac{\partial^2 \log H_2^2}{\partial x_1\,\partial x_2} + \frac{\partial \log H_1^2}{\partial x_2}\cdot\frac{\partial \log H_2^2}{\partial x_1} &= 0, \end{aligned} \tag{4.5}$$

where $g_{11} = e_1H_1^2$, $g_{22} = e_2H_2^2$ and e_i $(i = 1, 2)$ are ± 1 and determine the sign of g_{ii}. These equations imply that the metric coefficients can be written in the form

$$H_1^2 = X_1(\sigma_1 + \sigma_2), \qquad H_2^2 = X_2(\sigma_1 + \sigma_2), \tag{4.6}$$

where X_i and σ_i $(i = 1, 2)$ are functions of x_i only. For the purposes of studying the pseudo-Euclidean plane we take $e_1 = -e_2 = 1$. For the determination of the various types of coordinate system we need to specify the curvature of the space. For the pseudo-Euclidean plane this amounts to equating the one nonidentically zero component of the Riemannian curvature tensor R_{1221} to zero in order that the space be flat, i.e., have zero curvature.

This gives

$$R_{1221} = H_1^2\left(\frac{\partial^2 \log H_1}{\partial x_2^2} + \frac{\partial \log H_1}{\partial x_2}\cdot\frac{\partial}{\partial x_2}\log\frac{H_1}{H_2}\right) - H_2^2\left(\frac{\partial^2 \log H_2}{\partial x_1^2} + \frac{\partial \log H_2}{\partial x_1}\cdot\frac{\partial}{\partial x_1}\log\frac{H_2}{H_1}\right) = 0. \tag{4.7}$$

This equation together with the conditions (4.5) completely determines all the possible coordinate systems allowing separation of variables in (4.1). In evaluating these it is convenient to choose variables x_1, x_2 such that $\sigma_1 = x_1$, $\sigma_2 = -x_2$. (This is when σ_1, σ_2 are not constant functions.) We now enumerate the possibilities.

Type 1. X_i, σ_i $(i = 1, 2)$ *are not constant functions.* Substituting the expressions $H_1^2 = X_1(x_1 - x_2)$, $H_2^2 = X_2(x_1 - x_2)$ into equation (4.7), we get

$$2\left(\frac{1}{X_2} - \frac{1}{X_1}\right) + (x_1 - x_2)\left[\left(\frac{1}{X_1}\right)' + \left(\frac{1}{X_2}\right)'\right] = 0. \tag{4.8}$$

Differentiating twice with respect to x_2, this equation gives

$$(1/X_2)''' = 0 \tag{4.9}$$

so that $1/X_2 = ax_2^2 + bx_2 + c = f(x_2)$. Substituting this equation back into the first derivative of (4.8), we have that $1/X_1 = f(x_1)$. The general differential form of Type 1 is then

$$ds^2 = (x_1 - x_2)\left[\frac{dx_1^2}{f(x_1)} - \frac{dx_2^2}{f(x_2)}\right]. \tag{4.10}$$

The number of distinct forms of this type can be found by considering the various forms for the quadratic function $f(x)$. These possibilities are now enumerated.

(i) *$f(x)$ has two distinct roots.* $f(x)$ can then be taken in standard form $f(x) = 4x(x - 1)$ (we take the coefficient of x^2 to be positive with no loss of generality). The differential form is then

$$ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)}\right). \tag{4.11}$$

There are two possibilities for the ranges of the parameters x_1, x_2 such that we have a differential form which is locally pseudo-Euclidean. (*Note.* This is an extra requirement that should be born in mind throughout this derivation.) In other words, for small values of the parameters x_1, x_2 the differential form should be expressible in the form $ds^2 = A(u_1, u_2)(du_1^2 - du_2^2)$, $u_i = u_i(x_1, x_2)$, $i = 1, 2$. The two possibilities are:

(a) $x_1 > 1$, $x_2 < 0$;
(b) $1 > x_1, x_2 > 0$ or $1 < x_1, x_2 < \infty$.

The first of these systems is clearly a valid one as both $x_1 - x_2$ and $f(x)$ are positive. For the second possibility the relative sign between g_{11} and g_{22} is preserved (i.e., $\operatorname{sgn}(g_{11}/g_{22}) = -$) and this is sufficient to make this choice of coordinates locally pseudo-Euclidean. This is in contrast with the coordinate systems in the Euclidean plane in which the metric remains positive definite. This will be a feature of several of the coordinate systems we find for the pseudo-Euclidean plane.

(ii) *$f(x)$ has two equal roots.* We can then take $f(x) = 4x^2$ without loss of generality. The differential form is

$$ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1^2} - \frac{dx_2^2}{x_2^2}\right). \tag{4.12}$$

There are then two possibilities for x_1, x_2.

(a) $x_1, -x_2 > 0$;

(b) $x_1, x_2 > 0$.

The first corresponds to a differential form for which H_i^2, $i = 1, 2$, are always positive, the second type to a system which only preserves the relative sign of H_1^2 and H_2^2.

(iii) *$f(x)$ has two complex conjugate roots.* We can then take $f(x) = x^2 + 1$. The differential form is then

$$ds^2 = (x_1 - x_2)\left[\frac{dx_1^2}{x_1^2 + 1} - \frac{dx_2^2}{x_2^2 + 1}\right]. \tag{4.13}$$

There is no essential restriction on x_1, x_2 in this case other than that they be real.

(iv) *$f(x)$ is linear in x.* Then $f(x) = 4x$ and the differential form is

$$ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1} - \frac{dx_2^2}{x_2}\right), \tag{4.14}$$

where the restrictions on x_1, x_2 can be reduced to $x_1, x_2 > 0$.

(v) *$f(x)$ is a constant*, viz. 4. The differential form is then

$$ds^2 = \frac{x_1 - x_2}{4}(dx_1^2 - dx_2^2). \tag{4.15}$$

There is no restriction on the range of x_1, x_2.

This is the complete list of differential forms of Type 1.

There are a number of further possibilities to be considered.

Type 2. $\sigma_2 = \text{const.}$ Then the metric coefficients can be chosen without loss of generality to be $H_1^2 = 1$, $H_2^2 = X_2^2$. Substituting these into the equation (4.7) we get that $X_1'' = 0$. By suitable choice of variables the corresponding differential form becomes

$$ds^2 = dx_1^2 - x_1^2\, dx_2^2. \tag{4.16}$$

The case $\sigma_1 = \text{const.}$ does not yield a new differential form apart from a minus sign. The remaining possibilities for which σ_1 and σ_2 are constant or H_i^2 are constant yields the usual Cartesian metric

$$ds^2 = dt^2 - dx^2. \tag{4.17}$$

We now proceed to systematically tabulate the various coordinate systems whose differential forms we have evaluated. In doing this we give the relation of the variables to the Cartesian variables t and x, illustrate and find the coordinate curves, find the additional operator necessary to specify the coordinate system and find the solutions of the various separated forms of the Laplace equation and the special functions appearing in these solutions.

It is perhaps appropriate to mention at this point that we will consider only those operators derived in the previous section for the group $E(1, 1)$ which are also inequivalent under the discrete transformations R and I. This proves to be an added convenience and considerably reduces the amount of coordinate systems to be classified.

$$\text{(4.18)} \qquad \text{(I)} \quad ds^2 = dt^2 - dx^2.$$

This is the case of Cartesian coordinates in the pseudo-Euclidean plane. The solution of the Laplace equation is a linear combination of the solutions

$$\text{(4.19)} \qquad \psi = e^{i\alpha t} e^{i\beta x}, \qquad \alpha^2 - \beta^2 = K^2$$

for fixed $|\alpha|$ and $|\beta|$. The coordinate curves are just the lines $t =$ const. and $x =$ const. The additional operator specifying this coordinate system will be chosen as $L = P_1 P_2$ with eigenvalue $-\alpha\beta$ (see later in this section for a more complete discussion).

$$\text{(4.20)} \qquad \text{(II)} \quad ds^2 = dx_1^2 - x_1^2\, dx_2^2.$$

This is the case of polar coordinates

$$\text{(4.21)} \qquad t = x_1 \cosh x_2, \qquad x = x_1 \sinh x_2,$$

where $0 \leqq x_1 < \infty$, $-\infty < x_2 < \infty$.

The additional operator specifying this coordinate system is $L = M^2$. The separation equations have the form

$$\text{(4.22)} \qquad \left(\frac{d^2}{dx_1^2} + \frac{\tau^2}{x_1^2} + K^2\right)\psi_1 = 0, \qquad L\psi_2 = \frac{d^2\psi_2}{dx_2^2} = -\tau^2\psi_2.$$

The solution of the Laplace equation is then

$$\text{(4.23)} \qquad \psi = \psi_1\psi_2 = x_1^{1/2} C_\nu(K x_1)\, e^{i\tau x_2},$$

where $C_\nu(z)$ is a solution of Bessel's equation (i.e., a linear combination of $J_\nu(z)$, $Y_\nu(z)$) and $\nu^2 = \frac{1}{4} - \tau^2$. The parametrization (4.21) only covers that sector of the pseudo-Euclidean plane given by $x \pm t > 0$. The coordinate curves have the equations

$$\text{(4.24)} \qquad x = t \tanh x_2, \qquad t^2 - x^2 = x_1^2$$

and are illustrated in Fig. 1.

Similar coordinate curves can be set up in the three other quadrants and the Laplace equation separated. The various parametrizations of t and x are then related by the discrete transformations R and I written down previously.

$$\text{(4.25)} \qquad \text{(III)} \quad ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1} - \frac{dx_2^2}{x_2}\right).$$

It is convenient to write $x_1 = \xi^2$, $x_2 = \eta^2$. Equation (4.25) is then

$$\text{(4.26)} \qquad ds^2 = (\xi^2 - \eta^2)(d\xi^2 - d\eta^2)$$

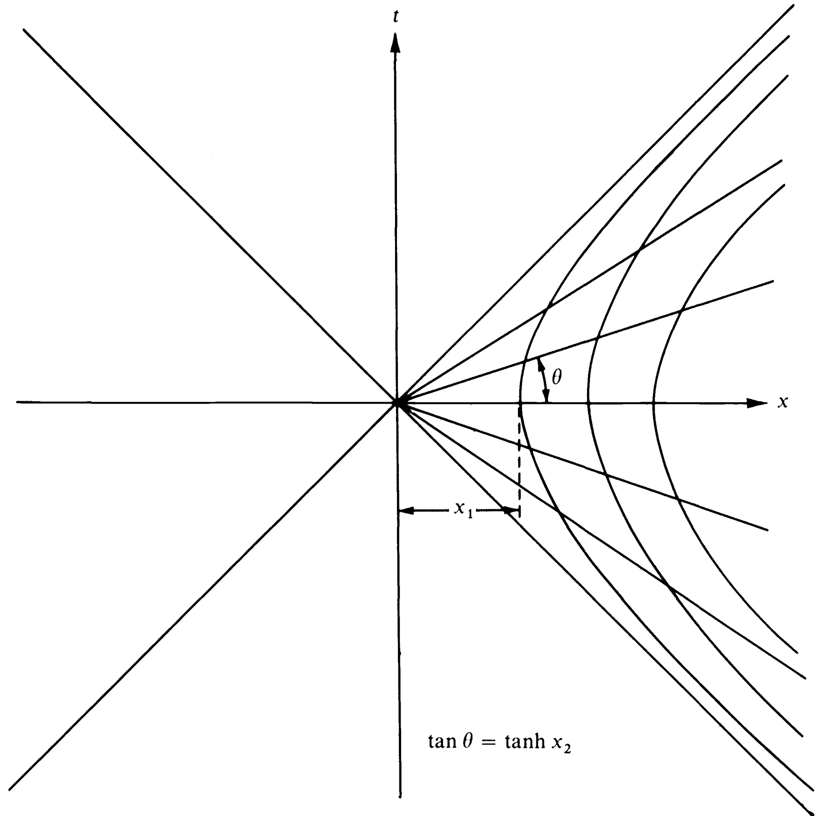

FIG. 1

and this corresponds to the choice of parabolic-type coordinates in the pseudo-Euclidean plane, viz.

(4.27) $t = \frac{1}{2}(\xi^2 + \eta^2), \quad x = \xi\eta, \qquad -\infty < \xi < \infty, \quad 0 \leqq \eta < \infty.$

The additional operator required to specify this coordinate system is

$$L = MP_2 + P_2M = \xi^2\frac{\partial^2}{\partial\eta^2} - \eta^2\frac{\partial^2}{\partial\xi^2}. \tag{4.28}$$

The separated Laplace equation then reduces to two ordinary differential equations of the form

$$\left(\frac{d^2}{dx^2} + K^2x^2 + B\right)\Phi(x) = 0, \tag{4.29}$$

where $x = \xi, \eta$. The separation constant B is generated by the eigenvalue equation of the operator L, i.e., $L\psi = -B\psi$. Equation (4.29) is a form of the parabolic cylinder equation having solutions

$$\Phi_A(x) = \sum_{\varepsilon = \pm 1} A_\varepsilon D_{(-1+i\lambda)/2}[\varepsilon(1 + i)\sqrt{K}x] \tag{4.30}$$

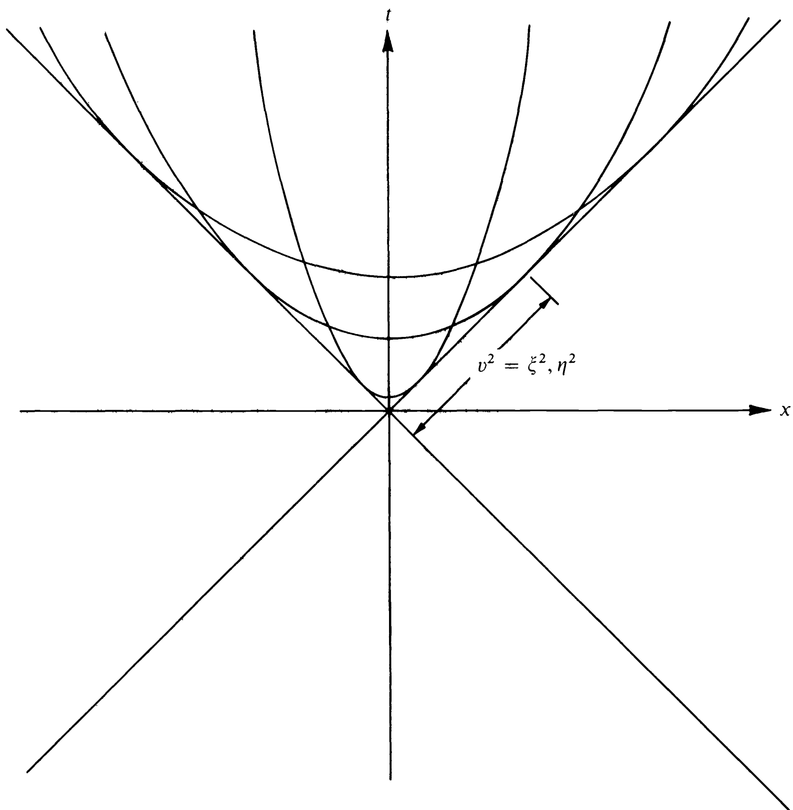

FIG. 2

with $\lambda = B/K$. The solution of the separated Laplace equation is then the product of two Φ-functions $\psi = \Phi_A(\xi)\Phi_B(\eta)$. The coordinate curves are parabolas with equations

$$2t = v^2 + x^2/v^2, \qquad v = \xi, \eta. \tag{4.31}$$

These coordinate curves are illustrated in Fig. 2. They form a set of parabolas with the t-axis as the axis of symmetry. The curves completely fill the quadrant $t \mp x > 0$ and also parametrize the bounding lines of this quadrant, i.e., the lines $t = \pm x$ $(t > 0)$. All that has been discussed thus far also holds true for the same set of coordinate curves in the x-axis reflected quadrant, i.e., under the reflection R. If, however, we choose to take this set of curves in the quadrant $x \pm t > 0$, the operator needed in addition to the Laplacian Δ is $L = MP_1 + P_1M$. This, however, is equivalent to $MP_2 + P_2M$ if we include the inversion operation I in our group.

$$\text{(IV)} \qquad ds^2 = \frac{x_1 - x_2}{4}(dx_1^2 - dx_2^2). \tag{4.32}$$

This metric corresponds to a parabolic-type coordinate system in which the Cartesian coordinates t and x are given by

$$\begin{aligned} t &= -\tfrac{1}{2}(x_1 - x_2)^2 - (x_1 + x_2), \\ x &= -\tfrac{1}{2}(x_1 - x_2)^2 + (x_1 + x_2), \end{aligned} \qquad -\infty < x_i < \infty, \quad i = 1, 2. \tag{4.33}$$

The additional operator necessary to specify the coordinate system is

$$L = MP_1 + P_1M + MP_2 + P_2M + (P_1 - P_2)^2 \tag{4.34}$$
$$= \frac{1}{(x_1 - x_2)}\left(x_1\frac{\partial^2}{\partial x_2^2} - x_2\frac{\partial^2}{\partial x_1^2}\right).$$

The separated equations then become

$$\left(\frac{\partial^2}{\partial x_i^2} + K^2x_i + B\right)\psi_i(x_i) = 0, \qquad i = 1,2, \tag{4.35}$$

where the eigenvalue equation of the additional operator is $L\psi = B\psi$. Here as usual $\psi = \psi_1\psi_2$. The general solution of (4.35) has the form

$$\psi_i = \left(x + \frac{B}{K}\right)^{1/2} C_{1/3}\left(\frac{2}{3}K\left(x + \frac{B}{K}\right)^{3/2}\right). \tag{4.36}$$

This solution can also be expressed in terms of the Airy functions $\mathrm{Ai}(-z)$ and $\mathrm{Bi}(-z)$ with $z = x + B/K$. The solution of the Laplace equation is then the product of two such solutions. The equations of the coordinate curves are

$$t + x = (2x_i + t - x)^2, \qquad i = 1,2. \tag{4.37}$$

These are parabolas with axis of symmetry parallel to the line $t = x$ and all touching the line $t = -x$ on the same side.

The situation is illustrated in Fig. 3.

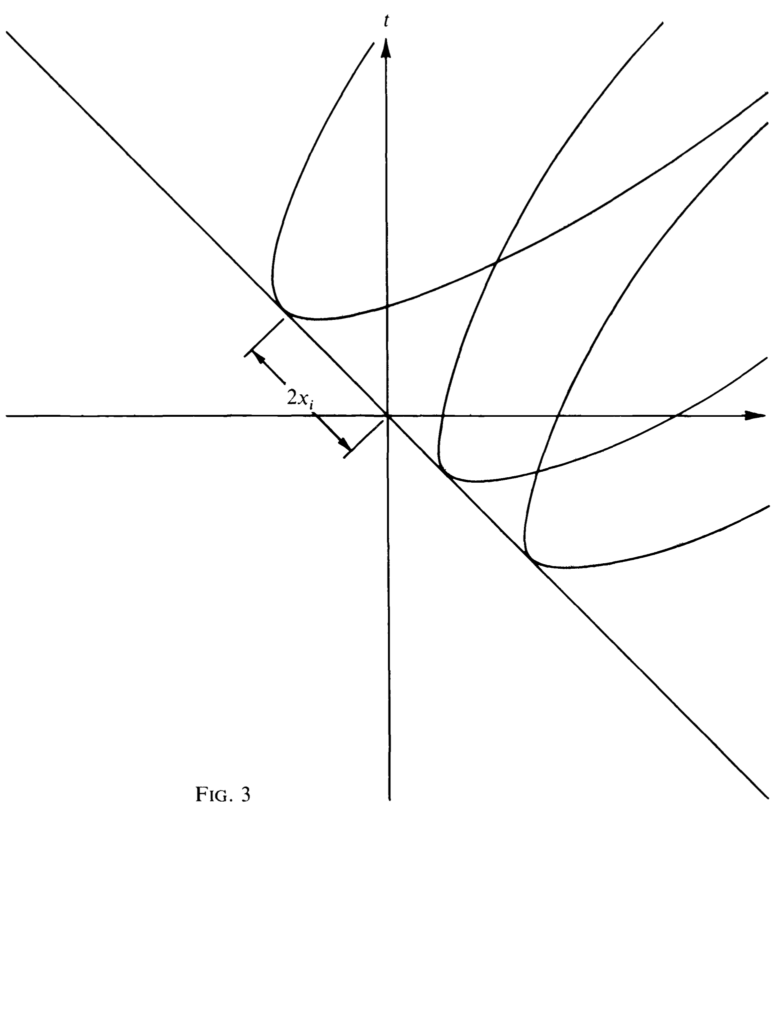

FIG. 3

For $-\infty < x_i < \infty$, $i = 1, 2$, the parametrization (4.33) covers the half-plane $t + x > 0$. The half-plane $t + x < 0$ can be covered by a similar set of curves by applying the inversion operator IRIR: $(t, x) \to (-t, -x)$.

$$\text{(4.38)} \qquad \text{(V)} \quad ds^2 = (x_1 - x_2)\left(\frac{dx_1^2}{x_1^2 + 1} - \frac{dx_2^2}{x_2^2 + 1}\right).$$

It is convenient to choose new variables y_i, $i = 1, 2$, such that $x_i = \sinh y_i$. The differential form is then

$$\text{(4.39)} \qquad ds^2 = (\sinh y_1 - \sinh y_2)(dy_1^2 - dy_2^2).$$

The Cartesian coordinates can then be written in terms of these variables as

$$\text{(4.40)} \qquad \begin{aligned} 2t &= \cosh \tfrac{1}{2}(y_1 - y_2) + \sinh \tfrac{1}{2}(y_1 + y_2), \\ 2x &= \cosh \tfrac{1}{2}(y_1 - y_2) + \sinh \tfrac{1}{2}(y_1 + y_2), \end{aligned} \qquad -\infty < y_i < \infty, \quad i = 1, 2.$$

The additional operator specifying this coordinate system is

$$\text{(4.41)} \quad L = M^2 - P_1 P_2 = \frac{4}{(\sinh y_1 - \sinh y_2)}\left(\sinh y_1 \frac{\partial^2}{\partial y_2^2} - \sinh y_2 \frac{\partial^2}{\partial y_1^2}\right).$$

The separated equations then become

$$\text{(4.42)} \qquad \left(\frac{d^2}{dy_i^2} + K^2 \sinh y_i + B\right)\psi_i(y_i) = 0, \qquad i = 1, 2,$$

where $L\psi = 4B\psi$.

Equation (4.42) is a form of the Mathieu equation having the general solution

$$\text{(4.43)} \qquad \psi_i(y_i) = A ce_m(iz, q) + B fe_m(iz, q),$$

where $2z = y_i - i\pi/2$, $q = 2iK^2$. Here we are using the notation for Mathieu functions as found, for instance, in the book by Moon and Spencer [17]. The solution ψ we seek is then the product of two solutions of the type (4.43). The coordinate curves for this coordinate system are given by the equations

$$\text{(4.44)} \qquad \begin{aligned} &4e^{-y_i}[e^{y_i}(t - x) + (t + x)][e^{y_i}(t + x) - (t - x)] \\ &\quad = (e^{2y_i} + 1)^2, \qquad i = 1, 2. \end{aligned}$$

These are rectangular hyperbolas with asymptotes rotated through an angle α given by $\tan \alpha = e^{y_i}$.

The situation is depicted in Fig. 4.

The coordinates in this case cover all of the pseudo-Euclidean plane.

$$\text{(4.45)} \qquad \text{(VI)} \quad ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1^2} - \frac{dx_2^2}{x_2^2}\right), \qquad x_1, -x_2 > 0.$$

It is convenient here to change the variables so that $x_1 = 4e^{u_1}$, $x_2 = -4e^{u_2}$. The differential form is then

$$\text{(4.46)} \qquad ds^2 = (e^{2u_1} + e^{2u_2})(du_1^2 - du_2^2).$$

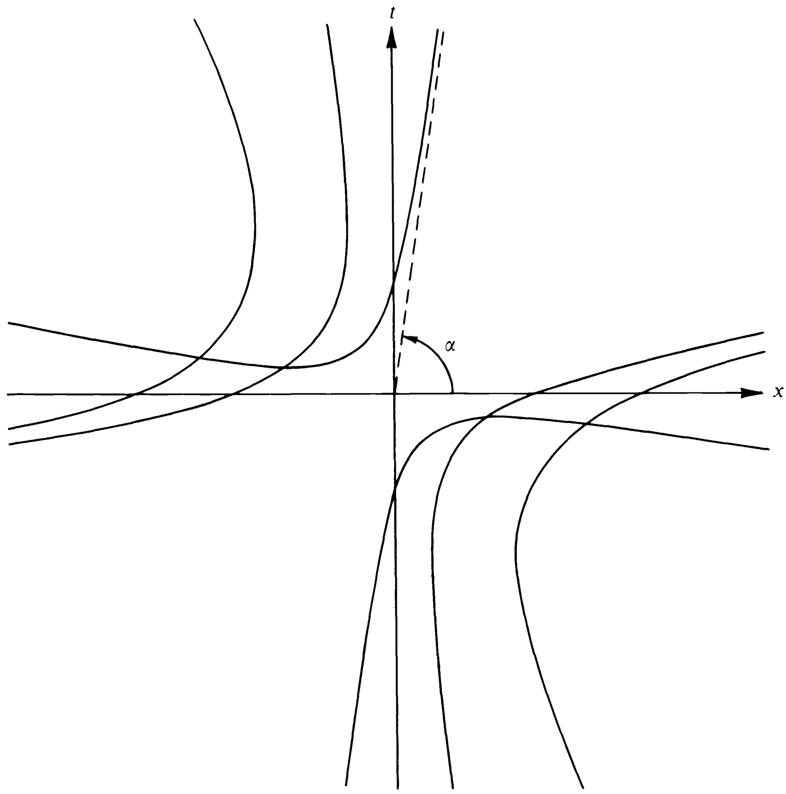

FIG. 4

The Cartesian coordinates in terms of these variables are

$$
(4.47)\qquad \begin{aligned} t &= \sinh(u_1 - u_2) + e^{u_1+u_2}, \\ x &= \sinh(u_1 - u_2) - e^{u_1+u_2}, \end{aligned} \qquad -\infty < u_i < \infty, \quad i = 1, 2.
$$

The additional operator specifying the coordinate system is

$$
(4.48)\qquad L = M^2 + (P_1 + P_2)^2 = \frac{1}{e^{2u_1} + e^{2u_2}}\left(e^{2u_2}\frac{\partial^2}{\partial u_1^2} + e^{2u_1}\frac{\partial^2}{\partial u_2^2}\right),
$$

and the separated equations become

$$
(4.49a)\qquad \left(\frac{d^2}{du_1^2} + K^2 e^{2u_1} - \nu^2\right)\psi_1(u_1) = 0,
$$

$$
(4.49b)\qquad \left(\frac{d^2}{du_2^2} - K^2 e^{2u_2} - \nu^2\right)\psi_2(u_2) = 0.
$$

The solution of these equations has the form

$$
(4.50)\qquad \psi = \psi_1\psi_2 = C_\nu(K e^{u_1})C_\nu(iK e^{u_2})
$$

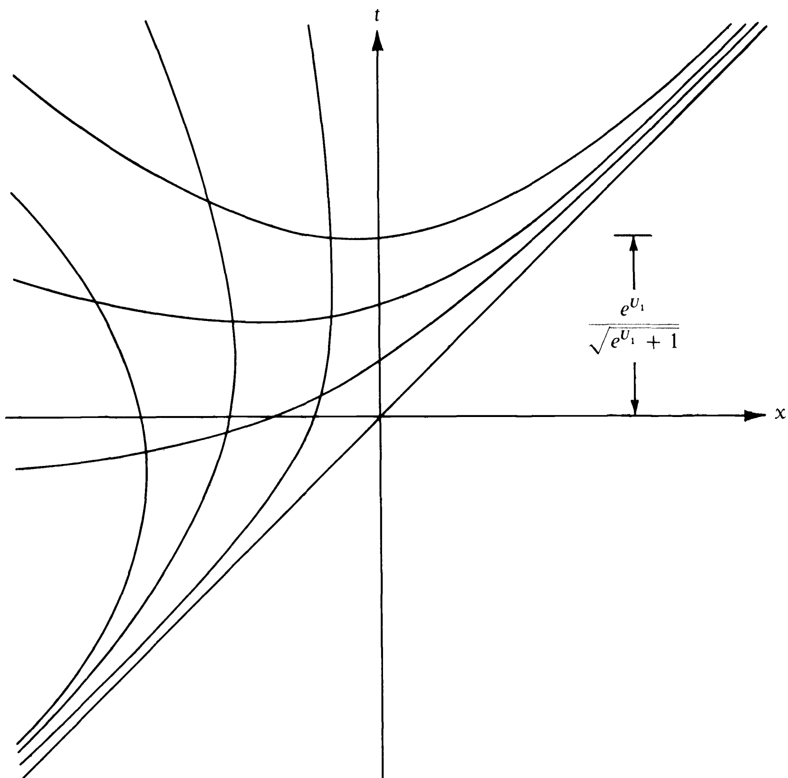

FIG. 5

with $C_\nu(z)$ as in (4.23). ν^2 is the eigenvalue of L (i.e., $L\psi = \nu^2\psi$). The equations of the coordinate curves are

$$\pm e^{u_i}(t^2 - x^2) = e^{2u_i} - (t-x)^2 \tag{4.51}$$

with the $+$ or $-$ sign being taken according as $i = 1$ or 2 respectively.

These coordinate curves are sections of hyperbolas as shown in Fig. 5. The parametrization (4.47) completely covers the region $t > x$, $-\infty < t + x < \infty$. A similar set of curves reflected about the line $t = x$ can parametrize the remaining half-plane. This merely involves the inversion IRIR: $(t, x) \to (-t, -x)$ of the Cartesian coordinates.

(VII) This coordinate system has the same differential form as (VI) with the exception that $x_1, x_2 > 0$.

Putting $e^{u_i} = x_i$, $i = 1, 2$, the differential form is

$$ds^2 = (e^{2u_1} - e^{2u_2})(du_1^2 - du_2^2) \tag{4.52}$$

with Cartesian coordinates

$$\begin{aligned} t &= \cosh(u_1 - u_2) + e^{u_1+u_2}, \\ x &= \cosh(u_1 - u_2) - e^{u_1+u_2}, \end{aligned} \qquad -\infty < u_i < \infty, \quad i = 1, 2. \tag{4.53}$$

The additional operator L required to specify the coordinate system is

$$L = (P_1 + P_2)^2 - M^2 = \frac{1}{(e^{2u_1} - e^{2u_2})}\left(e^{2u_2}\frac{\partial^2}{\partial u_1^2} - e^{2u_1}\frac{\partial^2}{\partial u_2^2}\right). \tag{4.54}$$

The separation equations then reduce to two Bessel-type equations of the form (4.49a) for both u_1 and u_2. So the solution of the Laplace equation has the form

$$\psi = C_\nu(K\,e^{u_1})C_\nu(K\,e^{u_2}), \tag{4.55}$$

where ν^2 is again the eigenvalue of the operator L. The coordinate curves are given by the equations

$$t^2 - x^2 = 2\,e^{2u_i} + \tfrac{1}{2}e^{-2u_i}(t - x)^2. \tag{4.56}$$

These are sections of hyperbolas and are illustrated in Fig. 6.

The parametrization as written in (4.53) only covers the region $t + x > 0$, $t > x + 1$. Further sets of such curves can be introduced by applying the operators I and R to equations (4.53). However, the strip $|t - x| < 1$ cannot be covered by such curves. This then is the first example of a coordinate system which cannot be made to cover all of the pseudo-Euclidean plane.

$$\text{(VIII)}\quad ds^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{x_1(x_1 - 1)} - \frac{dx_2^2}{x_2(x_2 - 1)}\right) \tag{4.57}$$

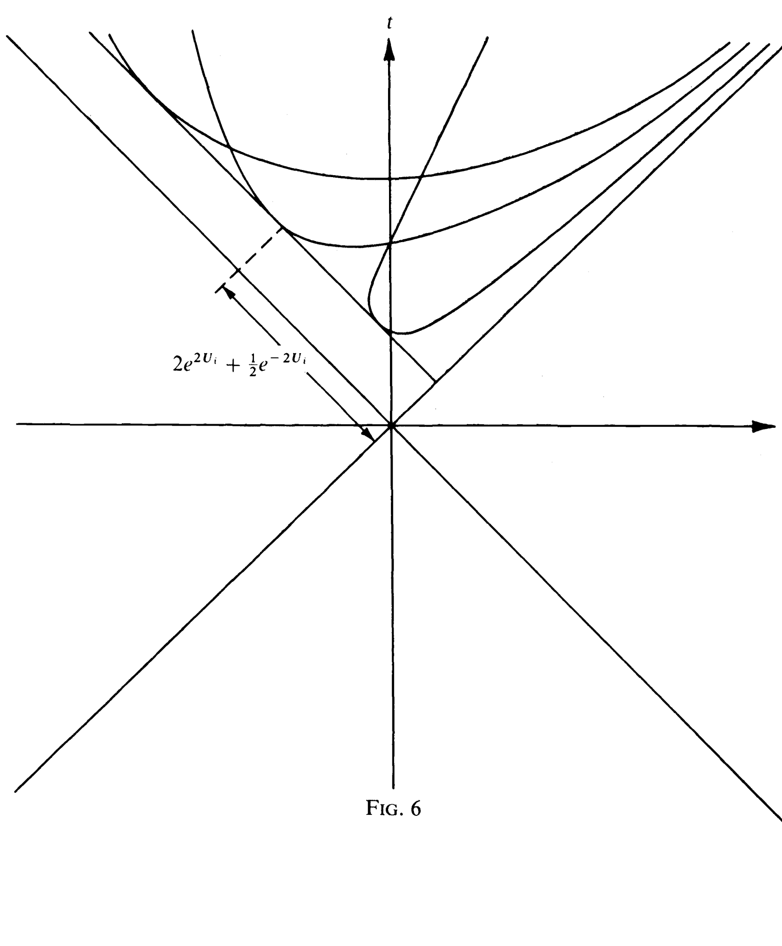

FIG. 6

with the parameters in the range $x_1 > 1$, $x_2 < 0$.

Putting $x_1 = \cosh^2 a$ and $x_2 = -\sinh^2 b$ the differential form can be written

$$ds^2 = (\cosh^2 a + \sinh^2 b)(da^2 - db^2) \tag{4.58}$$

and the corresponding Cartesian coordinates are

$$t = \sinh a \cosh b, \qquad x = \cosh a \sinh b, \qquad -\infty < a, b < \infty, \tag{4.59}$$

The additional operator L is

$$L = M^2 + P_2^2 = \frac{1}{\cosh^2 a + \sinh^2 b}\left(\cosh^2 a \frac{\partial^2}{\partial b^2} + \sinh^2 b \frac{\partial^2}{\partial a^2}\right) \tag{4.60}$$

and the separated equations have the form

$$\left(\frac{d^2}{da^2} + \frac{1}{2}K^2 \cosh 2a + \lambda\right)\psi_1(a) = 0, \tag{4.61a}$$

$$\left(\frac{d^2}{db^2} - \frac{1}{2}K^2 \cosh 2b + \lambda\right)\psi_2(b) = 0 \tag{4.61b}$$

with $\lambda = B + \frac{1}{2}K^2$.

These are Mathieu equations with solutions of the form (4.43). For (4.61a), $z = a$, $q = \frac{1}{4}K^2$, and for (4.61b), $z = b$, $q = -\frac{1}{4}K^2$. B is the eigenvalue of L. The general solution of the Laplace equation is the product of two solutions of the Mathieu equation. The coordinate curves are the families of hyperbolas with equations

$$\frac{t^2}{\sinh^2 a} - \frac{x^2}{\cosh^2 a} = 1, \qquad \frac{x^2}{\sinh^2 b} - \frac{t^2}{\cosh^2 b} = 1. \tag{4.62}$$

These curves are illustrated in Fig. 7. They clearly parametrize all the pseudo-Euclidean plane.

(IX) This is the second type of coordinate system with differential form as in (4.57). It is convenient to choose two different parametrizations of x_1, x_2.

(i) If $0 < x_1, x_2 < 1$, we choose $x_1 = \sin^2 \alpha$, $x_2 = \sin^2 \beta$; the differential form is

$$ds^2 = (\sin^2 \alpha - \sin^2 \beta)(d\alpha^2 - d\beta^2). \tag{4.63}$$

(ii) If $1 < x_1, x_2 < \infty$, we choose $x_1 = \sinh^2 a$, $x_2 = \sinh^2 b$; the differential form is

$$ds^2 = (\sinh^2 a - \sinh^2 b)(da^2 - db^2). \tag{4.64}$$

The Cartesian coordinates are

$$\begin{gathered} t = \cosh a \cosh b, \qquad x = \sinh a \sinh b, \\ -\infty < a < \infty, \quad 0 \leqq b < \infty, \end{gathered} \tag{4.65}$$

and

$$\begin{gathered} t = \cos \alpha \cos \beta, \qquad x = \sin \alpha \sin \beta, \\ 0 < \alpha < 2\pi, \quad 0 \leqq \beta < \pi, \end{gathered} \tag{4.66}$$

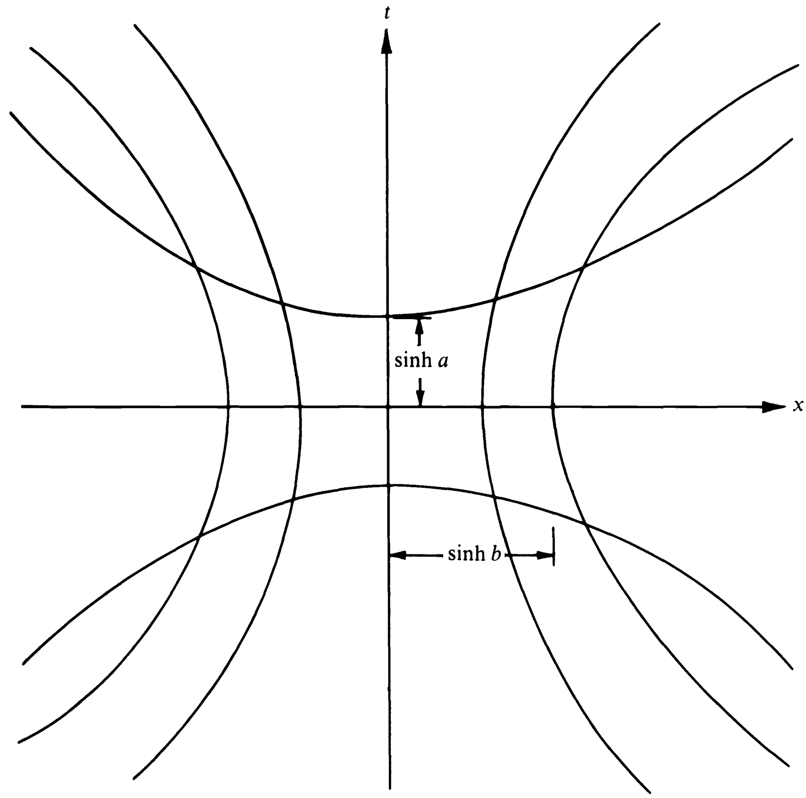

FIG. 7

The additional operator L which specifies the coordinate system is

$$(4.67)\quad \begin{aligned} L = M^2 - P_2^2 &= \frac{1}{\sinh^2 a - \sinh^2 b}\left(\sinh^2 a \frac{\partial^2}{\partial b^2} - \sinh^2 b \frac{\partial^2}{\partial a^2}\right) \\ &= \frac{1}{\sin^2 \alpha - \sin^2 \beta}\left(\sin^2 \beta \frac{\partial^2}{\partial \alpha^2} - \sin^2 \alpha \frac{\partial^2}{\partial \beta^2}\right). \end{aligned}$$

The separation equations assume the two forms

$$(4.68)\quad \left(\frac{d^2}{dx^2} + \frac{1}{2}K^2 \cosh 2x + \lambda_1\right)\phi(x) = 0, \qquad x = a, b,$$

$$(4.69)\quad \left(\frac{d^2}{dx^2} - \frac{1}{2}K^2 \cos 2x + \lambda_2\right)\theta(x) = 0, \qquad x = \alpha, \beta,$$

where $\lambda_1 = B - \frac{1}{2}K^2$, $\lambda_2 = B + \frac{1}{2}K^2$ and B is the eigenvalue of L. Equation (4.68) has a solution in terms of Mathieu functions of the form (4.43) with $q = \frac{1}{4}K^2$ and $z = x$. The solution of (4.69) is a linear combination of periodic solutions of the Mathieu equation of the form

$$(4.70)\quad \theta(x) = Ace_m(x, q) + Bfe_m(x, q)$$

with $q = \frac{1}{4}K^2$. The solution of Laplace's equation for this coordinate system is either the product of two nonperiodic solutions of the Mathieu equation of the form (4.43) or two periodic solutions of this equation of type (4.70). The coordinate curves are given by the equations

$$\frac{t^2}{\cosh^2 y} - \frac{x^2}{\sinh^2 y} = 1, \qquad y = a, b, \tag{4.71}$$

$$\frac{t^2}{\cos^2 y} + \frac{x^2}{\sin^2 y} = 1, \qquad y = \alpha, \beta. \tag{4.72}$$

The coordinate curves are illustrated in Fig. 8.

The nine orthogonal coordinate systems we have found allow the separation of variables for the Laplace equation in the pseudo-Euclidean plane. There are two cases in which the resulting parametrization cannot be made to cover the whole plane even with the addition of the discrete transformations R and I. The one feature to be noticed is that the basis defining operator we have called L does not run through all of the second column of Table 2 as all the orthogonal coordinate systems are considered. In addition, there appears to be no unique choice of additional operator for the Cartesian coordinate system. We now proceed to resolve this question by considering further nonorthogonal coordinate systems for which the Laplace equation separates variables. Such coordinates have previously been considered in the case of three-dimensional Euclidean space by Weatherburn [18]

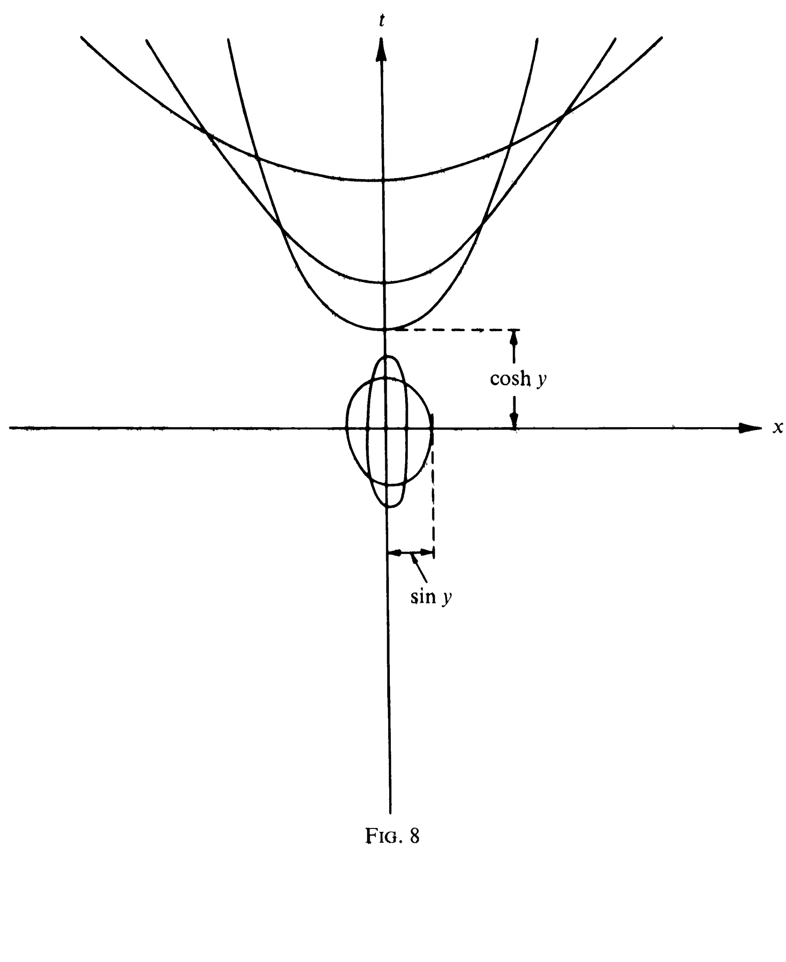

FIG. 8

but not from the point of view of the separation of variables. We now proceed to evaluate all such systems for the pseudo-Euclidean plane.

Consider a coordinate system with the differential form

$$ds^2 = a\,dx_1^2 + b\,dx_2^2 + 2h\,dx_1\,dx_2. \tag{4.73}$$

The Laplacian for this system is then (see, for instance, [19])

$$\nabla^2 = H_{11}\frac{\partial^2}{\partial x_1^2} + H_{12}\frac{\partial^2}{\partial x_1\,\partial x_2} + H_{22}\frac{\partial^2}{\partial x_2^2} + H_1\frac{\partial}{\partial x_1} + H_2\frac{\partial}{\partial x_2}, \tag{4.74}$$

where

$$\begin{gathered} H_{11} = b/g, \quad H_{12} = 2h/g, \quad H_{22} = a/g, \\ H_1 = \frac{1}{g^{1/2}}\left[\frac{\partial}{\partial x_2}\left(\frac{b}{g^{1/2}}\right) - \frac{\partial}{\partial x_2}\left(\frac{h}{g^{1/2}}\right)\right], \\ H_2 = \frac{1}{g^{1/2}}\left[\frac{\partial}{\partial x_2}\left(\frac{a}{g^{1/2}}\right) - \frac{\partial}{\partial x_1}\left(\frac{h}{g^{1/2}}\right)\right], \end{gathered} \tag{4.75}$$

and $g = |\det(g_{ij})| = |ab - h^2|$ (a, b and h are functions of x_1 and x_2, $h \neq 0$). If we seek a solution of (4.1) in the separable form $\psi = A(x_1)B(x_2)$, this requires that the equation

$$H_{11}\frac{A''}{A} + H_{12}\frac{A'B'}{AB} + H_{22}\frac{B''}{B} + H_1\frac{A'}{A} + H_2\frac{B'}{B} + K^2 = 0 \tag{4.76}$$

admit a separation of variables. This is only possible if $H_{11} = H_1 = 0$, i.e., if $b = 0$ (or equivalently $a = 0$). Equation (4.76) may then be written in the form

$$H_{12}\frac{A'}{A} = -\left(K^2 + H_2\frac{B'}{B} + H_{22}\frac{B''}{B}\right)\frac{B}{B'}. \tag{4.77}$$

Now for this equation to be in separable form we require that $a = a(x^2)$, $h = h(x^2)$. Without loss of generality the resulting differential form then can be written in the form

$$ds^2 = a(x_2)\,dx_1^2 + dx_1\,dx_2. \tag{4.78}$$

The requirement that the space be flat, i.e., $R_{1221} = 0$, gives $a'' = 0$ so that

$$a = cx_2 + d. \tag{4.79}$$

There are then three possible forms of ds^2, viz.

$$\text{(I)} \quad c = d = 0, \quad ds^2 = dx_1\,dx_2, \tag{4.80}$$

$$\text{(II)} \quad c = 0, \quad d \neq 0, \quad ds^2 = dx_1^2 + dx_1\,dx_2, \tag{4.81}$$

$$\text{(III)} \quad c, d \neq 0, \quad ds^2 = x_2\,dx_1^2 + dx_1\,dx_2. \tag{4.82}$$

We now give the coordinate systems corresponding to these differential forms and include the treatment of the conventional Cartesian coordinates.

1. *Cartesian coordinates of type* I (i.e., ordinary Cartesian coordinates). The Laplacian is $\Delta = \partial^2/\partial t^2 - \partial^2/\partial x^2$. A convenient choice of additional operator L

is $P_1P_2 = \partial^2/\partial t\,\partial x$. The separation equations for the solutions of (4.1) and $P_1P_2\psi = B\psi$ have the form

$$\frac{d\psi_1}{dt} = \alpha\psi_1, \quad \frac{d\psi_2}{dx} = \beta\psi_2, \quad \alpha\beta = B \tag{4.83}$$

with $\beta^2 - \alpha^2 = K^2$. These equations are equivalent to those obtained by the separation of the Laplace equation in the absence of any additional operator L viz. $\psi_1'' = \alpha^2\psi_1$, $\psi_2'' = \beta^2\psi_2$.

2. *Cartesian coordinates of type* II. In this case we choose coordinates such that

$$t' = t + x, \quad x' = t - x. \tag{4.84}$$

The Laplacian then has the form $\Delta = 4(\partial^2/\partial t'\,\partial x')$ and the generators of translations are

$$P_1 = \frac{\partial}{\partial t'} + \frac{\partial}{\partial x'}, \qquad P_2 = \frac{\partial}{\partial t'} - \frac{\partial}{\partial x'}. \tag{4.85}$$

The convenient choice of additional operator is $P_1 + P_2 = 2(\partial/\partial t')$. The separation equations for the Laplace equation and the eigenvalue equation $(P_1 + P_2)\psi = B\psi$ are the same as in (4.83) with $\alpha\beta = -\frac{1}{4}K^2$, $\alpha = \frac{1}{2}B$. The solution is then $\psi_1(t')\psi_2(x')$. This coordinate system is obtained from system 1 by a Euclidean rotation of 45 degrees.

3. *Cartesian coordinates of type* III. In this case we choose coordinates such that

$$t' = -2t, \quad x' = t + x. \tag{4.86}$$

The Laplacian then has the form $\Delta = \partial^2/\partial t'^2 + \partial^2/\partial t'\,\partial x'$ and the generators of translations are

$$P_1 = -2\frac{\partial}{\partial t'} + \frac{\partial}{\partial x'}, \qquad P_2 = \frac{\partial}{\partial x'}. \tag{4.87}$$

The convenient choice of additional operator is P_2. The separation equations for (4.1) and the equation $P_2\psi = \alpha\psi$ are then

$$\frac{d\psi_2}{dx'} = \alpha\psi_2, \qquad \frac{d^2\psi_1}{dt'^2} + \alpha\frac{d\psi_1}{dt'} + K^2\psi_1 = 0. \tag{4.88}$$

These are just the separation equations of the Laplace equation. The coordinate curves are illustrated in Fig. 9. All other coordinate systems of this type can be obtained from the choice (4.86) by a combination of the discrete transformations R and I.

4. *The coordinate system with differential form* (4.82). The Cartesian coordinates are such that

$$t = x_2\,e^{x_1/2} - e^{-x_1/2}, \qquad x = x_2\,e^{x_1/2} + e^{-x_1/2}. \tag{4.89}$$

The separation equations have the solution

$$\psi = e^{\alpha x_1/2}x_2^{\alpha/2}C_\alpha(2iKx_2^{1/2}). \tag{4.90}$$

TABLE 3

Coordinate systems which allow a separation of variables in the pseudo-Euclidean plane

Coordinate system	Basis defining operator	Solution of Laplace equation $\Delta\psi + K^2\psi = 0$	Comments
1. Cartesian coordinates (type I) t and x	P_1P_2	Exponential or plane wave solutions	This is the defining coordinate system; parametrization covers all of the pseudo-Euclidean plane.
2. Cartesian coordinates (type II) $t' = t + x, \quad x' = t - x$	$P_1 + P_2$	Exponential solutions	Rectangular Cartesian coordinates rotated by 45° (nonorthogonal)
3. Cartesian coordinates (type III) $t' = -2t, \quad x' = t + x$	P_2	Exponential solutions	One set of lines from type I and one from type II (nonorthogonal)
4. Polar coordinates $t = x_1 \cosh x_2, \quad x = x_1 \sinh x_2$	M^2	$x_1^{1/2}C_\nu(Kx_1)\,e^{i\tau x_2}$ $C_\nu(z)$ a solution of Bessel's equation	Together with I and R covers all the plane except the lines $t = \pm x$
5. Parabolic coordinates of type I $t = \frac{1}{2}(\xi^2 + \eta^2), \quad x = \xi\eta$	$MP_1 + P_1M$	Product of parabolic cylinder functions	Coordinate curves have identical equations. The coordinates do however parametrize the lines $t = \pm x$.
6. Parabolic coordinates of type II $t = -\frac{1}{2}(x_1 - x_2)^2 - (x_1 + x_2)$ $x = -\frac{1}{2}(x_1 - x_2)^2 + (x_1 + x_2)$	$MP_1 + P_1M + MP_2 + P_2M + (P_1 - P_2)^2$	Products of two linear combinations of the Airy functions $A_i(-Z_i)B_i(-Z_i)$ with $Z_i = X_i + B/K, \quad i = 1, 2$	Parametrization covers the half-plane $t + x \geqq 0$; by adding the inversion operator the other half-plane is also covered. Coordinate curves have identical equations.
7. Hyperbolic coordinates of type I $2t = \cosh \frac{1}{2}(y_1 - y_2) + \sinh \frac{1}{2}(y_1 + y_2)$ $2x = \cosh \frac{1}{2}(y_1 - y_2) - \sinh \frac{1}{2}(y_1 + y_2)$	$M^2 - P_1P_2$	Product of Mathieu equation solutions with argument displaced by $i\pi/2$	Coordinate curves are identical; parametrization covers all the plane.

TABLE 3 (*cont.*)

Coordinate system	Basis defining operator	Solution of Laplace equation $\Delta\psi + K^2\psi = 0$	Comments
8. Hyperbolic coordinates of type II $t = \sinh(y_1 - y_2) + e^{y_1+y_2}$ $x = \sinh(y_1 - y_2) - e^{y_1+y_2}$	$M^2 + (P_1 + P_2)^2$	Product of two solutions of Bessel's equation, one with real and one with imaginary argument	Coordinate curves have identical equations. Parametrization covers the region $t - x \geqq 0$. By adding I and R the remaining half-plane can be parametrized.
9. Hyperbolic coordinates of type III $t = \cosh(y_1 - y_2) + e^{y_1+y_2}$ $x = \cosh(y_1 - y_2) - e^{y_1+y_2}$	$M^2 - (P_1 + P_2)^2$	Product of two solutions of Bessel's equation $C_\nu(Ke^{y_1})C_\nu(Ke^{y_2})$	Identical coordinate curves. The parametrization does not cover all the plane, only the region $t + x > 0$, $t - x > 1$.
10. Elliptic coordinates of type I $t = \sinh y_1 \cosh y_2, \quad x = \cosh y_1 \sinh y_2$	$M^2 + P_2^2$	Product of two solutions of the non-periodic Mathieu equation	This parametrization does cover all the pseudo-Euclidean plane.
11. Elliptic coordinates of type II (i) $t = \cosh y_1 \cosh y_2$, $x = \sinh y_1 \sinh y_2$ (ii) $t = \cos y_1 \cos y_2$, $x = \sin y_1 \sin y_2$	$M^2 - P_2^2$	(i) as in coordinate system 10 (ii) product of two solutions of the periodic Mathieu equation	This parametrization does not cover the plane and cannot be made to do so by the addition of I and R.
12. Semihyperbolic coordinates $t = x_2 e^{x_1/2} - e^{-x_1/2}$ $x = x_2 e^{x_1/2} + e^{-x_1/2}$	M	Product of exponential and solution of modified Bessel equation	This parametrization covers the whole plane if I and R included (nonorthogonal).

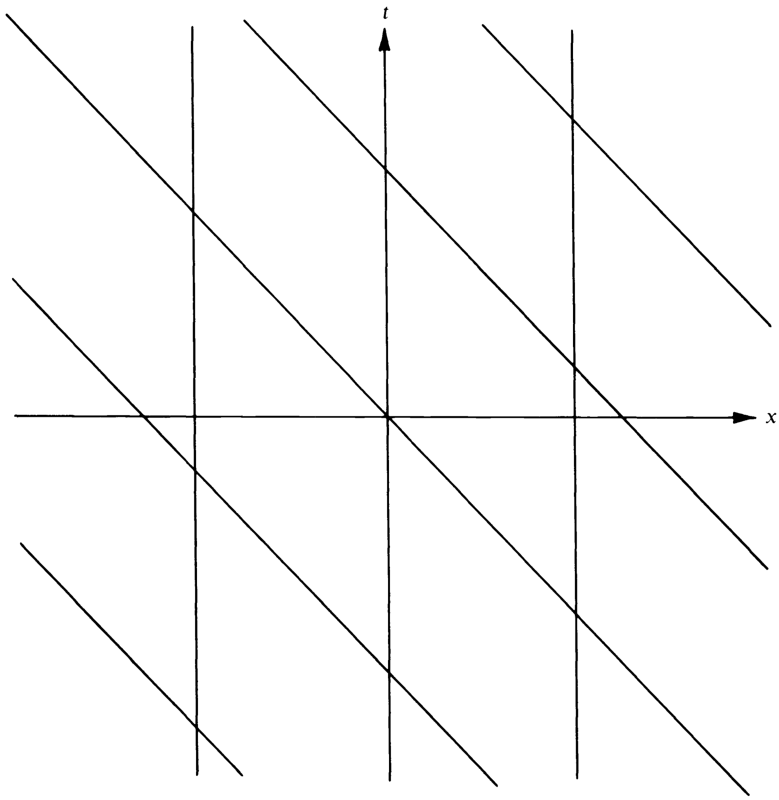

FIG. 9

Here α is the eigenvalue of the basis defining operator $L = M = \partial/\partial x_1$. The coordinate curves are given by the equations

$$t - x = -2e^{-x_1/2}, \qquad t^2 - x^2 = -4x_2. \tag{4.91}$$

These are illustrated in Fig. 10.

Comments. In this section we have evaluated all coordinate systems orthogonal and nonorthogonal which allow the separation of variables in the Laplace equation (4.1). All but one of these coordinate systems can be associated with a basis defining operator which can be written in the form (3.1) in terms of the generators of $E(1, 1)$. The results are summarized in Table 3. Only the operator $L = MP_1 + P_1M + MP_2 + P_2M$ does not correspond to a separable coordinate system for the Laplace equation (4.1). This operator, although it is a limiting case of an operator which does correspond to a separation of variables, does not give a separation of variables, when diagonalized, in the pseudo-Euclidean plane. We should also comment here that the nonorthogonal separable coordinate systems we have classified correspond in each case to the diagonalization of a first order operator in the lie algebra of $E(1, 1)$. This is a reflection of the fact that the diagonalization of such an operator does not uniquely determine a corresponding separable coordinate system.

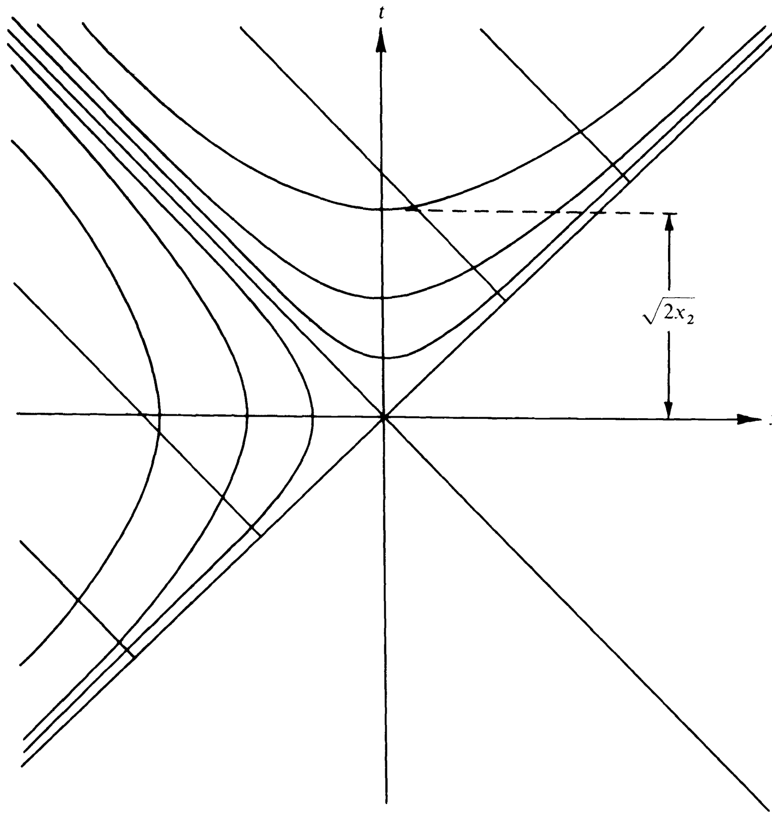

FIG. 10

5. The classification of all separable differential forms in three-dimensional Minkowski space. In this section we give a complete classification of the possible differential forms for which the Laplace operator admits a separation of variables in three-dimensional Minkowski space. In classifying these forms, we make use of the analysis of Eisenhart with the exception that the equations giving a zero curvature tensor are modified so as to correctly include the indefinite signature of the Minkowski space metric. Accordingly we do not go through the details of the derivation here. This is also appropriate from the point of view that many of the coordinate systems found have already been evaluated in this paper (for the case of $E(1, 1)$) or elsewhere [6]–[8]. Many of the coordinate systems have a subgroup Casimir operator in the basis defining operators. In addition, we do not touch on the problem of complete sets of basis defining operators for the associated group $E(2, 1)$ under equivalence relations of the type introduced for $E(1, 1)$. The only features of the group $E(2, 1)$, acting on three-dimensional Minkowski space, which will prove useful are the specification of its Lie algebra generators; $E(2, 1)$ is the group of proper transformations which preserve the relativistic distance S between two points (t_i, x_i, y_i), $i = 1, 2$, in Cartesian coordinates, where

$$S^2 = (t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2. \tag{5.1}$$

We denote the generators of translations along the t, x, y axes by P_1, P_2 and P_3 respectively, the generators of hyperbolic rotations in tx, ty planes by K_1, K_2 and the generator of the xy plane rotations by M_3. For the degenerate representations of $E(2, 1)$ the only nonzero Casimir operator is the Laplacian

$$\Delta = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \tag{5.2}$$

satisfying an equation of the form

$$\Delta\psi + \chi^2\psi = 0. \tag{5.3}$$

We now proceed to the enumeration of all separable differential forms in this space.

System 1. The defining case of Cartesian coordinates

$$dS^2 = dt^2 - dx^2 - dy^2. \tag{5.4}$$

The Laplace equation then has exponential type solutions.

Systems 2–9. These are the coordinate systems corresponding to $E(1, 1)$-type cylindrical coordinates, i.e., we have bases in Which P_3 and one of the operators L in the eight coordinate systems (II)–(IX) (in § 4) are diagonal. The special functions which are the solution of (5.3) then have the form of the product of $e^{i\tau y}$ and the corresponding solution of the two-dimensional Laplace equation (4.1) with $K^2 = \chi^2 - \tau^2$. No further comment is required here.

Systems 10–12. These are the three $E(2)$-type cylindrical coordinates for which P_1 and one of the operators L appearing in Table 1 are diagonal and define the basis (excluding of course the Cartesian case). The corresponding separated solutions then have the form of the product of $e^{i\tau t}$ and the corresponding solution of the two-dimensional Laplace equation (2.7) with $K^2 = \chi^2 - \tau^2$. These solutions are well known [19] and will not be reproduced here.

Systems 13–21. Coordinate systems in which one of the basis defining operators is the Casimir operator $K_1^2 + K_2^2 - M_3^2$ of the $SO(2, 1)$ subgroup of $E(2, 1)$ generated by K_1, K_2, M_3 and the remaining basis operator is symmetric and quadratic in these generators. These coordinate systems have been discussed in detail by Winternitz et al. [7]. For coordinate systems of this type the differential form is

$$dS^2 = dx_0^2 - x_0^2\, dw^2, \tag{5.5}$$

where dw^2 is one of the nine quadratic forms evaluated by Olevski for a space of constant negative curvature in two dimensions. The x_0 or radial part of the Laplace equation has the same form in all three coordinate systems, viz.

$$\left(\frac{d^2}{dx_0^2} + \frac{2}{x_0}\frac{d}{dx_0} - \frac{j(j+1)}{x_0^2} + \chi^2\right)\phi(x_0) = 0 \tag{5.6}$$

which has the general solution

$$\phi(x_0) = AJ_{j+1/2}(\chi x_0) + BJ_{-j-1/2}(\chi x_0). \tag{5.7}$$

The general solution of (5.3) is the product of $\phi(x_0)$ and the corresponding $SO(2, 1)$ basis functions. For the three subgroup-type bases for $SO(2, 1)$ these functions have already been evaluated [20]. Two of the nonsubgroup-type bases have been studied by MacFadyen and Winternitz. For the remaining four coordinate systems we now give the differential forms and Laplace equation solutions.

(i) *Semihyperbolic system.*

$$dw^2 = (x_2 - x_1)\left(\frac{dx_1^2}{(x_1 - a)(x_1^2 + 1)} - \frac{dx_2^2}{(x_2 - a)(x_2^2 + 1)}\right) \tag{5.8}$$

with $x_2 < a < x_1$. The separation equations are

$$\begin{aligned}\Bigg[\frac{d^2}{dy_i^2} + \frac{1}{2}\left\{\frac{1}{y_i} + \frac{1}{y_i - 1} + \frac{1}{y_i - K^{-2}}\right\}\frac{d}{dy_i} \\ + \frac{j(j+1)y_i + B}{4y_i(y_i - 1)(y_i - K^{-2})}\Bigg]F(y_i) = 0, \qquad i = 1, 2,\end{aligned} \tag{5.9}$$

where K is complex and $|K| = 1$. We also have $x_i = (1 + i)y_i$. This is just the Lamé equation with complex modulus of absolute value 1.

(ii) *Elliptic parabolic system.* From the differential form

$$dw^2 = \frac{(x_1 - x_2)}{4}\left(\frac{dx_1^2}{(x_1 - 1)x_1^2} - \frac{dx_2^2}{(x_2 - 1)x_2^2}\right), \tag{5.10}$$

where $0 < x_2 < 1 < x_1$, we have putting $x_1 = 1/\cos^2\theta$, $x_2 = 1/\cosh^2 a$ the differential form

$$dw^2 = \left(\frac{1}{\cos^2\theta} - \frac{1}{\cosh^2 a}\right)(d\theta^2 + da^2). \tag{5.11}$$

The separation equations then give differential equations whose solutions are $F(a)$, $G(\theta)$, where

$$F(a) = AP_j^\tau(\tanh a) + BQ_j^\tau(\tanh a), \tag{5.12a}$$

$$G(\theta) = CP_j^\tau(i\tan\theta) + DQ_j^\tau(i\tan\theta). \tag{5.12b}$$

τ^2 is the separation constant and $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ are first and second kind Legendre functions respectively.

(iii) *Hyperbolic parabolic system.* The differential form is the same as with (5.10) with the variables now subject to the restrictions $x_2 < 0 < 1 < x_1$. Then putting $x_1 = 1/\cos^2\theta$, $x_2 = -1/\sinh^2 b$ we get the differential form

$$dw^2 = \left(\frac{1}{\cos^2\theta} + \frac{1}{\sinh^2 b}\right)(d\theta^2 + db^2). \tag{5.13}$$

The separation equations then have solutions $F(b)$, $G(\theta)$ with $G(\theta)$ as in (5.12b) and

$$F(b) = AP_j^\tau(\coth b) + BQ_j^\tau(\coth b). \tag{5.14}$$

(iv) *Semicircular parabolic system.* From the differential form

$$dw^2 = (x_1 - x_2)\left(\frac{dx_1^2}{x_1^3} - \frac{dx_2^2}{x_2^3}\right) \tag{5.15}$$

putting $x_1 = \xi^{-2}$, $x_2 = -\eta^{-2}$ we get the form

$$dw^2 = \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)(d\xi^2 + d\eta^2). \tag{5.16}$$

The separation equations then have the solutions $F(\xi)G(\eta)$, where

$$F(\xi) = \xi^{1/2}(AJ_{j+1/2}(b\xi) + BJ_{-j-1/2}(b\xi),$$
$$G(\eta) = \eta^{1/2}(CI_{j+1/2}(b\eta) + DI_{-j-1/2}(b\eta). \tag{5.17}$$

Winternitz et al. [7] have also given the expressions for Cartesian coordinates in terms of the variables appearing in the differential forms as well as the extra operator required to specify the basis. We should note however that the Cartesian coordinates have only been given inside the light cone.

Systems 22–27. Coordinate systems for which the two basis defining operators are the generator of a hyperbolic or proper rotation and a basis operator of the $E(1, 1)$ or $E(2)$ subgroup. We now enumerate the possible number of differential forms:

$$\text{(i)} \quad ds^2 = (x_0^2 - x_1^2)(dx_0^2 - dx_1^2) - x_0^2x_1^2\, dx_2^2. \tag{5.18}$$

This corresponds to a choice of Cartesian coordinates

$$t = \pm\tfrac{1}{2}(x_0^2 + x_1^2), \quad x = x_0x_1 \cos x_2, \quad y = x_0x_1 \sin x_2. \tag{5.19}$$

The basis functions of (5.3) are then diagonal in $K_1P_2 + P_2K_1 + K_2P_3 + P_3K_2$ and M_3. The separated equations in x_0 and x_1 have the form

$$\left(\frac{d^2}{dx_i^2} + \frac{1}{x_i}\frac{d}{dx_i} + \chi^2x_i^2 - \frac{m^2}{x_i^2} + q^2\right)F_i(x_i) = 0, \qquad i = 0, 1. \tag{5.20a}$$

This is the Bessel wave equation with general solution

$$F_i(x_i) = A_iJ_m(\chi, q, x_i) + B_iJ_{-m}(\chi, q, x_i), \qquad i = 0, 1 \tag{5.20b}$$

(in the notation of Moon and Spencer [17]). The corresponding solution of Laplace's equation is then $\psi = F_0F_1H$, where $H = e^{imx_2}$.

$$\begin{aligned}\text{(ii)} \quad ds^2 &= (\cosh^2 x_0 + \sinh^2 x_1)(dx_0^2 - dx_1^2) \\ &\quad - \cosh^2 x_0 \sinh^2 x_1\, dx_2^2.\end{aligned} \tag{5.21}$$

This corresponds to a choice of Cartesian coordinates

$$t = \sinh x_0 \cosh x_1, \qquad x = \cosh x_0 \sinh x_1 \cos x_2, \tag{5.22}$$

$$y = \cosh x_0 \sinh x_1 \sin x_2. \tag{5.23}$$

The basis functions of this system are then diagonal in the operators $K_1^2 + K_2^2 + P_2^2 + P_3^2$ and M_3. The separation equations for the solution of the Laplace equation $F(x_0)G(x_1)H(x_2)$ are

$$\left[\frac{1}{\cosh x_0}\frac{d}{dx_0}\cosh x_0\frac{d}{dx_0} + \frac{m^2}{\cosh^2 x_0} + \chi^2 \cosh^2 x_0 - j(j+1)\right]F(x_0) = 0, \tag{5.24}$$

$$\left[\frac{1}{\sinh x_1}\frac{d}{dx_1}\sinh x_1\frac{d}{dx_1} - \frac{m^2}{\sinh^2 x_1} - \chi^2 \sinh^2 x_1 - j(j+1)\right]G(x_1) = 0, \tag{5.25}$$

with $H(x_2) = e^{imx_2}$. The equations (5.24)–(5.25) are both spheroidal equations with general solutions

$$F(x_0) = AP_j^m(i\chi, i \sinh x_0) + BQ_j^m(i\chi, i \sinh x_0), \tag{5.26}$$

$$G(x_1) = CP_j^m(i\chi, \cosh x_1) + DQ_j^m(i\chi, \cosh x_1), \tag{5.27}$$

where $P_\mu^\nu(a, z)$ and $Q_\mu^\nu(a, z)$ are solutions of the spheroidal equation (or Legendre wave equation) which reduce to the corresponding first and second kind Legendre functions when $a = 0$.

$$\text{(iii)} \quad ds^2 = (\sinh^2 x_0 - \sinh^2 x_1)(dx_0^2 - dx_1^2) - \sinh^2 x_0 \sinh^2 x_1\, dx_2^2, \tag{5.28}$$

$$ds^2 = (\sin^2 x_0 - \sin^2 x_1)(dx_0^2 - dx_1^2) - \sin^2 x_0 \sin^2 x_1\, dx_2^2. \tag{5.29}$$

These differential forms correspond to the two possible choices of Cartesian coordinates

$$\begin{gathered} t = \cosh x_0 \cosh x_1, \quad x = \sinh x_0 \sinh x_1 \cos x_2, \\ y = \sinh x_0 \sinh x_1 \sin x_2, \end{gathered} \tag{5.30}$$

$$t = \cos x_0 \cos x_1, \quad x = \sin x_0 \sin x_1 \cos x_2, \quad y = \sin x_0 \sin x_1 \sin x_2. \tag{5.31}$$

The basis functions are diagonal in the operators $K_1^2 + K_2^2 - P_2^2 - P_3^2$ and M_3. The separation equations for the Laplace equation solution $F(x_0)G(x_1)H(x_2)$ are

$$\left[\frac{1}{\sinh x_i}\frac{d}{dx_i}\sinh x_i\frac{d}{dx_i} - \frac{m^2}{\sinh^2 x_i} + \chi^2 \sinh^2 x_i - j(j+1)\right]\psi(x_i) = 0, \tag{5.32}$$

$$\left[\frac{1}{\sin x_i}\frac{d}{dx_i}\sin x_i\frac{d}{dx_i} - \frac{m^2}{\sin^2 x_i} + \chi^2 \sin^2 x_i + j(j+1)\right]\phi(x_i) = 0, \tag{5.33}$$

where $\psi(x_i)$, $\phi(x_i) = F(x_0)$ or $G(x_1)$. As with the previous coordinate system $H(x_2) = e^{imx_2}$. Equations (5.32)–(5.33) are both forms of the spheroidal equation and have general solutions

$$\psi(x_i) = AP_j^m(\chi, \cosh x_i) + BQ_j^m(\chi, \cosh x_i), \tag{5.34}$$

$$\phi(x_i) = CP_j^m(\chi, \cos x_i) + DQ_j^m(\chi, \cos x_i). \tag{5.35}$$

$$\text{(iv)} \quad ds^2 = x_1^2 x_2^2 dx_0^2 - (x_1^2 + x_2^2)(dx_1^2 + dx_2^2). \tag{5.36}$$

This corresponds to the choice of Cartesian coordinates

$$t = x_1 x_2 \cosh x_0, \quad x = x_1 x_2 \sinh x_0, \quad Y = \tfrac{1}{2}(x_1^2 + x_2^2). \tag{5.37}$$

The basis functions are diagonal in the operators K_1 and $K_2P_1 + P_1K_2 + M_3P_2 + P_2M_3$. The separated equations have the form (5.20b) with m replaced by $i\tau$. q^2 is real and positive for the equation in x_1 and negative for the equation in x_2. The corresponding solution of the Laplace equation is then the product of two solutions of the type (5.20b) (with the specified changes) and $e^{i\tau x_0}$.

$$\text{(v–vi)} \quad ds^2 = \sinh^2 x_1 \sin^2 x_2\, dx_0^2 - (\sinh^2 x_1 + \sin^2 x_2)(dx_1^2 + dx_2^2). \tag{5.38}$$

This corresponds to the choice of Cartesian coordinates

$$\begin{gathered} t = \sinh x_0 \sinh x_1 \sin x_2, \quad x = \cosh x_0 \sinh x_1 \sin x_2, \\ y = \cosh x_1 \cos x_2. \end{gathered} \tag{5.39}$$

The basis functions are diagonal in the operators K_1 and $K_2^2 - M_3^2 + P_0^2 - P_1^2$. The separation equations for the solution of the Laplace equation are for the x_1 dependence a solution of the type (5.32) and for the x_2 dependence a solution of the type (5.33), the only changes in the parameters in these equations being $\chi \to i\chi$, $m \to \tau$ (real). The x_0 dependence is given by $e^{i\tau x_0}$. Similar remarks apply to an additional coordinate system in which the Cartesian coordinates are given by

$$t = \sinh x_0 \cosh x_1 \cos x_2, \quad x = \cosh x_0 \cosh x_1 \cos x_2, \quad y = \sinh x_1 \sin x_2. \tag{5.40}$$

As this system differs very little from (5.38) we make no further comment on it.

Systems 28–32. Other coordinate systems. (See *Note added in proof.*)

System 28.

$$\begin{aligned} ds^2 = {} & \frac{(x_0 - x_1)(x_0 - x_2)}{(x_0 - a)(x_0 - 1)x_0} dx_0^2 - \frac{(x_1 - x_0)(x_1 - x_2)}{(x_1 - a)(x_1 - 1)x_1} dx_1^2 \\ & - \frac{(x_2 - x_0)(x_2 - x_1)}{(x_2 - a)(x_2 - 1)x_2} dx_2^2, \end{aligned} \tag{5.41}$$

where

$$x_0 > a > 1 > x_1 > 0 > x_2.$$

The separation equations have the form

$$\left[\frac{d^2}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i - a} + \frac{1}{x_i - 1} + \frac{1}{x_i}\right]\frac{d}{dx_i} + \varepsilon \frac{\chi^2 x_i^2 + A x_i + B}{(x_i - a)(x_i - 1)x_i}\right] F_i(x_i) = 0, \tag{5.42}$$

where $\varepsilon = +1$ if $i = 0, 1$ and -1 if $i = 2$. For $i = 0, 1$ this is the Lamé wave equation with solution

$$F_i(x_i) = A E_p^q(\bar{\chi}, x_i^{1/2}) + B F_p^q(\bar{\chi}, x_i^{1/2}), \tag{5.43}$$

where $B = \frac{1}{4}(1 + a^2)q$, $A = -\frac{1}{4}p(p + 1)$ and $\bar{\chi} = \frac{1}{2}\chi$. For $i = 3$ the solution is again of the form (5.43) with argument $ix_2^{1/2}$ instead of $x_i^{1/2}$.

System 29.

$$\begin{aligned} ds^2 = {} & \frac{(x_0 - x_1)(x_0 - x_2)}{(x_0 - 1)^2 x_0} dx_0^2 + \frac{(x_1 - x_0)(x_1 - x_2)}{(x_1 - 1)^2 x_1} dx_1^2 \\ & + \frac{(x_2 - x_0)(x_2 - x_1)}{(x_2 - 1)^2 x_2} dx_2^2, \end{aligned} \tag{5.44}$$

where $x_0 > 1 > x_1 > 0 > x_2$.

The separation equations are

$$\left[\frac{d^2}{dx_i^2} + \frac{1}{2}\left[\frac{1}{x_i} + \frac{2}{x_i - 1}\right]\frac{d}{dx_i} + \varepsilon \frac{\chi^2 x_i^2 + A x_i + B}{(x_i - 1)^2 x_i}\right] F_i(x_i) = 0 \tag{5.45}$$

with ε as in system 28. These equations are identical with those of (5.42) and hence the solutions of the Laplace equation are as in that case with the only modification being $B = \frac{1}{2}q$.

System 30.

$$ds^2 = \frac{(x_0 - x_1)(x_0 - x_2)}{x_0^2(x_0 - 1)} dx_0^2 + \frac{(x_1 - x_0)(x_1 - x_2)}{x_1^2(1 - x_1)} dx_1^2 + \frac{(x_2 - x_0)(x_2 - x_1)}{x_2^2(1 - x_2)} dx_2^2 \tag{5.46}$$

with $x_2 > 1 > x_1 > 0 > x_0$. The almost identical nature of this system to 29 enables us to make no further comment on its properties.

Systems 31–32.

$$ds^2 = (e^{2x_0} + e^{2x_1})(dx_0^2 - dx_1^2) - e^{2x_0} e^{2x_1} dx_2^2. \tag{5.47}$$

Separation of variables gives the equations

$$\left[r^2 \frac{d^2}{dr^2} + 2r\frac{d}{dr} + \varepsilon\left(\frac{\tau^2}{r^2} + x^2 r^2\right) - j(j + 1)\right]F(r) = 0, \tag{5.48}$$

where $\varepsilon = +1$ if $r = e^{x_0}$ and $\varepsilon = -1$ if $r = e^{x_1}$ is an asymptotic form of the Legendre wave equation for r large. We have at the time of writing found no standard notation for the solution of this equation. The second system of this type can be obtained from this system by the transformation $x_0 \to x_0 + i(\pi/2)$.

6. Concluding remarks. What has been achieved in this paper has been the classification of all coordinate systems for which the Laplace equation admits a separation of variables in the pseudo-Euclidean plane. This includes both orthogonal and nonorthogonal coordinates. With each such coordinate system we can associate an additional operator which uniquely determines the basis. We have shown that there are three coordinate systems in the pseudo-Euclidean plane which are nonorthogonal and no nonorthogonal system in the Euclidean plane which admits a separation of variables of the Laplace equation. It has been shown that with the addition of the discrete operations R and I that every class of second order symmetric operators in the generators of $E(1, 1)$ but one can be associated with a separable coordinate system of the Laplace operator. This exception is the operator $L = MP_1 + P_1M + MP_2 + P_2M$.

For the three-dimensional case we have merely evaluated all the differential forms and associated basis functions for the case of orthogonal coordinates which allow separation. The problem of the parametrization of the space has not been given in any detail. This is of particular interest for the case of bases defined according to the chain $E(2, 1) \supset SO(2, 1) \supset L$ with L some second order operator. One then has to parametrize inside, outside and on the light cone in terms of the various coordinate systems. For the case of the pseudo-Euclidean plane these regions (with the exception of the light cone) can be regarded as identical. In the three-dimensional case they are however distinct and the spectra of the Laplacian is also different inside and outside the light cone [20], [21]. We hope to return to some of these problems in the near future.

In terms of future work it would seem expedient to make a systematic study of the possible nonsubgroup bases of the compact groups, in particular the orthogonal and unitary groups. A study of these bases should prove relevant to associated physical problems (e.g., the group $SO(4)$ and the Coulomb problem). Related to this work will also be the study of the associated special functions and the evaluation of some new properties in this area. As far as the Euclidean groups are concerned we hope to be able to give a complete treatment of the Poincaré group $E(3, 1)$ and its separable coordinate systems as well as make the first steps in a systematic treatment of the Euclidean groups $E(p, q)$. Finally it should be mentioned that the study of nonorthogonal separable coordinate systems will of necessity be given a more detailed analysis.

Note added in proof. It should be noted that the classification of separable differential forms in § 5 is incomplete. A more detailed presentation of such systems is currently under preparation. The majority of separable systems have, however, been presented in this article.

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THE PERIODIC PART OF THE RESPONSE TO A PERIODIC EXCITATION*

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Abstract. The aim is to describe an algebraic procedure which gives explicit formulas for the steady state part of the response to a periodic excitation which may be a generalized function; the transfer function may have positive degree, and its poles may lie on the imaginary axis. The theorems are intended to form a rigorous basis for applications to network analysis.

Introduction. An important problem in network analysis is to determine the steady state part $[E]^\tau$ of the response to a periodic nonsinusoidal excitation E which vanishes on the negative axis. The steady state part $[E]^\tau$ can be obtained by two procedures: the Fourier procedure gives $[E]^\tau$ in the form of a Fourier series; the operational procedure gives $[E]^\tau$ by an explicit formula involving finitely many terms. This paper presents the operational procedure in a rigorous mathematical form and generalizes this procedure in various directions. More specifically, the periodic excitation E is allowed to be a generalized function which vanishes on the negative axis (for example, E could be a row of impulses, or just one impulse concentrated at the origin). The transfer function h is allowed to have positive degree; moreover, the poles of h may have arbitrary order and may lie anywhere outside a sequence of equidistant points situated on the imaginary axis (this sequence depends on the period of the excitation E). Some attention is given to minimizing the amount of work involved in calculating $[E]^\tau$; for example, $[E]^\tau$ can be obtained by repeated application of the formula used when the transfer function has only simple poles (see § 7 and the material following Remark 8.2); see Theorem 8.1 for an explicit formula giving $[E]^\tau$ in the case of multiple poles.

Motivation. Professor Clare McGillem (of the Department of Electrical Engineering of Purdue University) brought to my attention the practical interest of finding an efficient operational procedure for calculating $[E]^\tau$. The few textbooks dealing with the operational procedure (e.g., [2] and [12]) describe it by working out one example by reasonings which occasionally appeal to nonmathematical arguments (see Remark 7.3); none consider the case where the transfer function has a pole of order greater than one. Remark 7.6 deals with the following classical situation: the transfer function h is defined by the equation $h(s) = 1/(s - a)$, and the excitation E is a row of equidistant impulses starting at the time $t = 0$; it turns out that the steady state part $[E]^\tau$ of the response is the periodic extension of an exponential function: this form of the answer is clearly more useful (for example, for finding the extrema of $[E]^\tau$) than the Fourier series answer obtained in [13].

In this paper is defined a space $\mathscr{P}_\sigma$ of periodic generalized functions; the space $\mathscr{P}_\sigma$ contains the space of functions which vanish on $(-\infty, 0)$ and have on $(0, \infty)$ period $\sigma > 0$; moreover, if an element of $\mathscr{P}_\sigma$ is an ordinary function vanishing on $(-\infty, 0)$, then it is on $(0, \infty)$ a periodic function having period σ (see Remark 3.4; of course, $\mathscr{P}_\sigma$ contains, in particular, the infinite series of impulses concentrated at the points $k\sigma$, where $k = 0, 1, 2, 3, \cdots$; see Remark 7.4).

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Throughout, σ is an arbitrary number >0. The transfer function is a rational function whose poles lie outside the sequence $2\pi ki/\sigma$ ($k = 0, \pm 1, \pm 2, \pm 3, \cdots$). Given an excitation E of period σ which vanishes on $(-\infty, 0)$, the *periodic part* of the response is the unique periodic function (of period σ) differing from the response by a solution y of the differential equation

$$\mu\left(\frac{d}{dt}\right)y = 0 \quad \text{for } t > 0,$$

where μ is the denominator of the rational function h; if stability occurs (that is, if all the poles of h lie on the left-hand side of the imaginary axis), then the periodic part equals the steady state part (of the response); see Remark 5.1. The case where E is a generalized function seems to require a restriction on the degree of the transfer function h; see Theorem 8.3. Remark 6.2 involves a transfer function of positive degree.

Organization. Section 2 discusses the notion of "response" and "transfer function". Section 3 ("Periodic operators") deals with the space $\mathscr{P}_\sigma$. The purpose of § 4 ("Free responses") is to provide theorems to be used later (see Remark 4.2). Section 5 ("The periodic part") deals with some key properties subsuming the existence of the periodic part. Section 6 ("Sinusoidal excitations") presents a slight generalization of properties known under the heading of "frequency response". Sufficient conditions for the existence of the periodic part are given in §§ 7–8, which deal with nonsinusoidal excitations (the case of multiple poles is relegated to § 8). For the main results, see Theorem 8.2 and Remark 8.4.

1. Algebraic operational calculus. In order to avoid unnecessary growth conditions, it is best to replace the operational calculus based on the Laplace transformation by a direct approach such as the one described in [8]–[11] or [14]. This § 1 is devoted to a brief sketch of the relevant material. The notions of "response" and "transfer function" are discussed in § 2.

Notation. We denote by $\mathbb{R}$ the set $(-\infty, \infty)$ of all the real numbers; $\mathbb{C}$ denotes the complex field.

A function $f(\cdot)$ is called *piecewise-continuous* if $f(\cdot)$ is a mapping of $\mathbb{R}$ into $\mathbb{C}$ such that $f(\cdot)$ has only a finite number of discontinuities in each finite interval; moreover, it is required that both right- and left-hand limits of $f(\cdot)$ exist at each point.

DEFINITION 1.1. Let $(\mathscr{K})$ denote the family of all the piecewise-continuous functions $f(\cdot)$ such that $f(t) = 0$ for $t \leqq 0$.

Remark 1.1. If $f(\cdot) \in (\mathscr{K})$, then $|f(t\pm)| < \infty$ for every r in $\mathbb{R}$. As usual, we denote by $f(t-)$ the left-hand limit:

$$f(t-) = \lim f(u) \quad \text{for } u \to t \text{ and } u < t.$$

Remark 1.2. The direct operational calculus described in [8]–[11] is based on the following two notions. A *test function* is an infinitely differentiable mapping of $\mathbb{R}$ into $\mathbb{C}$ which, at the origin, vanishes together with all its derivatives (for example, the equation $\varphi(t) = \exp(-1/|t|)$ defines a test-function). An *operator* is a linear mapping of the space of test functions into itself. For example, the differentiator D

(which assigns to each test function its derivative) is an operator. If g_k $(k = 1, 2)$ are two operators, then $g_1 = g_2$ means that $g_1(\varphi) = g_2(\varphi)$ for every test function φ.

If $f(\cdot)$ and $g(\cdot)$ belong to the space $(\mathscr{K})$ we denote by $f \wedge g(\cdot)$ the function defined by

$$f \wedge g(t) = \int_0^t f(t-u)g(u)\,du \quad \text{for } t \in \mathbb{R}.$$

Let $\mathscr{A}$ be the family of all the operators A such that the equation

$$A(w_1 \wedge w_2)(\cdot) = (Aw_1) \wedge w_2(\cdot)$$

holds whenever both $w_1(\cdot)$ and $w_2(\cdot)$ are test functions. It can be shown (see [9, Thm. 1.22]) that $\mathscr{A}$ is a commutative subalgebra of the algebra of all the operators (multiplication is the usual operator product). The operator D, defined by $Dw(\cdot) = w'(\cdot)$ for every test function $w(\cdot)$, is an invertible element of the algebra $\mathscr{A}$ (in the sense that there exists a unique element D^{-1} of $\mathscr{A}$ such that DD^{-1} equals the unit element 1 of the algebra $\mathscr{A}$).

To any function $f(\cdot)$ in $(\mathscr{K})$ there corresponds an operator f, called the *operator of the function* $f(\cdot)$; the operator f is defined by the equation

$$f(w) = f \wedge w'(\cdot) \quad \text{for any test-function } w(\cdot).$$

The correspondence $f(\cdot) \mapsto f$ is a linear mapping of the space $(\mathscr{K})$ onto a subspace $\mathscr{K}$ of the algebra $\mathscr{A}$.

Remark 1.3. Suppose that $g_k(\cdot) \in (\mathscr{K})$ for $k = 1, 2$; if $g_1(t) = g_2(t)$ for $t > 0$, then $g_1 = g_2$.

Conversely, if $g_1 = g_2$, then $g_1(t) = g_2(t)$ whenever both $g_1(\cdot)$ and $g_2(\cdot)$ are continuous at the point t (see [8, p. 27]).

THEOREM 1.1. *Suppose that* $g_k(\cdot) \in (\mathscr{K})$ *for* $k = 1, 2$. *If* $g_1 = g_2$, *then* $g_1(t-) = g_2(t-)$ *for all* $t \in \mathbb{R}$.

Proof. The proof is immediate from Remark 1.3.

DEFINITION 1.2. We set $\mathscr{K} = \{g : g(\cdot) \in (\mathscr{K})\}$. Thus, $f \in \mathscr{K}$ if f equals the operator g of some function $g(\cdot)$ belonging to $(\mathscr{K})$. If $f \in \mathscr{K}$ and $t \in \mathbb{R}$ we denote by $f(t)$ the unique element of the set $\{g(t-) : g(\cdot) \in (\mathscr{K}) \text{ and } g = f\}$; in view of Theorem 1.1, this unique element exists.

Remark 1.4. *Consequently*, if $f = g$ and $g(\cdot) \in (\mathscr{K})$, then $f(t) = g(t-)$. Thus, to every f in $\mathscr{K}$ there corresponds a function $f(\cdot)$ belonging to $(\mathscr{K})$; this correspondence $f \mapsto f(\cdot)$ is linear.

THEOREM 1.2. *The space* $\mathscr{K}$ *is a linear subspace of the algebra* $\mathscr{A}$. *If* $f_k \in \mathscr{K}$ *and* $c_k \in \mathbb{C}$, *then*

$$\left[\sum_k c_k f_k\right](t) = \sum_k c_k f_k(t) \quad \textit{for } t > 0.$$

Proof. The proof is immediate from Remark 1.2 and Definition 1.2.

The derivative. Suppose that $f \in \mathscr{K}$. If $f(\cdot)$ is continuous on $\mathbb{R}$, then the equation $Df = f'$ holds whenever $f'(\cdot) \in (\mathscr{K})$ (that is, the operator product of the operators D and f equals the operator of the derivative of the function $f(\cdot)$: see [9, p. 203]).

The translation operator. Suppose $\alpha \geqq 0$; we denote by $\mathbf{T}_\alpha(\ )$ the function defined by

$$\mathbf{T}_\alpha(t) = \begin{cases} 0 & \text{for } t \leqq \alpha, \\ 1 & \text{for } t > \alpha. \end{cases} \tag{1.1}$$

In particular, $\mathbf{T}_0$ is the operator of the Heaviside step function. If $f \in \mathscr{K}$ then the operator product $\mathbf{T}_\alpha f$ (of the operators $\mathbf{T}_\alpha$ and f) belongs to $\mathscr{K}$; in fact,

$$\mathbf{T}_\alpha f(t) = \mathbf{T}_\alpha(t) f(t - \alpha) \quad \text{for } t > 0; \tag{1.2}$$

see [9, p. 206] or [8, p. 30].

Example 1.1. Suppose that $a \in \mathbb{C}$. If k is an integer $\geqq 1$, we denote by $D\mathbf{T}_0/(D - a)^k$ the operator A such that $D\mathbf{T}_0 = (D - a)^k A$ (that is, $D\mathbf{T}_0 = DA - aA$ in case $k = 1$); it turns out that $A \in \mathscr{K}$ and

$$\frac{D\mathbf{T}_0}{(D - a)^k}(t) = e^{at}\frac{t^{k-1}}{(k-1)!}, \quad \text{for } t > 0; \tag{1.3}$$

see [8, p. 49].

Convergence. Henceforth, the linear space $\mathscr{A}$ is endowed with the locally convex linear Hausdorff topology defined in [9, p. 209]. In case $\alpha > 0$, it turns out that the operator $\mathbf{T}_0 - \mathbf{T}_\alpha$ is an invertible element of the algebra $\mathscr{A}$; in fact,

$$\frac{1}{\mathbf{T}_0 - \mathbf{T}_\alpha} = \sum_{k=0}^{\infty} \mathbf{T}_{k\alpha}, \tag{1.4}$$

and $1/(\mathbf{T}_0 - \mathbf{T}_\alpha)$ is an element of $\mathscr{K}$ such that

$$\frac{1}{\mathbf{T}_0 - \mathbf{T}_\alpha}(t) = 1 + [t/\alpha] \quad \text{for } t > 0,$$

where $[t/\alpha]$ is the greatest integer smaller than t/α. See [8, pp. 250–251].

Impulses. Let $\alpha \geqq 0$. The operator product $D\mathbf{T}_\alpha$ (of the operators D and $\mathbf{T}_\alpha$) satisfies the equation

$$D\mathbf{T}_\alpha = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathbf{T}_\alpha - \mathbf{T}_{\alpha+\varepsilon});$$

note that

$$(\mathbf{T}_\alpha - \mathbf{T}_{\alpha+\varepsilon})(t) = \begin{cases} 1 & \text{for } \alpha < t < \alpha + \varepsilon, \\ 0 & \text{otherwise;} \end{cases}$$

consequently, $D\mathbf{T}_\alpha$ corresponds to the unit impulse applied at the time $t = \alpha$. Multiplying by D both sides of (1.4), we obtain

$$\frac{D}{\mathbf{T}_0 - \mathbf{T}_\alpha} = \sum_{k=0}^{\infty} D\mathbf{T}_{k\alpha}; \tag{1.5}$$

this is a row of impulses applied at the times $t = k\alpha$ ($k = 0, 1, 2, 3, \cdots$). See [8, p. 251].

Remark 1.5. In view of Remark 1.3, (1.2) gives $\mathbf{T}_0\mathbf{T}_\alpha = \mathbf{T}_\alpha$ and

$$f = \mathbf{T}_0 f \quad \text{whenever } f \in \mathscr{K}. \tag{1.6}$$

DEFINITION 1.3. An operator E is said to *agree with zero* on the interval $(-\infty, 0)$ if the equation $Ew(t) = 0$ holds whenever $t < 0$ and for each test function $w(\cdot)$. An operator E agrees with zero on the interval $(-\infty, 0)$ if (and only if) $E = E\mathbf{T}_0$ (see [10, 1.31 and 3.25]). It is easily verified that the operators (1.4) and (1.5) agree with zero on the interval $(-\infty, 0)$.

Convolution. If $f_k \in \mathscr{A}$ for $k = 1, 2$, we set

$$f_1 * f_2 \overset{\text{def}}{=} f_1 D^{-1} f_2. \tag{1.7}$$

Note that $D * f_2 = f_2$. If $f_k \in \mathscr{K}$, it turns out that the operator $f_1 * f_2$ belongs to $\mathscr{K}$; in fact,

$$f_1 * f_2(t) = \int_0^t f_1(t-u) f_2(u)\, du \quad \text{for } t > 0; \tag{1.8}$$

see [9, p. 204] or [8, p. 28]. Consequently, the operation (1.7) is a generalization of the operation of convolution of functions.

Suppose that $a \in \mathbb{C}$, and let f_1 be the operator $D\mathbf{T}_0/(D-a)$: Definition (1.7) gives, for $f_2 \in \mathscr{A}$,

$$\frac{D\mathbf{T}_0}{D-a} * f_2 = \frac{f_2}{D-a}; \tag{1.9}$$

since $f_1 = D\mathbf{T}_0/(D-a)$, it follows from (1.3) that $f_1(x) = e^{ax}$ for $x > 0$; from (1.9) and (1.8) it therefore results that the equation

$$\frac{f_2}{D-a}(t) = \int_0^t e^{at-au} f_2(u)\, du \quad \text{for } t > 0 \tag{1.10}$$

holds whenever $f_2 \in \mathscr{K}$.

Polynomials in D. Let p be a polynomial

$$p(s) = c_0 + c_1 s + \cdots + c_k s^k + \cdots; \tag{1.11}$$

since the coefficients c_k are determined by p, we are entitled to write

$$p(D) = c_0 + c_1 D + \cdots + c_k D^k + \cdots. \tag{1.12}$$

Remark 1.6. Let n be the degree of the polynomial p, and consider an element w of $\mathscr{K}$ such that $w^{(n)}(\cdot) \in (\mathscr{K})$ and such that $w^{(k)}(\cdot)$ is continuous on $\mathbb{R}$ for all non-negative integers $k < n$; if so, then

$$p(D)w = c_0 w + c_1 w' + \cdots + c_k w^{(k)} + \cdots;$$

see [9, p. 203]. Consequently, the operator $p(D)w$ belongs to $\mathscr{K}$ and the equation

$$p(D)w(t) = c_0 w(t) + c_1 w'(t) + \cdots + c_k w^{(k)}(t) + \cdots \tag{1.13}$$

holds for all $t > 0$. As usual, $w^{(k)}(\cdot)$ denotes the kth derivative of the function $w(\cdot)$.

THEOREM 1.3. *Let p be the polynomial* (1.11). *Suppose that p has degree $n \geqq 1$. If $F \in \mathscr{K}$ and $w_1 = F/p(D)$, then $p(D)w_1(\cdot) = F(\cdot)$ and the equation*

$$c_0 w(t) + c_1 w'(t) + \cdots + c_k w^{(k)}(t) + \cdots = F(t) \tag{1.14}$$

holds for $w(\cdot) = w_1(\cdot)$ *and* $t > 0$. *In fact,* $w_1(\cdot)$ *is the only solution of* (1.14) *which satisfies the conditions*

$$w^{(k)}(0) = 0 \quad \textit{for } 0 \leqq k < n. \tag{1.15}$$

Proof. It is well known (see, for example, [7, p. 194]) that there exists a unique function $w_0(\cdot)$ such that $w_0^{(k)}(\cdot)$ is continuous on $\mathbb{R}$ for each nonnegative integer $k < n$ and such that both (1.14) and (1.15) are satisfied by $w(\cdot) = w_0(\cdot)$; from (1.14) it follows easily that $w_0^{(n)}(\cdot) \in (\mathscr{K})$; from Remark 1.6 therefore follows the equation $p(D)w_0(\cdot) = F(\cdot)$. Since (by Remark 1.3) this equation implies $p(D)w_0 = F$ and $w_0 = F/p(D)$, it follows from our hypothesis ($w_1 = F/p(D)$) that $w_0 = w_1$; the conclusion $w_0(\cdot) = w_1(\cdot)$ now comes from Remark 1.4 and the fact that $w_0(\cdot)$ is continuous on $\mathbb{R}$.

2. Transfer function. Let μ and ψ be polynomials and suppose that $\mu \neq 0$. By definition, the operator $\psi(D)/\mu(D)$ is the operator A such that $\psi(D) = \mu(D)A$. Let E be an operator; we write

$$[E] \overset{\text{def}}{=} \frac{\psi(D)}{\mu(D)} E; \tag{2.1}$$

it obviously follows that

$$\mu(D)[E] = \psi(D)E. \tag{2.2}$$

Equations of the form (2.2) occur in filters (a *filter* is a black box into which is fed an excitation E and out of which comes a resulting response $[E]$). Set

$$G = \frac{\psi(D)}{\mu(D)} D\mathbf{T}_0. \tag{2.3}$$

Let us suppose that $E = E\mathbf{T}_0$ that is, suppose that E is an element of the algebra $\mathscr{A}$ which agrees with zero on $(-\infty, 0)$ (see Remark 1.5); from (2.1) it follows that

$$[E] = \frac{\psi(D)}{\mu(D)} \mathbf{T}_0 E = \left(\frac{\psi(D)}{\mu(D)} D\mathbf{T}_0 \right) D^{-1} E,$$

so that (2.3) and Definition (1.7) give

$$[E] = G * E. \tag{2.4}$$

Let us call $[E]$ the *response* to E; in case E is the unit impulse $D\mathbf{T}_0$, it follows from (2.1) and (2.3) that $[D\mathbf{T}_0] = G$: the operator G equals the response to $D\mathbf{T}_0$ (this response will be called the *impulse response*). Thus (2.4) states that $[E] = [D\mathbf{T}_0] * E$: the response to E is obtained by convoluting with E the impulse response.

In case the degree of ψ is smaller than the degree of μ, it turns out that $G \in \mathscr{K}$ and

$$\mathfrak{L}G = \psi/\mu, \tag{2.5}$$

where $\mathfrak{L}$ denotes the Laplace transformation. Thus, the impulse response equals the operator of the function $G(\cdot)$ satisfying (2.5) Since $[E] = G * E$, it follows from (2.4)–(2.5) that the function ψ/μ is the *transfer function* in the sense of [15, p. 190]. If, in addition to the restriction on the degrees, we suppose that both $[E]$ and

E are Laplace-transformable, then (2.4) gives $\mathfrak{L}[E] = (\mathfrak{L}G)(\mathfrak{L}E)$; consequently (2.5) gives

$$\mathfrak{L}[E] = \frac{\psi}{\mu}\,\mathfrak{L}E \quad \text{and} \quad \frac{\psi}{\mu} = \frac{\mathfrak{L}[E]}{\mathfrak{L}E};$$

from [6, p. 97] and [2, pp. 74–85] it therefore results that $[E]$ is what is commonly called the *response* to E, while the function ψ/μ is indeed the "transfer function" in the usual sense.

DEFINITION 2.1. Given the ratio $h = \psi/\mu$ of two polynomials (whose degrees need not be related) and an operator E, we shall call $h(D)E$ the *response* to E. As we have seen at the beginning of this § 2, this terminology is in accord with the standard terminology.

3. Periodic operators. Throughout σ is a fixed number such that $\sigma > 0$.

DEFINITION 3.1. We denote by Λ^σ the family of all the operators f in $\mathscr{K}$ such that

$$f(t + \sigma) = f(t) \quad \text{for all } t > 0. \tag{3.1}$$

Remark 3.1. Clearly Λ^σ is a linear subspace of the space $\mathscr{K}$. We shall define a space $\mathscr{P}_\sigma$ which contains (besides all elements of Λ^σ) rows of impulses; the space $\mathscr{P}_\sigma$ is such that, if it contains an element of $\mathscr{K}$, then this element belongs to Λ^σ (see Remark 3.3).

THEOREM 3.1. *If $G \in \mathscr{K}$, let $G^0(\cdot)$ be the function defined by*

$$G^0(u) = \begin{cases} G(u) & \textit{for } u \leqq \sigma, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.2}$$

If $G \in \Lambda^\sigma$, then the equation

$$G(t) = G^0(t - n\sigma) \quad \textit{whenever } n\sigma < t < n\sigma + \sigma \tag{3.3}$$

holds for any integer $n \geqq 0$.

Proof. If $n\sigma < t < n\sigma + \sigma$, then $0 < t - n\sigma < \sigma$ and

$$G(t) = G(\{t - n\sigma\} + n\sigma) = G(t - n\sigma);$$

Conclusion (3.3) is now immediate from (3.2).

THEOREM 3.2. *Suppose that $f \in \mathscr{K}$. The operator f belongs to Λ^σ if (and only if) there exists some operator w in $\mathscr{K}$ such that*

$$w(t) = 0 \quad \textit{for } t > \sigma \tag{3.4}$$

and

$$f = \frac{w}{\mathbf{T}_0 - \mathbf{T}_\sigma}. \tag{3.5}$$

Proof. Suppose that $f \in \Lambda^\sigma$; let us verify that (3.4)–(3.5) hold when $w(\cdot)$ is the function defined by

$$w(t) = \mathbf{T}_0(t)f(t) - \mathbf{T}_\sigma(t)f(t - \sigma) \quad \text{for } t \in \mathbb{R}; \tag{3.6}$$

note that $w \in \mathscr{K}$. From (3.6), (1.2) and Theorem 1.2, it follows that

$$w(t) = (\mathbf{T}_0 f - \mathbf{T}_\sigma f)(t) \quad \text{for } t \in \mathbb{R};$$

consequently, $w = \mathbf{T}_0 f - \mathbf{T}_\sigma f = (\mathbf{T}_0 - \mathbf{T}_\sigma)f$. Conclusion (3.5) is therefore immediate from (1.4). It remains to prove (3.4). To that effect, take $u > \sigma$ and note that (3.6) implies

$$w(u) = f(u) - f(u - \sigma) = f(\{u - \sigma\} + \sigma) - f(u - \sigma) = 0;$$

the last equation is from (3.1). This completes the proof of (3.4).

To prove the converse, suppose that (3.5) holds for some element w of $\mathscr{K}$ which satisfies (3.4). Multiplying both sides of (3.5) by $\mathbf{T}_0 - \mathbf{T}_\sigma$, we obtain $\mathbf{T}_0 f - \mathbf{T}_\sigma f = w$; from Theorem 1.2 and (1.2), we therefore have

$$f(u) - \mathbf{T}_\sigma(u) f(u - \sigma) = w(u) \quad \text{for } u > 0. \tag{3.7}$$

Since $f \in \mathscr{K}$ (by hypothesis), we only have to establish (3.1). To that effect, take $t > 0$ and substitute $u = t + \sigma$ into (3.7); this gives

$$f(t + \sigma) - \mathbf{T}_\sigma(t + \sigma) f(t) = w(t + \sigma) = 0. \tag{3.8}$$

The last equation is from our hypothesis (3.4) (note that $u > \sigma$). Since $\mathbf{T}_0(t + \sigma) = 1$ (by (1.1)), equation (3.8) gives the desired conclusion: $f(t + \sigma) = f(t)$.

DEFINITION 3.2. Let p be a polynomial; if $p \neq 0$, we shall denote by $|p|$ the degree of p; if $p = 0$, we set $|p| = -\infty$. It is understood that $-\infty < x$ for any $x \in \mathbb{R}$.

Let ψ and μ be polynomials and suppose that $\mu \neq 0$. The division algorithm guarantees the existence of a unique pair (p, λ) of polynomials such that

$$\psi/\mu = p + \lambda/\mu \quad \text{and} \quad |\lambda| < |\mu|;$$

we call p the *polynomial part* of ψ/μ. We denote by $\mathscr{P}_\sigma(\psi/\mu)$ the family of all the operators of the form $p(D)f_1 + f_2$, where $f_k \in \Lambda^\sigma$ for $k = 1, 2$:

$$\mathscr{P}_\sigma(\psi/\mu) = \{p(D)f_1 + f_2 : (f_1, f_2) \in \Lambda^\sigma \times \Lambda^\sigma\}.$$

Remark 3.2. Thus $G \in \mathscr{P}_\sigma(\psi/\mu)$ if the equation $G = p(D)f_1 + f_2$ holds for some pair (f_1, f_2) of elements of the space Λ^σ.

Remark 3.3. The family $\mathscr{P}_\sigma(\psi/\mu)$ is a linear subspace of the algebra $\mathscr{A}$; it contains rows of impulses (see Remark 7.4).

THEOREM 3.3. *If ψ and μ are polynomials such that $\mu \neq 0$, then*

$$\mathscr{K} \cap \mathscr{P}_\sigma(\psi/\mu) = \Lambda^\sigma.$$

Proof. If $f \in \Lambda^\sigma$, then $f = p(D)f_1 + f_2$, with $f_1 = 0$ and $f_2 = f$; consequently,

$$\Lambda^\sigma \subset \mathscr{K} \cap \mathscr{P}_\sigma(\psi/\mu). \tag{3.9}$$

To prove the converse, take

$$G \in \mathscr{K} \quad \text{and} \quad G \in \mathscr{P}_\sigma(\psi/\mu); \tag{3.10}$$

therefore, the property

$$G = p(D)f_1 + f_2 \quad \text{with } f_k \in \Lambda^\sigma \tag{3.11}$$

holds for $k = 1, 2$. In case $p = 0$, the conclusion $G \in \Lambda^\sigma$ is immediate. In case $|p| = 0$, we have $p(D) = c_0$ for some $c_0 \in \mathbb{C}$; the conclusion $G \in \Lambda^\sigma$ comes directly from the fact that Λ^σ is a linear space. The rest of this proof deals with the case $|p| \geqq 1$. Set

$$f = p(D)f_1; \tag{3.12}$$

therefore $G = f + f_2$, whence $f = G - f_2$ and $f_2 \in \mathscr{K}$ (since $f_2 \in \Lambda^\sigma$). Consequently it follows from $G \in \mathscr{K}$ (see (3.10)) that

$$f \in \mathscr{K} \quad \text{and} \quad G = f + f_2. \tag{3.13}$$

Since $f_2 \in \Lambda^\sigma$ (and by Remark 3.1), it follows from (3.13) that the conclusion $G \in \Lambda^\sigma$ can be obtained by proving that

$$f \in \Lambda^\sigma. \tag{3.14}$$

Since $f_1 \in \Lambda^\sigma$, it follows from Theorem 3.2 the existence of an operator w_1 in $\mathscr{K}$ such that

$$w_1(t) = 0 \quad \text{for } t > \sigma \tag{3.15}$$

and

$$\frac{1}{\mathbf{T}_0 - \mathbf{T}_\sigma} w_1 = f_1.$$

Multiplying by $p(D)$ both sides of this last equation, we obtain

$$\frac{1}{\mathbf{T}_0 - \mathbf{T}_\sigma} p(D)w_1 = p(D)f_1 = f; \tag{3.16}$$

the last equation is from (3.12). Calling $w = \mathbf{T}_0 f - \mathbf{T}_\sigma f$, it results from (3.16) that

$$p(D)w_1 = w, \tag{3.17}$$

so that (3.16) can be written

$$f = \frac{w}{\mathbf{T}_0 - \mathbf{T}_\sigma}; \tag{3.18}$$

moreover, since $w = \mathbf{T}_0 f - \mathbf{T}_\sigma f$, it follows from Theorem 1.2 and (1.2) that $w(t) = \mathbf{T}_0(t)f(t) - \mathbf{T}_\sigma(t)f(t - \sigma)$. Therefore $w \in \mathscr{K}$. From (3.18) and Theorem 3.2 we see that the conclusion (3.14) can be obtained by proving that

$$w(t) = 0 \quad \text{for } t > \sigma. \tag{3.19}$$

From (3.17) it follows that $w_1 = w/p(D)$; since $|p| \geqq 1$ we can apply Theorem 1.3 to infer that

$$c_0 w_1(t) + c_1 w_1'(t) + \cdots + c_k w_1^{(k)}(t) + \cdots = w(t), \tag{3.20}$$

where the c_k ($k = 0, 1, 2, 3, \cdots$) are the coefficients of the polynomial p; in view of (3.15), conclusion (3.19) is immediate from (3.20).

THEOREM 3.4. *Let ψ and μ be polynomials such that $\mu \neq 0$. If $|\psi| \leqq |\mu|$, then $\mathscr{P}_\sigma(\psi/\mu) = \Lambda^\sigma$.*

Proof. In view of (3.9), it suffices to prove that $\mathscr{P}_\sigma(\psi/\mu) \subset \Lambda^\sigma$. Since $|\psi| \leqq |\mu|$, the polynomial part p of ψ/μ is a number $p = c_0$; if $G \in \mathscr{P}_\sigma(\psi/\mu)$, it follows from Remark 3.2 that $G = c_0 f_1 + f_2$ with $f_k \in \Lambda^\sigma$: the conclusion $G \in \Lambda^\sigma$ is now immediate from Remark 3.1.

Remark 3.4. Let $\mathscr{P}_\sigma$ denote the family $\{f \in \mathscr{P}_\sigma(p) : p \text{ is a polynomial}\}$. If $f \in \mathscr{K} \cap \mathscr{P}_\sigma$, then there exists a polynomial p such that $f \in \mathscr{K} \cap \mathscr{P}_\sigma(p)$ the conclusion $f \in \Lambda^\sigma$ comes from Theorem 3.3.

4. Free responses. Let $(\mathscr{C}^\infty)$ be the family of all the functions which vanish on the interval $(-\infty, 0]$ and are infinitely differentiable on $(0, \infty)$. Let μ be a polynomial

$$\mu(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_k s^k + \cdots .$$

If $y(\cdot) \in (\mathscr{C}^\infty)$, we denote by $\mu(d)y(\cdot)$ the function defined by $\mu(d)y(t) = 0$ for $t \leqq 0$ and by

$$\mu(d)y(t) = c_0 y(t) + c_1 y'(t) + \cdots + c_k y^{(k)}(t) + \cdots$$

for $t > 0$. Let $(\square\mu)$ be the family of all the functions $y(\cdot)$ in $(\mathscr{C}^\infty)$ such that $\mu(d)y(t) = 0$ for $t > 0$:

$$(\square\mu) = \{y(\cdot) \in (\mathscr{C}^\infty) : \mu(d)y(t) = 0 \quad \text{for } t > 0\}.$$

We write

$$\square\mu = \{y : y(\cdot) \in (\square\mu)\}. \tag{4.1}$$

Note that $\mathbf{T}_0 \in \square\mu$ if (and only if) $\mu(0) = 0$.

LEMMA 4.1. *Let μ be a polynomial. If μ_1 is a polynomial such that μ/μ_1 is also a polynomial, then $\square\mu_1 \subset \square\mu$.*

Proof. The proof is easy; see [7, p. 210].

Remark 4.1. Clearly, $\square\mu$ is a linear subspace of $\mathscr{K}$ (see Definition 1.2); the elements of $\square\mu$ could be called *free responses* [6, pp. 35–36].

DEFINITION 4.1. A polynomial μ will be called *dissipative* if all the zeros of μ lie to the left of the imaginary axis. If μ is dissipative, it is easily verified that $(\square\mu)$ is a space of transients (a *transient* is a function of t which approaches zero as $t \to \infty$).

Remark 4.2. The aim of the present § 4 is to prove a generalization of the following uniqueness property. Suppose that μ is dissipative; if $\sigma > 0$, then the constant zero is the only element of $\Lambda^\sigma \cap \square\mu$. The dissipativity condition on μ will be replaced by the less restrictive condition $\mu \in \mathrm{V}_\sigma$ (see Definition 4.2); also, the space Λ^σ will be replaced by the larger space $\mathscr{P}_\sigma(\psi/\mu)$ (see (3.9)).

We shall also prove two theorems (4.1 and 4.2) which will be used later on. Most of the material in this § 4 comes directly from the theory of ordinary differential equations with constant coefficients.

DEFINITION 4.2. Suppose $\sigma > 0$. We shall write $\mu \in \mathrm{V}_\sigma$ to indicate that μ is a polynomial such that

$$\mu(2k\pi i/\sigma) \neq 0 \quad \text{whenever } k \in \mathbb{Z},$$

where $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$.

Remark 4.3. If $\sigma > 0$ and $\mu \in \nabla_\sigma$, then $\mu(a) = 0$ implies $e^{a\sigma} \neq 1$ and $a \neq 0$. Clearly, if μ is dissipative, then $\mu \in \nabla_\sigma$.

Remark 4.4. Unless otherwise stated, m denotes a nonzero polynomial and α denotes the set of its zeros. Thus

$$a \in \alpha \Leftrightarrow m(a) = 0; \tag{4.2}$$

note that α is a subset of the field $\mathbb{C}$ of complex numbers. If b is the leading coefficient of the polynomial m, there exists a family $\{ma : a \in \alpha\}$ of integers $\geqq 1$ such that

$$m(s) = b \prod_{a \in \alpha} (s - a)^{ma} \quad \text{for all } s \in \mathbb{C}. \tag{4.3}$$

Suppose that $a \in \mathbb{C}$. If k is an integer $\geqq 1$, we denote by $e_a^k(\cdot)$ the function defined for any $t \in \mathbb{R}$ by

$$e_a^k(t) = \mathbf{T}_0(t)\, e^{at} t^{k-1}/(k-1)!; \tag{4.4}$$

note that $e_a^k(\cdot) \in (\mathscr{C}^\infty) \subset (\mathscr{K})$; from (1.3) and Remark 1.3 we see that

$$\acute{e}_a^k = D\mathbf{T}_0/(D - a)^k. \tag{4.5}$$

LEMMA 4.2. *The linear space* ($\square m$) *is the linear span of the set* $\{e_a^k(\cdot) : a \in \alpha$ *and* $1 \leqq k \leqq ma\}$.

Proof. See [5, pp. 88–90] or [4, p. 72].

THEOREM 4.1. *If* $y \in \square m$, *there exists a polynomial* λ *such that*

$$y = \frac{\lambda(D)}{m(D)} D\mathbf{T}_0 \quad \text{and} \quad |\lambda| < |m|. \tag{4.6}$$

Proof. Since $y \in \square m$, it follows from (4.1) and Lemma 4.2 that there exist complex numbers c_a^k such that

$$y(\cdot) = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k e_a^k(\cdot); \tag{4.7}$$

from Remark 1.3 and (4.5) it therefore follows that

$$m(D)y = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k m(D) \frac{D\mathbf{T}_0}{(D - a)^k},$$

whence, by (4.3),

$$m(D)y = \sum_{a \in \alpha} \sum_{k=1}^{ma} b c_a^k \left[\left\{ \prod_{z \in \beta} (D - z)^{mz} \right\} (D - a)^{ma-k} \right] D\mathbf{T}_0,$$

where $\beta = \{z \in \alpha : z \neq a\}$; this can also be written

$$m(D)y = \left[\sum_{a \in \alpha} \left\{ \prod_{z \in \beta} (D - z)^{mz} \right\} \left(\sum_{k=1}^{ma} b c_a^k (D - a)^{ma-k} \right) \right] D\mathbf{T}_0. \tag{4.8}$$

Denoting by $\lambda(D)$ the operator inside the square brackets, conclusion (4.6) results from (4.8) by observing that

$$|\lambda| \leqq \max_{a \in \alpha} \{(|m| - ma) + (ma - 1)\} \leqq |m| - 1.$$

THEOREM 4.2. *Let λ be a polynomial such that $|\lambda| < |m|$; then*

$$\textit{if } y = \frac{\lambda(D)}{m(D)} D\mathbf{T}_0, \quad \textit{then } y \in \Box m.$$

Proof. Expanding λ/m into partial fractions [1, p. 25], it follows from (4.3) that there exist complex numbers c_a^k such that

$$\frac{\lambda(D)}{m(D)} = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k \frac{1}{(D-a)^k};$$

therefore

$$y = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k \frac{D\mathbf{T}_0}{(D-a)^k} = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k e_a^k.$$

The last equation is from (4.5). Since $e_a^k \in \mathscr{K}$, it now follows from Theorem 1.2 that

$$y(\cdot) = \sum_{a \in \alpha} \sum_{k=1}^{ma} c_a^k e_a^k(\cdot);$$

the conclusion $y \in \Box m$ is now immediate from Lemma 4.2.

LEMMA 4.3. *Let $\{p_a : a \in \alpha\}$ be a finite set of polynomials. If*

$$\sum_{a \in \alpha} p_a(t)\, e^{at} = 0 \quad \textit{for all } t > 0,$$

then $p_a = 0$ for all $a \in \alpha$.

Proof. The proof is straightforward; see [7, pp. 199–200].

THEOREM 4.3. *Suppose $\sigma > 0$. If $m \in \nabla_\sigma$, then*

$$\Lambda^\sigma \cap \Box m \subset \{0\}. \tag{4.9}$$

(Loosely speaking, the constant zero is the only free response which is periodic with period σ.)

Proof. By Remark 4.3, our hypothesis $m \in \nabla_\sigma$ implies

$$e^{a\sigma} \neq 1. \tag{4.10}$$

Suppose that $y \in \Lambda^\sigma \cap \Box m$. From Definition 3.1 it follows that

$$y(t + \sigma) - y(t) = 0 \quad \text{for } t > 0. \tag{4.11}$$

Since $y \in \Box m$, we can infer from (4.7) and (4.4) the existence of polynomials P_a such that

$$y(t) = \sum_{a \in \alpha} e^{at} P_a(t) \quad \text{for } t > 0. \tag{4.12}$$

We have to prove that $y = 0$. From (4.12) and (4.11) it follows that

$$\sum_{a \in \alpha} e^{at}[e^{a\sigma} P_a(t + \sigma) - P_a(t)] = 0. \tag{4.13}$$

Let p_a be the polynomial defined by

$$p_a(t) = e^{a\sigma} P_a(t + \sigma) - P_a(t); \tag{4.14}$$

from (4.13)–(4.14) and Lemma 4.3 it follows that $p_a = 0$ for all $a \in \alpha$. That is (by (4.14)),

(4.15) $$e^{a\sigma}P_a(t+\sigma) = P_a(t) \quad \text{for } t > 0$$

and for any $a \in \alpha$. We shall now prove that the assumption

(4.16) $$P_a \neq 0 \quad \text{for some } a \in \alpha$$

implies a contradiction; this will conclude the proof, since it then follows that $P_a = 0$ for all $a \in \alpha$, from which our conclusion $y = 0$ is now an immediate consequence of (4.12). Let

(4.17) $$P_a(s) = \sum_{k=0}^{n} b_k s^k \quad \text{with } b_n \neq 0$$

be the polynomial in assumption (4.16); from (4.17) it follows that

(4.18) $$e^{a\sigma}b_n(t+\sigma)^n + \lambda_1(t) = b_n t^n + \lambda_2(t) \quad \text{for } t > 0,$$

where λ_k are polynomials of degree less than n. Dividing both sides of (4.18) by t^n, we obtain

$$b_n e^{a\sigma}\left(1 + \frac{\sigma}{t}\right)^n + \frac{1}{t^n}\lambda_1(t) = b_n + \frac{1}{t^n}\lambda_2(t);$$

taking limits as $t \to \infty$, we obtain (since $|\lambda_k| < n$)

(4.19) $$b_n e^{a\sigma} + 0 = b_n + 0.$$

Since $b_n \neq 0$ (by (4.17)), equation (4.19) implies $e^{a\sigma} = 1$, which contradicts (4.10). Consequently, assumption (4.16) is false. This concludes the proof.

THEOREM 4.4. *Suppose that $\sigma > 0$ and $\mu \in \nabla_\sigma$. Let ψ be a polynomial. If A is an operator, there is at most one pair $(y^\tau, y^\square)$ such that*

$$A = y^\tau + y^\square \quad \textit{with } y^\tau \in \mathscr{P}_\sigma(\psi/\mu) \quad \textit{and} \quad y^\square \in \square\mu.$$

Proof. Suppose that $A = y_k^\tau + y_k^\square$ with $y_k^\tau \in \mathscr{P}_\sigma(\psi/\mu)$ and $y_k^\square \in \square\mu$ for $k = 1, 2$. Consequently $y_1^\tau - y_2^\tau = y_2^\square - y_1^\square$; if we set

(4.20) $$y = y_1^\tau - y_2^\tau,$$

it then follows that

(4.21) $$y = y_2^\square - y_1^\square.$$

Since $y_k^\tau \in \mathscr{P}_\sigma(\psi/\mu)$, it follows from (4.20) and Remark 3.3 that $y \in \mathscr{P}_\sigma(\psi/\mu)$; since $y_k^\square \in \square\mu$, we may infer from (4.21) and Remark 4.1 that $y \in \square\mu$; thus we have

(4.22) $$y \in \square\mu \quad \text{and} \quad y \in \mathscr{P}_\sigma(\psi/\mu).$$

Since $y \in \square\mu$, it follows from Remark 4.1 that $y \in \mathscr{K}$; therefore, (4.22) gives $y \in \mathscr{K} \cap \mathscr{P}_\sigma(\psi/\mu)$, whence $y \in \Lambda^\sigma$ (by Theorem 3.3). Having thus verified that $y \in \Lambda^\sigma$, it follows from (4.22) that $y \in \Lambda^\sigma \cap \square\mu$; Theorem 4.3 (with $m = \mu$) therefore gives $y = 0$. In view of (4.20)–(4.21), the conclusions $y_1^\tau = y_2^\tau$ and $y_1^\square = y_2^\square$ are now at hand.

5. The periodic part. Henceforth, σ is a fixed number > 0; also, h will denote the ratio ψ/μ of two polynomials ψ and μ such that $\mu \in \nabla_\sigma$ (see Definition 4.2). As before, $\mathbb{C}$ denotes the field of complex numbers.

DEFINITION 5.1. Recall that the *polynomial part* of h is the first element p of the unique pair (p, λ) of polynomials such that

$$h = p + \lambda/\mu \quad \text{and} \quad |\lambda| < |\mu|.$$

As in Definition 3.2, we denote by $\mathscr{P}_\sigma(h)$ the family of all the operators of the form $p(D)f_1 + f_2$ (where $f_k \in \Lambda^\sigma$ for $k = 1, 2$).

Orientation. Let E be an operator. From Theorem 4.4 it follows that there exists at most one operator y^τ belonging to $\mathscr{P}_\sigma(h)$ and such that $h(D)E - y^\tau$ belongs to $\square\mu$. If such an operator y^τ exists, we shall call it the *periodic part* of $h(D)E$. This § 5 deals with some general properties which depend on the existence of the periodic part of $h(D)E$. It will be convenient to denote by $[\sigma, h]$ the set of all the operators E such that the periodic part of $h(D)E$ exists; as we shall see in (8.35), the inclusion $\Lambda^\sigma \subset [\sigma, h]$ holds; moreover, $\mathscr{P}_\sigma(p_1) \subset [\sigma, h]$ whenever p_1 is a polynomial such that $|p_1\psi| \leqq |\mu|$ (see Remark 8.4). The example in Remark 5.6 is typical.

Notation. Recall that $h = \psi/\mu$ and $\mu \neq 0$; we set

$$|h| \stackrel{\text{def}}{=} |\psi| - |\mu|.$$

In case $\psi = 0$, we have $h = 0$ and $|h| = -\infty - |\mu| = -\infty$. from Theorem 3.4 it follows that

(5.1) $$\mathscr{P}_\sigma(h) = \Lambda^\sigma \quad \text{when} \quad |h| \leqq 0.$$

DEFINITION 5.2. Given an operator E, we shall write $E \in [\sigma, h]$ to indicate that there exists a pair $(y^\tau, y^\square)$ such that

(5.2) $$h(D)E = y^\tau + y^\square \quad \text{with } y^\tau \in \mathscr{P}_\sigma(h) \quad \text{and} \quad y^\square \in \square\mu.$$

Suppose that $E \in [\sigma, h]$; from Theorem 4.4 it follows that there exists exactly one pair $(y^\tau, y^\square)$ satisfying the relations (5.2). We write

(5.3) $$h(\tau)E = y^\tau \quad \text{and} \quad h(\square)E = y^\square.$$

Further, we call $h(\tau)E$ the *periodic part* of $h(D)E$ (recall that $h(D)E$ is *the response* to E: see Definition 2.1).

Remark 5.1. Suppose that $E \in [\sigma, h]$. From Definition 5.2 we see that $h(\tau)E$ is the unique operator y^τ such that $y^\tau \in \mathscr{P}_\sigma(h)$ and $h(D)E - y^\tau$ belongs to the space $\square\mu$:

(5.4) $$h(\tau)E \in \mathscr{P}_\sigma(h) \quad \text{and} \quad h(\square)E \in \square\mu$$

and

(5.5) $$h(D)E = h(\tau)E + h(\square)E.$$

If all the zeros of the polynomial μ lie to the left of the imaginary axis, then $\mu \in \nabla_\sigma$ and each element of $\square\mu$ is a transient (see Definition 4.1). Thus $h(\tau)E$ is the steady state part of $h(D)E$ (since $h(D)E - h(\tau)E$ is a transient).

Remark 5.2. Given an operator E, if there exists a pair $(y^\tau, y^\square)$ such that

$$h(D)E = y^\tau + y^\square \quad \text{with } y^\tau \in \mathscr{P}_\sigma(h) \quad \text{and} \quad y^\square \in \square\mu,$$

then $E \in [\sigma, h]$ (by Definition 5.2); further, $h(\tau)E = y^\tau$ and $h(\square)E = y^\square$.

Remark 5.3. If $|h| \leqq 0$ and $E \in [\sigma, h]$, it follows from (5.1) and Remark 5.1 that $h(\tau)E$ is the unique operator y^τ in Λ^σ such that $h(D)E - y^\tau$ belongs to the space $\square\mu$.

Remark 5.4. Suppose that $|h| \leqq 0$ and let E be an operator such that the equation $h(D)E = y^\tau + y^\square$ holds for some pair $(y^\tau, y^\square)$ such that $y^\tau \in \Lambda^\sigma$ and $y^\square \in \square\mu$; it then follows from (5.1) and Remark 5.2 that $E \in [\sigma, h]$ and $h(\tau)E = y^\tau \in \Lambda^\sigma$.

DEFINITION 5.3. Let $[\sigma, h]$ be the family of all the operators E such that $E \in [\sigma, h]$.

Remark 5.5. It is not hard to verify that $[\sigma, h]$ is a linear subspace of the algebra $\mathscr{A}$; moreover, if $E_k (k = 1, 2, 3, \cdots, m)$ is a finite sequence of elements of the space $[\sigma, h]$, then the equation

$$h(\tau)\left[\sum_{k=1}^{m} c_k E_k\right] = \sum_{k=1}^{m} c_k [h(\tau)E_k] \tag{5.6}$$

holds for any sequence $c_k (k = 1, 2, 3, \cdots)$ in $\mathbb{C}$; this is not hard to prove.

Remark 5.6. Suppose that $|h| < 0$. Let us verify that $D\mathbf{T}_0$ (the unit impulse) belongs to the space $[\sigma, h]$; in so doing, we shall also obtain the equations $h(\tau)D\mathbf{T}_0 = 0$ and $h(\square)D\mathbf{T}_0 = h(D)D\mathbf{T}_0$.

First, note that $\mathbf{T}_0 \in \Lambda^\sigma$ (see (1.1) and Definition 3.1). Consequently $D\mathbf{T}_0 \in \mathscr{P}_\sigma(p)$, where p is the polynomial $p(s) = s$ (see Definition 5.1); since $|h| < 0$, we have $h = \psi/\mu$ with $|\psi| < |\mu|$. Consequently

$$h(D)D\mathbf{T}_0 = y^\square, \quad \text{where } y^\square = \frac{\psi(D)}{\mu(D)} D\mathbf{T}_0.$$

It now follows from Theorem 4.2 that $y^\square \in \square\mu$; since $h(D)D\mathbf{T}_0 = 0 + y^\square$, it follows from Remark 5.2 that $D\mathbf{T}_0 \in [\sigma; h]$ and $h(\tau)D\mathbf{T}_0 = 0$.

Orientation. We conclude this § 5 with two lemmas which will be needed later.

LEMMA 5.1. *Suppose that h is the ratio ψ/μ of two polynomials ψ and μ such that $\mu \in \mathrm{V}_\sigma$. Let p be the polynomial part of h: in consequence of Definition* 5.1, *there exists a polynomial λ such that*

$$h = p + \lambda/\mu \quad \text{and} \quad |\lambda| < |\mu|. \tag{5.7}$$

If $E \in [\sigma, \lambda/\mu)$ and $E \in \Lambda^\sigma$, then $E \in [\sigma, h]$ and

$$h(\tau)E = p(D)E + \frac{\lambda(\tau)}{\mu(\tau)} E, \tag{5.8}$$

where

$$\frac{\lambda(\tau)}{\mu(\tau)} E \overset{\text{def}}{=} \frac{\lambda}{\mu}(\tau)E. \tag{5.9}$$

Proof. Since $E \in [\sigma, \lambda/\mu]$, it follows from Definition 5.2 that

$$\frac{\lambda(D)}{\mu(D)}E = y_1^\tau + y^\square \quad \text{with } y^\square \in \square\mu. \tag{5.10}$$

In view of (5.9) and Remark 5.3, it also results that

$$y_1^\tau = \frac{\lambda(\tau)}{\mu(\tau)}E \in \Lambda^\sigma. \tag{5.11}$$

On the other hand, (5.7) and (5.10) imply that

$$h(D)E = p(D)E + y_1^\tau + y^\square. \tag{5.12}$$

Since $y_1^\tau \in \Lambda^\sigma$ (by (5.11)), and since p is the polynomial part of h, it follows from Definition 5.1 and the hypothesis $E \in \Lambda^\sigma$ that the equation

$$y^\tau = p(D)E + y_1^\tau \tag{5.13}$$

defines an element y^τ of $\mathscr{P}_\sigma(h)$; therefore (5.12) becomes

$$h(D)E = y^\tau + y^\square \quad \text{with } y^\tau \in \mathscr{P}_\sigma(h) \quad \text{and} \quad y^\square \in \square\mu$$

(by (5.10)). From Remark 5.2 we may now conclude that $E \in [\sigma, h]$ and

$$h(\tau)E = y^\tau = p(D)E + y_1^\tau. \tag{5.14}$$

The second equation is from (5.13). Conclusion (5.8) is now immediate from (5.14) and (5.11).

LEMMA 5.2. *Suppose that h_0 is a ratio ψ_0/μ of two polynomials ψ_0 and μ such that $\mu \in \nabla_\sigma$ and*

$$\Lambda^\sigma \subset [\sigma, h_0]. \tag{5.15}$$

Further, let p be a polynomial such that $|ph_0| \leqq 0$ and

$$\Lambda^\sigma \subset [\sigma, ph_0]. \tag{5.16}$$

If $E = p(D)f_1 + f_2$ with $f_k \in \Lambda^\sigma$ (for $k = 1, 2$), then $E \in [\sigma, h_0]$ and $h_0(\tau)E \in \Lambda^\sigma$; moreover,

$$h_0(\tau)E = ph_0(\tau)f_1 + h_0(\tau)f_2. \tag{5.17}$$

Proof. Since $h_0 = \psi_0/\mu$, we have

$$ph_0 = p\psi_0/\mu \quad \text{and} \quad \mu \in \nabla_\sigma. \tag{5.18}$$

The hypothesis $|ph_0| \leqq 0$ clearly implies that $|h_0| \leqq 0$; from hypothesis (5.15) and $f_2 \in \Lambda^\sigma$, it follows that $f_2 \in [\sigma, h_0]$; we may therefore infer from Remark 5.3 that

$$h_0(\tau)f_2 \in \Lambda^\sigma \tag{5.19}$$

and

$$h_0(D)f_2 = h_0(\tau)f_2 + y_2^\square \quad \text{with} \quad y_2^\square \in \square\mu. \tag{5.20}$$

Since $f_1 \in \Lambda^\sigma$ it results from hypothesis (5.16) that $f_1 \in [\sigma, ph_0]$; since $|ph_0| \leqq 0$

(by hypothesis), we can again use Remark 5.3 (and (5.18)) to conclude that

$$ph_0(\tau)f_1 \in \Lambda^\sigma \tag{5.21}$$

and

$$ph_0(D)f_1 = ph_0(\tau)f_1 + y_1^\square \quad \text{with} \quad y_1^\square \in \square\mu. \tag{5.22}$$

Our hypothesis $E = p(D)f_1 + f_2$ clearly implies that

$$h_0(D)E = ph_0(D)f_1 + h_0(D)f_2. \tag{5.23}$$

Combining (5.23) with (5.22) and (5.20), we obtain

$$h_0(D)E = y^\tau + y_1^\square + y_2^\square, \tag{5.24}$$

where

$$y^\tau = ph_0(\tau)f_1 + h_0(\tau)f_2. \tag{5.25}$$

Since both $y_1^\square$ and $y_2^\square$ belong to $\square\mu$ (see (5.22) and (5.20)), we may conclude from Remark 4.1 that $y_1^\square + y_2^\square$ is an element $y^\square$ of $\square\mu$. Equation (5.24) becomes

$$h_0(D)E = y^\tau + y^\square \quad \text{with } y^\square \in \square\mu. \tag{5.26}$$

Since $y^\tau \in \Lambda^\sigma$ (by (5.25), (5.21), and (5.19), we can now obtain from Remark 5.4 the conclusions $E \in [\sigma, h_0]$ and $h_0(\tau)E = y^\tau \in \Lambda^\sigma$. Conclusion (5.17) is immediate from (5.25).

6. Sinusoidal excitations. This section culminates with Theorem 6.3, which is a slight extension of a result extensively used in electrical engineering textbooks (of course, we shall not assume the usual dissipativity condition; see Definition 4.1). Equation (6.22) shows what happens to the classical "phasor" equation (6.16) when the transfer function is allowed to have degree 1.

As before $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \cdots\}$. Unless otherwise specified, h denotes the ratio ψ/μ of two polynomials. When $a \in \mathbb{C}$, we denote by $F_a(\cdot)$ the function defined by

$$F_a(t) = \mathbf{T}_0(t)\, e^{at} \quad \text{for } t \in \mathbb{R}. \tag{6.1}$$

Note that $F_a \in \mathscr{K}$; from (1.3) it follows that

$$F_a = \frac{D\mathbf{T}_0}{D - a}. \tag{6.2}$$

It is easily verified that

$$DF_a = D\mathbf{T}_0 + aF_a. \tag{6.3}$$

LEMMA 6.1. *Suppose that* $|h| \leqq 0$. *If* $a \in \mathbb{C}$ *and* $\mu(a) \neq 0$, *then*

$$h(D)F_a = h(a)F_a + y^\square, \quad \text{where } y^\square \in \square\mu. \tag{6.4}$$

Proof. Let Φ be the polynomial defined by

$$\Phi(s) = \mu(a)\psi(s) - \psi(a)\mu(s). \tag{6.5}$$

Since $h = \psi/\mu$, (6.5) implies that

$$h(s) - h(a) = \frac{\Phi(s)}{\mu(a)\mu(s)}; \tag{6.6}$$

moreover, since $|\psi| \leqq |\mu|$ (by hypothesis), (6.5) gives

$$|\Phi| \leqq |\mu|. \tag{6.7}$$

Set $c = 1/\mu(a)$; since $c\Phi(a) = 0$, there exists a polynomial λ such that

$$\Phi(s)/\mu(a) = (s - a)\lambda(s); \tag{6.8}$$

consequently (6.6) implies

$$h(D)F_a - h(a)F_a = \frac{(D - a)\lambda(D)}{\mu(D)}F_a,$$

whence, by (6.2),

$$h(D)F_a - h(a)F_a = \frac{\lambda(D)}{\mu(D)}D\mathbf{T}_0. \tag{6.9}$$

From (6.8) we see that $|\lambda| < |\Phi|$; from (6.7) it therefore follows that $|\lambda| < |\mu|$. Conclusion (6.4) is now immediate from (6.9) and Theorem 4.2.

Notation. Let $g(\cdot)$ be a function such that $\mathbf{T}_0(\cdot)g(\cdot)$ is a function $g^+(\cdot)$ belonging to the space $(\mathscr{K})$; if $g^+ \in [\sigma, h]$ and if the operator $h(\tau)g^+$ is an element f of $\mathscr{K}$, it follows from Remark 4.1 that the number $f(t)$ is defined for any $t \in \mathbb{R}$. Under these circumstances, we shall write

$$h(\tau)g(t) = f(t) \quad \text{for } t > 0 \tag{6.10}$$

instead of

$$h(\tau)g^+ = f. \tag{6.11}$$

THEOREM 6.1. *Suppose that $\sigma > 0$ and $h = \psi/\mu$ with $|h| \leqq 0$ and $\mu \in \nabla_\sigma$. If $b \in \mathbb{C}$, then $b\mathbf{T}_0 \in [\sigma, h]$ and*

$$h(\tau)b\mathbf{T}_0(t) = bh(0) \quad \textit{for } t > 0. \tag{6.12}$$

Proof. Since $\mu \in \nabla_\sigma$, it follows from Remark 4.3 that $\mu(0) \neq 0$. Setting $a = 0$ in (6.2), we obtain $F_0 = \mathbf{T}_0$, and (6.4) becomes

$$h(D)\mathbf{T}_0 = h(0)\mathbf{T}_0 + y^\square, \quad \text{where } y^\square \in \square\mu. \tag{6.13}$$

Obviously, $h(0)\mathbf{T}_0 \in \Lambda^\sigma$. From (6.13) and Remark 5.4 it therefore follows that $\mathbf{T}_0 \in [\sigma, h]$ and $h(\tau)\mathbf{T}_0 = h(0)\mathbf{T}_0$; consequently, $b\mathbf{T}_0$ also belongs to $[\sigma, h]$ (by Remark 5.5), and the equation

$$h(\tau)b\mathbf{T}_0 = bh(0)\mathbf{T}_0 \tag{6.14}$$

is immediate from (5.6). Setting $g(t) = 1$ in (6.10)–(6.11), we find that (6.14) can be written

$$h(\tau)b\mathbf{T}_0(t) = bh(0)\mathbf{T}_0(t) = bh(0) \quad \text{for } t > 0.$$

Remark 6.1. As a consequence of Theorem 6.1, we have, for $|h| \leqq 0$,

$$b\mathbf{T}_0 \in [\sigma, h] \quad \text{and} \quad b\mathbf{T}_0 \in \Lambda^\sigma.$$

Thus, in view of Lemma 5.1, the restriction $|h| \leqq 0$ can be removed.

THEOREM 6.2. *As usual,* $i = \sqrt{-1}$. *Let* r *be a nonzero real number; moreover, suppose that*

$$\mu(ik|r|) \neq 0 \quad \text{for all } k \in \mathbb{Z}. \tag{6.15}$$

If $|h| \leqq 0$ *and* $\sigma = 2\pi/|r|$, *the operator* F_{ir} *belongs to* $[\sigma, h]$ *and*

$$h(\tau)\, e^{irt} = h(ir)\, e^{irt} \quad \text{for } t > 0. \tag{6.16}$$

Proof. Since $|r| = 2\pi/\sigma$, our hypothesis (6.15) guarantees that $\mu \in \mathrm{V}_\sigma$ (see Definition 4.2); since (6.15) also implies $\mu(ir) \neq 0$, we may apply Lemma 6.1 (with $a = ir$) to obtain

$$h(D)F_{ir} = h(ir)F_{ir} + y^\square, \quad \text{where } y^\square \in \square\mu. \tag{6.17}$$

Since $F_{ir}(t + \sigma) = F_{ir}(t)$ (by (6.1)), the operator $h(ir)F_{ir}$ belongs to Λ^σ; in view of (6.17), the equation

$$h(\tau)\, F_{ir} = h(ir)\, F_{ir} \tag{6.18}$$

follows directly from Remark 5.4. Conclusion (6.16) is obtained by setting $g(t) = e^{irt}$ in (6.10)–(6.11) (recall that $F_{ir}(t) = g(t)\mathbf{T}_0(t)$).

Remark 6.2. Let us remove the condition $|h| \leqq 0$ from the hypotheses of Theorem 6.2 If p is the polynomial part of h, it then follows from (5.8) that

$$h(\tau)F_{ir} = p(D)F_{ir} + \frac{\lambda(\tau)}{\mu(\tau)}F_{ir}. \tag{6.19}$$

Of course, $\lambda/\mu = h - p$. Setting $h = \lambda/\mu$ in (6.18), equation (6.19) becomes

$$h(\tau)F_{ir} = p(D)F_{ir} + \frac{\lambda(ir)}{\mu(ir)}F_{ir}. \tag{6.20}$$

In the particular case $p(D) = D$, we have $h(s) = s + \lambda(s)/\mu(s)$ and

$$h(\tau)F_{ir} = D\mathbf{T}_0 + \left[ir + \frac{\lambda(ir)}{\mu(ir)}\right]F_{ir}. \tag{6.21}$$

Equation (6.21) is derived from (6.20) by using (6.3); it can also be written

$$h(\tau)F_{ir} = D\mathbf{T}_0 + h(ir)F_{ir} \quad \text{when } |h| = 1. \tag{6.22}$$

Recall that $D\mathbf{T}_0$ is the unit impulse and compare with (6.18).

THEOREM 6.3. *Suppose that* h *is the ratio* ψ/μ *of two polynomials with real coefficients and such that* $|h| \leqq 0$. *Given a number* $\omega > 0$ *such that* $\mu(ki\omega) \neq 0$ *for all* $k \in \mathbb{Z}$, *let* α *be the principal value of the argument of the complex number* $h(i\omega)$. *If* $\theta \in \mathbb{R}$ *and* $g(t) = \sin(\omega t + \theta)$, *then*

$$h(\tau)g(t) = |h(i\omega)| \sin(\omega t + \theta + \alpha) \quad \text{for } t > 0. \tag{6.23}$$

Proof. Set $\sigma = 2\pi/\omega$. Obviously, $\mu \in \mathrm{V}_\sigma$ and $\mu(k\omega i) \neq 0$ for all $k \in \mathbb{Z}$; the hypotheses of Theorem 6.2 are satisfied for $r = \pm\omega$. It is now merely a question of applying (6.18) proving that

$$h(\tau)[aF_{i\omega} + \bar{a}F_{-i\omega}] = |h(i\omega)|f,$$

where $a = e^{i\theta}/2i$ and $f(t) = \sin(\omega t + \theta + \alpha)$.

7. Nonsinusoidal excitations. Throughout, σ is a fixed number > 0. If $w \in \mathscr{K}$ and $a \in \mathbb{C}$, it follows from (1.10) that

$$\frac{w}{D-a}(t) = e^{at}\int_0^t e^{-au}w(u)\,du \quad \text{for } t > 0. \tag{7.1}$$

If $G \in \mathscr{K}$, we define $G^0(\cdot)$ as in Theorem 3.1; consequently (7.1) gives

$$\frac{G^0}{D-a}(\sigma) = e^{a\sigma}\int_0^\sigma e^{-au}G(u)\,du. \tag{7.2}$$

DEFINITION 7.1. As in (4.4), let $e_a^1(\cdot)$ be the function defined by $e_a^1(t) = e^{at}\mathbf{T}_0(t)$. If $G \in \mathscr{K}$ and $a \neq 0$, we set

$$\frac{G}{\tau-a} = \frac{G}{D-a} - \left[\frac{G^0}{D-a}(\sigma)\right](e^{a\sigma}-1)^{-1}e_a^1. \tag{7.3}$$

Orientation. In case h is a rational function with simple poles, we shall, in this section, show that the periodic response $h(\tau)G$ can be expressed as a linear combination of operators of the form $G/(\tau - a)$.

LEMMA 7.1. *If $G \in \Lambda^\sigma$ and $a \neq 0$, then $G/(\tau - a)$ belongs to Λ^σ.*

Proof. Set

$$b = \frac{-e^{a\sigma}}{e^{a\sigma}-1}\int_0^\sigma e^{-au}G(u)\,du. \tag{7.4}$$

Set $y = G/(\tau - a)$; from (7.1), (7.3) and (7.2), we see that

$$y(t) = \frac{G}{\tau-a}(t) = e^{at}\left\{b + \int_0^t e^{-au}G(u)\,du\right\}. \tag{7.5}$$

In view of Definition 3.1, it will suffice to verify that $y(\theta + \sigma) = y(\theta)$ for all $\theta > 0$. To that effect, take any $\theta > 0$ and note that (7.5) gives

$$y(\theta+\sigma) = e^{a(\theta+\sigma)}\left\{b + \int_0^{\theta+\sigma} e^{-au}G(u)\,du\right\}. \tag{7.6}$$

Since

$$\int_0^{\theta+\sigma} e^{-au}G(u)\,du = \left[\int_0^\sigma e^{-au}G(u)\,du\right] + \int_\sigma^{\theta+\sigma} e^{-au}G(u)\,du, \tag{7.7}$$

the change of variable $\tau = u - \sigma$ gives

$$\int_\sigma^{\theta+\sigma} e^{-au}G(u)\,du = e^{-a\sigma}\int_0^\theta e^{-a\tau}G(\tau+\sigma)\,d\tau. \tag{7.8}$$

Since $G(\tau + \sigma) = G(\tau)$, we can combine (7.8) with (7.7) and (7.4) to obtain

$$\int_0^{\theta+\sigma} e^{-au}G(u)\,du = [-b + e^{-a\sigma}b] + e^{-a\sigma}\int_0^\theta e^{-a\tau}G(\tau)\,d\tau. \tag{7.9}$$

Substituting (7.9) into (7.6), we have

$$y(\theta + \sigma) = e^{a(\theta+\sigma)}\left\{0 + e^{-a\sigma}b + e^{-a\sigma}\int_0^\theta e^{-au}G(u)\,du\right\}; \tag{7.10}$$

the conclusion $y(\theta + \sigma) = y(\theta)$ is now immediate from (7.10) and (7.5).

THEOREM 7.1. *Let α be a subset of $\mathbb{C}$ containing none of the points of the form $2k\pi i/\sigma$, where $k \in \mathbb{Z}$. Given a number k_0 and a subset $\{c_a : a \in \alpha\}$ of $\mathbb{C}$; let h be the function defined by*

$$h(s) = k_0 + \sum_{a\in\alpha} c_a \frac{1}{s-a}. \tag{7.11}$$

If $G \in \Lambda^\sigma$, then $G \in [\sigma, h]$ and

$$h(\tau)G = k_0 G + \sum_{a\in\alpha} c_a \frac{G}{\tau - a}; \tag{7.12}$$

moreover, $h(\tau)G \in \Lambda^\sigma$, and

$$h(\square)G(t) = \sum_{a\in\alpha} c_a \left[\frac{G^0}{D-a}(\sigma)\right]\frac{e^{at}}{e^{a\sigma}-1}. \tag{7.13}$$

Proof. Equation (7.11) implies

$$h(D)G = k_0 G + \sum_{a\in\alpha} c_a \frac{G}{D-a};$$

consequently (7.3) gives

$$h(D)G = k_0 G + \sum_{a\in\alpha} c_a \frac{G}{\tau-a} + y^\square, \tag{7.14}$$

where

$$y^\square = \sum_{a\in\alpha} c_a \left[\frac{G^0}{D-a}(\sigma)\right](e^{a\sigma} - 1)^{-1}\, e_a^1. \tag{7.15}$$

It is easily seen that h is the ratio $h = \psi/m$ of two polynomials ψ and m such that $|\psi| \leqq |m|$. Indeed, m is defined by the equation $m(s) = b\prod_{a\in\alpha}(s-a)$ (see (4.3)); clearly our condition on the set α insures that $m \in \nabla\sigma$. Note that $ma^1 = 1$ (see Remark 4.4); from (7.15) it follows that $y^\square(\cdot)$ belongs to the linear span of the set $\{e_a^k(\,) : a \in \alpha \text{ and } 1 \leqq k \leqq ma\}$; in view of Lemma 4.2, this proves that $y^\square(\cdot)$ belongs to the space $(\square m)$. Consequently $y^\square \in \square m$ (see (4.1)); therefore if we set

$$y^\tau = k_0 G + \sum_{a\in\alpha} c_a \frac{G}{\tau-a}, \tag{7.16}$$

it follows from (7.14) that

$$h(D)G = y^\tau + y^\square \quad \text{with } y^\square \in \square m. \tag{7.17}$$

From (7.16), Lemma 7.1 and Remark 3.1 we see that $y^\tau \in \Lambda^\sigma$. As we remarked earlier, it follows from (7.11) that $|h| \leqq 0$; we may therefore apply Remark 5.4 to infer from (7.17) that $G \in [\sigma, h]$, $h(\tau)G = y^\tau \in \Lambda^\sigma$ and $h(\square)G = y^\square$. Conclusions (7.12) and (7.13) are now immediate from (7.16) and (7.15).

DEFINITION 7.2. A function h will be called a *simple σ-stable function* if h is the ratio ψ/μ of two polynomials ψ and μ such that $\mu \in \nabla_\sigma$ and such that μ has only simple zeros.

THEOREM 7.2. *Let h be a simple σ-stable function such that $|h| \leqq 0$, If $G \in \Lambda^\sigma$ then $G \in [\sigma, h]$ and $h(\tau) \in \Lambda^\sigma$.*

Proof. By Definition 7.2 the function h is the ratio ψ/μ of two polynomials ψ and μ such that $\mu \in \nabla_\sigma$ and such that μ has only simple zeros. Since $|h| \leqq 0$, the polynomial part of h (see Lemma 5.1) is a number k_0. We can therefore write $h = k_0 + \lambda/\mu$ with $|\lambda| < |\mu|$; further, the partial fraction expansion theorem enables us to write h in the form (7.11). If $G \in \Lambda^\sigma$, the conclusions $G \in [\sigma, h]$ and $h(\tau)G \in \Lambda^\sigma$ come directly from Theorem 7.1. If α is the set of zeros of the polynomial μ, then (7.12) gives

$$h(\tau)G = k_0 G + \sum_{a \in \alpha} c_a \frac{G}{\tau - a}.$$

It is not hard to verify that

$$c_a = \frac{\lambda(a)}{g_a(a)}, \quad \text{where } g_a(s) = \frac{\mu(s)}{s - a},$$

while the number k_0 is given by the equation

$$k_0 = \frac{\lambda(0)}{\mu(0)} + \sum_{a \in \alpha} a^{-1} c_a.$$

Remark 7.1. Just as in Remark 6.2, the restriction $|h| \leqq 0$ can be removed.

THEOREM 7.3. *Suppose that h_0 is a simple σ-stable function; further, let p be a polynomial such that $|ph_0| \leqq 0$. If $E = p(D)f_1 + f_2$ with $f_k \in \Lambda^\sigma$ (for $k = 1, 2$), then $E \in [\sigma, h_0]$ and $h_0(\tau)E \in \Lambda^\sigma$; moreover,*

$$h_0(\tau)E = ph_0(\tau)f_1 + h_0(\tau)f_2. \tag{7.18}$$

Proof. Clearly, the hypothesis $|ph_0| \leqq 0$ implies $|h_0| \leqq 0$; from Theorem 7.2 we therefore have the inclusion

$$\Lambda^\sigma \subset [\sigma, h_0]. \tag{7.19}$$

Set $h_0 = \psi/\mu$; therefore $\mu \in \nabla_\sigma$ and the polynomial μ has only simple zeros. Since $ph_0 = p\psi/\mu$, it follows that ph_0 is a simple σ-stable function; since $|ph_0| \leqq 0$ (by hypothesis), the inclusion

$$\Lambda^\sigma \subset [\sigma, ph_0] \tag{7.20}$$

now comes directly from Theorem 7.2. The conclusions are immediate from Lemma 5.2.

Remark 7.2. Let h be a simple σ-stable function such that $|h| \leqq 0$, and suppose that $f_k \in \Lambda^\sigma$ for $k = 1, 2$. Setting $p = 1$ in Theorem 7.3, we obtain

$$h(\tau)(f_1 + f_2) = h(\tau)f_1 + h(\tau)f_2. \tag{7.21}$$

In particular, if $b \in \mathbb{C}$ and $f_2 = -b\mathbf{T}_0$, we can use (6.10)–(6.11) and (6.12) to obtain

$$h(\tau)[f_1 - b\mathbf{T}_0](t) = h(\tau)f_1(t) - bh(0) \quad \text{for } t > 0. \tag{7.22}$$

If $E \in \Lambda^\sigma$ is such that the restriction of its graph to the interval $(0, \sigma)$ is a line having slope m, then

$$\frac{\tau}{\tau - a}E(t) = \frac{\sigma m\, e^{at}}{e^{a\sigma} - 1} - \frac{m}{a} \quad \text{for } 0 < t < \sigma. \tag{7.23}$$

Remark 7.3. The following is essentially Example 7-11 in the textbook [3, pp. 213–215]; the reasoning in [3] involves unnecessary physical principles (e.g., "the voltage across a capacitor cannot change instantaneously"). Let E be a voltage source in a simple electric circuit consisting of a resistance $R > 0$ and a capacitance $C > 0$; let b be the initial voltage across the capacitor. If y is the voltage across the resistance, then

$$E = y + \frac{1}{RCD}y + b\mathbf{T}_0;$$

consequently if $a = -1/RC$ and $h(s) = s/(s - a)$, then

$$y = h(D)(E - b\mathbf{T}_0). \tag{7.24}$$

Thus y can be considered as the response to $E - b\mathbf{T}_0$; from Remark 5.1 it follows that the steady state part of y equals the periodic part $y^\tau = h(\tau)(E - b\mathbf{T}_0)$ of y. From (7.22) and $h(0) = 0$, we obtain

$$y^\tau(t) = \frac{\tau}{\tau - a}E(t) + 0 \quad \text{for } t > 0. \tag{7.25}$$

Let E be the sawtooth function of period σ defined by

$$E(t) = \frac{\sigma - t}{\sigma} \quad \text{for } 0 < t < \sigma; \tag{7.26}$$

from (7.25)–(7.26) and (7.23) it follows that

$$y^\tau(t) = \frac{-e^{at}}{e^{a\sigma} - 1} + \frac{1}{a\sigma} = \frac{\tau}{\tau - a}E(t) \quad \text{for } 0 < t < \sigma. \tag{7.27}$$

Remark 7.4. Given a real number L, let G be the operator defined by

$$G = \sum_{k=0}^{\infty} LD\mathbf{T}_{k\sigma} = \frac{LD}{\mathbf{T}_0 - \mathbf{T}_\sigma}. \tag{7.28}$$

G represents a row of impulses of magnitude L applied at the times $k\sigma$ (where $k = 0, 1, 2, 3, \cdots$); see (1.5). Let p be the polynomial $p(s) = s$. To verify that $G \in \mathscr{P}_\sigma(p)$, it follows from Definition 3.2 that it will suffice to establish the equation

$$L^{-1}G = DE + (1/\sigma)\mathbf{T}_0, \tag{7.29}$$

where E is the sawtooth function of period σ defined by (7.26). It is not hard to verify that

$$E = \frac{-1}{\sigma D}\mathbf{T}_0 + \frac{1}{\mathbf{T}_0 - \mathbf{T}_\sigma};$$

see 5.43.8 in [8, p. 106]. Consequently

$$DE + \frac{1}{\sigma}\mathbf{T}_0 = \frac{D}{\mathbf{T}_0 - \mathbf{T}_\sigma} = L^{-1}G; \tag{7.30}$$

the second equation is from (7.28). Equation (7.29) is immediate from (7.30); it implies that $G \in \mathscr{P}_\sigma(p)$ when p is the polynomial $p(s) = s$.

Remark 7.5. Suppose that $a > 0$ and let h_0 be the simple σ-stable function defined by $h_0(s) = 1/(s - a)$; let p be the polynomial $p(s) = s$. Since $|ph_0| \leqq 0$, we can apply Theorem 7.3 with $E_1 = L^{-1}G$; from (7.29) and (7.18) it follows that

$$\frac{1}{\tau - a}L^{-1}G = \frac{\tau}{\tau - a}E + \frac{1}{\tau - a}\left(\frac{1}{\sigma}\right)\mathbf{T}_0.$$

That is, by (6.14),

$$\frac{1}{\tau - a}L^{-1}G = \frac{\tau}{\tau - a}E + \left(\frac{1}{\sigma}\right)\frac{1}{0 - a}\mathbf{T}_0. \tag{7.31}$$

In view of (7.27), equation (7.31) can be written

$$\left[\frac{1}{\tau - a}L^{-1}G\right](t) = \frac{-e^{at}}{e^{a\sigma} - 1} + \frac{1}{a\sigma} - \frac{1}{a\sigma} \quad \text{for } 0 < t < \sigma. \tag{7.32}$$

Remark 7.6. Let G be the row of impulses defined by (7.28); let it be the input voltage for a simple electric circuit consisting of a resistance R and an inductance L. The current y_1 satisfies the equation $Ry_1 + LDy_1 = G$. In view of Definition 1.4, the response to G is given by the equation

$$y_1 = \frac{1}{LD + R}G = \frac{1}{D - a}L^{-1}G, \quad \text{where } a = -R/L.$$

Since $a < 0$, it follows from Remark 5.1 that the steady state part of the current y_1 is given by

$$y_1^\tau(t) = \left[\frac{1}{\tau - a}L^{-1}G\right](t) = \frac{-e^{at}}{e^{a\sigma} - 1} \quad \text{for } 0 < t < \sigma;$$

the last equation is from (7.32). This answer is more informative than the Fourier series answer obtained for the same problem in [13, p. 174].

8. Poles of arbitrary orders. As usual, σ is a fixed number > 0 throughout.

Remark 8.1. Recall Definition 4.2; we write $\mu \in \mathrm{V}_\sigma$ to indicate that μ is a polynomial such that $\mu(2k\pi i/\sigma) \neq 0$ whenever $k \in \mathbb{Z}$.

DEFINITION 8.1. A function h will be called *σ-stable* if it is the ratio ψ/μ of two polynomials ψ and μ such that $\mu \in \mathrm{V}_\sigma$. As in Definition 7.2, we say that h is a *simple* σ-stable function if h is σ-stable and has only simple poles.

Orientation. The object of this § 8 is to extend the results of § 7 to σ-stable functions having poles of multiple order. We shall give two explicit procedures (see Remark 8.2 and Theorem 8.1) for calculating the periodic part of the response to an element of Λ^σ. Although we start with the restriction $|h| \leqq 0$, this restriction is removed in Theorem 8.2 (as we saw in (6.22), the case $|h| = 1$ implies that the periodic part has an impulse term).

LEMMA 8.1. *Let g_1 be a σ-stable function, and let g_2 be a simple σ-stable function; further, suppose that $|g_k| \leqq 0$ for $k = 1, 2$. If $E \in [\sigma, g_1]$, then $E \in [\sigma, g_2 g_1]$ and*

$$g_2 g_1(\tau)E = g_2(\tau)[g_1(\tau)E] \in \Lambda^\sigma. \tag{8.1}$$

Proof. Let $g_1 = \psi_1/\mu_1$ and $g_2 = \psi_2/\mu_2$. Then

$$g_1 g_2 = \psi_1\psi_2/\mu, \quad \text{where } \mu = \mu_1\mu_2 \tag{8.2}$$

Further, set

$$y_1^\tau = g_1(\tau)E. \tag{8.3}$$

From our hypotheses $E \in [\sigma, g_1]$ and $|g_1| \leqq 0$, we can infer from Remark 5.4 and (8.3) that

$$g_1(D)E = y_1^\tau + y_1^\square, \tag{8.4}$$

where

$$y_1^\tau = g_1(\tau)E \in \Lambda^\sigma \quad \text{and} \quad y_1^\square \in \square\mu_1. \tag{8.5}$$

From (8.4) it follows that

$$g_2 g_1(D)E = g_2(D)y_1^\tau + g_2(D)y_1^\square. \tag{8.6}$$

By hypothesis, g_2 is a simple σ-stable function with $|g_2| \leqq 0$. Since $y_1^\tau \in \Lambda^\sigma$ (by (8.5)), we can set $h = g_2$ and $G = y_1^\tau$ in Theorem 7.2 to infer that $y_1^\tau \in [\sigma, g_2]$; from Remark 5.3 we therefore have

$$g_2(D)y_1^\tau = y_2^\tau + y_2^\square, \tag{8.7}$$

where

$$y_2^\tau = g_2(\tau)y_1^\tau \in \Lambda^\sigma \quad \text{and} \quad y_2^\square \in \square\mu_2. \tag{8.8}$$

Substituting (8.7) into (8.6), we obtain

$$g_2 g_1(D)E = y_2^\tau + y^\square, \tag{8.9}$$

where

$$y^\square = g_2(D)y_1^\square + y_2^\square. \tag{8.10}$$

Since $|g_2 g_1| \leqq 0$ and $y_2^\tau \in \Lambda^\sigma$ (by (8.8)), we see from (8.9) and Remark 5.3 that it only remains to prove the property $y^\square \in \square\mu$. To that effect, we use the fact that $y_k^\square \in \square\mu_k$ (for $k = 1, 2$; see (8.5) and (8.8)); from Theorem 4.1 it therefore follows that there exists polynomials λ^1 and λ^2 such that $|\lambda^k| < |\mu_k|$ and $y_k = \lambda^k(D)D\mathbf{T}_0/\mu_k(D)$;

combining this with (8.10), we get

$$y^{\square} = g_2(D)\frac{\lambda^1(D)}{\mu_1(D)}D\mathbf{T}_0 + \frac{\lambda^2(D)}{\mu_2(D)}D\mathbf{T}_0. \tag{8.11}$$

But $g_2 = \psi_2/\mu_2$ with $|\psi_2| \leqq |\mu_2|$. Consequently, (8.11) gives

$$y^{\square} = \frac{(\psi_2\lambda^1 + \mu_1\lambda^2)(D)}{\mu_1\mu_2(D)}D\mathbf{T}_0\,; \tag{8.12}$$

it now only remains to observe that the degree of the polynomial $\psi_2\lambda^1 + \mu_1\lambda^2$ is smaller than $|\mu_1| + |\mu_2|$ (since $|\lambda^k| < |\mu_k|$ and $|\psi_2| \leqq |\mu_2|$). The conclusion $y^{\square} \in \square\mu_1\mu_2$ is now immediate from Theorem 4.2 and (8.12).

Since $\mu = \mu_1\mu_2$ (see (8.2)), we have just proved that $y^{\square} \in \square\mu$. On the other hand, we have (8.9) with $y_2^{\tau} \in \Lambda^{\sigma}$ (see (8.8)); since $|g_2g_1| \leqq 0$, we may set $h = g_2g_1$ in Remark 5.4 to conclude from (8.9) that $E \in [\sigma, g_2g_1]$ and

$$g_1g_1(\tau)E = y^{\tau} = g_2(\tau)[g_1(\tau)E] \in \Lambda^{\sigma}.$$

The last equation is from (8.8) and (8.5).

LEMMA 8.2. *Let h_k $(k = 1, 2, 3, \cdots)$ be a sequence of simple σ-stable functions such that $|h_k| \leqq 0$; for any integer $n \geqq 1$ we set*

$$H_n = \prod_{k=1}^{n} h_k. \tag{1}$$

If $G \in \Lambda^{\sigma}$ we set $G_1 = h_1(\tau)G$ and define recursively

$$G_{\nu+1} = h_{\nu+1}(\tau)G_\nu \quad \textit{for any integer } \nu. \tag{2}$$

If n is any integer $\geqq 1$, we have

$$G \in [\sigma, H_n] \quad \text{and} \quad H_n(\tau)G = G_n. \tag{3_n}$$

Proof. In case $n = 1$, we have $H_1 = h_1$ and we can get $g_2 = h_1$ in Theorem 7.2 to obtain $G \in [\sigma, H_1]$ and $G_1 = H_1(\tau)G \in \Lambda^{\sigma}$. We proceed by induction. Suppose that

$$G \in [\sigma, H_\nu] \quad \text{and} \quad H_\nu(\tau)G = G_\nu. \tag{3_ν}$$

It is easily verified that H_ν is σ-stable; since $H_{\nu+1} = h_{\nu+1}H_\nu$, we may therefore apply Lemma 8.1 (with $g_2 = h_{\nu+1}$ and $g_1 = H_\nu$). Since $G \in [\sigma, H_\nu]$ (by the induction hypothesis (3_ν)), Lemma 8.1 gives $G \in [\sigma, g_2H_\nu]$; therefore

$$G \in [\sigma, H_{\nu+1}]. \tag{8.13}$$

Moreover, (8.1) yields

$$h_{\nu+1}H_\nu(\tau)G = h_{\nu+1}(\tau)[H_\nu(\tau)G]. \tag{8.14}$$

Since $H_{\nu+1} = h_{\nu+1}H_\nu$ and $H_\nu(\tau)G = G_\nu$ (by (3_ν)), equation (8.14) can be written

$$H_{\nu+1}(\tau)G = h_{\nu+1}(\tau)G_\nu = G_{\nu+1}\,; \tag{8.15}$$

the last equation is from definition (2). In view of (8.14)–(8.15), we have proved that (3_ν) implies ($3_{\nu+1}$). Consequently, (3_n) holds for any integer $n \geqq 1$.

Remark 8.2. Let h be a σ-stable function such that $|h| \leqq 0$. Suppose that there exists a sequence h_k $(k = 1, 2, 3, \cdots, n)$ of simple σ-stable functions such that $|h_k| \leqq 0$ and

$$(8.16) \qquad h = \prod_{k=1}^{n} h_k\,;$$

if $G \in \Lambda^\sigma$, it follows from Lemma 8.2 (with $H_n = h$) that $G \in [\sigma, h]$ and $h(\tau)G = G_n$, where G_n is the operator defined recursively by the equations

$$G_1 = h_1(\tau)G, \quad G_2 = h_2(\tau)G_1, \quad \cdots, \quad G_n = h_n(\tau)G_{n-1}.$$

Remark 8.3. It is not hard to prove that it is always possible to construct a sequence h_k $(k = 1, 2, 3, \cdots, n)$ of simple σ-stable functions h_k with $|h_k| \leqq 0$ such that (8.16) holds for any σ-stable function h with $|h| \leqq 0$; the proof is left to the reader.

Orientation. We shall now give a different procedure for calculating $h(\tau)G$ when $G \in \Lambda^\sigma$.

Notation. Let a be a complex number such that

$$(8.17) \qquad a \neq 2k\pi i/\sigma \quad \text{for all } k \in \mathbb{Z}.$$

Let n be an integer $\geqq 1$. If $G \in \Lambda^\sigma$, we shall denote by $G/(\tau - a)^n$ the operator G_n defined recursively by the equations

$$(8.18) \qquad G_1 = \frac{G}{\tau - a}, \quad G_2 = \frac{G_1}{\tau - a}, \quad \cdots, \quad G_n = \frac{G_{n-1}}{\tau - a}.$$

Let p be the polynomial defined by $p(s) = s - a$; we set

$$(8.19) \qquad h_a^n = \prod_{k=1}^{n} h_k, \quad \text{where } h_k = 1/p.$$

Consequently

$$(8.20) \qquad h_a^n(s) = \frac{1}{p(s)^n} = \frac{1}{(s - a)^n}.$$

Since (8.17) implies $p(2k\pi i/\sigma) = (2k\pi i/\sigma) - a \neq 0$, we see that h_k is a simple σ-stable function. Suppose that $G \in \Lambda^\sigma$. From (8.19) and Remark 8.2 (with $h = h_a^n$) it follows that

$$(8.21) \qquad G \in [\sigma, h_a^n] \quad \text{and} \quad h_a^n(\tau)G = G_n \in \Lambda^\sigma,$$

where G_n is given by the recursion formula

$$(8.22) \qquad G_1 = h_1(\tau)G, \quad G_2 = h_2(\tau)G_1, \quad \cdots, \quad G_n = h_n(\tau)G_{n-1}.$$

Since $h_k = 1/p$, we have $h_k(s) = 1/(s - a)$. From Theorem 7.1 it therefore follows that $h_k(\tau)E = E/(\tau - a)$; consequently (8.22) is equivalent with (8.18):

$$(8.23) \qquad \frac{G}{(\tau - a)^n} = G_n = h_a^n(\tau)G.$$

The last equation is from (8.21).

THEOREM 8.1. *Let h be a σ-stable function ψ/μ such that $|h| \leqq 0$; let α be the set of zeros of the polynomial μ. From the partial fraction expansion theorem it follows that there exist complex numbers c_a^n and integers $ma > 0$ such that*

$$h(s) = \sum_{a\in\alpha} \sum_{n=1}^{ma} c_a^n \frac{1}{(s-a)^n}. \tag{8.24}$$

If $G \in \Lambda^\sigma$, then $G \in [\sigma, h]$ and

$$h(\tau)G = \sum_{a\in\alpha} \sum_{n=1}^{ma} c_a^n \frac{G}{(\tau-a)^n}. \tag{8.25}$$

Proof. From (8.24) and (8.20) we have

$$h(D)G = \sum_{a\in\alpha} \sum_{n=1}^{ma} c_a^n h_a^n(D)G. \tag{8.26}$$

where

$$h_a^n = 1/\mu_a^n \quad \text{and} \quad \mu_a^n(s) = (s-a)^n. \tag{8.27}$$

Suppose that $G \in \Lambda^\sigma$. From (8.21) it follows that $G \in [\sigma, h_a^n]$; we may therefore apply Remark 5.1 to write

$$h_a^n(D)G = h_a^n(\tau)G + y_a^n, \tag{8.28}$$

where

$$h_a^n(\tau)G \in \Lambda^\sigma \quad \text{and} \quad y_a^n \in \square\mu_a^n. \tag{8.29}$$

Substituting (8.28) into (8.26), we obtain

$$h(D)G = y^\tau + y^\square, \tag{8.30}$$

where

$$y^\tau = \sum_{a\in\alpha} \sum_{n=1}^{ma} c_a^n h_a^n(\tau)G \tag{8.31}$$

and

$$y^\square = \sum_{a\in\alpha} \sum_{n=1}^{ma} c_a^n y_a^n. \tag{8.32}$$

Recall that $h = \psi/\mu$. From (8.24) and (8.27) we see that μ/μ_a^n is a polynomial; from Lemma 4.1 it therefore follows that

$$\square\mu_a^n \subset \square\mu \tag{8.33}$$

Since $y_a^n \in \square\mu_a^n$ (see (8.29)), it follows from (8.33) that $y_a^n \in \square\mu$. From (8.32) and Remark 4.1, we therefore have

$$y^\square \in \square\mu \quad \text{and} \quad y^\tau \in \Lambda^\sigma; \tag{8.34}$$

the property $y^\tau \in \Lambda^\sigma$ comes from (8.29), (8.31) and Remark 3.1. Since h is a σ-stable function ψ/μ such that $|h| \leqq 0$, we can use Remark 5.3 to infer from (8.30) and (8.34) the properties $G \in [\sigma, h]$ and $h(\tau)G = y^\tau$; conclusion (8.25) is now immediate from (8.31) and (8.23).

THEOREM 8.2. *Suppose that h_1 is a σ-stable function with polynomial part p; in consequence of Definitions* 8.1 *and* 5.1, *there exists a σ-stable function h such that $h_1 = p + h$ and $|h| < 0$. If $G \in \Lambda^\sigma$, then $G \in [\sigma, h_1]$ and $h_1(\tau)G = p(D)G + h(\tau)G$.*

Proof. Since $|h| \leqq 0$, Theorem 8.11 gives $G \in [\sigma, h]$. The conclusion now comes from Lemma 5.1.

Remark 8.4. Suppose that h_1 is a σ-stable function. From Theorem 8.2 it follows immediately that $\Lambda^\sigma \subset [\sigma, h_1]$.

THEOREM 8.3. *Suppose that h_0 is a σ-stable function; further, let p be a polynomial such that $|ph_0| \leqq 0$. If $E = p(D)f_1 + f_2$ with $f_k \in \Lambda^\sigma$ (for $k = 1, 2$), then $E \in [\sigma, h_0]$ and $h_0(\tau)E \in \Lambda^\sigma$. Moreover, $h_0(\tau)E = ph_0(\tau)f_1 + h_0(\tau)f_2$.*

Proof. From (8.35) we have $\Lambda^\sigma \subset [\sigma, h_0]$. Set $h_0 = \psi/\mu$; therefore, $\mu \in \nabla_\sigma$ and $ph_0 = p\psi/\mu$. Consequently, ph_0 is σ-stable and the inclusion $\Lambda^\sigma \subset [\sigma, ph_0]$ is now immediate from (8.35). The conclusion comes directly from Lemma 5.2.

Remark 8.5. Suppose that h_0 is a σ-stable function, and let p be a polynomial such that $|ph_0| \leqq 0$. If $E \in \mathscr{P}_\sigma(p)$, then the equation $E = p(D)f_1 + f_2$ holds for some functions f_k $(k = 1, 2)$ in Λ^σ (see Remark 3.2; from Theorem 8.3 it now follows that $E \in [\sigma, h_0]$. Consequently

$$\mathscr{P}_\sigma(p) \subset [\sigma, h_0] \quad \text{when } |ph_0| \leqq 0.$$

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VARIATIONAL METHODS AND QUADRATIC FUNCTIONAL INEQUALITIES*

Dedicated to M. R. Hestenes at an Honoring Symposium, May 25, 1973, on the Occasion of his Retirement from the UCLA Mathematics Department

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Abstract. In the context of a self-adjoint generalized differential system that is equivalent to a type of linear vector Riemann–Stieltjes integral equation, certain functional inequalities are presented generalizing, in particular, the well-known Liapunov inequality $\int_a^b q^+(t)\,dt > 4/(b-a)$, which is satisfied by $q^+(t) = \frac{1}{2}[q(t) + |q(t)|]$ whenever $q(t)$ is a real-valued Lebesgue integrable function on the compact real interval $[a, b]$ which is such that the differential equation $u''(t) + q(t)u(t) = 0$ is oscillatory on $[a, b]$. In particular, some decided extensions of the results of the author's recent paper [19] are given.

1. Introduction. In the Sturmian theory for a real linear homogeneous differential equation of the second order

$$[r(t)u'(t)]' - p(t)u(t) = 0, \qquad a \leqq t \leqq b, \tag{1.1}$$

the importance of an associated quadratic functional

$$p_a\eta^2(a) + p_b\eta^2(b) + \int_a^b \{r(t)[\eta'(t)]^2 + p(t)\eta^2(t)\}\,dt \tag{1.2}$$

is well known, the central feature being that (1.1) is the Euler equation for (1.2), while the involved boundary conditions consist of certain "basic restraints" on $\eta(a), \eta(b)$ and the associated transversality (natural boundary) conditions for (1.2) subject to the basic restraints (see, for example, Reid [10], [17, Chaps. V, VI, VII]). Also, for nonparametric variational problems in the plane with separated end conditions the second variation functional along an extremal arc is of the form (1.2), and (1.1) is the associated Jacobi, or accessory, differential equation.

The generalized differential system to be considered here is equivalent to a type of linear vector Riemann–Stieltjes integral equation, to which the author's attention was first directed by the following facts.

(i) If for a nonparametric variational problem in the plane $\mathscr{E}$ is an extremaloid (broken extremal arc) with a finite number of corners determined by values t_j, $j = 1, \cdots, m$, where $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, then the special second variation functional along $\mathscr{E}$ computed for an embedding family of admissible arcs with no corners except possibly at the same intermediate values t_j is of the form (see Reid [9], [16])

$$\sum_{\alpha=0}^{m+1} p_\alpha\eta^2(t_\alpha) + \int_a^b \{r(t)[\eta'(t)]^2 + p(t)\eta^2(t)\}\,dt. \tag{1.3}$$

In this case, for the functional (1.3) subject to the basic restraints at $t = a$ and $t = b$ the Jacobi condition yields equation (1.1) on each of the intermediate open intervals $(t_\alpha, t_{\alpha+1})$, $\alpha = 0, 1, \cdots, m$, the natural boundary conditions at $t = a$ and

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$t = b$, and also the transition equations

$$(1.4) \qquad r(t_j^+)u'(t_j^+) - r(t_j^-)u'(t_j^-) - p_j u(t_j) = 0, \qquad j = 1, \cdots, m,$$

at the intermediate values $t_1, \cdots, t_m$.

(ii) If $a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, then for a set of linear second order difference equations, or recursive relations, of the form

$$(1.5) \quad \rho_j \frac{u(t_{j+1}) - u(t_j)}{t_{j+1} - t_j} - \rho_{j-1} \frac{u(t_j) - u(t_{j-1})}{t_j - t_{j-1}} - \pi_j u(t_j) = 0, \qquad j = 1, \cdots, m,$$

with $\rho_\alpha > 0$, $\alpha = 0, 1, \cdots, m$, there exists a theory analogous to that of the Sturmian theory for equation (1.1) (see Bôcher [3, Chap. II], Whyburn [23], Fort [5, Chap. X], Atkinson [1, Chaps. 1–7], Harris [6]). Indeed, according to Sturm [21, p. 186], his initial consideration of comparison and oscillation theorems was in the context of difference relations.

Generalized linear differential systems of the sort considered here have been treated in the earlier papers [12], [13], [14] of the author. The consideration is variational in nature, with central feature the extremizing character of Hermitian functionals as exemplified in the accessory boundary problems for variational problems (see, for example, Birkhoff and Hestenes [2], Hestenes [7], Reid [10], [17, Chap. VII] and [18]). Also, the real scalar generalized second order differential equations occurring in the works of Sz.-Nagy [22], [20, pp. 247–254] and Feller [4] are particular instances of the general system considered here. In particular, the theorems of § 3 provide decided extensions of the results of the author's papers [15] and [19].

Matrix notation is used throughout; in particular, matrices of one column are called vectors, and for a vector (w_α), $\alpha = 1, \cdots, n$, the norm $|w|$ is given by $(|w_1|^2 + \cdots + |w_n|^2)^{1/2}$. The vector space of ordered n-tuples of complex numbers, with complex scalars, is denoted by $\mathbf{C}_n$. If W is a linear subspace of $\mathbf{C}_n$, the orthogonal complement $\{z | z^*y = 0 \text{ for } y \in W\}$ of W in $\mathbf{C}_n$ is denoted by $W^\perp$. The $n \times n$ identity matrix is denoted by E_n, or merely by E if there is no ambiguity, and 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix is designated by M^*. If $M = [M_{\alpha\beta}]$ and $N = [N_{\alpha\beta}]$ are $n \times r$ matrices, then the $2n \times r$ matrix $H = [H_{\sigma\beta}]$, $\sigma = 1, \cdots, 2n$; $\beta = 1, \cdots, r$, with $H_{\alpha\beta} = M_{\alpha\beta}$ and $H_{n+\alpha,\beta} = N_{\alpha\beta}$, is denoted by $[M; N]$. If M is an $n \times n$ matrix, the symbol $|M|$ is used for the supremum of $|Mw|$ on the unit ball $\{w | |w| \leqq 1\}$ of $\mathbf{C}_n$. The notation $M \geqq N$ $\{M > N\}$ is used to signify that M and N are Hermitian matrices of the same dimensions, and $M - N$ is a nonnegative $\{$positive$\}$ Hermitian matrix. If an Hermitian matrix function $M(t)$, $t \in [a, b]$, is such that $M(s) - M(t) \geqq 0$ $\{\leqq 0\}$ for $a \leqq s < t \leqq b$, then $M(t)$ is called nonincreasing $\{$nondecreasing$\}$ Hermitian on $[a, b]$. A matrix function is called continuous, integrable, etc., when each element of the matrix possesses the specified property.

If a matrix function $M(t)$ is a.c. $\{$absolutely continuous$\}$ on $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values where these derivatives exist, and zero elsewhere. Similarly, if $M(t)$ is $\{$Lebesgue$\}$ integrable on $[a, b]$, then $\int_a^b M(t)\, dt$ denotes the matrix of integrals of respective elements of $M(t)$. For a given interval $[a, b]$ the symbols $\mathfrak{C}$, $\mathfrak{L}$, $\mathfrak{L}^2$, $\mathfrak{L}^\infty$, $\mathfrak{BV}$ and $\mathfrak{A}$ are used to denote the class of finite-dimensional matrix functions $M(t) = [M_{\alpha\beta}(t)]$ on $[a, b]$ which are respectively continuous, {Lebesgue} integrable, {Lebesgue} measurable and $|M_{\alpha\beta}(t)|^2 \in \mathfrak{L}$, measurable and essentially bounded, of bounded variation, and absolutely continuous on $[a, b]$. If M, S and T are matrix functions of respective dimensions $m \times n$, $r \times m$ and $n \times s$ on $[a, b]$, and $S \in \mathfrak{C}$, $T \in \mathfrak{C}$, $M \in \mathfrak{BV}$, then $\int_a^b S[dM]T$ denotes the $r \times s$ matrix with elements given by the Riemann–Stieltjes integrals

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \int_a^b S_{i\alpha}(t) T_{\beta j}(t)\, dM_{\alpha\beta}(t);$$

also, $\int_a^b [dM]T$ and $\int_a^b S[dM]$ designate $\int_a^b E_m[dM]T$ and $\int_a^b S[dM]E_n$, respectively. If $M(t)$ and $N(t)$ are matrix functions such that $M(t) = N(t)$ a.e. {almost everywhere} on $[a, b]$, we write simply $M(t) = N(t)$. Moreover, for $M(t)$ an $n \times r$ matrix function on $[a, b]$ the $2n \times r$ matrix $[M(a); M(b)]$ is denoted by $\hat{M}$.

2. Some basic results for generalized differential systems. In the following the $n \times n$ matrix functions A_0, A_1, B, C, M are defined on a given compact interval $[a, b]$ on the real line, and satisfy the following condition.

($\mathfrak{H}$) *$B(t)$, $C(t)$, $M(t)$ are Hermitian, and $A_1(t)$ is nonsingular for $t \in [a, b]$; the matrix functions A_0, A_1, A_1^{-1}, B, C belong to $\mathfrak{L}^\infty$, and $M \in \mathfrak{BV}$.*

The symbol $\mathfrak{D} = \mathfrak{D}[a, b]$ will denote the class of n-dimensional vector functions y which are a.c. on $[a, b]$, and for which there exists a $z \in \mathfrak{L}^2$ such that $L[y, z] \equiv L[y] - Bz = 0$ on $[a, b]$, where $L[y] = A_1 y' + A_0 y$; the fact that z is thus associated with y is denoted by $y \in \mathfrak{D}:z$. The subclass of $\mathfrak{D}$ on which $y(b) = 0$ is denoted by $\mathfrak{D}_{*0}$, the subclass of $\mathfrak{D}$ on which $y(a) = 0$ is denoted by $\mathfrak{D}_{0*}$, and $\mathfrak{D}_0 = \mathfrak{D}_{0*} \cap \mathfrak{D}_{*0}$, with corresponding meanings of the symbols $y \in \mathfrak{D}_{*0}:z$, $y \in \mathfrak{D}_{0*}:z$ and $y \in \mathfrak{D}_0:z$. Attention will be restricted to operators L with domain D a linear manifold satisfying $\mathfrak{D}_0 \subset D \subset \mathfrak{D}$. In particular, $\mathfrak{D}_0 = \{y | y \in \mathfrak{D}, \hat{y} = 0\}$, and if S denotes the set of $2n$-dimensional vectors ξ for which there exists a $y \in D$ with $\hat{y} = \xi$, then S is a linear manifold in $\mathbf{C}_{2n}$ and $D = \{y | y \in \mathfrak{D}, \hat{y} \in S\}$. Finally, we shall denote by $\mathfrak{D}^*$ the class of n-dimensional vector functions $z \in \mathfrak{L}^2$ such that $z = (A_1^*)^{-1} v_z$ with $v_z \in \mathfrak{BV}$.

As in Reid [14], we shall consider certain self-adjoint boundary problems involving the vector generalized differential system

$$\begin{aligned} \Delta[y, z] &\equiv -dv_z + (Cy + A_0^* z)\, dt + [dM]y = 0, \\ L[y, z] &\equiv L[y] - Bz = 0. \end{aligned} \tag{2.1}$$

By a solution $(y; z)$ of (2.1) is meant a pair of n-dimensional vector functions $(y, z) \in \mathfrak{A} \times \mathfrak{D}^*$ satisfying $L[y, z] = 0$ on $[a, b]$ and the Riemann–Stieltjes integral equation

$$v_z(t) = v_z(\tau) + \int_\tau^t \{C(s)y(s) + A_0^*(s)z(s)\}\, ds + \int_\tau^t [dM(s)]y(s) \tag{2.2}$$

for $(\tau, t) \in [a, b] \times [a, b]$. In particular, for a solution $(y; z)$ the vector function Ω

of $\mathfrak{BV}$ defined as

$$(2.3)\qquad \Omega(t; y, z) = -v_z(t) + \int_\tau^t \{C(s)y(s) + A_0^*(s)z(s)\}\, ds + \int_\tau^t [dM(s)]y(s)$$

is equal to the constant vector function $-v_z(\tau)$. In general, a pair of $n \times r$ matrix functions Y, Z is a solution of the corresponding matrix generalized differential system

$$(2.1_M)\qquad \Delta[Y, Z] = 0, \qquad L[Y, Z] = 0$$

if each column vector of $(Y; Z)$ is a solution of (2.1).

For $(y_\alpha, z_\alpha) \in \mathfrak{C} \times \mathfrak{L}^2$, $\alpha = 1, 2$, we introduce the notation

$$(2.4)\qquad \begin{aligned} J_0[y_1 : z_1, y_2 : z_2] &= \int_a^b \{z_2^* B z_1 + y_2^* C y_1\}\, dt + \int_a^b y_2^*[dM]y_1, \\ J[y_1 : z_1, y_2 : z_2] &= \hat{y}_2^* Q \hat{y}_1 + J_0[y_1 : z_1, y_2 : z_2], \end{aligned}$$

where Q is a $2n \times 2n$ Hermitian matrix. Also, for brevity we write $J_0[y : z]$ and $J[y : z]$ for the respective functionals $J_0[y : z, y : z]$ and $J[y : z, y : z]$. In particular, if $y_\alpha \in \mathfrak{D} : z_\alpha$, $\alpha = 1, 2$, then the values of the functionals (2.4) are independent of the choice of the z_α satisfying with y_α the definitive relation $L[y_\alpha, z_\alpha] = 0$, and we write $J_0[y_1, y_2]$ for the more complicated notation $J_0[y_1 : z_1, y_2 : z_2]$, with similar meanings for $J[y_1, y_2]$, $J_0[y]$ and $J[y]$.

The following results may be established by the methods of § 2 of Reid [13]; they also appear as Lemmas 2.1, 2.2 and 2.3 of [14].

LEMMA A. *If* $(y, z) \in \mathfrak{C} \times \mathfrak{D}^*$ *and* $\eta \in \mathfrak{D} : \zeta$, *then*

$$(2.5)\qquad J[y; z, \eta : \zeta] = \hat{\eta}^* T[y, z] + \int_a^b \eta^*(s)[d\Omega(s; y, z)],$$

where

$$(2.6)\qquad T[y, z] = Q\hat{y} + [\operatorname{diag}\{-E_n, E_n\}]\hat{v}_z.$$

LEMMA B. *If* $(y, z) \in \mathfrak{C} \times \mathfrak{L}^2$, *the following conditions are equivalent*:

(a) $J[y : z, \eta : \zeta] = 0$ *for* $\eta \in D : \zeta$;

(b) *there exists a* $z_0 \in \mathfrak{D}^*$ *such that* $B[z - z_0] = 0$, $T[y, z_0] \in S^\perp$ *and* $\Delta[y, z_0] = 0$.

LEMMA C. *If* $y \in \mathfrak{A}$, *then there exists a* z *such that* (y, z) *is a solution of*

$$(2.7a)\qquad \Delta[y, z] = 0, \qquad L[y, z] = 0,$$

$$(2.7b)\qquad \hat{y} \in S, \qquad T[y, z] \in S^\perp,$$

if and only if there exists a z_0 *such that* $y \in D : z_0$ *and* $J[y : z_0, \eta : \zeta] = 0$ *for* $\eta \in D : \zeta$.

The results of the above Lemmas B and C may be described as the condition that (2.7) is the "Euler–Lagrange system" for the Hermitian variational integral

$$(2.8)\qquad J[y] = \hat{y}^* Q \hat{y} + \int_a^b \{z^* B z + y^* C y\}\, dt + \int_a^b y^*[dM]y$$

subject to the restraint $y \in D : z$.

For a nondegenerate interval $I \subset [a, b]$ the symbol $\Lambda(I)$ will denote the linear space of n-dimensional vector functions v which are solutions of

$$-v' + A_0^* A_1^{*-1} v = 0, \qquad BA_1^{*-1} v = 0 \quad \text{on } I;$$

that is, $v \in \Lambda(I)$ if and only if (y, z) with $y \equiv 0, v_z = v$ is a solution of (2.7a) on I. If $\Lambda(I)$ is zero-dimensional, the system (2.7a) is said to be *normal* on I, or to have *order of normality zero* on I, whereas if $\Lambda(I)$ has dimension $d = d(I) > 0$, then (2.7a) is called *abnormal*, with *order of abnormality* d *on* I. If $I = [r, s]$, for brevity $\Lambda[r, s]$ and $d[r, s]$ are written instead of the more precise $\Lambda([r, s])$ and $d([r, s])$, with similar contractions in case I is of the form $[r, s)$, $(r, s]$ or (r, s). The system (2.7a) is said to be *identically normal* on I if it is normal on arbitrary nondegenerate subintervals of I. If $(\eta, \zeta) \in \mathfrak{A} \times \mathfrak{L}^2$ and $L[\eta, \zeta] = 0$ on a subinterval I of $[a, b]$, then for $v \in \Lambda(I)$ it follows readily that $v^*\eta$ is constant on I.

The subspace of $\Lambda[a, b]$ on which the $2n$-dimensional vector (w_σ) with $(w_i) = -v(a)$, $(w_{n+i}) = v(b)$ belongs to $S^\perp$ will be denoted by $\Lambda\{S\}$. Obviously $v \in \Lambda\{S\}$ if and only if $(y, z) = (0, A_1^{*-1} v)$ is a solution of the system (2.7). If $\dim \Lambda\{S\} = 0$, then the boundary problem (2.7) is said to be *normal*, or to have *order of abnormality zero*, whereas if $\dim \Lambda\{S\} = d > 0$, then (2.7) is called *abnormal*, with *order of abnormality* d. In the latter case, by a method used originally for accessory boundary problems for variational problems of Bolza type (see [14, § 2]), one may determine a subspace S_1 of $\mathbf{C}_{2n}$ such that $S_1 \supset S$, $\dim S_1 = d + \dim S$, and the boundary problem (2.7$_1$) consisting of (2.7a) and the boundary conditions

$$(2.7_1\text{b}) \qquad \hat{y} \in S_1, \qquad T[y, z] \in S_1^\perp$$

is normal, and if $y \in \mathfrak{D}$, then $y \in D_1 = \{y | y \in \mathfrak{D}, \hat{y} \in S_1\}$ if and only if $y \in D$. Moreover, (2.7$_1$) is equivalent to (2.7) in the following sense: if $(y; z)$ is a nontrivial solution of (2.7$_1$), then $y(t) \not\equiv 0$ on $[a, b]$ and $(y; z)$ is a solution of (2.7), whereas if $(y; z)$ is a solution of (2.7) and v_β, $\beta = 1, \cdots, d_a$, is a basis for $\Lambda[a, b]$, then there exist unique constants c_β such that $(y, z + \sum_\beta c_\beta A_1^{*-1} v_\beta)$ is a solution of (2.7$_1$). In particular, if $\eta \in D : \zeta$ and $\eta \in D_1 : \zeta_1$, then $\zeta_1 - \zeta \in \Lambda[a, b]$, and hence $B[\zeta_1 - \zeta] = 0$ and $J_0[\eta : \zeta] = J_0[\eta : \zeta_1]$. The boundary problem (2.7$_1$) is called the *normal boundary problem equivalent to* (2.7).

Distinct values t_1 and t_2 on $[a, b]$ are said to be (*mutually*) *conjugate* with respect to the generalized differential system (2.7a) if there exists a solution $(y; z)$ of this system such that $y(t_1) = 0 = y(t_2)$ and $y(t) \not\equiv 0$ on the subinterval with endpoints t_1, t_2. The system (2.7a) is called *disconjugate* on a subinterval $I \subset [a, b]$ if there exists no pair of distinct values on this subinterval which are conjugate. A basic result on disconjugacy is presented in the following theorem, for the proof of which reference is made to [13, § 5] and [14, Thms. 3.1, 3.2].

THEOREM 2.1. *If $J_0[\eta]$ is nonnegative on $\mathfrak{D}_0$, then $B(t) \geqq 0$ a.e. on $[a, b]$. Moreover, if $B(t) \geqq 0$ a.e. on $[a, b]$, then the following conditions are equivalent:*

(i) *$J_0[\eta]$ is positive definite on $\mathfrak{D}_0$;*

(ii) *the system (2.7a) is disconjugate on $[a, b]$;*

(iii) *there exists a solution $(Y; Z)$ of (2.1$_M$) with $Y(t)$ nonsingular on $[a, b]$, and the constant matrix $V_Z^* Y - Y^* V_Z = Z^* A_1 Y - Y^* A_1^* Z$ the zero matrix.*

For the boundary problem (2.7) we have the following result.

THEOREM 2.2. *If $J[\eta]$ is nonnegative on D, then $B(t) \geqq 0$ a.e. on $[a, b]$, and exactly one of the following conditions holds:*

(a) *there exists a solution $(y; z)$ of (2.7) with $y \not\equiv 0$ on $[a, b]$;*

(b) *there exists a $\kappa > 0$ such that if $\eta \in D : \zeta$ and $t \in [a, b]$, then*

$$J[\eta] \geqq \kappa[|\eta(a)|^2 + |\eta(b)|^2 + |\eta(t)|^2 + \int_a^b \{|\eta'|^2 + |\eta|^2\} \, dt]; \tag{2.9}$$

moreover, if $\Pi(t)$ is an $n \times n$ nondecreasing Hermitian matrix function which is nonconstant on $[a, b]$, then

$$J[\eta] \geqq \frac{\kappa}{V[a, b|\Pi]} \int_a^b \eta^*[d\Pi]\eta \quad \textit{for } \eta \in D, \tag{2.10}$$

where $V[a, b|\Pi]$ is the supremum of $\sum_{\alpha=1}^m |\Pi(t_\alpha) - \Pi(t_{\alpha-1})|$ for all partitions $a = t_0 < t_1 < \cdots < t_m = b$ of $[a, b]$.

As $\mathfrak{D}_0 \subset D$, if $J[\eta]$ is nonnegative on D, then this functional is nonnegative on $\mathfrak{D}_0$, and by a result of the above theorem we have $B(t) \geqq 0$ a.e. on $[a, b]$. If $J[\eta]$ is nonnegative on D, but not positive definite on D, then there exists a $y \in D$ such that $y \not\equiv 0$ on $[a, b]$ and $J[y] = 0$. If $y \in D : z_0$ and $\eta \in D : \zeta$, then for arbitrary real c we have that $y_c = y + c\eta$, $z_c = z_0 + c\zeta$ satisfies $y_c \in D$: z_c and $0 \leqq J[y_c]$, from which it follows that $J[y : z_0, \eta : \zeta] = 0$, and hence by Lemma C there exists a z such that $(y; z)$ is a solution of (2.7).

In the alternate case we have that $J[\eta]$ is positive definite on D, and from the results of Lemma 5.1 and Corollary of [14] for the associated normal boundary problem (2.7_1) it follows that there exists a positive constant κ such that (2.9) holds for all $\eta \in D_1$ and $t \in [a, b]$, and consequently from the above remarks on the normal boundary problem equivalent to (2.7) we have that (2.9) is valid for all $\eta \in D$ and $t \in [a, b]$.

Since (2.9) implies that $J[\eta] \geqq \kappa|\eta(t)|^2$ for arbitrary $\eta \in D$ and $t \in [a, b]$, if Π is an $n \times n$ nondecreasing Hermitian matrix function on $[a, b]$, then

$$\int_a^b \eta^*[d\Pi]\eta \leqq \frac{1}{\kappa} V[a, b|\Pi] J[\eta],$$

and consequently (2.10) holds since $V[a, b|\Pi] > 0$ for a nondecreasing Hermitian matrix function Π which is nonconstant on $[a, b]$.

Corresponding to (2.12) of [14], let

$$\begin{aligned} A^0 &= A_1^{-1} B A_1^{*-1} - A_1^{-1} A_0, \qquad B^0 = A_1^{-1} B A_1^{*-1}, \\ C^0 &= C + A_0^* A_1^{*-1} M + M A_1^{-1} A_0 - M A_1^{-1} B A_1^{*-1} M. \end{aligned} \tag{2.11}$$

Similar to results of [13, § 2] and [14, § 2], it may be established that if $f \in \mathfrak{L}$ on $[a, b]$ then $(y; z)$ is a solution of the system

$$\Delta[y, z] = f\, dt, \qquad L[y, z] = 0, \tag{2.12a}$$

$$\hat{y} \in S_1, \qquad T[y, z] \in S_1^{\perp}, \tag{2.12b}$$

if and only if $(y^0; v^0) = (y, v_z - My)$ is a solution of the ordinary differential equation system

$$(2.12^0)\qquad \begin{aligned} L_1^0[y^0, v^0] &\equiv -v^{0\prime} + C^0 y^0 - A^{0*} v^0 = f, \\ L_2^0[y^0, v^0] &\equiv y^{0\prime} - A^0 y^0 - B^0 v^0 = 0, \\ \hat{y} \in S_1, \qquad T^0[y^0, v^0] &\equiv Q^0 \hat{y}^0 + [\operatorname{diag}\{-E_n, E_n\}]\hat{v}^0 \in S_1^\perp, \end{aligned}$$

where $Q^0 = Q + \operatorname{diag}\{-M(a), M(b)\}$.

Now if (2.7_1) has only the identically vanishing solution, then the system

$$(2.13a)\qquad L_1^0[y^0, v^0] = 0, \qquad L_2^0[y^0, v^0] = 0,$$

$$(2.13b)\qquad \hat{y}^0 \in S_1, \qquad T^0[y^0, v^0] \in S_1^\perp$$

has only the identically zero solution. Also for arbitrary $f \in \mathfrak{L}$ the solution of (2.12^0) is given by

$$(2.14)\qquad y^0(t) = \int_a^b G(t,s) f(s)\, ds, \qquad v^0(t) = \int_a^b G_0(t,s) f(s)\, ds,$$

where the $n \times n$ matrix functions G, G_0 belong to a classical Green's matrix for the incompatible system (2.13), and which on $\square = [a,b] \times [a,b]$ satisfy the following conditions:

(i) *G is continuous on $\square$, is a.c. in each argument on $[a,b]$ for fixed values of the other argument, and $G(t,s) \equiv [G(s,t)]^*$ on $\square$;*

(ii) *G_0 is continuous on each of the sets $\square_1 = \{(t,s) \in \square | s < t\}$ and $\square_2 = \{(t,s) \in \square | t < s\}$, is bounded on $\square$, and for $\alpha = 1, 2$ the restriction of G_0 to $\square_\alpha$ has a finite limit at each $(t,t) \in \square$* ($G_{0,\alpha}$ will denote the uniquely determined continuous matrix function on the closure of $\square_\alpha$ which is equal to G_0 on $\square_\alpha$);

(iii) *if $s \in [a,b]$ and ξ is an arbitrary vector in $\mathbf{C}_n$, then $y^0(t) = G(t,s)\xi$, $v^0(t) = G_0(t,s)\xi$ satisfy the differential equations* (2.13a) *on each nondegenerate subinterval $[a,s)$ and $(s,b]$, and $\hat{y}^0 \in S_1$ so that $G(\cdot,s)\xi \in D : G_0(\cdot,s)\xi$; moreover, if $s \in (a,b)$, then also $T^0[y^0, v^0] \in S_1^\perp$.*

As an extension of the result of Theorem 2.2 we have the following theorem, corresponding to Theorem 2.1 of [14].

THEOREM 2.3. *If (2.7_1) has only the identically vanishing solution, then for arbitrary $\psi \in \mathfrak{BV}$ the system*

$$(2.15a)\qquad \Delta[y,z] = d\psi, \qquad L[y,z] = 0,$$

$$(2.15b)\qquad \hat{y} \in S_1, \qquad T[y,z] \in S_1^\perp$$

has a unique solution, which is given by

$$(2.16)\qquad \begin{aligned} y(t) &= \int_a^b G(t,s)[d\psi(s)], \\ v_z(t) &= \int_a^t G_{0,1}(t,s)[d\psi(s)] + \int_t^b G_{0,2}(t,s)[d\psi(s)]. \end{aligned}$$

As a ready consequence of Lemma A, we have the following result.

COROLLARY. *If the hypotheses of Theorem* 2.3 *are satisfied, and* $(y; z)$ *is the solution of* (2.15) *for a* $\psi \in \mathfrak{BB}$, *then*

$$J[y] = \int_a^b \int_a^b [d\psi^*(t)]G(t,s)[d\psi(s)]. \tag{2.17}$$

3. Generalized Liapunov inequalities. As a ready consequence of Theorem 2.3 and its corollary we have the following result.

THEOREM 3.1. *Suppose that* (2.7_1) *has only the identically vanishing solution, and* $\Pi(t)$ *is an* $n \times n$ *nondecreasing Hermitian matrix function on* $[a, b]$. *Then a vector function* $y(t)$ *belongs to a solution* $(y; z)$ *of the boundary problem*

$$\begin{gathered} \Delta[y,z] = \lambda[d\Pi]y, \qquad L[y,z] = 0, \\ \hat{y} \in S_1, \qquad T[y,z] \in S_1^{\perp}, \end{gathered} \tag{3.1}$$

for a value λ *if and only if* $y(t)$ *is continuous on* $[a, b]$ *and*

$$y(t) = \lambda \int_a^b G(t,s)[d\Pi(s)]y(s), \qquad t \in [a,b]. \tag{3.2}$$

Moreover, for such a $y(t)$ *we have*

$$\begin{aligned} J[y] &= \lambda^2 \int_a^b \int_a^b y^*(t)[d\Pi(t)]G(t,s)[d\Pi(s)]y(s) \\ &= \lambda \int_a^b y^*(t)[d\Pi(t)]y(t). \end{aligned} \tag{3.3}$$

If $\Pi(t)$ is a nondecreasing Hermitian matrix function on $[a, b]$, its domain of definition will be understood to be extended to $(-\infty, \infty)$ by defining $\Pi(t) = \Pi(a)$ on $(-\infty, a)$ and $\Pi(t) = \Pi(b)$ on (b, ∞). Moreover, we shall denote by $\sigma(\Pi)$ the points of increase of Π; that is, the set of points s such that for each $h > 0$ the nonnegative Hermitian matrix $\Pi(s + h) - \Pi(s - h)$ is nonzero. The set $\sigma(\Pi)$ is a closed set which is nonempty if Π is nonconstant on $[a, b]$, and for $\tilde{y}(t)$ a vector function continuous on $[a, b]$ we have that

$$\int_a^b G(t,s)[d\Pi(s)]\tilde{y}(s) = \int_{\sigma(\Pi)} G(t,s)[d\Pi(s)]\tilde{y}(s). \tag{3.4}$$

Indeed, if $\tilde{y}$ is any bounded function on $\sigma(\Pi)$ such that the integral of the right-hand member of (3.4) exists as a Lebesgue–Stieltjes integral for $t \in \sigma(\Pi)$, then in view of the uniform continuity of G on $[a, b] \times [a, b]$ it follows that

$$w(t) = \int_{\sigma(\Pi)} G(t,s)[d\Pi(s)]\tilde{y}(s), \qquad t \in \sigma(\Pi),$$

defines a continuous vector function w on $\sigma(\Pi)$. In particular, if $\tilde{y}$ is a bounded function on $\sigma(\Pi)$ such that

$$\tilde{y}(t) = \lambda \int_{\sigma(\Pi)} G(t,s)[d\Pi(s)]\tilde{y}(s), \qquad t \in \sigma(\Pi), \tag{3.5}$$

then $\tilde{y}$ is continuous on $\sigma(\Pi)$. Consequently, such a $\tilde{y}$ admits continuous extensions on $[a, b]$, and a particular continuous extension of $\tilde{y}$ on $[a, b]$ is given by

$$y(t) = \lambda \int_{\sigma(\Pi)} G(t, s)[d\Pi(s)]\tilde{y}(s), \qquad t \in [a, b]. \tag{3.6}$$

Moreover, for z the vector function such that the associated v_z is given by

$$v_z(t) = \lambda \int_a^t G_{0,1}(t, s)[d\Pi(s)]y(s) + \lambda \int_t^b G_{0,2}(t, s)[d\Pi(s)]y(s),$$

we have that $(y; z)$ is a solution of (3.1) and

$$\begin{aligned} J[y] &= \lambda^2 \int_{\sigma(\Pi)} \int_{\sigma(\Pi)} y^*(t)[d\Pi(t)]G(t, s)[d\Pi(s)]y(s) \\ &= \lambda \int_{\sigma(\Pi)} y^*(t)[d\Pi(t)]y(t). \end{aligned} \tag{3.7}$$

If $J[\eta]$ is positive definite on D, and $\Pi(t)$ is an $n \times n$ nondecreasing Hermitian matrix function on $[a, b]$ which is nonconstant, and the class

$$D[\Pi] = \left\{ \eta | \eta \in D, \int_a^b \eta^*[d\Pi]\eta \neq 0 \right\} \tag{3.8}$$

is nonempty, then from Theorem 5.1 of [14] it follows that the boundary problem (3.1) does possess proper values; that is, values λ for which there exist nonidentically vanishing solutions $(y; z)$ of this system. Moreover, in view of (3.7) and the extremizing properties of proper values as established in Theorem 5.1 of [14], all proper values are positive, and the largest constant κ such that

$$J[\eta] \geqq \kappa \int_a^b \eta^*(t)[d\Pi(t)]\eta(t) \quad \text{for } \eta \in D \tag{3.9}$$

is given by $\kappa = \lambda_1$, the smallest proper value of (3.1). Moreover, if $\eta \in D$ and equality holds in (3.9), then there exists a solution $(y; z)$ of (3.1) for $\lambda = \lambda_1$ and a constant c such that $\eta(t) \equiv cy(t)$. In view of the above established relations between solutions of (3.5) and solutions of (3.1), we have the following result.

THEOREM 3.2. *Suppose that $J[\eta]$ is positive definite on D, and $\Pi(t)$ is an $n \times n$ nondecreasing Hermitian matrix function on $[a, b]$ such that the class $D(\Pi)$ of* (3.8) *is nonempty. Then there exist $\mu > 0$ such that the integral equation*

$$\mu\tilde{y}(t) = \int_{\sigma(\Pi)} G(t, s)[d\Pi(s)]\tilde{y}(s), \qquad t \in \sigma(\Pi), \tag{3.10}$$

possesses nonidentically vanishing bounded solutions $\tilde{y}$, and the largest value κ such that the inequality (3.9) *holds is $\kappa = 1/\mu_M$, where $\mu = \mu_M$ is the largest value such that* (3.10) *has a nonidentically vanishing bounded solution. Moreover, for $\eta \in D$ equality in* (3.9) *holds only if $\eta(t) \equiv cy(t)$, where $y(t)$ is for $\lambda = 1/\mu_M$ the extension* (3.6) *of a solution $\tilde{y}$ of* (3.10) *for $\mu = \mu_M$.*

If $J[\eta]$ is positive definite on D, then $J[\eta]$ is positive definite on $\mathfrak{D}_0 \subset D$, and there is no pair of points on $[a, b]$ which are conjugate; also, if $a \leqq s_1 < s_2 \leqq b$

while (2.7a) is normal on the subinterval $[s_1, s_2]$ and ξ_1, ξ_2 are arbitrary vectors of $\mathbf{C}_n$, then there is a unique solution (y^0, z^0) of this system such that $y^0(s_\alpha) = \xi_\alpha$, $\alpha = 1, 2$. In particular, $\eta = y^0$ is an element of $\mathfrak{D}$ joining the points (s_α, ξ_α), $\alpha = 1, 2$. If K is a nonzero $n \times n$ matrix that is nonnegative Hermitian, for $s \in [a, b]$ let Π_s denote the nondecreasing Hermitian matrix function on $[a, b]$ defined as: if $s \in (a, b]$, then $\Pi_s(t) = 0$ for $t \in [a, s)$, $\Pi_s(t) = K$ on $[s, b]$, while for $s = a$ we set $\Pi_a(a) = 0$, $\Pi_a(t) = K$ for $t \in (a, b]$. For such a matrix function Π_s we have $D[\Pi_s] = \{\eta | \eta \in D, \eta^*(s)K\eta(s) \neq 0\}$. Moreover, if $G(t, s)$ is the matrix function of Theorem 2.3, then since $K \geqq 0$ and $KG(s, s)K$ is Hermitian, the proper values of $G(s, s)K$ are all real and its largest proper value $\mu_{\max}(s)$ is equal to the maximum of $\xi^*KG(s, s)K\xi$ on $\{\xi | \xi \in \mathbf{C}_n, \xi^*K\xi = 1\}$.

These properties are consequences of well-known results for symmetrizable completely continuous linear transformations in Hilbert space, and, in particular, are direct consequences of results in Reid [11, § 7]. An equivalent characterization of $\mu = \mu_{\max}(s)$ is that it is the largest proper value of the Hermitian matrix $K^{1/2}G(s, s)K^{1/2}$, where $K^{1/2}$ denotes the unique nonnegative Hermitian square root of K. If (2.7a) is normal on all subintervals $[a, s]$ and $[s, b]$ for $s \in (a, b)$, then for $s \in (a, b)$ the class $D[\Pi_s]$ is nonempty and $\mu_{\max}(s)$ is positive; however, for $s = a$ or $s = b$ the class may be empty, in which case $G(s, s)K = 0$ and $\mu_{\max}(s)$ is zero.

In view of Theorem 3.2 and the above remarks we have the following generalization of results of Theorems 2.2 and 2.3 of [19], where it is understood that if $s = a$ or $s = b$ and $D[\Pi_s]$ is empty, then $\kappa(s) = +\infty$ and the right-hand member of (3.11) is set equal to zero.

THEOREM 3.3. *Suppose that $J[\eta]$ is positive definite on D, and the system* (2.7a) *is normal on all subintervals $[a, s]$ and $[s, b]$ with $s \in (a, b)$. If K is a nonzero $n \times n$ matrix which is nonnegative Hermitian, then for $s \in [a, b]$ the largest constant $\kappa(s)$ such that*

$$J[\eta] \geqq \kappa(s)\eta^*(s)K\eta(s) \quad \textit{for } \eta \in D[a, b] \tag{3.11}$$

is equal to $1/\mu_{\max}(s)$, where $\mu = \mu_{\max}(s)$ is the largest proper value of the symmetrizable $n \times n$ matrix $G(s, s)K$ and $G(t, s)$ is the matrix function of Theorem 2.3; *moreover, if $D[\Pi_s]$ is nonempty, then equality in* (3.11) *holds if and only if there exist a constant c and a $\xi_0 \in \mathbf{C}_n$ such that $\eta(t) \equiv cG(t, s)K\xi_0$. Correspondingly, the largest constant κ such that*

$$J[\eta] \geqq \kappa\eta^*(s)K\eta(s) \quad \textit{for } \eta \in D[a, b] \textit{ and } s \in (a, b) \tag{3.12}$$

is $\kappa = 1/\max\{\mu_{\max}(s) | s \in [a, b]\}$.

A corresponding application of Theorem 3.2 to the case of a nondecreasing Hermitian matrix function on $[a, b]$ with only a finite number of points of increase yields the following result.

THEOREM 3.4. *Suppose that $J[\eta]$ is positive definite on D, and the system* (2.7a) *is normal on arbitrary nondegenerate subintervals of $[a, b]$. If $\mathfrak{s}$ denotes a k-tuple of values s_β, $\beta = 1, \cdots, k$, satisfying $a \leqq s_1 < s_2 < \cdots < s_k \leqq b$, and K_β, $\beta = 1, \cdots, k$, are $n \times n$ nonzero matrices which are nonnegative Hermitian, then the* *largest constant* $\kappa(\mathfrak{s})$ *such that*

$$J[\eta] \geqq \kappa(\mathfrak{s}) \sum_{\beta=1}^{k} \eta^*(s_\beta) K_\beta \eta(s_\beta) \quad \textit{for } \eta \in D[a, b]$$

is equal to $1/\mu_{\max}(\mathfrak{s})$, *where* $\mu = \mu_{\max}(\mathfrak{s})$ *is the largest proper value of the symmetrizable* $kn \times kn$ *matrix* $[G(s_\alpha, s_\beta)K_\beta]$, α, $\beta = 1, \cdots, n$, *whose element in the* $[n(\alpha - 1) + i]$*-th row and* $[n(\beta - 1) + j]$*-th column is equal to the element in the* i*-th row and* j*-th column of* $G(s_\alpha, s_\beta)K_\beta$.

4. Special inequalities. With the aid of the results of § 3 one may readily obtain various results for special types of differential systems (2.7), and particular results of the sort presented by A. Ju. Levin [8]; in this latter connection, see also [19, § 4]. For brevity, attention will be limited to one such result.

THEOREM 4.1. *Suppose that* $J[\eta]$ *is positive definite on* D, *the system* (2.7a) *is normal on all subintervals* $[a, s]$ *and* $[s, b]$ *with* $s \in (a, b)$, *while* $C_0(t)$ *is an* $n \times n$ *Hermitian matrix function of class* $\mathfrak{L}^\infty$ *on* $[a, b]$, *and such that*

$$J[\eta : C_0] \equiv J[\eta] - \int_a^b \eta^*(t) C_0(t) \eta(t)\, dt \tag{4.1}$$

is not positive definite on D. *If* K *is a nonzero* $n \times n$ *matrix which is nonnegative Hermitian and such that there exists a nonnegative Lebesgue integrable function* $\gamma(t)$ *on* $[a, b]$ *such that* $\gamma(t)K - C_0(t) \geqq 0$ *for* t *a.e. on* $[a, b]$, *then*

$$\int_a^b \gamma(s)\, ds > 1/\max\{\mu_{\max}(s) | s \in [a, b]\}, \tag{4.2}$$

where $\mu = \mu_{\max}(s)$ *is the largest proper value of the symmetrizable matrix* $G(s, s)K$.

Since by hypothesis $J[\eta : C_0]$ is not positive definite on D there exists an $\eta_0 \in D$ with $\eta_0 \not\equiv 0$ and $J[\eta_0 : C_0] \leqq 0$. From Theorem 3.3 it follows that for $s \in [a, b]$ we have

$$\int_a^b \eta_0^*(t) C_0(t) \eta_0(t)\, dt \geqq J[\eta] \geqq [1/\mu_{\max}(s)] \eta_0^*(s) K \eta_0(s), \tag{4.3}$$

where if for $s = a$ or $s = b$ the class $D[\Pi_s]$ is empty then $\eta^*(s)K\eta(s) = 0$ for all $\eta \in D$ and $\mu_{\max}(s) = 0$, in which case the last term in (4.3) is to be interpreted as zero. Also, since $J[\eta_0] > 0$ we have

$$0 < \int_a^b \eta_0^*(t) C_0(t) \eta_0(t)\, dt \leqq \int_a^b \gamma(t) \eta_0^*(t) K \eta_0(t)\, dt, \tag{4.3'}$$

and the nonnegative continuous function $\eta_0^*(t)K\eta_0(t)$ is not identically zero on $[a, b]$. If $s_0 \in [a, b]$ is such that $\eta_0^*(t)K\eta_0(t)$ attains its maximum value on $[a, b]$ at $t = s_0$, then from (4.3), (4.3′) it follows that

$$\int_a^b \gamma(t)\, dt \geqq 1/\mu_{\max}(s_0) \geqq 1/\max\{\mu_{\max}(s) | s \in [a, b]\}. \tag{4.4}$$

Now if $J[\eta : C_0]$ fails to be nonnegative on D one may choose η_0 so that $J[\eta_0 : C_0] < 0$, in which case the first inequality in (4.3) becomes strict, and the first inequality in (4.4) is also strict. On the other hand, if $J[\eta : C_0]$ is nonnegative, but not positive definite, on D, from Theorem 2.2 it follows that there is a solution $(y; z)$ of the system

$$\Delta[y, z] = C_0 y\, dt, \qquad L[y, z] = 0,$$
$$\hat{y} \in S_1, \qquad T[y, z] \in S_1^{\perp},$$

with $y \not\equiv 0$ on $[a, b]$. Then $(\eta_0, \zeta_0) = (y, v_z - My)$ is a solution of the corresponding differential system (2.12^0) with $f = C_0 \eta_0$. Consequently, for this $\eta_0 \in D$ the first inequality in (4.3) is equality. Also, for s such that $\mu_{\max}(s) > 0$ the second inequality in (4.3) is a strict one unless there exist a constant c and a vector $\xi_0 \in \mathbf{C}_n$ such that $\eta_0(t) \equiv cG(t, s)K\xi_0$, in which case the normality of (2.13a) on each of the subintervals $[a, s), (s, b]$ which is nondegenerate, and the fact that $(y^0; z^0) = (G(\cdot, s)K\xi_0, G_0(\cdot, s)K\xi_0)$ is a solution of (2.13a) on such subintervals, implies that $C_0(t)\eta_0(t) = 0$ for $t \neq s$ and hence the contradictory result

$$\int_a^b \eta_0^*(t) C_0(t) \eta_0(t)\, dt = 0.$$

In particular, if s_0 is such that the maximum of $\eta_0^*(t)K\eta_0(t)$ on $[a, b]$ is attained for $t = s_0$, then $\mu_{\max}(s_0) > 0$ and

$$J[\eta_0] > [1/\mu_{\max}(s_0)]\eta_0^*(s_0)K\eta_0(s_0),$$

so that the first inequality in (4.4) is strict. Thus it has been established that (4.4) holds with the first inequality strict, which is equivalent to (4.2).

THEOREM 4.2. *Suppose that $J[\eta]$ is positive definite on $\mathfrak{D}_0$, the system* (2.7a) *is normal on all subintervals $[a, s]$ and $[s, b]$ with $s \in (a, b)$, while $C_0(t)$ is an $n \times n$ matrix function of the form $[q(t)\delta_{\alpha r}\delta_{\beta r}]$ for some r of the set $1, \cdots, n$, with q of class $\mathfrak{L}^\infty$ and such that relative to the system*

$$\Delta[y, z](t) - C_0(t)y(t)\, dt = 0, \qquad L[y, z](t) = 0 \tag{4.5}$$

there exists on $[a, n]$ a pair of conjugate points. Then $q^+(t) = \frac{1}{2}[q(t) + |q(t)|]$ must satisfy the integral relation

$$\int_a^b q^+(t)\, dt > 1/\max\{G_{rr}(s, s) | s \in [a, b]\}, \tag{4.6}$$

where $G(t, s) = [G_{\alpha\beta}(t, s)]$, $\alpha, \beta = 1, \cdots, n$, is the matrix belonging to the pair G, G_0 of Theorem 2.3 for the incompatible normal boundary problem

$$\Delta[y, z] = 0, \quad L[y, z] = 0, \quad \hat{y} = 0. \tag{4.7}$$

Note. One of the referees has kindly called the attention of the author to the paper by Zeev Nehari in the American Journal of Mathematics, 76 (1954), pp. 689–697, dealing with the zeros of solutions of second order linear differential equations in the complex plane, and wherein the Green's function is employed to establish a Liapunov-type inequality.

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AN INTEGRAL EQUATION FORMULATION OF A MIXED BOUNDARY VALUE PROBLEM ON A SPHERE*

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Abstract. The paper considers the boundary value problem (I): $\nabla^2 w(\rho, \varphi) = 0$, $0 \leqq \rho < 1$, $0 \leqq \varphi \leqq \pi$; $w(1, \varphi) = H_1(\varphi)$, $0 \leqq \varphi < \varphi_0$; $(\partial w/\partial \rho)(1, \varphi) = H_2(\varphi)$, $\varphi_0 < \varphi \leqq \pi$ on the unit sphere. A solution is sought in the form $w(\rho, \varphi) = (1/\pi)\int_0^\pi u(\rho \cos \varphi, \rho \sin \varphi \cos \theta)\, d\theta$, where u satisfies $u_{xx} + u_{yy} = 0$. A Fredholm integral equation of the second kind with a weakly singular kernel is obtained for a function $g = g(\varphi)$ which determines w. Besides the derivation of the integral equation, the principal results are the following: (i) the solution w is unique, (ii) if $H_1 \in C^4$, $H_2 \in C^2$, and if other explicit conditions are satisfied by H_1 and H_2, then the character of the solution w at $(1, \varphi_0)$ is obtained.

1. Introduction. In this paper, we derive a Fredholm integral equation of the second kind with a weakly singular kernel; the solution of this equation (equation (3.25)) will enable one to find a solution, $w = w(\rho, \varphi)$, of the mixed boundary value problem

$$\nabla^2 w = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial w}{\partial \rho}\right) + \frac{1}{\sin \varphi}\left(\frac{\partial}{\partial \varphi}\left(\sin \varphi \frac{\partial w}{\partial \varphi}\right)\right)\right] = 0,$$

$$\text{(I)}\qquad \begin{aligned} &(\rho, \varphi) \in G^+ = \{(\rho, \varphi) | 0 \leqq \rho < 1, 0 \leqq \varphi \leqq \pi\}, \\ &w(1, \varphi) = H_1(\varphi), \quad \varphi \in I_1 = \{\varphi | 0 \leqq \varphi < \varphi_0\}, \\ &\frac{\partial w}{\partial \rho}(1, \varphi) = H_2(\varphi), \quad \varphi \in I_2 = \{\varphi | \varphi_0 < \varphi \leqq \pi\}. \end{aligned}$$

Martin Schechter has proved [6] a Fredholm alternative theorem for a large class of mixed elliptic boundary value problems with a nonhomogeneous differential equation and homogeneous boundary conditions. His results applied to problem (I) guarantee that if the boundary functions satisfy an orthogonality condition with respect to the eigenfunctions of an associated adjoint problem, then at least one solution w exists, and $w \in C^\infty(F)$, where F is any closed subset of $G^+ - \{\varphi_0\}$. Our formal analysis leads us to seek a solution in a certain class of functions and to assume certain smoothness properties of the boundary functions, H_1 and H_2. Under these hypotheses, our principal conclusions, besides the derivation of the integral equation, are the following:

(i) The solution is unique.

(ii) The solution w and its first partial derivatives are continuous at $(1, \varphi_0)$.

2. Some basic theorems. We begin by observing two known facts about solutions of $\nabla^2 w = 0$. First of all, setting $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, it follows that

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial w/\partial y}{y} = 0. \tag{2.1}$$

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Second, a solution of (2.1) exists in the form

$$w = w(x, y) = \frac{1}{\pi}\int_0^\pi u(x, y\cos\theta)\,d\theta, \tag{2.2}$$

where

$$\Delta_2 u = u_{\xi\xi}(\xi, \eta) + u_{\eta\eta}(\xi, \eta) = 0$$

(cf. [3]). Let $\delta = \xi + i\eta$. The function u can be continued harmonically as an even function of η into $|\delta| < 1$. Let $f = u + iv$ be analytic in $|\delta| < 1$, where v is normalized by $v(0, 0) = 0$. It can be shown that $v(\xi, -\eta) = -v(\xi, \eta)$. Therefore, (2.2) can be rewritten as

$$w(x, y) = \frac{1}{\pi}\operatorname{Re}\left(\int_0^\pi f(x + iy\cos\theta)\,d\theta\right).$$

Let $z = x + iy\cos\theta = \rho\cos\varphi + i\rho\sin\varphi\cos\theta$. The boundary conditions of (I) can then be written as

$$\lim_{\rho\to 1^-} \operatorname{Re}\left(\frac{1}{\pi}\int_0^\pi f(z)\,d\theta\right) = H_1(\varphi), \quad \varphi \in I_1, \tag{2.3a}$$

$$\lim_{\rho\to 1^-} \operatorname{Re}\left(\frac{1}{\pi}\int_0^\pi f'(z)z\,d\theta\right) = H_2(\varphi), \quad \varphi \in I_2. \tag{2.3b}$$

Using the integral representation of an analytic function inside the unit circle and interchanging orders of integration, one can prove the following theorem.

THEOREM 1. *If* $g = g(z)$ *is analytic in* $|z| < 1$ *and* $\operatorname{Re} g(z) = \operatorname{Re} g(\bar{z})$, *then*

$$\begin{aligned}\lim_{\rho\to 1^-} \frac{1}{\pi}\operatorname{Re}\left(\int_0^\pi g(z)\,d\theta\right) &= A(\operatorname{Re} g(e^{is}))(\varphi)\\ &= (-E + F_c + G_s)(\operatorname{Re} g(e^{is}))(\varphi),\end{aligned}$$

where

$$\begin{aligned}E(u)(\varphi) &= \frac{1}{\pi}\int_0^\pi u(s)\,ds,\\ F_c(u)(\varphi) &= \frac{1}{\pi}\int_0^\varphi \frac{u(s)\cos(s/2)\,ds}{K(\varphi, s)},\\ G_s(u)(\varphi) &= \frac{1}{\pi}\int_\varphi^\pi \frac{u(s)\sin(s/2)\,ds}{K(\varphi, s)},\\ K(\varphi, s) &= (|\sin^2(\varphi/2) - \sin^2(s/2)|)^{1/2}.\end{aligned}$$

It can easily be shown that

$$F_c(1)(\varphi) = G_s(1)(\varphi) = E(1)(\varphi) = 1. \tag{2.4}$$

For later purposes, we define

$$F_s(u)(\varphi) = \frac{1}{\pi}\int_0^{\varphi} \frac{u(s)\sin(s/2)\,ds}{K(\varphi, s)},$$

$$G_c(u)(\varphi) = \frac{1}{\pi}\int_{\varphi}^{\pi} \frac{u(s)\cos(s/2)\,ds}{K(\varphi, s)}$$

and introduce the family of functions $(\mathrm{P.C.})^{k,p}(I)$, $I = I_1 \cup I_2$, defined to be those functions $v = v(\varphi)$ with the properties:

(i) $v, v', \cdots, d^{k-1}v/d\varphi^{k-1}$ are continuous on I;

(ii) the jump of $v^{(j)}$, $j = 0, 1, \cdots, k-1$, is finite at $\varphi = \varphi_0$;

(iii) $v^{(k)} \in C(I) \cap L_p(I)$, $1 < p \leqq 2$.

As a consequence of Theorem 1, the boundary conditions (2.3) take the form

$$A(u)(\varphi) = H_1(\varphi), \quad \varphi \in I_1, \tag{2.5a}$$

$$A\left(\frac{dv}{d\varphi}\right)(\varphi) = H_2(\varphi), \quad \varphi \in I_2, \tag{2.5b}$$

where $u(\varphi) + iv(\varphi) = f(e^{i\varphi})$. It is well known [4, p. 119] that $u(\varphi)$ and $v(\varphi)$ are related by the equation

$$u(\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(s)\cot\left(\frac{s-\varphi}{2}\right) ds. \tag{2.6}$$

Although (2.6) may be used to rewrite (2.5) in terms of v and v', we shall write (2.5) in terms of u and u' as a consequence of our next result.

THEOREM 2. *Let $f(e^{i\varphi}) = u(\varphi) + iv(\varphi)$ be the boundary values of a function analytic in $|z| < 1$, with u an even and v an odd function of φ. Let $v \in (\mathrm{P.C.})^{1,p}(I)$. Then*

$$F_c(u)(\varphi) = G_c(v)(\varphi) + E(u) + [v](\varphi_0)w_4(\varphi),$$

$$G_s(u)(\varphi) = -F_s(v)(\varphi) + E(u) + [v](\varphi_0)w_5(\varphi),$$

where

$$[v](\varphi_0) = v(\varphi_0 + 0) - v(\varphi_0 - 0),$$

$$w_4(\varphi) = F_c(w_1)(\varphi) + w_2(\varphi) + E(w_1),$$

$$w_5(\varphi) = G_s(w_1)(\varphi) + w_3(\varphi) + E(w_1),$$

$$w_1(\varphi) = -\frac{1}{\pi}\ln|\cos\varphi - \cos\varphi_0|, \qquad \varphi \neq \varphi_0,$$

$$w_2(\varphi) = \frac{2}{\pi}\ln\left(\frac{\sin(\varphi_0/2) + K(\varphi_0, \varphi)}{\sin(\varphi/2)}\right)(1 - u_{\varphi_0}(\varphi)),$$

$$w_3(\varphi) = \frac{2}{\pi}\ln\left(\frac{\cos(\varphi_0/2) + K(\varphi_0, \varphi)}{\cos(\varphi/2)}\right)u_{\varphi_0}(\varphi),$$

$u_{\varphi_0}(\varphi) =$ *unit step function with jump at* $\varphi = \varphi_0$.

Outline of proof. Let $f_0(e^{i\varphi}) = u_0(\varphi) + iv_0(\varphi) \in C^\infty(-\pi < \varphi < \pi)$ with $u_0(-\varphi) = u_0(\varphi)$, $v_0(-\varphi) = -v_0(\varphi)$. The functions u_0 and v_0 can be represented by the Fourier series

$$u_0(\varphi) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\varphi), \tag{2.7a}$$

$$v_0(\varphi) = \sum_{n=1}^{\infty} a_n \sin(n\varphi), \tag{2.7b}$$

where $|a_n| < O(1/n^2)$. Mehler's formula for the Legendre polynomial $P_n(\cos\varphi)$ [7, p. 57] asserts that

$$\begin{aligned} P_n(\cos\varphi) &= F_c(\cos(ns))(\varphi) - F_s(\sin(ns))(\varphi) \\ &= G_c(\sin(ns))(\varphi) + G_s(\cos(ns))(\varphi), \qquad n = 1, 2, \cdots. \end{aligned} \tag{2.8}$$

Forming $\sum_{n=1}^{\infty} a_n P_n(\cos\varphi)$, using the fact that the series (2.7) converge uniformly, and using (2.4), one obtains

$$\begin{aligned} \sum_{n=0}^{\infty} a_n P_n(\cos\varphi) &= F_c(u_0)(\varphi) - F_s(v_0)(\varphi) \\ &= G_c(v_0)(\varphi) + G_s(u_0)(\varphi). \end{aligned} \tag{2.9}$$

Laplace's representation of $P_n(\cos\varphi)$ [2, p. 343] and Theorem 1 imply that

$$P_n(\cos\varphi) = A(\cos(ns))(\varphi), \qquad n \geqq 0. \tag{2.10}$$

Multiplying (2.10) by a_n and summing, one obtains

$$\sum_{n=0}^{\infty} a_n P_n(\cos\varphi) = F_c(u_0) + G_s(u_0) - E(u_0). \tag{2.11}$$

Equations (2.9) and (2.11) imply that

$$G_s(u_0)(\varphi) = -F_s(v_0)(\varphi) + E(u_0), \tag{2.12a}$$

$$F_c(u_0)(\varphi) = G_c(v_0)(\varphi) + E(u_0). \tag{2.12b}$$

When $v \in (\mathrm{P.C.})^{1,p}(I)$, we start with

$$u(\rho, \varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{2\rho v(\sigma) \sin(\sigma - \varphi)\, d\sigma}{1 + \rho^2 - 2\rho\cos(\sigma - \varphi)} \right), \tag{2.13}$$

integrate the right-hand side of (2.13) by parts, take the limit as $\rho \to 1^-$, and use the fact that $C^\infty(I)$ is dense in $(\mathrm{P.C.})^{1,p}(I)$ in the norm of $L_p(I)$, $1 < p \leqq 2$. The result of these operations and equations (2.12) is Theorem 2.

Returning now to the boundary conditions (2.5), we seek a solution pair $u \in (\mathrm{P.C.})^{2,p}(I)$ and $v \in (\mathrm{P.C.})^{2,p}(I)$. Observing that $v'(\varphi) - iu'(\varphi) = \lim_{z \to e^{i\varphi}} (zf'(z))$ and that $E(v') = (1/\pi)[v](\varphi_0)$, we write the boundary condition (2.5b) in the form

$$F_s(u')(\varphi) - G_c(u')(\varphi) = H_2(\varphi) - \frac{1}{\pi}[v](\varphi_0) + [u'](\varphi_0)(w_4 + w_5)(\varphi), \qquad \varphi \in I_2. \tag{2.14}$$

3. Derivation of desired integral equation. Equations (2.5a) and (2.14) are weakly singular integral equations of the first kind for the functions u and u'. From these equations, we shall formally derive a weakly singular integral equation of the second kind, equation (3.25); the solution of (3.25) will determine all other functions of the problem.

To derive this equation, we begin by integrating (2.5a) by parts and then differentiating with respect to φ. The result is

$$\cot(\varphi/2)F_s(u')(\varphi) + \tan(\varphi/2)G_c(u')(\varphi) = H_1'(\varphi) - \frac{[u](\varphi_0)\tan(\varphi/2)\cos(\varphi_0/2)}{K(\varphi,\varphi_0)}, \quad \varphi \in I_1. \tag{3.1}$$

Under the assumption that $\lim_{\varphi-\varphi_0^-} H_1'(\varphi)$ exists and that $u \in (\mathrm{P.C.})^{2,p}(I)$, equation (3.1) implies that $[u](\varphi_0) = 0$. The jump in v at φ_0 must also be zero, as stated by the following lemma.

LEMMA 1. *If* $[u](\varphi_0) = 0$, $u \in (\mathrm{P.C.})^{2,p}(I)$, *and u is even, then* $[v](\varphi_0) = 0$.

Proof. Integrating the integral representation of $v(\rho, \varphi)$ in terms of $u(\varphi)$ by parts and taking the limit as $\rho \to 1^-$, it is easy to show that

$$v(\varphi) = \frac{1}{\pi}\int_0^\pi u'(s)\ln\left|\frac{\sin((s-\varphi)/2)}{\sin((s+\varphi)/2)}\right| ds. \tag{3.2}$$

It follows from (3.2) that v is continuous at φ_0.

The jump in u' at φ_0 is also zero, as stated by our next lemma.

LEMMA 2. *If* $u \in (\mathrm{P.C.})^{2,p}(I)$, *then* $[u'](\varphi_0) = 0$.

Outline of proof. We set $[u](\varphi_0) = 0$ in (3.1), integrate by parts, multiply the result by $\sin\varphi$, and then differentiate with respect to φ. These operations lead to an equation which implies that $[u'](\varphi_0) = 0$.

The boundary conditions (3.1) and (2.14) now take the form

$$\cot(\varphi/2)F_s(u')(\varphi) + \tan(\varphi/2)G_c(u')(\varphi) = H_1'(\varphi), \quad \varphi \in I_1, \tag{3.3}$$

$$F_s(u')(\varphi) - G_c(u')(\varphi) = H_2(\varphi), \quad \varphi \in I_2. \tag{3.4}$$

The next step is to derive singular integral equations from (3.3) and (3.4). To do so, we set $\varphi = t$ in (3.3), multiply by $1/K(\varphi, t)$, integrate from 0 to φ, interchange the order of integration, simplify, differentiate with respect to φ, and use (3.2). The result is

$$\cos(\varphi/2)u'(\varphi) + \frac{\sin(\varphi/2)}{2\pi}\int_0^\pi \frac{u'(s)\sin(s)\,ds}{K_1(s,\varphi)} = \frac{\sec(\varphi/2)v(\varphi)}{2} + h_1(\varphi), \quad \varphi \in I_1, \tag{3.5}$$

where

$$h_1(\varphi) = \cos(\varphi/2)\frac{d}{d\varphi}\left(\sin(\varphi/2)\int_0^\varphi \frac{H_1'(t)\,dt}{K(\varphi,t)}\right) \tag{3.6}$$

and $K_1(s, \varphi) = \sin^2(s/2) - \sin^2(\varphi/2)$. In (3.4), we set $\varphi = t$, multiply by $\sin(t)/[2K(\varphi, t)]$, integrate from φ to π, interchange the order of integration, and differentiate with respect to φ. The result is

$$\cos(\varphi/2)u'(\varphi) + \frac{\sin(\varphi/2)}{2\pi}\int_0^\pi \frac{u'(s)\sin(s)\,ds}{K_1(s,\varphi)} = h_2(\varphi), \quad \varphi \in I_2, \tag{3.7}$$

where

$$h_2(\varphi) = \frac{d}{d\varphi}\int_\varphi^\pi \frac{H_2(t)\sin(t)\,dt}{2K(\varphi,t)}. \tag{3.8}$$

Equations (3.5) and (3.7) can be rewritten as a single singular integral equation with the standard singular kernel, $1/(t-x)$, by setting $x = \sin^2(\varphi/2)$, $t = \sin^2(s/2)$, $U(x) = u'(\varphi)$, and

$$G(x) = g(\varphi) = \begin{cases} \dfrac{v(\varphi)}{2\cos(\varphi/2)} + h_1(\varphi), & 0 < \varphi < \varphi_0, \\ h_2(\varphi), & \varphi_0 < \varphi < \pi. \end{cases} \tag{3.9}$$

Making these substitutions, we obtain

$$\sqrt{1-x}\,U(x) + \frac{\sqrt{x}\,i}{\pi i}\int_0^1 \frac{U(t)\,dt}{t-x} = G(x), \qquad 0 < x < 1, \quad x \neq \sin^2(\varphi_0/2). \tag{3.10}$$

Applying the technique of reducing a singular integral equation to a Hilbert problem and then solving this problem, one finds that

$$u'(\varphi) = g(\varphi)\cos(\varphi/2) - \frac{\sin(\varphi/2)\gamma(\varphi)}{\pi}\int_0^\pi \frac{g(s)\sin(s)\,ds}{2\gamma(s)K_1(s,\varphi)}, \tag{3.11}$$

where $\gamma(\varphi) = \sec(\varphi/2)\exp[(2/\pi)\int_0^\pi \ln K(s,\varphi)\,ds]$. Using a well-known fact about integrals of periodic functions, one can show that $\int_0^\pi \ln K(s,\varphi)\,ds$ is a constant. Equation (3.11) then becomes

$$\begin{aligned} u'(\varphi) &= g(\varphi)\cos(\varphi/2) + \frac{\tan(\varphi/2)}{\pi}\int_0^\pi g(s)\sin(s/2)\,ds \\ &\quad - \frac{\sin\varphi}{2\pi}\int_0^\pi \frac{g(s)\sin(s/2)\,ds}{K_1(s,\varphi)}, \quad \varphi \in I. \end{aligned} \tag{3.12}$$

In order that $\lim_{\varphi\to\pi} u'(\varphi) = 0$, it is necessary that

$$\int_0^\pi g(s)\sin(s/2)\,ds = 0. \tag{3.13}$$

Since u' is to be continuous at φ_0, it follows that g must be continuous at φ_0. Substituting (3.13) into (3.12), setting $\varphi = t$, integrating from $0 < t < \varphi$, and interchanging the order of integration, we obtain

$$u(\varphi) = u(0) - \frac{2}{\pi}\int_0^\pi g(s)\sin(s/2)\ln\sin(s/2)\,ds$$

$$+\int_0^{\varphi} g(s)\cos(s/2)\,ds$$

$$+\frac{2}{\pi}\int_0^{\pi} g(s)\sin(s/2)\ln K(\varphi,s)\,ds. \tag{3.14}$$

Since $g(\varphi)$ contains $v(\varphi)$ in its definition, we proceed to eliminate $u(\varphi)$ from (3.14). First, note that

$$v(\varphi) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} u(\sigma)\cot\left(\frac{\sigma-\varphi}{2}\right)d\sigma = -\frac{\sin\varphi}{2\pi}\int_0^{\pi}\frac{u(\sigma)\,d\sigma}{K_1(\sigma,\varphi)} \tag{3.15}$$

and that

$$\int_{-\pi}^{\pi}\cot\left(\frac{\sigma-\varphi}{2}\right)d\sigma = 0. \tag{3.16}$$

It follows from (3.14) that

$$v(\varphi) = 2\int_0^{\pi} g(s)\cos(s/2)K_2(\varphi,s)\,ds + \frac{1}{\pi}\int_0^{\pi} g(s)\sin(s/2)K_3(\varphi,s)\,ds, \tag{3.17}$$

where

$$K_2(\varphi,s) = \frac{1}{2\pi}\ln\left|\frac{\sin[(s-\varphi)/2]}{\sin[(s+\varphi)/2]}\right| = -\sum_{n=1}^{\infty}\frac{\sin(n\varphi)\sin(ns)}{n\pi}, \tag{3.18}$$

$$K_3(\varphi,s) = -\frac{\sin(\varphi)}{2\pi}\int_0^{\pi}\frac{2\ln(K(\sigma,s))\,d\sigma}{K_1(\sigma,\varphi)}. \tag{3.19}$$

To simplify the expression for $K_3(\varphi, s)$, we note that the Fourier series for $2\ln(K(\sigma, s))$ is

$$2\ln(K(\sigma,s)) = -\ln(2) + \sum_{n=1}^{\infty} a_n(s)\cos(n\sigma), \tag{3.20}$$

where $a_n(s) = -2\cos(ns)/n$, $n \geqq 1$. It is an easy calculation from (3.20) to show that

$$\begin{aligned} \ln(2K^2(\sigma,s)) &= \lim_{\substack{|z|<1 \\ z \to e^{i\sigma}}} \operatorname{Re}\operatorname{Log}(1 - 2z\cos(s) + z^2) \\ &= \operatorname{Re}\operatorname{Log}[2(\cos(\sigma) - \cos(s))\,e^{i\sigma}], \end{aligned} \tag{3.21}$$

where Log denotes the principal branch of the complex log function. It follows from (3.21) and (3.15) that

$$K_3(\varphi,s) = \begin{cases} \varphi - \pi, & \varphi > s, \\ \varphi, & \varphi < s. \end{cases} \tag{3.22}$$

As a consequence of (3.22) and (3.13), equation (3.17) can be written as

$$v(\varphi) = 2\int_0^{\pi} g(s)\cos(s/2)K_2(\varphi,s)\,ds - \int_0^{\varphi} g(s)\sin(s/2)\,ds, \quad \varphi \in I. \tag{3.23}$$

Differentiating (3.23), using definition (3.9), and dividing by $\cos(\varphi/2)$ leads to

$$\frac{d}{dt}(v(t)\sec(t/2)) = 2\sec(t/2)\int_0^\pi g(s)\cos(s/2)\frac{\partial}{\partial t}K_2(t,s)\,ds - h_1(t)\tan(t/2), \tag{3.24}$$

$$t \in I_1.$$

We integrate (3.24) from 0 to φ, interchange orders of integration, and use (3.13) to find that

$$g(\varphi) = -1\int_0^{\varphi_0} K_4(\varphi, s)g(s)\,ds + h_3(\varphi), \quad \varphi \in I_1, \tag{3.25}$$

where

$$K_4(\varphi, s) = -\frac{1}{2\pi}\ln\left|\frac{\sin(\varphi/2) - \sin(s/2)}{\sin(\varphi/2) + \sin(s/2)}\right| = K_4(s, \varphi) \geqq 0, \tag{3.26}$$

$$h_3(\varphi) = -\int_{\varphi_0}^\pi K_4(\varphi, s)h_2(s)\,ds + h_1(\varphi) - \int_0^\varphi \frac{h_1(s)\tan(s/2)\,ds}{2}. \tag{3.27}$$

We proceed to analyze the Fredholm integral equation (3.25).

4. Properties of equation (3.25). In this section we assume that $0 \leqq \varphi_0 < \pi$.

Property 1. *If $h_3 \in L_2(I_1)$, then there exists a unique solution g of* (3.28) *in $L_2(I_1)$.*

Proof. The kernel K_4 is positive almost everywhere, and it is symmetric. It is known [1, p. 285] that the characteristic values of (3.25) are positive. Hence, -1 is not a characteristic value of (3.25). The Fredholm alternative theorem for L_2 kernels implies Property 1.

Property 2. *If $H_1 \in C^2(\bar{I}_1)$ and $H_2 \in C^1(\bar{I}_2)$, then the solution g of* (3.25) *is continuous on $\bar{I}_1$.*

Proof. Under the above hypothesis, it follows that $h_3 \in C(\bar{I}_1)$. The logarithmic character of K_4 implies that

$$\int_0^{\varphi_0} K_4(\varphi, s)g(s)\,ds \in C(\bar{I}_1).$$

Property 3. *If $H_1 \in C^3(\bar{I}_1)$ and $H_2 \in C^2(\bar{I}_2)$, then $g \in H^\alpha(I_1)$, where $H^\alpha(I_1)$, $0 < \alpha < 1$, denotes the class of Hölder continuous functions on I_1.*

Proof. Under these hypotheses $h_3 \in C^1(I_1)$. It is a straightforward technical lemma to prove that $\int_0^{\varphi_0} K_4(\varphi, s)g(s)\,ds \in H^\alpha(I_1)$ if $g \in C(\bar{I}_1)$.

Property 4. *If $H_1 \in C^3(\bar{I}_1)$ and $H_2 \in C^2(\bar{I}_2)$, then $g \in C^1(I_1) \cap L_2(I_1)$.*

Proof. The fact that $g \in C^1(I_1)$ follows immediately from Property 3. and a well-known lemma [5, p. 31] about derivatives of logarithmic integrals. The behavior of singular integrals at the endpoints of the interval implies that g has at most a logarithmic singularity at $\varphi = 0$ and $\varphi = \varphi_0$ [5, p. 85].

Equation (3.25) can now be differentiated and the resulting integrals integrated by parts. If

$$g(\varphi_0^-) = g(\varphi_0^+) = h_2(\varphi_0^+), \tag{4.1}$$

$$H_1'(0) = H_2(\pi) = 0, \tag{4.2}$$

then one obtains

$$g_1(\varphi) = g'(\varphi)\sec(\varphi/2) = \int_0^{\varphi_0} K_5(\varphi, s) g_1(s)\, ds + h_4(\varphi), \tag{4.3}$$

where

$$K_5(\varphi, s) = \frac{1}{\pi}\ln(K(\varphi, s)), \tag{4.4}$$

$$\begin{aligned} h_4(\varphi) = {} & \int_0^{\varphi_0} K_5(\varphi, s) g(s) \frac{d}{ds}\sec(s/2)\, ds + \int_{\varphi_0}^{\pi} K_5(\varphi, s)\frac{d}{ds}(h_2 \sec(s/2))\, ds \\ & + \left(\frac{dh_1}{ds} - h_1 \frac{\tan(\varphi/2)}{2}\right)\sec(\varphi/2). \end{aligned} \tag{4.5}$$

Arguments similar to those mentioned in Properties 2, 3 and 4 when applied to (4.3) lead to the following property.

Property 5. *If* $H_1 \in C^4(\bar{I}_1)$ *and* $H_2 \in C^3(\bar{I}_2)$, *then* $g \in C^2(I_1) \cap L_2(I_1)$.

Let $\mathscr{B} = \{(H_1, H_2) | H_1 \in C^4(I_1),\ H_2 \in C^3(I_2),\ g(\varphi_0^-) = h_2(\varphi_0^+),\ H_1'(0) = H_2(\pi) = 0\}$.

COROLLARY 1. *If* $(H_1, H_2) \in \mathscr{B}$, *then the function* v *defined by* (3.23) *and the function* u *defined by* (3.14) *are elements of* $(P.C.)^{2,p}(I)$.

If the constant $u(0)$ in (3.14) is selected so that (2.5a) is satisfied, then the function $u(\varphi)$ generates a harmonic function $u(x, y)$ which in turn generates a solution $w = w(\rho, \varphi)$ of the boundary value problem (I).

THEOREM 3. $w(1, \varphi)$, $(\partial w/\partial\varphi)(1, \varphi)$ *and* $(\partial w/\partial\rho)(1, \varphi)$ *are continuous at* $\varphi = \varphi_0$.

The proof depends on the fact that if $u(\varphi)$ is continuous at φ_0, then $A(u)(\varphi)$ is continuous at φ_0, and the abovementioned continuity properties of u and v.

Theorem 3 implies that $w(\rho, \varphi)$ has a normal mode expansion

$$w(\rho, \varphi) = \sum_{n=0}^{\infty} c_n(\rho^n P_n(\cos(\varphi))). \tag{4.6}$$

If $w = w_n(\rho, \varphi) = \rho^n P_n(\cos(\varphi))$, then $u = u_n(\varphi) = \cos(n\varphi)$, $v = v_n(\varphi) = \sin(n\varphi)$ and $g = g_n(\varphi) = -n\sin((n + 1/2)\varphi)$. It is easy to check that the orthogonality condition (3.13) is satisfied for $g = g_n$. Since our problem is linear, it follows that $g(s) = \sum_{n=0}^{\infty} c_n g_n(s)$ and that

$$\int_0^{\pi} g(s)\sin(s/2)\, ds = \sum_{n=0}^{\infty} c_n \int_0^{\pi} g_n(s)\sin(s/2)\, ds = 0.$$

Thus, condition (3.13) is satisfied automatically under the hypothesis that $(H_1, H_2) \in \mathscr{B}$.

The zero flux condition $\int_0^{\pi}(\partial w/\partial\rho)(1, \varphi)\sin(\varphi)\, d\varphi = 0$, which is a consequence of the differential equation $\nabla^2 w = 0$ and continuity conditions, is automatically satisfied when the boundary functions H_1 and H_2 are elements of $\mathscr{B}$. Without the requirement that $g(\varphi_0^-) = h_2(\varphi_0^+)$, v' would have a logarithmic singularity at φ_0. This in turn would imply that $(\partial^2/\partial\varphi\partial\rho)w(1, \varphi)$ would have a logarithmic singularity at φ_0.

Our uniqueness theorem for (3.25) leads to a uniqueness theorem for the boundary value problem (I).

THEOREM 4. *If* $(H_1, H_2) \in \mathscr{B}$, *then the boundary value problem* (I) *has a unique solution for all angles* φ_0, $0 \leqq \varphi_0 \leqq \pi$.

Proof. Every solution of $\nabla^2 w = 0$ in G^+ can be put in the form (2.2) for some two-dimensional harmonic function u. Property 1 of the integral equation (3.25) implies that u is unique for $0 \leqq \varphi_0 < \pi$. For $\varphi_0 = \pi$ the boundary value problem (I) is simply the interior Dirichlet problem for a sphere which is known to have a unique solution.

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ON SCHWARTZ'S HANKEL TRANSFORMATION OF CERTAIN SPACES OF DISTRIBUTIONS*

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Abstract. To extend the result of A. L. Schwartz's Hankel transformations to distributions, we define two new testing function spaces F_ν and G_ν on $(0, \infty)$. These two testing function spaces are characterized by their behavior near the origin and at infinity. We then prove that the Schwartz's Hankel transformation is a continuous linear mapping from F_ν into G_ν, and therefore the generalized Schwartz's Hankel transformation is a continuous linear mapping from the dual space G'_ν into the dual space F'_ν.

1. Introduction. A. H. Zemanian [8] first investigated the distributional Hankel transformation which is a generalization to distributions of the conventional Hankel transformation $\mathscr{H}_\mu$ defined for $\mu \geqq -1/2$ by

$$[\mathscr{H}_\mu \varphi(x)](y) \triangleq \int_0^\infty \mathscr{E}(x)\sqrt{xy}J_\mu(xy)\,dx. \tag{1.1}$$

Here $J_\mu(xy)$ is the Bessel function of the first kind of order μ. In order to do that, he defined the testing function space H_μ, and then proved that the Hankel transformation (1.1) is an isomorphism of H_μ onto itself, and therefore the generalized Hankel transformation $\mathscr{H}'_\mu$ defined by

$$\langle \mathscr{H}'_\mu f, \varphi \rangle = \langle f, \mathscr{H}_\mu \varphi \rangle,$$

where f belongs to the dual space H'_μ, is an isomorphism of H'_μ onto itself. Later on E. L. Koh and A. H. Zemanian [11] extended it to the complex plane using a different testing function space $\mathscr{J}_a$. W. Y. Lee [5] did a similar work which is a counterpart of the Fourier transformation of spaces of type S of I. M. Gel'fand and G. E. Shilov [3]. Recently A. L. Schwartz [6] defined his Hankel transformation $\hbar_\nu$ for $\nu \geqq -1/2$ as follows:

$$[\hbar_\nu \varphi(x)](y) \triangleq \int_0^\infty \varphi(x) m'(x) \mathscr{J}_\nu(xy)\,dx, \tag{1.2}$$

where

$$m'(x) = [2^\nu \Gamma(\nu + 1)]^{-1} x^{2\nu+1}, \qquad \mathscr{J}_\nu(x) = 2^\nu \Gamma(\nu + 1) x^{-\nu} J_\nu(x).$$

Note that the kernel function in (1.2) is quite different from that of (1.1). Then he proved the following inversion theorem [6, pp. 713–714].

THEOREM 1.1. *If $f \in L^1$ satisfies the inequality*

$$\int_0^1 |f(x)| x^{\nu+1/2}\,dx < \infty$$

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for $x > 0$ and if f is of bounded vatiation in a neighborhood of x, then for $\nu \geqq -1/2$,

$$\frac{1}{2}\{f(x+0)+f(x-0)\} = \lim_{\lambda\to\infty}\int_0^\infty \mathscr{J}_\nu(xu)\,dm(u)\int_0^\infty f(y)\mathscr{J}_\nu(uy)\,dm(y). \tag{1.3}$$

Subsequently, L. S. Dube and J. N. Pandey [1] generalized it to distributions using two testing function spaces $G_{\alpha,\delta}$ and $H_{\alpha,\delta}$, where $0 < \alpha \leqq \nu + 1/2$ and $\delta \geqq 0$. They then proved that the Schwartz's Hankel transformation (1.2) is a continuous linear mapping from $G_{\alpha,\delta}$ into $H_{\alpha,\delta}$ and therefore the generalized Hankel transformation is a continuous linear mapping from the dual space $H'_{\alpha,\delta}$ into the dual space $G'_{\alpha,\delta}$. The drawback of the two testing function spaces $G_{\alpha,\delta}$ and $H_{\alpha,\delta}$ is that they are not closed under differentiation. However, closedness of testing function spaces under differential operation is understood to be true in distribution theory. The motivation of the present work is to modify $G_{\alpha,\delta}$ and $H_{\alpha,\delta}$ so that differentiation in the given space is allowed. Furthermore it is essential to define new testing function spaces to satisfy the assumptions of Schwartz's theorem so that we can generalize his result to distributions. We define two testing function spaces F_ν and G_ν, and prove that the Hankel transformation (1.2) is a continuous linear mapping from F_ν into G_ν, and hence the generalized Hankel transformation is a continuous linear mapping of the dual space G'_ν into the dual space F'_ν. Here all the dual spaces are equipped with strong dual topology.

2. The testing function spaces F_ν and G_ν. The space F_ν consists of smooth functions φ on $0 < x < \infty$ for which

$$\gamma_{m,k}(\varphi) \triangleq \sup_{0<x<\infty} |x^m \Delta^k_{\nu,x}\varphi(x)| < \infty, \qquad m, k = 0, 1, 2 \cdots,$$

where $\Delta_{\nu,x} = D_x^2 + (2\nu + 1)x^{-1}D_x$. The space G_ν consists of smooth functions φ on $0 < x < \infty$ for which

$$\gamma_k(\varphi) \triangleq \sup_{0<x<\infty} |\Delta^k_{\nu,x}\varphi(x)| < \infty, \qquad k = 0, 1, 2 \cdots,$$

where $\Delta_{\nu,x}$ is the same as before. We equip the spaces F_ν and G_ν with the topology generated by the seminorms $\{\gamma_{m,k}\}_{m,k=0}^\infty$ and $\{\gamma_k\}_{k=0}^\infty$ respectively. It is easy to see that both F_ν and G_ν are Fréchet spaces. Obviously F_ν is a proper subspace of G_ν algebraically and the topology of F_ν is stronger than the one induced by G_ν. For instance $\varphi(x) = 1/(1 + x^2)$ belongs to G_ν but not to F_ν. From the definitions we have the following inclusion relations:

$$\mathscr{D}(0,\infty) \subset \mathscr{S}(0,\infty) \subset F_\nu \subset G_\nu \subset \mathscr{E}(0,\infty).$$

Note that all the above inclusion relations are proper algebraically and topologically.

Remark 1. Every testing function in F_ν and G_ν satisfies the assumptions of Schwartz's Theorem 1.1.

Remark 2. Let H_ν be the Zemanian space [10, pp. 129–130]. Then $\varphi \in F_\nu$ if and only if $x^{\nu+1/2}\varphi \in H_\nu$.

Suppose the Taylor expansion of φ near the origin has a form for each $k = 0, 1, 2, \cdots,$

$$\varphi(x) = a_0 + a_1 x^2 + \cdots + a_k x^{2k} + o(x^{2k}), \tag{2.1}$$

where the $a_p (0 \leqq p \leqq k)$ are constant depending on φ. An inductive argument on k shows

$$\Delta_{\nu,x}^k = b_0 D_x^{2k} + b_1 x^{-1} D_x^{2k-1} + b_2 x^{-2} D_x^{2k-2} + \cdots + b_{2k-1} x^{-(2k-1)} D_x,$$

where the b_p $(0 \leqq p \leqq 2k - 1)$ are constant $(b_0 = 1)$ depending on ν. Thus we have the following.

THEOREM 2.1. (i) *A nonconstant smooth function φ on $0 < x < \infty$ belongs to the space F_ν if and only if φ has the form* (2.1) *near the origin and is rapidly decreasing as $x \to \infty$.*

(ii) *A nonconstant smooth function φ on $0 < x < \infty$ belongs to the space G_ν if and only if φ has the form* (2.1) *near the origin and $\varphi(x) = O(x^{-\varepsilon})$ for any $\varepsilon > 0$ as $x \to \infty$.*

Note that Theorem 2.1 is similar to [10, Lemma 5.2.1, p. 130]. This is conceivable from Remark 2.

3. The Schwartz's Hankel transformation. Now we prove the main theorem.

THEOREM 3.1. *For $\nu \geqq -1/2$, the Hankel transformation* (1.2) *is a continuous linear mapping from the space F_ν into the space G_ν.*

Proof. Let B be a bounded set in F_ν and let φ be any element in B. Then since $\Delta_{\nu,y}^k \mathscr{J}_\nu(xy) = (-1)^k x^{2k} \mathscr{J}_\nu(xy)$ (differentiation with respect to y) and since

$$\begin{aligned}\Phi(y) = [\hbar_\nu \varphi(x)](y) &= \int_0^\infty \varphi(x) m'(x) \mathscr{J}_\nu(xy)\, dx \\ &= \int_0^\infty \varphi(x) x^{2\nu+1} (xy)^{-\nu} J_\nu(xy)\, dx,\end{aligned}$$

we have

$$\begin{aligned}\sup_{0<y<\infty} |\Delta_{\nu,y}^k \Phi(y)| &\leqq \int_0^\infty |\varphi(x) x^{2k+\nu+1}| \sup_{0<y<\infty} (|y^{-\nu} J_\nu(xy)|)\, dx \\ &\leqq \int_0^1 \sup_{0<x<1} \{|\varphi(x) x^{2k+\nu+1}| \sup_{0<y<\infty} (|y^{-\nu} J_\nu(xy)|)\}\, dx \\ &\quad + \int_1^\infty \sup_{1<x<\infty} \{|\varphi(x) x^{2k+\nu+1}| \sup_{0<y<\infty} (|y^{-\nu} J_\nu(xy)|)\}\, dx \\ &\leqq \gamma_{2k,0}(\varphi) + \gamma_{2k+q,0}(\varphi) < \infty,\end{aligned} \tag{3.1}$$

where q is chosen to be a positive integer greater than $\nu + 1/2$. Note that $\sup_{0<y<\infty} |y^{-\nu} J_\nu(xy)| = O(|x|^{-1/2})$. Since the inequality (3.1) does not depend on the choice of φ in B, it follows that the Hankel transformation $\hbar_\nu$ maps a bounded set in F_ν into a bounded set in G_ν, and therefore is continuous. This proves the theorem.

Now let f belong to the dual space G'_ν and let φ belong to F_ν. Define the generalized Hankel transformation $\hbar'_\nu$ by

$$\langle \hbar'_\nu f, \varphi \rangle = \langle f, \hbar_\nu \varphi \rangle. \tag{3.2}$$

The above definition is meaningful from Theorem 3.1. Invoking [10, p. 29], we obtain the following.

THEOREM 3.2. *For $\nu \geqq -1/2$, the generalized Hankel transformation h'_ν defined by* (3.2) *is a continuous linear mapping from the dual space G'_ν into the dual space F'_ν.*

Theorem 3.2 is a generalization of Schwartz's Theorem 1.1 to distributions. Since F_ν and the Zemanian space H_ν are isomorphic under the isomorphism $\varphi \mapsto x^{\nu+1/2}\varphi$, Theorems 3.1 and 3.2 still hold for φ in H_ν replaced by $x^{-(\nu+1/2)}\varphi$.

For f in $\mathscr{E}'(0, \infty)$, define F by

$$
\begin{aligned}
F(y) &= \langle f(x), m'(x)J_\nu(xy)\rangle \\
&= \langle f(x), x^{2\nu+1}(xy)^{-\nu}J_\nu(xy)\rangle.
\end{aligned}
\tag{3.3}
$$

Since $x^{2\nu+1}(xy)^{-\nu}J_\nu(xy)$ belongs to $\mathscr{E}(0, \infty)$, the equality (3.3) is well-defined. Notice that we are not allowed to choose f either in F'_ν or G'_ν because $x^{2\nu+1}(xy)^{-\nu}$ $J_\nu(xy)$ belongs to neither F_ν nor G_ν. Here $x^{2\nu+1}(xy)^{-\nu}J_\nu(xy)$ is understood to be the principal value. In view of [10, p. 146] and the Cauchy integral formula the following theorem is not hard to prove.

THEOREM 3.3. *For any $f \in \mathscr{E}'(0, \infty)$, $F(y)$ defined by* (3.3) *is a smooth function.*

Theorem 3.3 is similar to [10, Thm. 5.6-1, p. 146], but this is expected because of Remark 2. Moreover, $F(y)$ behaves like a polynomial. Indeed we have the following.

THEOREM 3.4. *Let $f \in \mathscr{E}'(0, \infty)$. Then $F(y)$ defined by* (3.3) *satisfies the following inequality*:

$$
|F(y)| \leqq \begin{cases} C, & 0 < y \leqq 1, \\ Cy^k, & 1 < y < \infty, \end{cases}
$$

for some constant C and a sufficiently large positive integer k.

Proof. Since f in $\mathscr{E}'(0, \infty)$ has a compact support according to [10, pp. 37–38]· set $A = \operatorname{supp} f$. Choose a smooth function $\lambda(x)$ with compact support such that $\lambda(x) \equiv 1$ on a neighborhood of A. Then there exists a positive integer k and a constant C ([10, pp. 18–19]) such that

$$
\begin{aligned}
|F(y)| &= |\langle f(x), (xy)^{-\nu}J_\nu(xy)\lambda(x)x^{2\nu+1}\rangle| \\
&\leqq C \sup_{x\in A} \max_{0\leqq r\leqq k} |\Delta^r_{\nu,x}\{(xy)^{-\nu}J_\nu(xy)\lambda(x)x^{2\nu+1}\}|.
\end{aligned}
\tag{3.4}
$$

An inductive argument on r shows us that

$$
\begin{aligned}
&\Delta^r_{\nu,x}\{(xy)^{-\nu}J_\nu(xy)\lambda(x)x^{2\nu+1}\} \\
&\quad = (-1)^r y^{2r} \sum_{p=0}^{r} a_{r,p}(xy)^{-(\nu+p)}J_{\nu+p}(xy)\lambda(x)x^{2\nu+1} \\
&\quad + (-1)^{r-1} \sum_{p=0}^{r-1} a_{r-1,p}(xy)^{-(\nu+p)}J_{\nu+p}(xy)\lambda(x)x^{2(\nu-1)+1} \\
&\quad + \cdots + a_{0,0}(xy)^{-\nu}J_\nu(xy)\lambda(x)x^{2(\nu-r)+1},
\end{aligned}
\tag{3.5}
$$

where the $a_{q,p}$ $(0 \leqq q \leqq r, 0 \leqq p \leqq r)$ are constants $(a_{r,0} = 1)$ depending on ν.

In forming (3.5) we have used the following equalities:

$$\Delta_{\nu,x}\{(xy)^{-\nu}J_\nu(xy)\} = (-1)^r y^{2r}(xy)^{-\nu}J_\nu(xy),$$

$$D_x\{(xy)^{-\nu}J_\nu(xy)\} = -y(xy)^{-\nu}J_{\nu+1}(xy),$$

$$D_x\{(xy)^{-\nu}J_\nu(xy)\}D_x\{\lambda(x)x^{2\nu+1}\} = -(2\nu+1)y^2(xy)^{-(\nu+1)}J_{\nu+1}(xy)\lambda(x)x^{2\nu+1},$$

$$\lambda^{(r)}(x) = 0 \quad (r \geqq 1) \quad \text{on } A.$$

Utilizing [7, p. 45], we can easily prove the following inequality:

$$\sup_{x\in A} |(xy)^{-(\nu+p)}J_{\nu+p}(xy)\lambda(x)x^{2(\nu-q)+1}| < C_{q,p}. \tag{3.6}$$

It follows from inequalities (3.4), (3.5) and (3.6) that

$$\begin{aligned}|F(y)| &\leqq C \max_{0\leqq r\leqq k}\left\{y^{2r}\sum_{p=0}^{r}|a_{r,p}|C_{0,p} + y^{2(r-1)}\sum_{p=0}^{r-1}|a_{r-1,p}|C_{1,p} + \cdots + |a_{0,0}|C_{r,0}\right\}\\ &\leqq \begin{cases} C', & 0 < y \leqq 1,\\ C'y^{2k}, & 1 < y < \infty.\end{cases}\end{aligned}$$

This completes the proof.

Theorems 3.3 and 3.4 together yield the following.

THEOREM 3.5. *Suppose $f \in \mathscr{E}'(0,\infty)$. Then the generalized Hankel transformation $\mathscr{h}'_\nu f$ is a regular distribution in F_ν generated by F defined by* (3.3).

4. Some open problems. In this paper we extended the Schwartz's Hankel transformation (1.2) for $\nu \geqq -1/2$ to distributions. But it is unknown whether our results still hold for $\nu < -1/2$. We shall state some problems related to this paper.

Problem 1. How can we extend Schwartz's Hankel transformation for $\nu < -1/2$? Once this is done we may easily extend it to distributions.

Problem 2. How can we extend our results to the complex plane? This problem is equivalent to finding appropriate complex testing function spaces suitable for our theory. Note that we cannot use the same argument as Koh and Zemanian's [11] because the kernel function $x^{2\nu+1}(xy)^{-\nu}J_\nu(xy)$ does not belong to the testing function space F_ν.

Problem 3. Solve the Cauchy problem of the differential operator $\Delta_{\nu,x} = D_x^2 + (2\nu+1)x^{-1}D_x$:

$$\frac{\partial u(x,t)}{dt} = P(\Delta_{\nu,x})u(x,t),$$

$$u(x,0) = u_0(x),$$

where $u(x,t)$ is an $m \times 1$ column vector, and P is an $m \times m$ polynomial matrix with constant coefficients. Can we apply the Hankel transformation (1.2) to attack the problem?

In [13], we gave a μ^2 uniqueness class of the Cauchy problem of differential operator $S_\mu = D_x^2 - (4\mu - 1)/(4x^2)$ by using the Hankel transformation (1.1). Gelfand and Shilov [14] solved the same Cauchy problem of differential operator $i(\partial/\partial x)$ by using the Fourier transformation.

Problem 4. Since the Zemanian space H_ν is isomorphic to our testing function space F_ν via the mapping $\varphi \mapsto x^{-(\nu+1/2)}\varphi$, and since the Hankel transformation (1.1) ((1.2)) maps $H_\nu(F_\nu)$ onto (into) $H_\nu(G_\nu)$ respectively, is there an isomorphism from H_ν into G_ν, or equivalently, from G'_ν into H'_ν? This problem is equivalent to finding a relation between two Hankel transformations (1.1) and (1.2).

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AN OPERATOR EQUATION AND BOUNDED SOLUTIONS OF INTEGRO-DIFFERENTIAL SYSTEMS*

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Abstract. The main result gives conditions under which (locally) a one-to-one, bicontinuous correspondence exists between bounded solutions (or bounded solutions tending to zero as $t \to +\infty$) of a linear, integro-differential system of Volterra type and such solutions of perturbations of the system. The perturbations are allowed to be of any functional type which satisfy a local Lipschitz condition near the origin. Certain recently proved stability results for such systems are special cases. The results also constitute a generalization of similar results for ordinary differential equations, which motivate the approach and proofs. The proofs rely on an abstract lemma proved for a certain operator equation. In order to apply the perturbation theorems some results are also given concerning bounded solutions of linear integro-differential systems. An application is made to Volterra's predator-prey population dynamics model with hereditary effects where it is shown, for certain specific, but reasonable hereditary kernels, that the critical (or saturation point) of the system is unstable.

Introduction. Our main purpose is to investigate the existence of bounded solutions of the $n \times n$ system of integro-differential equations

$$\text{(P)} \qquad x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)\,ds + h(t)(x) + g(t), \qquad t \geqq t_0,$$

where $h(t)(0) \equiv 0$, $t \geqq t_0$, which is to be considered a perturbation of the linear homogeneous system

$$\text{(H)} \qquad y'(t) = A(t)y(t) + \int_{t_0}^{t} B(t,s)y(s)\,ds, \qquad t \geqq t_0.$$

The goal is to place conditions on h and on (H), or more precisely on the related nonhomogeneous system

$$\text{(NH)} \qquad z'(t) = A(t)z(t) + \int_{t_0}^{t} B(t,s)z(s)\,ds + g(t), \qquad t \geqq t_0,$$

under which it is possible to assert that (locally) there is a one-to-one correspondence between the bounded solutions of (P) and those of (H). The perturbation term h is to be thought of as an operator which maps the set of functions defined for $t \geqq t_0$ into itself and which is in some sense small; this will be made precise below. Typical perturbations are of the form

$$h(t)(x) = h(t, x(t)) \quad \text{or} \quad \int_{t_0}^{t} K(t,s,x(s))\,ds \quad \text{or} \quad x(t)\int_{t_0}^{t} K(t,s,x(s))\,ds.$$

Obviously perturbations of Fredholm type could also be considered. Specific conditions will be placed on the $n \times n$ matrices A and B below.

First, in § 1 we will state and prove a result for an abstract operator equation which, when applied to (P) in § 2 will lead to our main results. In § 3 we will study the linear nonhomogeneous system (H) with regard to the hypotheses for our

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main results. Finally in § 4 the important special case $A(t) \equiv A =$ const. and $B(t, s) \equiv B(t - s)$, which is the form of the vast majority of applications, will be considered. Included in § 4 is an application to Volterra's equations for predator-prey population dynamics with hereditary effects where, amongst other things, it is shown that under certain reasonable assumptions on the hereditary factor such populations will not have a stable critical or saturation point.

Our approach and results for (P) are motivated by (and generalize) certain results for differential equations ($B \equiv 0$). See, for example, [3], [8]. They also serve to generalize certain stability results for integro-differential equations [6], [7], [9], [11]. Our Lemma 1 below bears an interesting relationship to the idea of admissibility of linear operators and the results of Corduneanu and Miller [4], [10], [12]. Lemma 1 is independent of the abstract results of Miller in [10] and the admissibility approach in general in that it deals with closed operators on spaces which are not necessarily complete as opposed to continuous linear operators on Banach spaces.

1. An operator equation. Consider the equation

$$Lx = f(x), \tag{1.1}$$

where L is a linear operator with domain $D(L)$ and range $R(L)$ contained in Banach spaces X and Y respectively and where $f(x)$ is an operator from X into Y. Let $N(L)$ denote the null space of L. Our goal is to obtain a correspondence between solutions of (1.1) and $N(L)$ by making suitable assumptions on L and f. First, we assume the following.

H1. L is closed on $D(L)$ and there exists a subspace $S \subseteq D(L)$ such that the restriction of L to S (denoted L_s) is closed and one-to-one and has closed range.

Here the domain $D(L)$ and the subspace S are purposively not assumed to be complete as this will be the case in our applications to (P) below. Let $R(L_s)$ be the range of L_s and set $\Sigma(r) = \{x \in X : |x|_X \leqq r\}$. Concerning the operator f we assume, without loss of generality, that $f(x) = h(x) + g$ where $h(0) = 0$ and $g \in Y$; in addition we assume the following hypothesis.

H2. h maps $D(L)$ into $R(L_s)$ continuously in such a way that for some constants θ and r, $0 \leqq \theta < +\infty$, $0 < r \leqq +\infty$, we have $|h(x) - h(y)|_Y \leqq \theta |x - y|_X$ for all $x, y \in D(L) \cap \Sigma(r)$.

Under H1 it follows by the closed graph theorem that L_s has a bounded inverse L_s^{-1}. From this we can conclude the following basic lemma.

LEMMA 1. *Suppose* H1 *and* H2 *hold. Suppose also that* θ *in* H2 *satisfies* $\theta|L_s^{-1}| < 1$. *Then there exists a constant* $c > 0$ *such that for each* $g \in R(L_s)$ *satisfying* $|g|_Y \leqq cr$, *a one-to-one bicontinuous mapping* Q *exists from the set* $N(L) \cap \Sigma(cr)$ *into the set of solutions of* (1.1) *contained in* $D(L) \cap \Sigma(r)$.

Proof. We first show that Q is well-defined. Given $n \in N(L) \cap \Sigma(cr)$, define the operator $T: D(L) \to D(L)$ by $Tx \equiv n + L_s^{-1} f(x)$. For $x \in \Sigma(r)$ it follows from H2 (with $y = 0$) that

$$|Tx|_X \leqq [c(1 + |L_s^{-1}|) + |L_s^{-1}|\theta] r.$$

Thus, if we choose $c < (1 - |L_s^{-1}|\theta)(1 + |L_s^{-1}|)^{-1}$, then T maps $D(L) \cap \Sigma(r)$

into itself. Moreover, by H2 we have $|Tx - Ty|_Y \leqq |L_s^{-1}|\theta|x - y|_X$ and hence T is a contraction on $D(L) \cap \Sigma(r)$. Choosing $x_1 \in D(L) \cap \Sigma(r)$ and setting

$$x_n = Tx_{n-1} \in D(L) \cap \Sigma(r), \quad n \geqq 2,$$

we know that $x_n \to x_0$ in X and that $f(x_n) \to f(x_0) \in R(L_s)$ in Y (by H2). Since $Lx_n = LTx_{n-1} = LL_s^{-1}f(x_{n-1}) = f(x_{n-1}) \to f(x_0)$ in Y and since L is closed on $D(L)$, it follows that the point x_0 lies in $D(L) \cap \Sigma(r)$ and $Lx_0 = f(x_0)$; that is, x_0 solves (1.1) and we have a well-defined function $Q: Q(n) = x_0$.

Suppose $n_i \in N(L) \cap \Sigma(cr)$ for $i = 1, 2$ and $Q(n_1) = Q(n_2) = x$; i.e., $x = n_i + L_s^{-1}f(x)$ for $i = 1, 2$. By subtraction we find that $n_1 = n_2$ and that Q is consequently one-to-one. Finally, Q and Q^{-1} are continuous as is shown by the following inequalities:

$$\begin{aligned}|Q(n_1) - Q(n_2)|_X &= |x_1 - x_2|_X = |Tx_1 - Tx_2|_X \leqq |n_1 - n_2|_X + |L_s^{-1}|\theta|x_1 - x_2|_X \\ &= |n_1 - n_2|_X + |L_s^{-1}|\theta|Q(n_1) - Q(n_2)|_X\end{aligned}$$

or

$$|Q(n_1) - Q(n_2)|_X \leqq (1 - |L_s^{-1}|\theta)^{-1}|n_1 - n_2|_X;$$

and

$$\begin{aligned}|Q^{-1}(x_1) - Q^{-1}(x_2)|_X &= |n_1 - n_2|_X = |x_1 - Tx_1 - x_2 + Tx_2|_X \\ &\leqq |x_1 - x_2|_X + |L_s^{-1}|\theta|x_1 - x_2|_X \\ &= (1 + |L_s^{-1}|\theta)|x_1 - x_2|_X. \qquad \square\end{aligned}$$

Remarks. (a) If $R(L)$ is closed in Y and $N(L)$ admits a projection, then we may take $S = M$ in H1 and Lemma 1 where $D(L) = N(L) \oplus M(L)$. Then L_s^{-1} is the pseudo-inverse of L on $D(L)$ and, by the closed graph theorem, L is continuous on $D(L)$. These circumstances do not hold, however, in our application to (P) below.

(b) We can further assert that if $|g|_Y \leqq cr/|L_s^{-1}|$, then there exists a constant $r^* > 0$ such that the range of Q contains *all* solutions of (1.1) contained in $D(L) \cap \Sigma(r^*)$. To see this, choose r^* so small that $|x - L_s^{-1}f(x)|_X \leqq cr$ for $|x|_X \leqq r^*$. This is possible by the way g is chosen since $I - L_s^{-1}f$ is continuous. Let $x \in D(L) \cap \Sigma(r^*)$ be a solution of (1.1) and define $n = s - L_s^{-1}f(x)$ which, by the assumption made, lies in $N(L) \cap \Sigma(cr)$. Thus, there exists a unique solution $x' \in D(L) \cap \Sigma(r)$ of (1.1) such that $x' = Q(n)$; i.e., $x' = n + L_s^{-1}f(x')$. But then $x' - x = L_s^{-1}(f(x') - f(x))$ which implies $|x' - x|_X \leqq |L_s^{-1}|\theta|x' - x|_X$ or $x' = x$ in as much as $|L_s^{-1}|\theta < 1$. Hence, x is in the range of Q.

(c) If it is assumed that S is a Banach space and that $R(L)$ is contained in a Banach subspace Y^* of Y, then Theorem 1 can be proved with X and Y taken as Fréchet spaces (instead of Banach spaces) whose respective topologies induce topologies on S and Y^* weaker than their respective norm topologies. In this case L is continuous by the closed graph theorem and one obtains from this modification of Theorem 1 an alternate statement of a theorem of Miller [10, Thm. 1]. Again, however, in our applications S is not complete.

2. Integro-differential equations. We return now to systems (P), (H) and (NH) where we assume the following.

H3. $g(t)$ and $A(t)$ are locally integrable in $t \geqq t_0$ and $B(t, s)$ is locally in L^1 in (t, s), $t \geqq s \geqq t_0$.

Under these conditions, the Volterra integral equation obtained from (NH) by integration has a kernel $k(t, s) = A(s) + \int_s^t B(r, s)\,dr$ for which, it is not difficult to see, conditions sufficient for the existence and uniqueness of a continuous solution for $t \geqq t_0$ (as given by Miller in [12]) are fulfilled for each initial vector $x(t_0) = x_0 \in R^n$ and each $g(t)$. This continuous solution, by virtue of the fact that it solves this integral equation and that $k(t, s)$ has the properties described in H3, is in fact absolutely continuous and consequently is the unique solution of (NH) for $t \geqq t_0$ and $x(t_0) = x_0$. Our goal now is to apply Lemma 1 to the perturbed system (P).

Let

$$BC = \{x(t) \in C[t_0, +\infty) : |x|_0 = \sup_{t \geqq t_0} |x(t)| < +\infty\}$$

and L^p, $1 \leqq p < +\infty$, be the Banach space of functions defined and measurable for $t \geqq t_0$ for which $|x|_p = \int_{t_0}^{+\infty} |x|^p\,dx < +\infty$. For convenience, we let L^∞ also denote BC and $|x|_\infty = |x|_0$. We take $X = BC$ and $Y = L^p$, $1 \leqq p \leqq +\infty$, in Lemma 1. Define the linear operator L by

$$Lx \equiv x' - A(t)x - \int_{t_0}^t B(t, s)x(s)\,ds$$

whose domain we take to be the linear subspace $D^p(L) = \{x \in BC : x(t)$ is absolutely continuous for $t \geqq t_0$ and $Lx \in L^p\}$. (By a solution of (P), (H), or (N) we mean an absolutely continuous function satisfying the corresponding system for almost all $t \geqq t_0$.) Define X_1 to be those vectors in R^n which, as initial conditions at t_0, give rise to bounded solutions of (H); X_1 is clearly a linear subspace of R^n. Let X_2 be any space supplementary to X_1: $R^n = X_1 \oplus X_2$; and let P_i be the projection of R^n onto X_i. In H1 we take S to be $S^p = \{x \in D^p(L) : x(t_0) \in X_2\}$ which is easily seen to be a subspace of $D^p(L)$. In order to fulfill H1 we assume the following.

H4^p. for each $g(t) \in L^p$, $1 \leqq p \leqq +\infty$, there exists at least one bounded solution $z \in BC$ of (NH).

Under this hypothesis the range R^p of L restricted to S^p is all of $Y = L^p$ and hence is closed. For by H4^p, given $g \in Y$ there exists $z \in D^p(L)$ such that $Lx = g$ and if $y(t) \in BC$ is the unique solution of (H) satisfying $y(t_0) = P_1 z(t_0)$, then $x = z - y \in S^p$ and $Lx = g$. Moreover, L is one-to-one on the subspace S^p for if $Lx_1 = Lx_2$ for $x_1, x_2 \in S^p$, then $L(x_1 - x_2) = 0$ and $x_1 - x_2 \in S^p$, which means $y = x_1 - x_2$ is a bounded solution of (H) with initial state in X_2. Since X_2 is supplementary to X_1 it must be that $y = 0$.

Finally all that remains in order to show that H1 is fulfilled is that L is closed on $D^p(L)$ and S^p. To this end suppose $x_n \in D^p(L)$ and $g_n = Lx_n$ converge in BC and L^p respectively to $x^0 \in BC$ and $g^0 \in L^p$. Integrating $g_n = Lx_n$, we have

$$x_n(t) = x_n(t_0) + \int_{t_0}^t \left[A(s) + \int_s^t B(r, s)\,dr\right] x_n(s)\,ds + \int_{t_0}^t g_n(s)\,ds.$$

For fixed, but arbitrary $t \geqq t_0$, we find (using H3 and the dominated convergence theorem) that

$$x^0(t) = x^0(t_0) + \int_{t_0}^{t} \left[A(s) + \int_{s}^{t} B(r, s)\, dr \right] x^0(s)\, ds + \int_{t_0}^{t} g^0(s)\, ds$$

and consequently $x^0(t)$ is absolutely continuous and solves $Lx^0 = g^0$. This proves that L is closed on $D^p(L)$. If on the other hand $x_n(t) \in S^p$, then in addition to $x^0(t) \in D^p(L)$, it is obvious that $x_n(t_0) \in X_2$ implies that $x^0(t_0) \in X_2$ and hence $x^0(t) \in S^p$; i.e., L is also closed on S^p.

Having fulfilled H1, we can assert the conclusion of Lemma 1 for (P) provided the perturbation term h and the nonhomogeneous term $g(t)$ satisfy the necessary conditions. The following two theorems contain our main results.

THEOREM 1. *Suppose* H3 *and* H4^p *hold. Further suppose* $h(t)(x)$ *maps* BC *into* L^p, $1 \leqq p \leqq +\infty$, *in such a way that for some constant* r, $0 < r \leqq +\infty$,

$$|h(t)(x) - h(t)(y)|_p \leqq \theta |x - y|_0 \tag{2.1}$$

holds for all $x, y \in BC$ *satisfying* $|x|_0, |y|_0 \leqq r$. *Then there exist three positive constants* a, b *and* θ^0 *with the following properties: if* $\theta \leqq \theta^0$ *then*

(i) *for every* $g \in L^p$, $|g|_p \leqq a$, *there exists a one-to-one bicontinuous correspondence* Q *between the bounded solutions* $y \in BC$ *of* (H) *satisfying* $|y(t_0)| \leqq b$ *and the bounded solutions* $x \in BC$ *of* (P) *satisfying* $|x|_0 \leqq r$, $|P_1 x(t_0)| \leqq b$; *and*

(ii) *the correspondence* Q *is such that if* $x = Qy$, *then* $P_1 x(t_0) = y(t_0)$.

Proof. For all the bounded solutions $y(t)$ of the linear homogeneous system (H) it is possible to assert that $|y|_0 \leqq M|y(t_0)|$ for some constant $M > 0$. The stated assumption (2.1) on h allows us to apply Lemma 1 in the context described above. Let c be the constant whose existence is guaranteed by Lemma 1 and take $\theta^0 = \frac{1}{2}|L_s^{-1}|^{-1}$ (where $S = S^p$), $a = cr$, and $b = M^{-1}cr$. Given a bounded solution $y(t)$ of (H), $|y(t_0)| \leqq b$ (hence, $y(t_0) \in X_1$), it follows that $|y|_0 \leqq cr$ and by Lemma 1 there exists a unique corresponding solution $x = Qz$ of (P) satisfying $|x|_0 \leqq r$; moreover, Q is invertible and bicontinuous. Referring to the proof of Lemma 1, $x = y + L_s^{-1} f(t)(x)$ and, hence, $P_1 x(t_0) = P_1 y(t_0) = y(t_0)$ since y being bounded implies $y(t_0) \in X_1$ and since $L_s^{-1} f(t)(x) \in S^p$ implies that $L_s^{-1} f(t)(x)$ at t_0 lies in X_2. Finally, Q is onto the set of solutions of (P) as described in the theorem, for if x is such a solution of (P) ($|x|_0 \leqq r$ and $|P_1 x(t_0)| \leqq b$) then we may define $y = x - L_s^{-1} f(t)(x)$ and find that y is a bounded solution of (H) satisfying $|y(t_0)| = |P_1 y(t_0)| = |P_1 x(t_0)| \leqq b$. Hence, $|y|_0 \leqq cr$ and $x' = Qy$ exists. But then $x' = y + L_s^{-1} f(t)(x')$ and hence $x - x' = L_s^{-1}[f(t)(x) - f(t)(x')]$ and $|x - x'|_0 \leqq |L_s^{-1}|\theta^0 |x - x'|_0$ by (2.1). Since $|L_s^{-1}|\theta^0 = \frac{1}{2}$, we conclude $x = x'$ and that Q is onto. □

As a second application of Lemma 1 we consider the question of the existence of bounded solutions of (P) which in addition tend to zero as $t \to +\infty$. We define $BC_0 = \{x \in BC : |x(t)| \to 0 \text{ as } t \to +\infty\}$ and let $X_1^0 \subseteq R^n$ be the linear space of initial vectors at $t = t_0$ which give rise to solutions of (H) in BC_0. Let $X_2^0 \subseteq R^n$ be such that $X_1^0 \oplus X_2^0 = R^n$ and P_i^0 be the projections of R^n onto X_i^0. If we take $X = BC_0$ and $Y = BC_0 \cap L^p$ under the norms $|x|_0$ and $|x|_Y = \frac{1}{2}(|x|_0 + |x|_p)$ respectively, and if we consider L as defined above on the domain $D_0^p(L)$

$= \{x \in BC_0 : x$ is absolutely continuous in $t \geqq t_0$ and $Lx \in BC_0 \cap L^p\}$, then setting $S = S_0^p = \{x \in D_0^p(L) : x(t_0) \in X_2^0\}$ it is not difficult to modify the argument for Theorem 1 to obtain the following.

THEOREM 2. *Suppose* H3 *and* $\mathrm{H4}_0^p$ *hold where*

$\mathrm{H4}_0^p$. *for each* $g(t) \in BC_0 \cap L^p$ *there exists at least one solution* $z(t) \in BC_0$ *of* (H).

If h satisfies the condition (2.1) *with* L^p *replaced by* $BC_0 \cap L^p$, *then the conclusions of Theorem* 1 *hold with* L^p *replaced by* $BC_0 \cap L^p$, BC *replaced by* BC_0, *and* P_1 *replaced by* P_1^0.

Remarks. (a) Inequality (2.1) is satisfied if for example $|h(t)(x) - h(t)(y)| \leqq \theta(t)|x - y|_0, t \geqq t_0$, where $\theta \in L^p$ or $\theta \in BC_0 \cap L^p$. Note also that the hypotheses on h are satisfied when h is "higher order" in x from BC to L^p (or $BC_0 \cap L^p$ to L^p); i.e., if given any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $|h(t)(x) - h(t)(y)|_Y \leqq \varepsilon|x - y|_0$ for all $|x|_0, |y|_0 \leqq \delta$. In this case we simply take $r = \delta(\theta^0)$. Such perturbations appear frequently in the theory of differential, integral and integro-differential systems [1], [3], [6], [7], [9], [11]. As a simple illustration (applicable to our application below) h may have the form

$$h \equiv a(t, x) + \int_{t_0}^{t} k_1(t, s)b(s, x(s))\, ds + \int_{t_0}^{t_1} k_2(t, s)c(s, x(s))\, ds,$$

where $a(t, \xi)$, $k_1(t, \xi)$ and $k_2(t, \xi)$ are all $o(|\xi|)$ uniformly in $t \geqq t_0$ and

$$\sup_{t \geqq t_0} \int_{t_0}^{t} |k_1(t, s)|\, ds < +\infty, \qquad \sup_{t \geqq t_0} \int_{t_0}^{t_1} |k_2(t, s)|\, ds < +\infty.$$

(b) The spaces X_2 and X_2^0 (and hence S^p and S_0^p) are not uniquely determined (unless $X_1 = R^n$ or $X_1^0 = R^n$ in which case $X_2 = \{0\}$ or $X_2^0 = \{0\}$). It is to be noted that the constant c whose existence is asserted by Theorems 1 and 2 depends on X_2 or X_2^0 respectively.

Many recent papers [1], [6], [7], [9], [10], [11] (and the references therein) have dealt with stability properties of integro-differential and Volterra integral equations. Theorems 1 and 2 have implications about the stability or instability of (P). To point this out explicitly we make the following definitions for (P) (and its special case (NH) and (H)). System (P) (or more precisely the zero solution of (P) corresponding to $g \equiv 0$) is called *conditionally stable on* L^p if there exists a set of vectors $M \subseteq R^n$ whose closure contains the origin for which, to any $\varepsilon > 0$, there corresponds a $\delta = \delta(\varepsilon, t_0) > 0$ such that $|g|_p \leqq \delta$, $g \in L^p$, and $x_0 \in M$, $|x_0| \leqq \delta$, implies the solution of (P) satisfying $x(t_0) = x_0$ exists for all $t \geqq t_0$ and satisfies $|x|_0 \leqq \varepsilon$. If, in addition to possessing conditional stability on L^p, (P) has the property that all solutions $x(t)$ corresponding to $x_0 \in M$, $|x_0| \leqq \delta_0$, and $|g|_p = \frac{1}{2}(|g|_p + |g|_0) \leqq \delta_0$, $g \in BC_0 \cap L^p$ for some fixed constant $\delta_0 > 0$ tend to zero as $t \to +\infty$, then (P) is called *conditionally asymptotically stable on* $BC_0 \cap L^p$. If M is an entire n-dimensional sphere in R^n, then (P) is called *stable* or *asymptotically stable* on the corresponding space. (These definitions of stability are special cases of more general definitions given for Volterra integral equations in [1].) System (P) is called *unstable on* L^p if it is not stable on L^p; i.e., if there exists a $\delta^* > 0$ and an $\varepsilon^* > 0$ such that for *every* initial vector x_0, $|x_0| \leqq \varepsilon^*$, and every

$g \in L^p$, $|g|_p \leqq \varepsilon^*$, the corresponding solution of (P) satisfies $|x(t)| > \varepsilon^*$ for some $t \geqq t_0$. Finally, we say that (H) *preserves* a given stability (or instability) property under the perturbation h if (P) has this stability (or instability) property. From Theorems 1 and 2 we can assert the following.

COROLLARY. *Under hypotheses* H3 *and* H4^p *system* (H) *preserves conditional stability, stability and instability on* L^p *for perturbations* h *satisfying the conditions of Theorem* 1. *Under* H3 *and* H$_0^p$, (H) *preserves conditional asymptotic stability and asymptotic stability on* $BC_0 \cap L^p$ *for* h *satisfying the conditions of Theorem* 2.

The case of stability and asymptotic stability preservation (which corresponds to the special case $X_1 = R^n$ or $X_1^0 = R^n$), under these conditions, is known (although proved and usually stated quite differently); see [1], [6], [11]. The preservation of instability and conditional stability is a generalization of known results for differential equations, where $B(t, s) = 0$ [3], [8].

3. The hypotheses H4^p and H4$_0^p$. We wish now to discuss briefly the assumptions H4^p and H4$_0^p$ for the linear system (NH) in order to give some insight into when they are fulfilled and how this can be determined. Further discussion of this question appears in § 4 below for the very important special case when $A(t) \equiv A$ = const. and $B(t, s) \equiv B(t - s)$. If H4^p holds, then given $g(t) \in L^p$, there exists a *unique* bounded solution $z(t)$ of (NH) satisfying $z_0 = z(t_0) \in X_2$; for, if z_1 and z_2 are two such solutions, then $y = z_1 - z_2$ is a bounded solution of (H) with $y(t_0) \in X_2$ and, hence, by the way X_1 and X_2 are defined, it follows that $y(t) \equiv 0$. This then establishes a function from L^p into $X_2 \in R^n$. The Corollary above in § 2 applied (with $h = 0$) to (NH) implies that this function is continuous. For $1 \leqq p < +\infty$ it follows easily from well-known theorems in functional analysis that there exists an $n \times n$ matrix $P(t)$, $|P(t)| \in L^q[t_0, +\infty)$, $q^{-1} + p^{-1} = 1$ for $p \neq 1$ and $q = +\infty$ for $p = 1$, such that

$$z_0 = -\int_{t_0}^{+\infty} P(s)q(s)\,ds. \tag{3.1}$$

The solution of (NH) is given by the variation of constants formula

$$z(t) = Y(t, t_0)z_0 + \int_{t_0}^{t} Y(t, s)g(s)\,ds, \qquad t \geqq t_0, \tag{VC}$$

where $Y(t, s)$ is the so-called fundamental solution matrix (or differential resolvent) of (NH); i.e., Y is the solution of the matrix equation

$$Y_t(t, s) = A(t)Y(t, s) + \int_{s}^{t} B(t, r)Y(r, s)\,dr, \qquad t \geqq s \geqq t_0,$$

$$Y(s, s) = I,$$

$I = n \times n$ identity matrix. This can be seen by straightforward substitution into (NH) (also see [6]). Thus, under H4^p, the unique bounded solution of (NH) with $z_0 \in X_2$ is given by

$$z(t) = -Y(t, t_0)\int_{t_0}^{+\infty} P(s)g(s)\,ds + \int_{t_0}^{t} Y(t, s)g(s)\,ds, \tag{3.2}$$

or

$$z(t) = \int_{t_0}^{t} V(t, s)g(s)\, ds + \int_{t}^{+\infty} W(t, s)g(s)\, ds \tag{3.3}$$

where

$$\begin{aligned} V(t, s) &= Y(t, s) - Y(t, t_0)P(s), \qquad && t_0 \leqq s \leqq t, \\ W(t, s) &= -Y(t, t_0)P(s), && t_0 \leqq t \leqq s. \end{aligned} \tag{3.4}$$

Using standard arguments (see for example [3], where the arguments used for differential equations carry over almost verbatim), one can show that (recall $p \neq +\infty$)

$$\int_{t_0}^{t} |V(t, s)|^q\, ds + \int_{t}^{+\infty} |W(t, s)|^q\, ds \leqq K, \qquad t \geqq t_0, \quad p \neq 1, \tag{3.5}$$

$$|V(t, s)| \leqq K \quad \text{for } t_0 \leqq s \leqq t \quad \text{and} \quad |W(t, s)| \leqq K \quad \text{for } t_0 \leqq t \leqq s, \quad p \neq 1, \tag{3.6}$$

for some constant $K > 0$.

It is the converse of this fact which interests us here. If a $P(t)$ can be found such that (3.5) or (3.6) holds, $1 \leqq q \leqq +\infty$, for W and V defined by (3.4), then given any $g(t) \in L^p$, $1 \leqq p \leqq +\infty$ (including now the case $p = +\infty$ corresponding to $q = 1$) it follows that the function $z(t)$ defined by (3.3) is a bounded solution of (NH) and, hence, H4^p holds. That $z(t)$ is a solution follows from the fact that it can be rewritten in the form (3.2) and that it is bounded follows from a simple application of Hölder's inequality.

THEOREM 3. *If an $n \times n$ matrix $P(t)$ can be found such that* (3.5) *holds for some integer q, $1 \leqq q < +\infty$, for W and V defined by* (3.4), *then* H4^p *holds for p such that $p^{-1} + q^{-1} = 1$ if $q \neq 1$ or $p = +\infty$ if $q = 1$. If* (3.6) *holds, then* H4^p *holds for $p = 1$.*

In the differential equations case ($B \equiv 0$), it turns out that $P(t) = -P_2 Y^{-1}(t)$, where $Y(t)$ is a fundamental solution matrix. Since $Y(t, s) = Y(t)Y^{-1}(s)$ in this case, one easily finds that $W(t, s) = Y(t)P_2 Y^{-1}(s)$ and $V(t, s) = Y(t)P_1 Y^{-1}(s)$ (note: $P_1 + P_2 = I$). The conditions (3.5), (3.6) are, in this case, familiar in the study of bounded solutions [3], [8]. In the autonomous case $A(t) \equiv A$, the projections P_1 and P_2, roughly speaking, "select out" the eigenvalues of A with nonpositive and nonnegative real parts respectively. Thus, H4^p for $p = 1$ is satisfied if those eigenvalues with zero real parts are simple; and, if all eigenvalues of A have nonzero real parts, then H4^p holds for all $1 \leqq p \leqq +\infty$. In the next section § 4 we indicate how these features roughly carry over to (NH) in the case $B \neq 0$ but $B(t, s) \equiv B(t - s)$.

For the hypothesis H4_0^p we have the following result.

THEOREM 4. *Suppose an $n \times n$ matrix $P(t)$ can be found such that, in addition to* (3.5) *for some q, $1 \leqq q < +\infty$, the condition*

$$\int_{t_0}^{T} |V(t, s)|\, ds \to 0 \quad \text{as } t \to +\infty \tag{3.7}$$

for every fixed $T \geqq t_0$. Then H4_0^p *holds for p such that $p^{-1} + q^{-1} = 1$ if $q \neq 1$ or*

$p = +\infty$ if $q = 1$.

If (3.6) *holds and in addition*

$$|V(t, s)| \to 0 \quad \text{as } t \to +\infty \tag{3.8}$$

for every $s \geqq t_0$, then $\mathrm{H4}_0^p$ *holds for $p = 1$.*

Proof. If (3.5) holds for $q \neq 1$, then $z(t)$ defined by (3.3) (or by (3.2)) is a bounded solution of (NH), as pointed out above, for each $g \in BC_0 \cap L^p$. We want to show in addition that $|z(t)| \to 0$ as $t \to +\infty$. Given any $\varepsilon > 0$ choose $T = T(\varepsilon) > 0$ so large that $(\int_T^{+\infty} |g|^p\, ds)^{1/p} \leqq \varepsilon/2K$. Then from

$$z(t) = \int_{t_0}^{T} V(t, s)g(s)\, ds + \int_{T}^{t} V(t, s)g(s)\, ds + \int_{t}^{+\infty} W(t, s)g(s)\, ds$$

for $t \geqq T$, follows (after an application of Hölder's inequality and (3.6)) that

$$|z(t)| \leqq \int_{t_0}^{T} |V(t, s)|\, ds\, |g|_0 + \varepsilon$$

which implies, upon letting $t \to +\infty$, that $\limsup_{t \to +\infty} |z(t)| \leqq \varepsilon$. Inasmuch as $\varepsilon > 0$ was arbitrary it follows that $|z(t)| \to 0$ as $t \to +\infty$. The case $p = +\infty$ is similar.

If (3.6) holds, then we have

$$|z(t)| \leqq \int_{t_0}^{T} |V(t, s)|\, |g(s)|\, ds + \varepsilon,$$

where $T \geqq t_0$ is chosen so that $\int_T^{+\infty} |g|\, ds \leqq \varepsilon/2K$. Using the dominated convergence theorem and (3.8) we again conclude that $|z(t)| \to 0$ as $t \to +\infty$. □

4. The convolution case and an application. Suppose $A(t) \equiv A =$ const. and $B(t, s) \equiv B(t - s)$ in (NH). Take $t_0 = 0$. Most applications of (P) have this form so it is important to develop techniques for testing $\mathrm{H4}^p$ and $\mathrm{H4}_0^p$ in this case; specifically, we wish to determine some conditions on A and B under which $\mathrm{H4}^p$ and $\mathrm{H4}_0^p$ hold for $p = 1$ and $+\infty$. In this case $Y(t, s) = Y(t - s)$, where $Y(t)$ solves the matrix equation

$$\begin{aligned} &Y'(t) = AY(t) + \int_0^t B(t - r)Y(r)\, dr, \\ &Y(0) = I. \end{aligned} \tag{4.1}$$

We assume from now on that $B \in L^1[0, +\infty)$. Letting * denote the Laplace transform, we have from (4.1) the equation

$$(sI - AB^*(s))Y^*(s) = I \tag{4.2}$$

for Y^*. A straightforward application of Gronwall's lemma to the equivalent integral equation for (4.1) together with the assumption $B \in L^1$ implies Y is exponentially bounded. A necessary and sufficient condition for what Miller [7], [9], [11] calls uniform asymptotic stability of (NH) is that

$$p(s) \equiv \det(sI - A - B^*(s)) \neq 0 \quad \text{for } \operatorname{Re} s > 0. \tag{4.3}$$

In terms of our hypotheses above, this result supplies necessary and sufficient conditions for the stability of (NH) on $L^\infty = BC$ (i.e., for the validity of hypothesis H4$^\infty$ in its strongest form: *all* solutions are bounded for each choice of g). This is because, as shown in [7], this condition implies $Y \in L^1$ and hence the stability on BC (see (VC)). For conditional stability and instability we expect the eigenvalue-like condition (4.2) to be relaxed in such a way as to allow for roots in the right half-plane.

Let us suppose then that $p(s)$ has roots in the right half-plane. We do not intend to study this situation in depth here, but instead to restrict our attention to remarks appropriate to our application below. See [13] for a more extensive study of this problem. Let r be any root of $p(s)$. We say that r has *algebraic multiplicity* $\mu \geqq 1$ if $p^{(i)}(r) = 0$ for $i = 0, 1, \cdots, \mu - 1, p^{(\mu)}(r) \neq 0$, and has *geometric multiplicity* $m \geqq 1$ if the $n \times n$ matrix $sI - A - B^*(s)$ at $s = r$ has rank $n - m$.

Let k, $0 \leqq k < +\infty$, be the number of roots of $p(s)$ such that $\operatorname{Re} s \geqq 0$. For simplicity assume all of these roots r_p^+, $1 \leqq p \leqq k$, have algebraic multiplicity $\mu = 1$. Let r_p^-, $p = 1, 2, \cdots$, denote the remaining roots (which may be infinite in number); $\operatorname{Re} r_p^- < 0$. Set $Y(t) = [y_{ij}(t)]$, $e_i = \operatorname{col}(\delta_{ij})$, and $p_{ij}(s)$ equal to the cofactor of the ijth entry of the matrix $sI - A - B^*(s)$. Solving (4.2) for the jth column of $Y^*(s)$ using Cramer's rule, we find $y_{ij}^*(s) = p_{ij}(s)/p(s)$, $1 \leqq i, j \leqq n$. We now assume (i) $B^*(s)$ is meromorphic in the entire complex plane and (ii) the estimate $|p_{ij}(s)/p(s)| \leqq K/|s|^\alpha$ holds for some constants K, $\alpha > 0$ and all s, $\operatorname{Re} s \geqq s_0 \geqq \max\{\operatorname{Re} r_p^+\}$. Since Y and hence y_{ij} are exponentially bounded, the complex inversion formula of p_{ij}/p along $\operatorname{Re} s = s_0$ exists and represents $Y(t)$, $t > 0$ [2, p. 183]. Also (ii) guarantees the validity of the residue series expansion for $y_{ij}(t)$ [2, p. 193]. Thus, if $\rho_{ij}(r_p^\pm)$ is the residue of $e^{st}p_{ij}(s)/p(s)$ at $s = r_p^\pm$ (for the simple roots $s = r_p^+$, $\rho_{ij}(r_p^+) = e^{r_p^+ t}p_{ij}(r_p^+)/p'(r_p^+)$), then

$$(4.4) \qquad y_{ij}(t) = \sum_{p=1}^{\infty} \rho_{ij}(r_p^-) + \sum_{p=1}^{k} p_{ij}(r_p^+)\, e^{r_p^+ t}/p'(r_p^+), \qquad t > 0,$$

for $1 \leqq i, j \leqq n$. The residues $\rho_{ij}(r_p^-)$ are all of the order $t^{\mu_p - 1}\, e^{r_p^- t}$. We can write then $Y(t) = Y^-(t) + Y^+(t)$, where $Y^-(t) = \sum_{p=1}^{\infty} \rho_{ij}(r_p^-)$ and $Y^+(t) = \sum_{p=1}^{k} \gamma_{ij}^p\, e^{r_p^+ t}$ with $\gamma_{ij}^p = p_{ij}(r_p^+)/p'(r_p^+)$.

Now we wish to construct $P(s)$ as in Theorem 3. Referring to (3.4) we have

$$V(t,s) = [Y^-(t-s) - Y^-(t)P(s)] + [Y^+(t-s) - Y^+(t)P(s)],$$
$$W(t,s) = -[Y^-(t)P(s) + Y^+(t)P(s)].$$

If we choose P such that

$$(4.5) \qquad Y^+(t-s) - Y^+(t)P(s) = 0$$

(and only if we do this), then as we will point out below, V and W will satisfy (3.5) with $q = 1$ if none of the r_p^+ lie on the imaginary axis and (3.6) in case some r_p^+ are on the imaginary axis.

To solve (4.5) for P we first consider the equation

$$(4.6) \qquad [\gamma_{ij}^p]P(s) = [\gamma_{ij}^p]\, e^{-r_p^+ s}$$

for fixed p, $1 \leqq p \leqq k$. Now $sI - AB^*(s)$ is singular at $s = r_p^+$ which implies that the sum of the products of the cofactors of any row times the corresponding elements of any row is zero. This means all rows of $r_p^+ I - A - B^*(r_p^+)$ are in the kernel of $[\gamma_{ij}^p]$ which in turn implies that the nullity of $[\gamma_{ij}^p]$ is greater than or equal to the rank, $n - m_p$, of $r_p^+ I - A - B^*(r_p^+)$, where $m_p \geqq 0$ is the geometric multiplicity of r_p^+. Thus, $rk[\gamma_{ij}^p] = n -$ nullity $\leqq n - (n - m_p) = m_p$. Assume now that all roots r_p^+ in the right half-plane also have geometric multiplicity one: $m_p = 1$ for all $p = 1, \cdots, k$. Then $rk[\gamma_{ij}^p] \leqq 1$. It is obvious in (4.6) that $[\gamma_{ij}^p]$ and the augmented matrix $[\gamma_{ij}^p | \gamma_{ij}^p e^{-r_p^+ s}]$ have the same rank. Thus, if $p_q(s)$ is the qth column of $P(s)$, we can eliminate all but one equation for each p from (4.6) and obtain a $k \times n$ system to be solved for p_q. This can be done for every column $1 \leqq q \leqq n$ of $P(s)$. Notice that $P(s)$ is a linear combination of exponents $e^{-r_p^+ s}$ and thus $P(s) \in BC$ or L^1 depending on whether or not $\operatorname{Re} r_p^+ = 0$ for some p. Moreover we have

$$V(t, s) = Y^-(t - s) - Y^-(t)P(s), \qquad W(t, s) = -[Y^-(t)P(s) + Y^+(t)P(s)],$$

and it is easily seen from the properties of $P(s)$ and Y^- that (3.6) (or (3.5) with $q = 1$) holds depending on whether $\operatorname{Re} r_p^+ = 0$ for some p (or not).

As an application of this approach and the perturbation Theorems 1 and 2 we consider the system

$$N_1' = N_1(e_1 - \gamma_1 N_2),$$
$$N_2' = N_2\left(-e_2 + \gamma_2 N_1 + \int_{-\infty}^{t} b(t - s) N_1(s)\, ds\right),$$

where $e_i > 0, \gamma_1 > 0$ and $\gamma_2 \geqq 0$ are constants and where $b(t) \geqq 0, b \in L^1[0, +\infty)$. This system is Volterra's model of a predator-prey population with hereditary effects [14, Chap. 4]. Besides initial conditions for N_1 and N_2 at $t_0 = 0$, it is assumed that N_1 is known for $t \in (-\infty, 0]$; say $N_1(t) = N_1^0(t)$. Defining $K_1 = e_1/\gamma_1$, $K_2 = e_2(\gamma_2 + |b|_1)^{-1}$, $x_1 = \log(N_1/K_1)$, $x_2 = \log(N_2/K_2)$ and $x = \operatorname{col}(x_1, x_2)$, we find that this system transforms to (P) with

$$A = \begin{bmatrix} 0 & -\gamma_1 K_2 \\ \gamma_2 K_1 & 0 \end{bmatrix}, \qquad B(t - s) = \begin{bmatrix} 0 & 0 \\ K_1 b(t - s) & 0 \end{bmatrix},$$

$$h(t)(x) = \begin{bmatrix} -\gamma_1 K_2(q(x_2) - x_2) \\ \gamma_2 K_1(q(x_1) - x_1) + K_1 \int_0^t b(t - s)(q(x_1) - x_1)\, ds \end{bmatrix},$$

$$g(t) = \begin{bmatrix} 0 \\ K_1 \int_{-\infty}^{0} b(t - s) q(N_1^0)\, ds \end{bmatrix},$$

where $q(x) = e^x - 1$. Observe that $|h(t)(x)| \leqq L|x|_0^2$ for all $x \in BC$ and some constant $L > 0$. Thus, h maps $\Sigma(r) \cap L^\infty$ $(= BC)$ into BC in such a way that (2.1) holds for small θ $(\leqq \theta^0)$ provided r is small; that is, the hypotheses of Theorems 1 and 2 for the perturbation term h are fulfilled with $p = +\infty$. (Note that $|g|_0$ is small if $|N_1^0|_0$ is small since $b \in L^1$.) Thus, to draw the conclusions of Theorems 1 and 2 and the Corollary we need only verify H4$'$ and H4$_0''$ respectively (H3 is obviously fulfilled). This we will do by using Lemma 2 above. We will only consider the following case: $b(t) = (\alpha t + \beta)e^{-\delta t}$ for $\alpha \geqq 0$, $\beta \geqq 0$, $\delta > 0$ and $\alpha^2 + \beta^2 \neq 0$. The ecological interpretation of $b(t)$ can be found in Volterra's original work [14]: $b(t) = b_1(t)b_2(t)$, where $b_1(t)$ is the probability of a predator born at time $t = 0$ surviving to time t and $b_2(t)$ is the expected number of offspring (per predator) per unit time born to the population of predators at time t per unit encounter with prey at time $t = 0$. The case $\alpha = \delta = 0$ was considered numerically by Davies in [5], where instability was found. However, this case (where $b(t) \equiv$ const.) is not particularly realistic in view of the physical interpretation of $b(t)$ (and also not amenable to our analysis since we demand, as does Volterra, that $b \in L^1$). The case $\alpha = 0$, $\delta > 0$ falls into the case of monotonically non-increasing kernels $b(t)$ considered by Miller [11]; however, his results appear to have a mistake (Corollary 3 on p. 264 seems to be false) and are in fact contradicted by our results below. This case is more reasonable than the one considered by Davies in that the hereditary effects represented by $b(t)$ decrease monotonically with time. Perhaps an even more reasonable case is $\alpha \neq 0$, where the full measure of the hereditary effects on predator births due to past encounters with prey are not instantaneously felt, but gradually increase to a maximum before decreasing monotonically to zero with time.

Proceeding as above, we must investigate the matrix $sI - A - B^*(s)$ which in this application is

$$\begin{bmatrix} s & \gamma_1 K_2 \\ -\gamma_2 K_1 - K_1 b^*(s) & s \end{bmatrix},$$

where $b^*(s) = (\beta s + \alpha + \beta\delta)/(s + \delta)^2$. Obviously $b \in L^1$ and $b^*(s)$ is meromorphic in the complex s-plane. Now $p(s) = s^2 + a_1 a_2 + a_2 b^*(s)$ or

$$p(s) = \frac{n(s)}{(s + \delta)^2}, \tag{4.7}$$

$$n(s) = s^4 + 2\delta s^3 + (\delta^2 + a_1 a_2)s^2 + (2\delta a_1 a_2 + a_2\beta)s + \delta^2 a_1 a_2 + a_2(\alpha + \beta\delta),$$

where $a_1 = \gamma_2 \geqq 0$, $a_2 = \gamma_1 K_1 K_2 > 0$. It is not difficult to check that condition (ii) above holds (with $\alpha = 1$). Thus, we have only to investigate the roots of $p(s)$ lying in the right half-plane; these roots coincide with those of the numerator $n(s)$ in (4.7). An application of the Hurwitz criteria to $n(s)$ shows that not all roots lie in the left half-plane (the third Hurwitzian determinant is negative for $\beta^2 + \alpha^2 \neq 0$ and zero for $\beta = \alpha = 0$). Moreover, making the substitution $\bar{s} = -s$ in $n(s)$ we find that $n(\bar{s})$ also cannot have all of its roots in the left half-plane (since the coefficient of $\bar{s}^3$ is $-2\delta < 0$); i.e., $n(s)$ cannot have all of its roots in the right half-plane. Finally it is easy to check that $n(s)$ has no roots on the imaginary axis nor on the positive real axis. Hence, we conclude that $n(s)$ has two roots in the left half-plane and $k = 2$ complex conjugate roots in the right half-plane both of algebraic multiplicity one. Thus, $p(s)$ has two such roots in the right half-plane. Whether the roots of n in the left half-plane are roots of p depends on whether either of these roots equals $-\delta$ or not. It is not difficult to investigate $n(s)$ at $s = -\delta$. We find that (a) $p(s)$ has two conjugate roots ($\neq \delta$) in the left half-plane if $\alpha \neq 0$ or (b) $p(s)$ has one negative root of multiplicity one ($\neq \delta$) if $\alpha = 0$. Since $n = 2$ and $sI - A - B^*(s)$ at any of these roots is singular but not identically zero, the geometric multiplicity of all (and in particular the two roots in the right half-plane) is also one. One can verify that the system (4.6) reduced by elimination of all but one equation for each p is solable for the matrix P. Hence, the linearized, nonhomogeneous system for this example satisfies both $\mathrm{H4}^\infty$ and $\mathrm{H4}_0^\infty$. This means, by the preservation Theorems 1 and 2, that the *nonlinear Volterra model above preserves the instability of its linearized system.* To be more specific about this instability we must determine X_1. The general solution of the linearized system is $Y(t)y_0$, where $Y(t)$ is given by (4.4), $n = 2$. Clearly, y_0 can be chosen so that $Y(t)y_0 \in BC$ or BC_0 if and only if $[p_{ij}(r_1^+)]y_0 = [p_{ij}(r_2^+)]y_0 = 0$. Recalling the definition of p_{ij} and that $r_2^+ = \bar{r}_1^+$ in our example and referring to A above, we can show without difficulty that the only solution to these simultaneous systems is $y_0 = 0$; i.e., $X_1 = \{0\}$ and no solutions of the linear system exist in BC or BC_0 (except the zero solution). This means (cf. Theorem 1) that there exist constants r, b', and $a > 0$ such that for any $g \in BC$, $|g|_0 \leqq a$, all solutions of the Volterra system (except $x \equiv 0$) satisfying $|x(t_0)| \leqq b'$ must satisfy $|x(t)| > r$ for some $t > t_0$. Or, in other words, if the initial size of the prey population $N_1^0(t)$ is small enough, then no solution $N_1(t)$, $N_2(t)$ exists to Volterra's model which for all t remain close to the "critical points" K_1, K_2 respectively, no matter how close the initial populations $N_1(0)$, $N_2(0)$ are taken to K_1, K_2 respectively.

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CONTINUOUS DEPENDENCE OF SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS*

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Abstract. The nonlinear Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds$$

is considered. We discuss topologies on the collection of functions g such that the solution of the equation varies continuously with the data g and f, where the topology on f is the uniform convergence on compact intervals. We give a necessary and sufficient condition (on such a topology) for the continuous dependence to hold. In a particular case where a Lipschitz condition is added we show that there exists a smallest topology which satisfies the condition, and characterize it.

1. Introduction. We consider the nonlinear Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds, \tag{E}$$

where t belongs to a half open (possibly unbounded) interval $I = [0, c)$, and x, f, and g have values in E_n, the n-dimensional Euclidean space. We assume that f is continuous and that g satisfies assumptions which are somewhat weaker but similar to those given in Miller's book [6]. The assumptions will be given in § 2. We will be concerned with the problem of how the solutions $x(t)$ of (E) vary with changes of the data f and g. The topology on the functions f will be the topology of uniform convergence on compact intervals. In his book [6], Miller defines a topology on the functions g which, briefly speaking, implies that the solutions of (E) vary continuously with respect to these topologies on f and g. In [5], Kelley proved the same type of continuity, with the same topology but under less restrictive assumptions.

In this paper we discuss possible topologies on the functions g. The main result is obtaining necessary and sufficient conditions for the continuous dependence to hold. The topology given by Miller satisfies (of course) the conditions but it is not the smallest topology with this property. We show that in general, there is no smallest topology.

We shall deal with subcollections of the functions g. Some properties of such a collection and topologies on it will be given in § 3, while § 8 is devoted to more discussion of the related literature.

The main theorem will be presented in § 4.

Uniqueness of the solution of (E) is not assumed. In § 5, we give a reformulation of the main theorem in terms of continuous dependence of the set of solutions. Another version of the theorem is its formulation as a theorem of continuous dependence on parameters. This will be done in § 6.

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If a Lipschitz condition is added, then the conditions for convergence have a simpler form that generalizes a theorem for ordinary differential equations. Again, we are able to show that the conditions are also necessary conditions and here a smallest topology does exist. We will do it in § 7.

In Appendix A, an example which answers a question by Miller is constructed. A proof of the inexistence of the smallest topology is postponed to Appendix B.

2. Assumptions and preliminary results. We denote by $|x|$ the norm of x in E_n. The space $C[0, b]$ is the space of E_n-valued continuous functions on $[0, b]$ with the sup norm, i.e., if ϕ belongs to $C[0, b]$, then its norm is

$$\|\phi\| = \sup\{|\phi(t)| : 0 \leqq t \leqq b\}.$$

Recall that $I = [0, c)$ is the domain of the variables t and s. We will assume that the function g in equation (E) satisfies the following.

(G1) g maps $I \times I \times E_n$ into E_n, g is measurable in s and continuous in x, and (without loss of generality) $g(t, s, x) = 0$ if $t < s$.

(G2) For each $b \in I$ and each positive N there is a function $m(t, s)$, integrable in s for a fixed t, such that $|g(t, s, x)| \leqq m(t, s)$ if $|x| \leqq N$ and $0 \leqq t \leqq b$.

(G3) For each $b \in I$ and each positive N, if $t \in I$ then the following expression

$$\sup\left\{\left|\int_0^b (g(t, s, \phi(s)) - g(\tau, s, \phi(s)))\, ds\right| : \phi \in C[0, b], \|\phi\| \leqq N\right\}$$

tends to zero when τ tends to t.

Our assumptions differ from hypotheses (H2)–(H4) which were used by Miller [6] (and Kelley [5]) in two respects. Miller assumes that the function $m(t, s)$ in (G2) satisfies $\sup\{\int_0^t m(t, s)\, ds : 0 \leqq t \leqq b\} < \infty$. He also states (H4), (which is similar to (G3)), with the integration of the absolute value rather than the absolute value of the integral. However, the author (of this paper) checked the proofs of Theorems 1.1 and 2.2 in [6, Chap. II] and found that essentially only assumptions (G1)–(G3) were used. Therefore, we have from [6] the following information about equation (E), when f is continuous and g satisfies (G1)–(G3).

(i) Equation (E) admits a local solution, i.e., a $b > 0$ and a continuous function $x(t)$ on $[0, b]$ exist such that $x(t)$ satisfies (E) for $0 \leqq t \leqq b$.

(ii) If $x(t)$ is a solution on $[0, b)$ where $b < c = \sup I$, and $x(t)$ is bounded on $[0, b)$, then $x(t)$ can be extended to a continuous solution on $[0, d]$ where $b < d < c$. Therefore, if $x(t)$ is a solution on $[0, b)$ which cannot be extended, then either $b = c$ or $\limsup_{t \to b-} |x(t)| = \infty$.

In Appendix A, we shall construct an example where the lim sup in (ii) cannot be replaced by lim. The example will satisfy also (H1)–(H4) and (H7) of [6]. Thus it is an answer to the open problem [6, Problem 15, p. 145].

3. The space $\mathscr{G}$ and jointly continuous topologies on it. We shall be concerned with subcollections of the collection of all functions g that satisfy (G1)–(G3). Briefly speaking each subcollection is formed by considering (G3) being fulfilled uniformly.

DEFINITION 3.1. Let $U(t, \tau, b, N)$ be a function defined on t, τ, b in I and positive numbers N, with values in $[0, \infty]$ such that

$$\lim_{\tau \to t} U(t, \tau, b, N) = 0.$$

The space $\mathscr{G}$ associated with U is the collection of all functions g that satisfy (G1), (G2) and the following uniform version of (G3):

(G3, U) If $\phi \in C[0, b]$ and $\|\phi\| \leqq N$, then

$$\left| \int_0^b (g(t, s, \phi(s)) - g(\tau, s, \phi(s)))\, ds \right| \leqq U(t, \tau, b, N).$$

DEFINITION 3.2. A topology $\mathscr{T}$ defined on $\mathscr{G}$ is *jointly continuous* if for every fixed t the mapping

$$(g, \phi) \to \int_0^t g(t, s, \phi(s))\, ds$$

from $\mathscr{G} \times C[0, t]$ to E_n is continuous with respect to the product topology of $\mathscr{T}$ and the sup-norm in $C[0, t]$.

We shall see in the coming section that a necessary and sufficient condition for the continuous dependence to hold is that the topology on the functions g will be jointly continuous. The natural question arises, namely, is there a smallest jointly continuous topology? The answer is the following.

PROPOSITION 3.3. *There is no smallest jointly continuous topology on* $\mathscr{G}$.

The proof is postponed to Appendix B.

4. The main result. Equation (E) was defined in the Introduction. If the functions f and g appear with an index, i.e., f_k and g_k, we shall use $(E)_k$ for

$$(E)_k \qquad x(t) = f_k(t) + \int_0^t g_k(t, s, x(s))\, ds.$$

Recall that the convergence of the functions f is the uniform convergence on compact subintervals of I. We shall use the Moore–Smith convergence of nets (sometimes called generalized sequences). Of course a simple sequence will do as well. A reference for convergence and nets is Kelley [4].

THEOREM A. *Let* $\mathscr{T}$ *be a topology on* $\mathscr{G}$. *A necessary and sufficient condition for the following property,*(C) *to hold is that* $\mathscr{T}$ *is jointly continuous.*

(C) *Suppose that the net* g_k *converges to* g *in the topology* $\mathscr{T}$. *Then for every net* f_k *converging to* f *the following holds. Let* $x_k(t)$ *be a maximally defined solution of* $(E)_k$. *Then there exist a maximally defined solution* $x(t)$ *of* (E), *with domain* $[0, \alpha)$, *and a subnet* $x_m(t)$ *of* $x(t)$ *such that* $x_m(t)$ *converges to* $x(t)$ *uniformly on compact subintervals of* $[0, \alpha)$. *In particular, if* $[0, \alpha_m)$ *is the domain of* $x_m(t)$, *and if* $0 < d < \alpha$, *then for* m *large enough*, $d \leqq \alpha_m$.

Proof. (a) *The sufficiency part.* Let α be defined by

$$\alpha = \sup \{d: \text{For } k \geqq k_0 \text{ the functions } x_k(t) - f_k(t) \text{ are defined and equicontinuous on } [0, d]\}.$$

First, we show that $0 < \alpha$. Let $b \in I$. Since $f_k(t)$ converge to $f(t)$ uniformly on $[0, b]$ it follows that a bound N exists such that $|f_k(t)| \leqq N$ for every large k, and every $t \in [0, b]$. Let d be such that $U(0, \tau, b, N + 1) < 1$ if $0 \leqq \tau \leqq d$. We claim that $x_k(t)$ exists and satisfies $|x_k(t)| < N + 1$ on $[0, d]$. If not then let $\tau \in [0, d]$ be the smallest τ such that $|x_k(\tau)| = N + 1$. Then $|x_k(s)| \leqq N + 1$ for $0 \leqq s \leqq \tau$ and the following inequality leads to a contradiction:

$$\begin{aligned} |x_k(\tau)| &\leqq |f_k(\tau)| + \left| \int_0^\tau g_k(\tau, s, x_k(s))\, ds \right| \\ &\leqq N + U(0, \tau, b, N + 1) < N + 1. \end{aligned}$$

Since $x_k(t)$ exist on $[0, d]$ and are uniformly bounded there by $N + 1$ it follows that for t, τ in $[0, d]$,

$$\begin{aligned} &|x_k(t) - f_k(t) - (x_k(\tau) - f_k(\tau))| \\ &\leqq \left| \int_0^d (g_k(t, s, x_k(s)) - g_k(\tau, s, x_k(s)))\, ds \right| \leqq U(t, \tau, d, N + 1) \end{aligned}$$

and this implies the equicontinuity of $x_k(t) - f_k(t)$ on $[0, d]$.

Next, we show that a subnet of $x_k(t)$, say $x_l(t)$, exists such that either for every l the function $x_l(t)$ is not defined on $[0, \alpha]$ or every subnet of $x_l(t) - f_l(t)$ is not equicontinuous on $[0, \alpha]$. Suppose that for every k the function $x_k(t)$ is defined on $[0, \alpha]$ and there is no such subnet $x_l(t)$. Since $x_k(0) = f_k(0)$ is bounded it follows that $\{x_k(t)\}$ is a precompact collection in $C[0, \alpha]$. Therefore, it is bounded, say by N. Let $\alpha < b < c$ and suppose that N is also a bound for $|f_k(t)|$ if $0 \leqq t \leqq b$. Let $\alpha < \tau < b$ be such that $U(\alpha, t, b, 3N + 1) < 1$ if $\alpha \leqq t \leqq \tau$. We claim that $x_k(t)$ is defined for $[0, \tau]$ and bounded there by $3N + 1$. The argument is similar to what was used in the preceding paragraph. Let $\alpha < t \leqq \tau$ be the smallest t such that $|x_k(t)| = 3N + 1$. Then

$$\begin{aligned} |x_k(t) - x_k(\alpha)| &\leqq |f_k(t) - f_k(\alpha)| + \left| \int_0^b (g_k(t, s, x_k(s)) - g_k(\alpha, s, x_k(s)))\, ds \right| \\ &< 2N + 1. \end{aligned}$$

This implies $|x_k(t)| < 3N + 1$, a contradiction. Since $x_k(t)$ are bounded on $[0, \tau]$ it follows from (G3, U) that $x_k(t) - f_k(t)$ are equicontinuous on $[0, \tau]$, and the sup property of α is a contradiction to $\alpha < \tau$.

Let $x_l(t)$ be the net given in the last paragraph. A diagonal process will now show the existence of $x(t)$ on $[0, \alpha)$ and a subset of $x_l(t)$ which will converge to $x(t)$ uniformly on every interval $[0, d]$ with $d < \alpha$. To see this let $t_m \to \alpha$. The net $x_l(t) - f_l(t)$ is equicontinuous on every $[0, t_m]$, and bounded there, therefore it is compact in $C[0, t_m]$. Let $x_{l,1} - f_{l,1}$ be a subnet of it which converges on $[0, t_1]$ uniformly to a certain $x(t) - f(t)$. Since $f_l(t)$ converges to $f(t)$ it follows that $x_{l,1}(t)$ converges to $x(t)$. Similarly a subnet $x_{l,2}(t)$ of $x_{l,1}(t)$ exists which converges uniformly on $[0, t_2]$ to an extension of $x(t)$, and denote this extension again by $x(t)$. Inductively, $x_{l,j}(t)$ is a subnet of $x_{l,j-1}(t)$ which converges uniformly to $x(t)$ on $[0, t_j]$. The desired subnet is the diagonal net defined as follows. The net consists of all the elements $x_{l,j}(t)$ for $j = 1, 2, \cdots$. The order is defined by (l_1, j)

$\geqq (l_2, i)$ if $j \geqq i$, and if $l'_1 \geqq l_1$, then the element (l'_1, j) when it appears in the net (l, i) has index (l', i) with $l' \geqq l_2$. (Recall that (l, j) is a subnet of (l, i) if $j \geqq i$.) It is easy to check that this diagonal process defines a net, and that in our case this net converges to $x(t)$ uniformly on compact intervals. (If the nets are actually sequences, then the standard diagonal process will show the existence of a converging subsequence.) Denote the subnet constructed in this paragraph by $x_m(t)$.

We claim that $x(t)$ is a solution of (E) on $[0, \alpha)$. Indeed, if $t < \alpha$, then

$$x_m(t) = f_m(t) + \int_0^t g_m(t, s, x_m(s))\, ds. \tag{4.1}$$

Since $x_m(s)$ converges in $C[0, t]$ to $x(s)$ and since g_m converges to g in the topology $\mathscr{T}$, it follows from the jointly continuous property of $\mathscr{T}$ (Definition 3.2) that the limit of (4.1) is

$$x(t) = f(t) + \int_0^t g(t, s, x(s))\, ds.$$

In order to complete the proof of (a) we have to show that $x(t)$ cannot be extended to $[0, \beta]$, where $\alpha < \beta$. If $\alpha = c$, there is nothing to prove. Suppose that $\alpha < c$ and that $|x(t)|$ is bounded on $[0, \alpha)$, say by N. Assume also that N is a bound for $|f_m(t)|$ when $0 \leqq t \leqq \alpha$. Let $\tau < \alpha$ be such that $\tau \leqq t \leqq \alpha$ implies $U(t, \tau, \alpha, 3N + 3) < 1$. For m large enough, $|x_m(t)| < N + 1$ for all $0 \leqq t \leqq \tau$. An application of the inequality

$$\begin{aligned} |x_m(t) - x_m(\tau)| &\leqq |f_m(t) - f_m(\tau)| + \left| \int_0^\alpha (g_m(t, s, x_m(s)) - g_m(\tau, s, x_m(s)))\, ds \right| \\ &< 2N + 2 \end{aligned}$$

for $\tau \leqq t \leqq \alpha$ and $|x(s)| \leqq 3N + 3$ shows that there is no smallest $t \in [\tau, \alpha]$ such that $|x_m(t)| = 3N + 3$. Since the integral part of the inequality is bounded by $U(t, \tau, \alpha, 3N + 3)$ and since without the absolute values equality holds, it follows that $x_m(t) - f_m(t)$ are equicontinuous on $[0, \alpha]$, in contradiction to the definition of the net $x_l(t)$.

(b) *The necessity.* Let $\mathscr{T}$ be a topology on $\mathscr{G}$ which is not jointly continuous. Then a $t \in I$ exists, and a net $\phi_k(s)$ of continuous functions, that converges uniformly on $[0, t]$ to a function $\phi(s)$ and such that

$$\int_0^t g_k(t, s, \phi_k(s))\, ds \quad \text{do not tend to} \quad \int_0^t g(t, s, \phi(s))\, ds.$$

Let g_l be a subnet of g_k such that

$$\left| \int_0^t (g_l(t, s, \phi_l(s)) - g(t, s, \phi(s)))\, ds \right| \geqq \varepsilon, \tag{4.2}$$

where $\varepsilon > 0$ is fixed. Define the net $f_l(\tau)$ in $C[0, t]$ by

$$f_l(\tau) = \phi_l(\tau) - \int_0^\tau g_l(\tau, s, \phi_l(s))\, ds.$$

We claim that $\{\phi_l(\tau) - f_l(\tau)\}$ is a precompact collection in $C[0, t]$. Its equicontinuity follows from the inequality

$$|\phi_l(\tau_1) - f_l(\tau_1) - (\phi_l(\tau_2) - f_l(\tau_2))| \leqq U(\tau_1, \tau_2, t, N),$$

where N is a bound for $\|\phi_k\|$ in $C[0, t]$. Since $f_l(0) = \phi_l(0)$ is a bounded net, it follows that $\|f_l\|$ is bounded. A bounded and equicontinuous family is precompact. Let $f_m(\tau)$ be a subnet of $f_l(\tau)$ which converges uniformly on $[0, t]$ to a certain function $f(\tau)$. Such a subnet exists since $\phi_l - f_l$ is compact and ϕ_l converges to ϕ. Notice that $\phi_m(\tau)$ is a solution of the equation $(\mathrm{E})_m$. If property (C) holds for the topology $\mathscr{T}$, then it follows that a subnet of $\phi_m(\tau)$ converges uniformly on $[0, t]$ to a solution of (E), but since ϕ_m converges to $\phi(\tau)$ it follows that $\phi(\tau)$ should be a solution of (E). This is not possible since going to the limit where $m \to \infty$ in (4.2) gives

$$\left| f(t) - \phi(t) - \int_0^t g(t, s, \phi(s))\, ds \right| \geqq \varepsilon,$$

a contradiction. Thus property (C) does not hold.

Remark. Notice that if the elements of the net f_k in Theorem A belong to a compact set, then during the proof we could conclude that $x_l(t)$ (and not only $x_l(t) - f_l(t)$) are equicontinuous. A particular case is where we refer only to sequences f_k rather than generalized sequences or nets. In this case, it is enough to deal with topologies which are jointly continuous on compacta, and then the compact-open topology is the smallest topology which has this property. See [4, Thm. 5, Chap. 7].

5. Continuous dependence of the solutions. In the special case where equation (E) in Theorem A has a unique solution, a conclusion of the theorem is that the net $x_k(t)$ converges to this solution uniformly on compact intervals. Without uniqueness we can only conclude that $x_k(t)$ converges to the set of solutions of (E). To make this idea precise we introduce the following distance between two continuous functions. Let $\underset{\sim}{x} = x(t)$ be a continuous function with domain $[0, \alpha)$ and let $\underset{\sim}{y} = y(t)$ be a continuous function. We define

$$d(\underset{\sim}{x}, \underset{\sim}{y}) = \sup \{\min (|x(t) - y(t)|, h(t)) : 0 \leqq t < \alpha\},$$

where $h(t) = \alpha - t$ if $\alpha < \infty$ and $h(t) = 1/(1 + t)$ if $\alpha = \infty$. (If $y(t)$ is not defined for a certain t, then we set $|x(t) - y(t)| = \infty$.) Notice that $d(\cdot, \cdot)$ is not symmetric, i.e., in general $d(\underset{\sim}{x}, \underset{\sim}{y}) \neq d(\underset{\sim}{y}, \underset{\sim}{x})$, but it is a metric on collections of functions with the same domain. Convergence of $d(\underset{\sim}{x}, \underset{\sim}{y}_k)$ to zero is equivalent to convergence of $y_k(t)$ to $x(t)$ uniformly on compact subsets of $[0, \alpha)$. If $\underset{\sim}{X}$ is a collection of functions we denote $d(\underset{\sim}{X}, \underset{\sim}{y}) = \inf \{d(\underset{\sim}{x}, \underset{\sim}{y}) : \underset{\sim}{x} \in \underset{\sim}{X}\}$. The following theorem is an immediate consequence of Theorem A.

THEOREM B. *Let $\mathscr{T}$ be a topology on $\mathscr{G}$. A necessary and sufficient condition for the following property* (C′) *to hold is that $\mathscr{T}$ is jointly continuous.*

(C′) *Suppose that the net g_k converges to g in the topology $\mathscr{T}$. Then for every net f_k converging to f the following holds. Let $\underset{\sim}{x}_k$ be a maximally defined solution of* $(\mathrm{E})_k$ *and let $\underset{\sim}{X}$ be the collection of maximally defined solutions of* (E). *Then $d(\underset{\sim}{X}, \underset{\sim}{x}_k)$ tends to zero when k tends to ∞.*

6. Continuous dependence on parameters. Let Λ be a topological space. For each $\lambda \in \Lambda$ let $f(t, \lambda)$ be a continuous function on $[0, c)$, let $g(t, s, x, \lambda)$ belong to $\mathscr{G}$, and let

$$x(t) = f(t, \lambda) + \int_0^t g(t, s, x(s), \lambda)\, ds. \tag{E$_\lambda$}$$

We shall investigate the continuity of the solutions of $(\mathrm{E})_\lambda$ on the parameter λ. Since uniqueness of the solution is not assumed we shall consider again the distances $d(\underset{\sim}{x}, \underset{\sim}{y})$ and $d(\underset{\sim}{X}, \underset{\sim}{y})$ which were introduced in the preceding section. The following theorem is a reformulation of Theorem B.

THEOREM C. (a) *Suppose that $\mathscr{T}$ is a jointly continuous topology on $\mathscr{G}$. Suppose that the mapping $\lambda \to g(\cdot, \lambda)$ is continuous with respect to $\mathscr{T}$, and that $\lambda \to f(\cdot, \lambda)$ is continuous with respect to uniform convergence on compact sets. If the net λ_k converges to λ, if x_k is a maximally defined solution of* $(\mathrm{E})_{\lambda_k}$ *and if $\underset{\sim}{X}$ is the collection on maximally defined solutions of* $(\mathrm{E})_\lambda$, *then $d(\underset{\sim}{X}, \underset{\sim}{x}_k)$ converge to zero.*

(b) *If $\mathscr{T}$ is not jointly continuous, then the conclusion of* (a) *does not hold.*

The continuity of $\lambda \to g(\cdot, \lambda)$ with respect to a jointly continuous topology has the following characterization which is easily verified.

PROPOSITION 6.1. *Let $\lambda \to g(\cdot, \lambda)$ be a function from a topological space Λ into $\mathscr{G}$. If it is continuous with respect to a jointly continuous topology, then for every t the expression*

$$Q(\phi, \lambda) = \int_0^t g(t, s, \phi(s), \lambda)\, ds$$

is jointly continuous in $\lambda \in \Lambda$ and $\phi \in C[0, T]$.

7. The results with a Lipschitz condition. Suppose that $\mathscr{G}_1$ is a subcollection of $\mathscr{G}$ and that we want to obtain continuous dependence theorems for $\mathscr{G}_1$ of the same form as Theorems A, B and C. Since the proof of Theorem A uses only generalized sequences of the space, it is clear that the theorem holds also when a subcollection $\mathscr{G}_1$ replaces $\mathscr{G}$. Namely, a sufficient and necessary condition for the continuous dependence properties (C), (C′) to hold is that the topology on $\mathscr{G}_1$ will be jointly continuous with respect to the members of $\mathscr{G}_1$. For some subcollections this necessary and sufficient condition has a nice representation. One case is as follows, where a Lipschitz condition is assumed. Moreover, in this case there exists a smallest jointly continuous topology.

Let $K(t, s, N)$ be a real-valued function, defined for $s \leqq t$ in I and for positive numbers N, and such that $K(t, \cdot, N)$ is integrable on $[0, t]$, for fixed t, and N. Let $\mathscr{G}_1$ be the subcollection of $\mathscr{G}$ of the functions g which satisfy $|g(t, s, x) - g(t, s, y)| \leqq K(t, s, N)$ if $t \leqq b$ and $|x|, |y| \leqq N$.

PROPOSITION 7.1. *A topology $\mathscr{T}$ on $\mathscr{G}_1$ is jointly continuous if and only if for every t and every continuous function $\phi(s)$ on $[0, t]$ the expression*

$$\int_0^t g(t, s, \phi(s))\, ds$$

is continuous in g.

Proof. If $\mathscr{T}$ is jointly continuous, then the expression is continuous in (g, ϕ), and therefore for a fixed ϕ it is continuous in g. This completes the "only if" part. In order to show the "if" part let g_k converge to g in $\mathscr{T}$ and let ϕ_k converge to ϕ in $C[0, t]$. We have to show that

$$\int_0^t g_k(t, s, \phi_k(s))\, ds \quad \text{converge to} \quad \int_0^t g(t, s, \phi(s))\, ds.$$

In order to verify this observe that

$$\begin{aligned}\left|\int_0^t (g_k(t, s, \phi_k(s)) - g(t, s, \phi(s)))\, ds\right| &\leqq \left|\int_0^t (g_k(t, s, \phi_k(s)) - g_k(t, s, \phi(s)))\, ds\right| \\ &\quad + \left|\int_0^t (g_k(t, s, \phi(s)) - g(t, s, \phi(s)))\, ds\right| \\ &= I_1 + I_2.\end{aligned}$$

The expression I_2 tends to zero by the convergence in the proposition. The magnitude of I_1 is less than $\|\phi_k - \phi\| \cdot \int_0^t K(t, s, N)\, ds$, where N is a fixed bound for $|\phi_k(s)|$, $0 \leqq s \leqq t$, and therefore also I_1 tends to zero.

As was mentioned before, on the subcollection $\mathscr{G}_1$ defined above, there is a smallest jointly continuous topology. The following theorem also gives a representation of it.

THEOREM 7.2. *Let $\mathscr{G}_1$ be the subcollection of $\mathscr{G}$ which satisfies the above Lipschitz condition. There exists a smallest jointly continuous topology on $\mathscr{G}_1$. This smallest topology is the weak topology induced by the functionals*

$$g \to \int_0^t g(t, s, \phi(s))\, ds,$$

where $t \in [0, \infty)$ and $\phi \in C[0, t]$, namely it is the smallest topology with respect to which all these functionals are continuous. (*A base of this topology is the collection of finite intersection of sets of the form $\{g : \int_0^t g(t, s, \phi(s)\, ds \in Q\}$, where Q is an open set in E_n, $t \in [0, \infty)$ and $\phi \in C[0, t]$.*)

Proof. In view of Proposition 7.1, the continuity of $g \to \int_0^t g(t, s, \phi(s))\, ds$ for $t \in R$ and $\phi \in C[0, t]$ is a characterization for the joint continuity of the topology. Obviously the described weak topology is the smallest one which satisfies this condition.

8. Remarks on the literature. Continuous dependence theorems for differential and integral equations have been extensively studied. See Miller [6] for an extensive bibliography and [6, p. 138] for historical remarks. Continuous dependence without uniqueness in the form of Theorem A appears in [3, Thm. 3.2, p. 14], and this form was also used by Kelley [5] and Miller [6; II, Thm. 4.2]. We formulated Theorem B in order to show that the continuous dependence of the set of solutions has meaning too. The topology used by Miller is defined on all the functions g which satisfy (H2)–(H7) (see [6, Def. 4.1, p. 106]) but it is easy to verify that if g_k converge to g in Miller's topology then almost all the members of the sequence belong to a fixed class $\mathscr{G}$, and moreover the bounds $U(t, \tau, b, N)$

which correspond to g_k converge to the bound that corresponds to g. Jointly continuous topologies were used already in the form of Proposition 6.1, by Hale [2] and Neustadt [7, Thm. 6.7], and in the form of uniform convergence on compact sets by Neustadt [7, p. 154]. The only place known to the author where necessary conditions for continuous dependence were discussed is [1].

Appendix A. We construct here an integral equation on $[0, \infty)$ with a maximally defined solution $x(t)$ with domain $[0, 1)$ such that $|x(t)|$ does not converge to infinity when $t \to 1$. The equation will satisfy (H1)–(H4) and (H7) of [6] (and thus also (G1)–(G3) of § 2), and therefore the equation is an answer to problem 15, p. 145 in [6].

The equation will be a scalar equation. Let $r(t)$ be a real-valued function on $[0, 1)$, continuous, increasing, with $r(0) = 0$ and $r(t) \to \infty$ when $t \to 1$. For instance $r(t) = t/(1 - t)$. Let $x(t)$ be a continuously differentiable function on $[0, 1)$ with the following properties. (i) $x(0) = 0$, (ii) $x(\tau_i) = 0$ for a sequence $\tau_i \to 1$. (iii) $x(\tau_j) \to \infty$ for a sequence $\tau_j \to 1$, (iv) For every $0 < t$ there is a $\tau < t$ such that $x(\tau) > r(t)$. For instance $x(\tau) = (r(\tau) + 1) \sin r(\tau)$. Define $g_1(t, s, y)$ by

$$g_1(t, s, y) = x'(s) \quad \left(= \frac{dx}{d\tau}(s) \right).$$

Then g_1 is continuous in the three arguments if $s < 1$ and $x(t)$ satisfies

$$x(t) = \int_0^t g_1(t, s, x(s))\, ds. \tag{A.1}$$

Define $g_2(t, x, y)$ as follows. If $0 < t < 1$, then

$$g_2(t, s, y) = \begin{cases} \alpha(t)(y - r(t)) & \text{if } y - r(t) \geqq 0, \\ 0 & \text{if } y - r(t) \leqq 0, \end{cases}$$

where $\alpha(t)$ is defined by the equality

$$x(t) = \alpha(t) \int_0^t \max(0, x(s) - r(t))\, ds.$$

Property (iv) implies that $\alpha(t)$ is well-defined, and obviously $\alpha(t)$ is continuous on $(0, 1)$. We now extend g_2 to $1 \leqq t$ by setting

$$g_2(t, s, y) = 0 \quad \text{if } 1 \leqq t.$$

Since $g_2(t, s, y) = 0$ if $t < 1$ and $y \leqq r(t)$, and since $r(t) \to \infty$ when $t \to 1$, it follows that $g_2(t, s, y)$ is continuous in its arguments when $0 < t$. Furthermore, the definition of $\alpha(t)$ implies

$$x(t) = \int_0^t g_2(t, s, x(s))\, ds. \tag{A.2}$$

Let $\theta(t)$ be the function given by

$$\theta(t) = \begin{cases} 0 & \text{if } 0 \leqq t \leqq \frac{1}{4}, \\ 2(t - \frac{1}{4}) & \text{if } \frac{1}{4} \leqq t \leqq \frac{3}{4}, \\ 1 & \text{if } \frac{3}{4} \leqq t. \end{cases}$$

Define $g(t, s, y)$ by

$$g(t,s,y) = (1 - \theta(t))g_1(t,s,y) + \theta(t)g_2(t,s,y).$$

Since $\theta(t)$ is 0 on an interval containing the origin, and $1 - \theta(t)$ is 0 on an interval containing 1, it follows that g is continuous in all its arguments for $t \geqq s$. This implies that g satisfies (G1)–(G3) and as a matter of fact (H2)–(H4) of [6] are satisfied as well (see [6, p. 87]). Also (H7) of [6] is satisfied. To see this, notice that if $\|\phi\| \leqq N$, then

$$\int_t^{t+h} |g(t+h,s,\phi(s))|\,ds \leqq h[N \max\{\alpha(\tau):\tfrac{1}{4} \leqq \tau, r(\tau) \leqq N\} + \max\{x'(s):0 \leqq s \leqq \tfrac{3}{4}\}].$$

The definition of g as a convex combination, together with (A.1) and (A.2) imply

$$x(t) = \int_0^t g(t,s,x(s))\,ds,$$

which means that $x(t)$ is a solution to the equation.

Appendix B.

Proof of Proposition 3.3. We shall use the Moore–Smith convergence as a characterization of a topology (see Kelley [4, Chap. 2]). Let us define the following convergence concept on $\mathscr{G}$.

The net g_k converges to g if for every t, every subnet g_l and every convergent net ϕ_l in $C[0,t]$ with limit ϕ, the net $\int_0^t g_l(t,s,\phi_l(s))\,ds$ converges (in E_n) to $\int_0^t g(t,s,\phi(s))\,ds$.

We claim that if a smallest jointly continuous topology does exist, then the above convergence is the Moore–Smith convergence relative to this topology, and in particular it induces a convergence class (see [4, p. 73]). In order to prove this claim notice that if g_k converges to g in a jointly continuous topology, then it fulfills the above convergence criterion. On the other hand if g_k converges to g in the above sense, then we can define a topology on $\mathscr{G}$ as follows. Every point different from g is an isolated point. A neighborhood of g is a set which contains a residual set of the net g_k. The convergence above assures us that this topology is jointly continuous. Therefore the smallest jointly continuous topology is smaller than this one, and hence g_k has to converge to g also in this smaller topology..

In order to conclude the argument of the inexistence of the smallest jointly continuous topology we show that the convergence defined above fails to satisfy property (d) of [4, p. 74] and thus it is not a convergence class. This will be demonstrated by a counterexample.

Let $f_m(t)$ (for $m = 1, 2, 3 \cdots$) be the function from $[0, \infty)$ to $[0, 1]$ defined as follows. The function $f_m(t)$ is a piecewise linear function where the "pieces" connect the points $(0,0)$, $(1/m, 1)$, $(2/m, 1)$, $(2/m, 0)$, $(3/m, 1)$, $(4/m, 0)$, $(5/m, 1) \cdots$. Let $\phi_{n,m}$ (for $n = 1, 2, \cdots, m = 1, 2, \cdots$) be the functions $\phi_{n,m}(t) = (1/n)f_m(t)$.

Let $g_{n,m}(t,x)$ be the continuous function defined on $[0,\infty) \times R$ into R as follows. We set $g_{n,m}(t,x) = 1$ if $x = \phi_{n,m}(t)$. Also $g_{n,m}(t,x) = 0$ if $|x - \phi(t)| \geqq 1/m$.

Now $g_{n,m}$ is the continuous extension with values in $[0, 1]$ given by the Tietze theorem. (An analytic formulation can be easily given in this case.) It is easy to verify that our functions $g(s, x)$ satisfy conditions (G1)–(G3) and (G3, U) and thus belong to a fixed collection $\mathscr{G}$. (As a matter of fact, the corresponding equations will be ordinary differential equations.)

We now claim that for a fixed n the sequence $g_{n,m}$ converges in the sense given at the beginning of the proof, to the identically zero function. In order to show this suppose that $\phi(s)$ is a continuous function on $[0, t]$. We have to show that if ϕ_1 is close to ϕ and if m is large enough, then $\int_0^t g_{n,m}(s, \phi_1(s))\, ds$ is close to zero. We shall do it (w.l.o.g) for $t = 1$ and $n = 1$. Also, without loss of generality, ϕ is from $[0, 1]$ into $[0, 1]$. Let Q be the open set in $[0, 1] \times [0, 1]$ defined by $Q = \{(t, x) \cdot |x - \phi(t)| < \varepsilon\}$. The area of Q is less than 2ε. From the definition of $f_m(s)$ it is clear that the proportion of the graph of f_m which intersects Q to the total length of the graph tends to the area of Q as m tends to infinity. Since the slope of f_m is fixed (for a fixed m) it is clear that the measure of $\{t : (t, f_m(t)) \in Q\}$ tends to the area of Q as $m \to \infty$. If now ϕ_1 satisfies $|\phi_1(s) - \phi(s)| < \varepsilon/2$ and if $1/m < \varepsilon/2$, then the measure of $\{t : g_{n,m}(s, \phi_1(s)) \neq 0\}$ is less than the area of Q, thus less than ε. Since $g_{n,m}$ has values in $[0, 1]$ it follows that for such a ϕ_1 and for m large enough, $|\int_0^1 g_{n,m}(s, \phi_1(s))\, ds| \leqq \varepsilon$. This proves our claim.

So far we showed that for a fixed n the sequence $g_{n,m}$ converges to 0. Thus the iterated limit exists and

$$\lim_n \lim_m g_{n,m} \equiv 0.$$

On the other hand our definitions imply that for each n, m,

$$\int_0^1 g_{n,m}(s, \phi_{n,m}(s))\, ds = 1.$$

Since the product net $\phi_{n,m}$ converges to the function 0 it follows that the product net $g_{n,m}$ does not converge to the zero function. Thus the condition on the iterated limit [4, (d), p. 74] does not hold and our convergence is not a convergence class.

We proved earlier that if a smallest jointly continuous topology exists then the convergence defined at the beginning of this proof does define a topology, i.e., it is a convergence class in the sense of [4]. Therefore, our contradiction shows that such a smallest topology does not exist.

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ELEMENTARY CONVOLUTION INEQUALITIES*

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Abstract. We formulate inequalities of the kind discussed in Chapter IX of Hardy, Littlewood and Pólya's *Inequalities* [5] in the language of integration on a locally compact Abelian group. We illustrate the effectiveness of this approach by deriving a number of superficially disparate inequalities, some of which appear to be new.

1. Introduction. A number of useful inequalities that involve convolutions, explicitly or implicitly, are discussed in [5, Chap. IX]. We present a different approach which brings out the underlying structure of these inequalities and can be used systematically to generate new inequalities as well as some old ones that seem not to have been recognized as belonging to the same type. This approach also lets us regard some inequalities that seem to be merely analogues of one another as being actually equivalent. Within its limitations our method is quite successful, although it is ineffective for more refined inequalities such as the two-parameter inequality for fractional integrals ([5, p. 290]; [9]) or the inequalities of Prékopa [13] and Leindler [8].

Our work is quite elementary, although we state our basic theorem in the language of topological groups. A reader who is not familiar with this language has only to interpret the theorem in one of the ways explained below.

THEOREM 1.1. *Let G be a unimodular locally compact Abelian group (written multiplicatively), with Haar measure μ. Let E be a measurable subset of G. Let λ be a nonnegative measure, not identically zero, on G, with $\lambda(E) < \infty$. Let f be a real-valued function on G, measurable with respect to λ. Let φ be continuous and convex on the convex cover of the range of f. Then*

$$\int_G d\mu(x)\varphi\left\{\frac{\int_E f(xt)\,d\lambda(t)}{\int_E d\lambda(t)}\right\} \leqq \int_G \varphi(f(x))\,d\mu(x). \tag{1.1}$$

If φ is concave, (1.1) *holds with the inequality in the opposite sense.*

We make the usual convention that when we write an inequality it holds whenever the large side is finite.

We list some of the particular interpretations of (1.1).

(a) G is the integers under addition; then (1.1) says that

$$\sum_{n=-\infty}^{\infty} \varphi\left\{\frac{\sum_{m\in E} a_{n+m}b_m}{\sum_{m\in E} b_m}\right\} \leqq \sum_{-\infty}^{\infty} \varphi(a_n). \tag{1.2}$$

(b) G is the circle under addition and μ is Lebesgue measure:

$$\int_0^{2\pi} \varphi\left\{\frac{\int_0^{2\pi} f(x+t)\,d\lambda(t)}{\int_0^{2\pi} d\lambda(t)}\right\} dx \leqq \int_0^{2\pi} \varphi(f(x))\,dx. \tag{1.3}$$

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(c) G is the real line under addition and μ is Lebesgue measure:

$$\int_{-\infty}^{\infty} \varphi\left\{\frac{\int_{-\infty}^{\infty} f(x+t)\, d\lambda(t)}{\int_{-\infty}^{\infty} d\lambda(t)}\right\} dx \leqq \int_{-\infty}^{\infty} \varphi(f(x))\, dx. \tag{1.4}$$

(d) G is the positive half-line under multiplication and $d\mu(x) = x^{-1}\, dx$:

$$\int_{0}^{\infty} \varphi\left\{\frac{\int_{0}^{\infty} f(xt)\, d\lambda(t)}{\int_{0}^{\infty} d\lambda(t)}\right\} \frac{dx}{x} \leqq \int_{0}^{\infty} \varphi(f(x)) \frac{dx}{x}. \tag{1.5}$$

(Compare [1], [2].)

(e) G consists of the real numbers greater than 1 under the operation $x^{\log y}$:

$$\int_{1}^{\infty} \varphi\left\{\frac{\int_{1}^{\infty} f(x^{\log t})\, d\lambda(t)}{\int_{1}^{\infty} d\lambda(t)}\right\} \frac{dx}{x \log x} \leqq \int_{1}^{\infty} \varphi(f(x)) \frac{dx}{x \log x}. \tag{1.6}$$

There are multi-dimensional cases corresponding to all of these; for example, the two-dimensional analogue of (1.5) is

$$\int_{0}^{\infty}\int_{0}^{\infty} \varphi\left\{\frac{\int_{0}^{\infty}\int_{0}^{\infty} f(xt, yu)\, d\lambda(u,t)}{\int_{0}^{\infty}\int_{0}^{\infty} d\lambda(u,t)}\right\} \frac{dx}{x}\frac{dy}{y} \leqq \int_{0}^{\infty}\int_{0}^{\infty} \varphi(f(x,y)) \frac{dx}{x}\frac{dy}{y}. \tag{1.7}$$

In § 2 we prove Theorem 1.1 and discuss some of the general inequalities that arise by specializing φ or λ. The question of equality in (1.1) and of the exactness of the (unit) "constant" is discussed in § 3. In §§ 4–7 we illustrate the theory with examples.

2. General inequalities. We need a version of Jensen's inequality for convex functions.

LEMMA 2.1. *Let λ be a nonnegative measure, E a set measurable with respect to λ, and $\int_E d\lambda > 0$. If φ is convex over the convex cover of the range of f, then*

$$\varphi\left\{\frac{\int_E f(t)\, d\lambda(t)}{\int_E d\lambda(t)}\right\} \leqq \frac{\int_E \varphi(f(t))\, d\lambda(t)}{\int_E d\lambda(t)}; \tag{2.1}$$

if φ is concave the inequality is reversed.

Any standard proof of a special case of (2.1) can be adapted to our situation, e.g., the one given in [15, p. 24]. It is also possible to take (2.1) as a definition of convexity [6].

To prove Theorem 1.1, apply (2.1) to $f_x(t) = f(xt)$, and integrate the resulting inequality over G with respect to Haar measure:

$$\int_G d\mu(x)\varphi\left\{\frac{\int_E f(xt)\, d\lambda(t)}{\int_E d\lambda(t)}\right\} \leqq \frac{1}{\int_E d\lambda(t)} \int_G d\mu(x) \int_E \varphi(f(xt))\, d\lambda(t). \tag{2.2}$$

By Fubini's theorem we can integrate in the opposite order on the right of (2.2); then by the invariance of Haar measure,

$$\int_G \varphi(f(xt))\, d\mu(t) = \int_G \varphi(f(x))\, d\mu(x),$$

and (1.1) follows.

Most applications deal with more special situations. We first note the form that Theorem 1.1 takes when $\varphi(u) = u^p$; we then require $f(x) \geqq 0$, and this condition will be tacitly assumed in all theorems dealing with this choice of φ.

THEOREM 2.1. *With the hypotheses of Theorem* 1.1,

$$\int_G d\mu(x)\left\{\int_E f(xt)\,d\lambda(t)\right\}^p \leqq \left\{\int_E d\lambda(t)\right\}^p \int_G f(x)^p\,d\mu(x), \qquad p < 0 \text{ or } p > 1; \tag{2.3}$$

the inequality is reversed when $0 < p < 1$.

Inequalities with $0 < p < 1$ have usually received less attention than those with $p > 1$ (and inequalities with $p < 0$ have received almost no attention at all). However, it is easy to put (2.3), with $0 < p < 1$, into a form involving an index greater than 1. We have only to write $\gamma = 1/p$ and replace f by f^γ; then we get

$$\int_G d\mu(x)\left\{\int_E f(xt)^\gamma\,d\lambda(t)\right\}^{1/\gamma} \geqq \left\{\int_E d\lambda(t)\right\}^{1/\gamma} \int_G f(x)\,d\mu(x), \qquad \gamma > 1. \tag{2.4}$$

If we now specialize G and λ we can read off a number of known inequalities from (2.3) or (1.2)–(1.7).

When $a_k \geqq 0$, $b_k \geqq 0$, we have

$$\sum_{n=-\infty}^{\infty}\left\{\sum_{m\in E} a_{m+n}b_m\right\}^p \leqq \left\{\sum_{m\in E} b_m\right\}^p \sum_{-\infty}^{\infty} a_n^p, \qquad p < 0 \text{ or } p > 1. \tag{2.5}$$

When $0 < p < 1$ the inequality is reversed. The case $p > 1$ was applied to the theory of entire functions by Plancherel and Pólya [12, p. 135].

Now take $d\lambda(t) = K(t)\,dt$ in (1.3) or (1.4), and $E = G$; we get

$$\int dx\left\{\int f(x+t)K(t)\,dt\right\}^p \leqq \left\{\int K(t)\,dt\right\}^p \int f(t)^p\,dt, \qquad p < 0 \text{ or } p > 1; \tag{2.6}$$

when $0 < p < 1$, the inequality is reversed. Here integration is over either $(-\infty, \infty)$ or $(0, 2\pi)$; in the second case f has period 2π.

Similarly from (1.5) (or by an exponential change of variable in (2.6)) we have

$$\begin{aligned} &\int_0^\infty x^{-1}\,dx\left\{\int_0^\infty f(xt)K(t)\,dt\right\}^p \\ &\qquad \leqq \left\{\int_0^\infty K(t)\,dt\right\}^p \int_0^\infty x^{-1}f(x)^p\,dx, \qquad p < 0 \text{ or } p > 1; \end{aligned} \tag{2.7}$$

when $0 < p < 1$ the inequality is reversed. (Cf. [4].)

In (2.7) replace xt by u in the inner integral on the left, replace $f(x)$ by $x^{1/p}f(x)$, and replace $K(t)$ by $t^{-1/p}K(t)$. We get

$$\int_0^\infty dx\left\{\int_0^\infty f(u)x^{-1}K\left(\frac{u}{x}\right)du\right\}^p \leqq \left\{\int_0^\infty K(t)t^{-1/p}\,dt\right\}\int_0^\infty f(x)^p\,dx, \qquad p > 1,$$

which can also be written

$$\int_0^\infty dx\left\{\int_0^\infty f(u)K(x,u)\,du\right\}^p \leqq \left\{\int_0^\infty K(t,1)t^{-1/p}\,dt\right\}\int_0^\infty f(x)^p\,dx, \tag{2.8}$$

where K is homogeneous of degree -1. This is one form of Theorem 3.19 of [5], on which most of Chapter IX of [5] is based. It can be given alternative forms corresponding to the other forms of Theorem 3.19.

In (2.6) and (2.7) we can interchange f and K, and replace $x + t$ by $-x + t$, xt by t/x. This remark is almost too trivial to make, but it is justified by leading to special inequalities that have not always been recognized as coming from the same source (see § 5). We quote the general inequalities for reference: if $p < 0$ or $p > 1$ (with reversed inequality for $0 < p < 1$),

$$\int dx\left\{\int f(t)K(x+t)\,dt\right\}^p \leqq \left\{\int f(t)\,dt\right\}^p \int_{-\infty}^{\infty} K(t)^p\,dt \tag{2.9}$$

(integration over a period for periodic functions, or over $(-\infty, \infty)$);

$$\int_0^\infty x^{-1}\,dx\left\{\int_0^\infty K(xt)f(t)\,dt\right\}^p \leqq \left\{\int_0^\infty f(t)\,dt\right\}^p \int_0^\infty t^{-1}K(t)^p\,dt. \tag{2.10}$$

We can also use Theorem 1.1 to obtain a general version of Young's two-parameter inequality ([5, Thm. 276]); various forms of this, some of them still more general than the one we present, are well known when the parameters all exceed 1 ([10], [11, Chap. 4]).

THEOREM 2.2. *With the hypotheses of Theorem* 1.1, *and* $d\lambda(t) = K(t)\,dt$, *let* $r^{-1} = p^{-1} + q^{-1} - 1$ (*i.e.*, $r = pq/(p + q - pq)$). *Then*

$$\begin{aligned}&\left\{\int_G d\mu(x)\left[\int_G f(xt)K(t)\,d\mu(t)\right]^r\right\}^{1/r}\\ &\qquad\leqq \left\{\int_G K(t)^q\,d\mu(t)\right\}^{1/q}\left\{\int_G f(t)^p\,d\mu(t)\right\}^{1/p},\end{aligned} \tag{2.11}$$

provided that $p > 1$, $p' > q > 1$ (*or equivalently* $q' > p > 1$). *The inequality is reversed if* $p < 1$ *and* $p' < q < 1$ (*or* $q' < p < 1$).

Here, as usual, $p' = p/(p-1)$. We can change the appearance of (2.11) considerably by taking rth powers (reversing the inequality when $r < 0$), replacing p, q, r respectively by α/γ, β/γ, $1/\gamma$, and replacing f, K by f^γ, K^γ. We then have the following result.

THEOREM 2.3. *With the hypotheses of Theorem* 2.2, *we have*

$$\int_G d\mu(x)\left\{\int_G f(xt)^\gamma K(t)^\gamma\,d\mu(t)\right\}^{1/\gamma} \leqq \left\{\int_G K(t)^\alpha\,d\mu(t)\right\}^{1/\alpha}\left\{\int_G f(x)^\beta\,d\mu(x)\right\}^{1/\beta}, \tag{2.12}$$

$\gamma^{-1} = \alpha^{-1} + \beta^{-1} - 1$, *provided that either* $0 < \alpha < 1$ *and* $0 < \beta < 1$; *or* $0 < \alpha < 1$, $\beta < 0$, $\alpha > \beta'$; *or* $0 < \beta < 1$, $\alpha < 0$, $\beta > \alpha'$. *We have* (2.12) *with the inequality reversed if* $\alpha > 1$, $\beta > 1$ (*and hence* $\gamma > 1$).

We first establish (2.11) when $p > 1$, $p' > q$. Put

$$I = \int_G f(xt)K(t)\,d\mu(t),$$

and write

$$I = \int_G f(xt)K(t)^{1-q}K(t)^q\,d\mu(t). \tag{2.13}$$

The proof involves four steps.

Step 1. We use Jensen's inequality (2.1) with index p, $d\lambda(t) = K(t)^q\,d\mu(t)$, and $f(t)$ replaced by $f(xt)K(t)^{1-q}$, to get

$$I^p \leqq \left\{\int_G K(t)^q\,d\mu(t)\right\}^{p-1}\int_G f(xt)^p K(t)^{(1-q)p+q}\,d\mu(t). \tag{2.14}$$

Step 2. Since $r > 0$ we preserve the inequality by raising both sides of (2.14) to the positive power r/p (positive since $p/(p-1) > q$). We then have

$$I^r \leqq \left\{\int_G K(t)^q\,d\mu(t)\right\}^{r-r/p}\left\{\int_G f(xt)^p K(t)^{p+q-pq}\,d\mu(t)\right\}^{r/p}. \tag{2.15}$$

Since $q > 1$ we have $r > p$.

Step 3. Apply Jensen's inequality with index r/p and $d\lambda(t) = f(xt)^p\,d\mu(t)$, obtaining (by the invariance of Haar measure)

$$\begin{aligned} I^r &\leqq \left\{\int_G K(t)^q\,d\mu(t)\right\}^{r-r/p}\left\{\int_G f(xt)^p\,d\mu(t)\right\}^{(r/p)-1}\int_G K(t)^{(p+q-pq)r/p} f(xt)^p\,d\mu(t) \\ &= \left\{\int_G K(t)^q\,d\mu(t)\right\}^{r-r/p}\left\{\int_G f(t)^p\,d\mu(t)\right\}^{(r/p)-1}\int_G K(t)^q f(xt)^p\,d\mu(t). \end{aligned} \tag{2.16}$$

Step 4. Integrate over G, use Fubini's theorem, and take $(1/r)$th powers to get (2.12).

If we try to repeat the argument with different values of the parameters we observe that the signs and magnitudes of the parameters are relevant at each of the four numbered steps. Thus, in Step 1 we could have either a $\geqq$ sign ($p > 1$ or $p < 0$), or a $\leqq$ sign ($0 < p < 1$). In Step 2 we preserve whichever inequality we had as a result of Step 1 provided that $r/p > 0$, but reverse it if $r/p < 0$. In Step 3 we are operating on only one side of the inequality, so we must preserve the sense of the inequality that we had in Step 2; thus we need $r/p > 1$ or $r/p < 0$ if Step 2 had $\leqq$, but $0 < r/p < 1$ if Step 2 had $\geqq$. Finally Step 4 preserves the sense of the inequality resulting from Step 3 if $r > 0$, and reverses it if $r < 0$.

We now have to see which values of q (if any) are admissible in Steps 2, 3, 4 for each possibility in Step 1 except when $p > 1$ and $q > 1$ (which has already been taken care of). A detailed analysis (which we omit) shows that the possible cases are as stated in Theorem 2.3.

We use Theorem 2.2 to obtain an inequality of a type that seems to be exemplified in the literature by only one instance ([3, Thm. 339], [6]), although there is an inequality of this kind corresponding to every choice of K in Theorem 2.2. We start from the well-known fact (a natural generalization of [5, Thm. 3.19]) that an inequality of the form

$$\int_G d\mu(x)\left\{\int_G H(yx^{-1})f(y)\,d\mu(y)\right\}^p \leqq k^p\int_G f(x)^p\,d\mu(x), \qquad p > 1, \tag{2.17}$$

is equivalent (by Hölder's inequality and its converse) to

$$\begin{aligned} &\int_G\int_G H(yx^{-1})f(x)g(y)\,d\mu(x)\,d\mu(y) \\ &\qquad \leqq k\left\{\int_G f(x)^p\,d\mu(x)\right\}^{1/p}\left\{\int_G g(y)^{p'}\,d\mu(y)\right\}^{1/p'}. \end{aligned} \tag{2.18}$$

We generalize (2.18) by introducing a parameter λ, $0 < \lambda < 1$, and consider

$$\int_G \int_G H(yx^{-1})^\lambda f(x)g(y)\,d\mu(x)\,d\mu(y) = \int_G \int_G H(t)^\lambda g(ty)f(y)\,d\mu(t)\,d\mu(y).$$

By Hölder's inequality with index $1/\lambda$, the right-hand side does not exceed

$$\left\{\int_G H(t)\,d\mu(t)\right\}^\lambda \left\{\int d\mu(t)\left[\int_G g(ty)f(y)\,d\mu(y)\right]^{1/(1-\lambda)}\right\}^{1-\lambda}.$$

Now apply (2.11) with $r = 1/(1-\lambda)$, $K = f$, and we get, with $1 - \lambda = p^{-1} + q^{-1} - 1$, i.e., $\lambda = 2 - p^{-1} - q^{-1} = 1/p' + 1/q'$,

$$\begin{aligned}(2.19)\qquad &\int_G \int_G H(yx^{-1})^\lambda f(x)g(y)\,d\mu(x)\,d\mu(y)\\ &\leqq \left\{\int_G H(t)\,d\mu(t)\right\}^\lambda \left\{\int_G f(x)^p\,dx\right\}^{1/p} \left\{\int_G g(y)^q\,dy\right\}^{1/q}.\end{aligned}$$

3. Exactness of constants. There is equality in Theorem 1.1 only in rather trivial ways. (The question is investigated in detail in [6].) When G is compact, there can be equality in Theorem 1.1 for any λ, since G has finite measure and we can take $f(x) \equiv 1$. When G is not compact we cannot in general have equality in (1.1). However, in this case, when $p > 1$ the constant $\{\int_E d\lambda(t)\}^p$ in (2.3) is best possible for each λ, at least when $E = G$, the case that occurs in most applications. In establishing this it is convenient to write G additively. We consider measurable subsets E of G which contain the identity and contain $-x$ when they contain x. By $E + H$ we mean the union of all translates $a + H$ for $a \in E$.

Now suppose that for a particular λ and for every f in $L^p(G)$ we had

$$\int_G d\mu(x)\left\{\int_G f(x+t)\,d\lambda(t)\right\}^p < c\int_G d\lambda(t)\int_G f(x)^p\,d\mu(x),$$

with $c < (\int_G d\lambda)^{p-1} < \infty$. Take $\varepsilon > 0$ so small that $\varepsilon/(1-\varepsilon) < 1$. Let E and H be compact; then

$$\int_H d\mu(x)\left\{\int_E f(x+t)\,d\lambda(t)\right\}^p < c\int_G d\lambda(t)\int_G f(x)^p\,d\mu(x).$$

Let f be the characteristic function of $E + H$; then $f(x+t) = 1$ for $t \in E$ and $x \in H$, and we have

$$\mu(H)\left\{\int_E d\lambda(t)\right\}^p < c\left(\int_G d\lambda\right)\mu(E+H),$$

$$\mu(H)/\mu(E+H) < c\left(\int_G d\lambda\right)\Big/\left(\int_E d\lambda\right)^p.$$

We can take E so large that

$$\left(\int_E d\lambda\right)^p > (1-\varepsilon)\left(\int_G d\lambda\right)^p,$$

and then

$$\frac{\mu(H)}{\mu(E+H)} < \frac{c}{1-\varepsilon}\left(\int_G d\lambda\right)^{1-p} = s < 1.$$

When G is the integers, the line, or the plane (or any Euclidean space) it is easy to see that H can be chosen so large that $\mu(H)/\mu(E+H) > s$, and we have a contradiction. We omit the proof in the general case, since the only applications of (2.3) in this paper are to Euclidean spaces.

The proof breaks down for $p < 1$ and we do not know whether the constant in (2.3) is best possible.

For Theorem 2.2, the constant in (2.11) is not best possible for every g. This is shown for $p > 1$ by Theorem 383 of [5] (see also [10]); and for $0 < p < 1$, $0 < q < 1$, by the sharper inequalities of Prékopa [13] and Leïndler [8]. However, (2.11) is best possible in a weaker sense when $p > 1$: no smaller constant makes it hold for all pairs (f, g).

We give the proof when G is the line. Suppose that

$$\left\{\int_{-\infty}^{\infty} dx\left[\int_{-\infty}^{\infty} f(x+t)g(t)\,dt\right]^r\right\}^{1/r} \leqq c\left\{\int_{-\infty}^{\infty} g(t)^q\,dt\right\}^{1/q}\left\{\int_{-\infty}^{\infty} f(x)^p\,dx\right\}^{1/p},$$

with $c < 1$, $p > 1$, $q > 1$, $1/r = 1/p + 1/q - 1$, for all f and g for which the right-hand side is finite. Take $f(x) = 1$ on $[-R-1, R+1]$ and 0 elsewhere, $g(x) = 1$ on $[-R, R]$ and 0 elsewhere; then the inequality would say

$$(2R)^{1+1/r} \leqq c(2R)^{1/q}(2R+2)^{1/p},$$

and hence $R/(R+1) \leqq c^p$, a contradiction for large R.

4. Variations on Hardy's inequality. In the rest of this paper we give some illustrations to indicate the scope of the general theorems in §§ 1 and 2. We begin with Hardy's inequality ([5, Thm. 327]), which states that

$$\int_0^\infty \left\{\frac{1}{x}\int_0^x f(t)\,dt\right\}^p dx \leqq (p')^p \int_0^\infty f(t)^p\,dx, \tag{4.1}$$

where the constant is best possible.

We begin with an inequality involving a general convex φ; this inequality resembles (4.1) superficially, but does not reduce to it when $\varphi(u) = u^p$. We start from (1.5), specialized to the case $d\lambda(t) = K(t)\,dt$, $E = G$, the half-line, and put $xt = u$ on the left. The result is

$$\int_0^\infty \frac{dx}{x}\varphi\left\{\frac{x^{-1}\int_0^\infty f(u)K(u/x)\,du}{\int_0^\infty K(t)\,dt}\right\} \leqq \int_0^\infty \varphi(f(x))\frac{dx}{x}. \tag{4.2}$$

If we now take $K(t) = 1$ for $0 < t < 1$, $K(t) = 0$ for $t > 1$, we get

$$\int_0^\infty \frac{dx}{x}\varphi\left\{\frac{1}{x}\int_0^x f(u)\,du\right\} \leqq \int_0^\infty \varphi(f(x))\frac{dx}{x}, \tag{4.3}$$

where φ is convex. (Compare [1], [2].)

To get (4.1) we specialize K differently. First take $\varphi(u) = u^p$:

$$\int_0^\infty \frac{dx}{x}\left\{\frac{1}{x}\int_0^\infty f(u)K\left(\frac{u}{x}\right)du\right\}^p \leqq \left\{\int_0^\infty K(t)\,dt\right\}^p \int_0^\infty f(x)\,\frac{dx}{x}, \qquad p > 1. \tag{4.4}$$

Now replace $f(u)$ by $u^{1/p}f(u)$:

$$\int_0^\infty \frac{dx}{x}\left\{\frac{1}{x}\int_0^\infty u^{1/p}f(u)K\left(\frac{u}{x}\right)du\right\}^p \leqq \left\{\int_0^\infty K(t)\,dt\right\}^p \int_0^\infty f(x)^p\,dx, \qquad p > 1. \tag{4.5}$$

This inequality persists when $p < 0$ and is reversed for $0 < p < 1$, provided that the integrals on the large side converge. If we now take $p > 1$, $K(u) = u^{-1/p}$ for $0 < u < 1$, $K(u) = 0$ for $u > 1$, we obtain (4.1). There is no analogue of (4.1) for $0 < p < 1$ since $\int K(u)\,du$ would diverge; but (4.1) remains valid for $p < 0$, as was noticed by Knopp [7].

The dual of Hardy's inequality,

$$\int_0^\infty dx\left\{\int_x^\infty f(u)\frac{du}{u}\right\}^p \leqq p^p\int_0^\infty f(x)\,dx, \qquad p > 1, \tag{4.6}$$

is (4.5) with $K(u) = u^{-1-1/p}$ for $u > 1$, $K(u) = 0$ for $0 < u < 1$. Here the inequality is reversed when $0 < p < 1$. The series analogue of (4.6) (which is deducible from the integral inequality) has an application in the theory of orthogonal series [14].

More generally, take $K(u) = u^\alpha$, $0 < u < 1$; $K(u) = 0$, $u > 1$. Then

$$\int_0^\infty K(u)\,du < \infty$$

if $\alpha > 0$. Replacing $f(t)$ by $t^{1-\alpha}f(t)$ in (4.4), and writing $r = 1 + \alpha p$, we get

$$\int_0^\infty x^{-r}\,dx\left\{\int_0^x f(t)\,dt\right\}^p \leqq (p')^p\int_0^\infty t^{-r}(tf(t))^p\,dt, \tag{4.7}$$

provided that $r > 1$ if $p > 1$ or $r < 1$ if $p < 0$. If $0 < p < 1$ and $r > 1$ we have the inequality in the opposite sense. The case $p > 1$ is half of [5, Thm. 330]; the other half corresponds to $K(u) = u^\alpha$ for $u > 1$, $K(u) = 0$ for $0 < u < 1$.

When $K(u) = u^{-1/p}$ for $0 < a < u < b$, and $K(u) = 0$ otherwise, $\int_0^\infty K(u)\,du$ is finite for all p, and we get

$$\begin{aligned} &\int_0^\infty dx\left\{\frac{1}{x}\int_{ax}^{bx} f(t)\,dt\right\}^p \leqq (p')^p(b^{1/p'} - a^{1/p'})\int_0^\infty f(t)^p\,dt, \qquad p > 1 \text{ or } p < 0;\\ &\int_0^\infty dx\left\{\frac{1}{x}\int_{ax}^{bx} f(t)\,dt\right\}^p \geqq (p')^p(a^{1/p'} - b^{1/p'})\int_0^\infty f(t)^p\,dt, \qquad 0 < p < 1. \end{aligned} \tag{4.8}$$

The convergence theorems arising from (4.8) are interesting in themselves. By the usual technique ([5, Chap. IX]) one can deduce the corresponding theorems for series: If $0 < p < 1$, $0 < a < b$ and $a_n \geqq 0$, then $\sum a_n^p$ converges if

$$\sum_{n=1}^\infty \left\{\frac{1}{n}\sum_{na}^{nb} a_k\right\}^p$$

converges; if $p > 1$ and $\sum a_n^p$ converges, then

$$\sum_{n=1}^{\infty} \left\{ \sum_{na}^{nb} a_k \right\}^p$$

converges. For example, take $p = \frac{1}{2}$ and replace a_n by a_n^2; then $\sum a_n$ converges if

$$\sum_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{n}^{2n} a_k^2 \right\}^{1/2} \tag{4.9}$$

converges.

Next we obtain some illustrations of (2.10) that are suggested by Hardy's inequality. Accordingly these inequalities involve $(\int f(x)\,dx)^p$ instead of $\int f(x)^p\,dx$.

The K used in Hardy's inequality is $K(u) = u^{-1/p}$ for $0 < u < 1$, $K(u) = 0$ elsewhere. Since $\int_0^\infty t^{-1} K(t)^p\,dt$ diverges there is no direct analogue. However (cf. (4.7)) we can use $K(u) = u^\alpha$ on $(0, 1)$ or $K(u) = u^{-\alpha}$ on $(1, \infty)$ if $\alpha > 0$. The first yields

$$\int_0^\infty x^{\alpha p - 1}\,dx \left\{ \int_0^x t^\alpha f(t)\,dt \right\}^p \leqq (\alpha p)^{-1} \left\{ \int_0^\infty f(t)\,dt \right\}^p, \qquad p > 1, \quad \alpha > 0,$$

with the inequality reversed if $0 < p < 1$. The second choice yields

$$\int_0^\infty x^{-\alpha p - 1} \left\{ \int_x^\infty t^{-\alpha} f(t)\,dt \right\}^p \leqq (\alpha p)^{-1} \left\{ \int_0^\infty f(t)\,dt \right\}^p, \qquad p > 1, \quad \alpha > 0.$$

When $p < 0$ we can take $K(u) = u^{-\alpha}$ on $(0, 1)$ or $K(u) = u^\alpha$ on $(1, \infty)$, $\alpha > 0$. We get, for example,

$$\int_0^\infty x^{\alpha|p| - 1}\,dx \left\{ \int_0^x t^{-\alpha} f(t)\,dt \right\}^{-|p|} \leqq (\alpha|p|)^{-1} \left\{ \int_0^\infty f(t)\,dt \right\}^{-|p|}.$$

In particular, when $p = -1$, this is

$$\int_0^\infty x^{\alpha - 1}\,dx \left\{ \int_0^x t^{-\alpha} f(t)\,dt \right\}^{-1} \leqq \alpha^{-1} \left\{ \int_0^\infty f(t)\,dt \right\}^{-1};$$

and when $\alpha = 1$,

$$\int_0^\infty \frac{dx}{\int_0^x t^{-1} f(t)\,dt} \leqq \frac{1}{\int_0^\infty f(t)\,dt}.$$

We now obtain a two-dimensional version of Hardy's inequality. Up to this point we would have been about as well off with Theorem 319 of [5], but it does not seem to be obvious how that theorem should generalize to higher dimensions. Here we take the group G to be the Cartesian product of the half-line with itself. We take $K(s, t) = s^{-1/p} t^{-1/p}$ for (s, t) in a convex set S whose boundary contains the origin, and $K(s, t) = 0$ outside S. Let S_{xy} be the set of points (xu, yv) with $(u, v) \in S$. Proceeding as in obtaining (4.1), we get

$$\int_0^\infty \int_0^\infty dx\,dy \left\{ \frac{1}{xy} \int_{S_{xy}} f(s, t)\,ds\,dt \right\}^p \leqq L^p \int_0^\infty \int_0^\infty f(s, t)^p\,ds\,dt, \qquad p > 1, \tag{4.10}$$

where

$$L = \iint_S u^{-1/p} r^{-1/p}\, du\, dv.$$

Finally we obtain a generalization of Hardy's inequality by using Theorem 2.2. An inequality of the form

$$\int d\mu(x) \left\{ \int K(x,y) f(y)\, d\mu(y) \right\}^p \leqq k^p \int f(x)^p\, d\mu(x)$$

is, as we observed at the end of § 2, equivalent to

$$\iint K(x,y) f(x) g(y)\, d\mu(x)\, d\mu(y) \leqq k \left\{ \int f(x)^p\, d\mu \right\}^{1/p} \left\{ \int g(y)^{p'}\, d\mu \right\}^{1/p'}.$$

To put Hardy's inequality into this form we start from (4.4),

$$\int_0^\infty \frac{dx}{x} \left\{ \frac{1}{x} \int_0^\infty f(y) K\left(\frac{y}{x}\right) dy \right\}^p \leqq k^p \int_0^\infty f(x)^p \frac{dx}{x}, \qquad k = \int_0^\infty K(t)\, dt,$$

and write it in the form

$$\int_0^\infty \frac{dx}{x} \left\{ \int_0^\infty f(y) \left(\frac{y}{x}\right) K\left(\frac{y}{x}\right) \frac{dy}{y} \right\}^p \leqq k^p \int_0^\infty f(x)^p \frac{dx}{x},$$

which is like (2.17) but with $H(u) = uK(u)$. Then (2.19) gives us

$$\int_0^\infty \int_0^\infty \left\{ \left(\frac{y}{x}\right) K\left(\frac{y}{x}\right) \right\}^\lambda f(x) g(y) \frac{dx}{x} \frac{dy}{y}$$
$$\leqq \left\{ \int_0^\infty K(t)\, dt \right\}^\lambda \left\{ \int_0^\infty f(x)^p \frac{dx}{x} \right\}^{1/p} \left\{ g(y)^q \frac{dy}{y} \right\}^{1/q}.$$

Replacing $f(x)$ by $x^{1/p} f(x)$, $g(y)$ by $y^{1/q} g(y)$, we find

$$\int_0^\infty \int_0^\infty \left\{ \left(\frac{y}{x}\right) K\left(\frac{y}{x}\right) \right\}^\lambda f(x) g(y) x^{(1/p)-1} y^{(1/q)-1}\, dx\, dy$$
$$\leqq \left\{ \int_0^\infty K(t)\, dt \right\}^\lambda \left\{ \int_0^\infty f(x)^p\, dx \right\}^{1/p} \left\{ \int_0^\infty g(y)^q\, dy \right\}^{1/q}.$$

Taking $K(u) = u^{-1/p}$ for $0 < u < 1$, we then obtain

$$\int_0^\infty dx \int_0^x f(x) g(y) x^{-\lambda/p' - 1/p'} y^{\lambda/p' - 1/q'}\, dy$$

(4.11)

$$\leqq (p')^\lambda \left\{ \int_0^\infty f(x)^p\, dx \right\}^{1/p} \left\{ \int_0^\infty g(y) q\, dy \right\}^{1/q}.$$

When $\lambda = 1$ we have $q = p'$, $q' = p$, and so

$$\int_0^\infty x^{-2+2/p} f(x)\, dx \int_0^x y^{1-2/p} g(y)\, dy,$$

which is an alternative form of Hardy's inequality.

One can of course give corresponding generalizations of any other special case of Theorem 1.1.

5. Hausdorff means and Laplace transforms. Hausdorff means are transforms of the form

$$F(x) = \int_0^1 f(xt)\,d\lambda(t); \tag{5.1}$$

we are concerned with the case when f is positive and λ is nondecreasing. Since (5.1) is of the form considered in (1.5) and (2.7) we can read off several inequalities. For example, if $p > 1$, (2.7) says that

$$\int_0^\infty F(x)^p \frac{dx}{x} \leqq \left\{\int_0^1 d\lambda(t)\right\}^p \int_0^\infty x^{-1} f(x)^p\,dx.$$

Replacing $f(x)$ by $f(x)^{1/p}$ and $d\lambda(t)$ by $t^{-1/p}\,d\lambda(t)$, we have

$$\int_0^\infty F(x)^p\,dx \leqq \left\{\int_0^1 t^{-1/p}\,d\lambda(t)\right\}^p \int_0^\infty f(x)^p\,dx \tag{5.2}$$

([3, p. 277]). For $0 < p < 1$ the inequality is reversed.

We can treat the Laplace transform

$$F(x) = \int_0^\infty e^{-xt}\,d\lambda(t) \tag{5.3}$$

in the same way, with modifications required by the divergence (in general) of both sides of (2.7) when $f(x) = e^{-x}$. When $\alpha > 0$ we can consider

$$x^\alpha F(x) = \int_0^\infty e^{-xt}(xt)^\alpha t^{-\alpha}\,d\lambda(t),$$

take $f(u) = u^\alpha e^{-u}$ in (2.7), and replace $d\lambda(t)$ by $t^{-\alpha}\,d\lambda(t)$. Then we have

$$\int_0^\infty x^{\alpha p - 1} F(x)^p\,dx \leqq \left\{\int_0^\infty t^{-\alpha}\,d\lambda(t)\right\}^p \int_0^\infty e^{-pt} t^{\alpha p - 1}\,dt,$$

that is,

$$\int_0^\infty x^{\alpha p - 1} F(x)^p\,dx \leqq \Gamma(\alpha p) p^{-\alpha p}\left\{\int_0^\infty t^{-\alpha}\,d\lambda(t)\right\}^p, \qquad p > 1, \quad \alpha > 0. \tag{5.4}$$

In particular, when $\alpha = 1/p$ we have an inequality parallel to (5.2),

$$\int_0^\infty F(x)^p\,dx \leqq p^{-1}\left\{\int_0^\infty t^{-\alpha}\,d\lambda(t)\right\}^p, \qquad p > 1. \tag{5.5}$$

On the other hand, we can put $xt = u$ and then replace x by $1/x$, so that

$$x^{1/p} \cdot x^{-1} F(x^{-1}) = \int_0^\infty u^{-1/p} e^{-u} f(xu)(xu)^{1/p}\,du, \qquad p > 1.$$

Here we take $d\lambda(t) = e^{-t}\,dt$ and $\varphi(u) = u^p$ in (1.5), and obtain

$$\int_0^\infty \frac{dx}{x}\{x^{(1/p)-1}F(x^{-1})\}^p \leqq \int_0^\infty x^{-1}\{x^{1/p}f(x)\}^p\,dx,$$

that is,

$$\int_0^\infty x^{p-2}F(x)^p\,dx \leqq \int_0^\infty f(x)^p\,dx \tag{5.6}$$

(see [4]).

We can also get inequalities for Laplace transforms from Theorem 2.2. With $F(x)$ still defined by (5.3), we can write

$$x^{-\alpha}F(x^{-1}) = \int_0^\infty (xu)^{1-\alpha}f(xu)(u^\alpha e^{-u})\frac{du}{u}, \qquad \alpha > 0,$$

where the integral has the form of the inner integral in (2.11). Hence

$$\begin{aligned}\int_0^\infty y^{\alpha r-1}F(y)^r\,dy &= \int_0^\infty x^{-1-\alpha r}F(x^{-1})^r\,dx\\ &= \left\{\int_0^\infty x^{p-\alpha p-1}f(x)^p\,dx\right\}^{r/p}\left\{\int_0^\infty u^{\alpha q-1}\,e^{-qu}\,du\right\}^{r/q}\\ &= \left\{\int_0^\infty x^{p-\alpha p-1}f(x)^p\,dx\right\}^{r/p}\{\Gamma(\alpha q)q^{-\alpha q}\}^{r/q}\end{aligned}$$

when $p > 1, q > 1, r^{-1} = p^{-1} + q^{-1} - 1, p' > q, \alpha > 0$. The result is particularly simple when $\alpha r = 1$ and $p - \alpha p = 1$, that is, $r = p', \alpha = 1/p', q = p'/2$, and we get

$$\int_0^\infty F(y)^{p'}\,dy \leqq \frac{2\pi}{p'}\left\{\int_0^\infty f(x)^p\,dx\right\}^{p'/p}, \qquad 1 < p \leqq 2, \tag{5.7}$$

([5, Thm. 352]). For the case $r = p$, $\alpha = 1/p$, see [4]. Further inequalities can be generated from other cases of Theorem 2.2.

6. Fractional integrals. Our method is not powerful enough to get the deeper inequalities ([5, Thm. 383], [10]), but it does give a convenient approach to the more elementary ones. We start from (4.5) and consider the two cases

$$\begin{aligned}&K_1(u) = u^{\alpha-1}(1-u)^{r-1}, \quad 0 < u < 1; \qquad K_1(u) = 0, \quad u > 1;\\ &K_2(u) = u^{\alpha-1}(u-1)^{r-1}, \quad u > 1; \qquad K_2(u) = 0, \quad 0 < u < 1.\end{aligned}$$

It is customary to suppose that $0 < r < 1$, but all we really need is $r > 0$. Then $k_s = \int_0^\infty K_s(u)\,du$ is finite provided either $s = 1$ and $\alpha > 0$ or $s = 2$ and $\alpha < 1 - r$; we have $k_1 = \Gamma(\alpha)\Gamma(r)/\Gamma(\alpha + r)$, $k_2 = \Gamma(1 - \alpha - r)\Gamma(r)/\Gamma(1 - \alpha)$. Using K_1, we then have, for $p > 1$ and $0 < r < 1$,

$$\int_0^\infty \frac{dx}{x}\left\{\frac{1}{x}\int_0^\infty u^{1/p}f(u)\left(\frac{u}{x}\right)^{\alpha-1}\left(\frac{1-u}{x}\right)^{r-1}du\right\}^p \leqq k_1^p\int_0^\infty f(x)^p\,dx,$$

and therefore

$$\int_0^\infty x^{-1-p(\alpha+r-1)}\,dx\left\{\int_0^\infty (x-u)^{r-1}f(u)u^{\alpha-1+1/p}\,du\right\}^p \leqq k_1^p\int_0^\infty f(x)^p\,dx.$$

Now replace $f(u)$ by $f(u)u^{1-\alpha-1/p}$, and we get

$$\int_0^\infty x^{-1-p(\alpha+r-1)}\,dx\left\{\int_0^x (x-u)^{r-1}f(u)\,du\right\}^p \leqq k_1^p\int_0^\infty f(x)^p x^{p-1-\alpha p}\,dx. \tag{6.1}$$

Proceeding similarly with K_2, we have

$$\int_0^\infty x^{-1-p(\alpha+r-1)}\,dx\left\{\int_x^\infty (u-x)^{r-1}f(u)\,du\right\}^p \leqq k_2^p\int_0^\infty f(x)^p x^{p-1-\alpha p}\,dx. \tag{6.2}$$

To get an inequality with a simple right-hand side, take $\alpha = 1/p'$, and then

$$\int_0^\infty dx\left\{x^{-r}\int_0^x (x-t)^{r-1}f(t)\,dt\right\}^p \leqq k_1^p\int_0^\infty f(t)^p\,dt, \qquad p>1. \tag{6.3}$$

This is half of [5, Thm. 329]. There is an inequality with a simple right-hand side for $0<p<1$, but it comes from (6.2) rather than (6.1):

$$\int_0^\infty dx\left\{x^{-r}\int_x^\infty (t-x)^{r-1}f(t)\,dt\right\}^p \geqq k_1^p\int_0^\infty f(t)^p\,dt, \qquad 0<p<1. \tag{6.4}$$

There is no exact analogue of (6.3) when $0<p<1$.

On the other hand, if we want a simple left-hand side we take $\alpha = (1/p') - r$ and then we have

$$\int_0^\infty dx\left\{\int_x^\infty (t-x)^{r-1}f(t)\,dt\right\}^p \leqq k_2^p\int_0^\infty \{t^r f(t)\}^p\,dt; \tag{6.5}$$

this is the other half of [5, Thm. 329]. We also have

$$\int_0^\infty dx\left\{\int_0^x (x-t)^{r-1}f(t)\,dt\right\}^p \geqq k_1^p\int_0^\infty \{t^r f(t)\}^p\,dt, \qquad 0<p<1, \quad p>\frac{1}{1-r} \tag{6.6}$$

(hence not when $0<r<1$).

One can interpret (6.3) loosely as saying that the fractional integral of f, of order r, is "more integrable" than f itself in a neighborhood of 0, since when $f\in L^p$, $p>1$, the fractional integral belongs to L^p even after being multiplied by the large factor x^{-r}. This is, of course, true whether or not f is positive, since the inequality holds for $|f|$. On the other hand, if f is positive, (6.4) says that the other fractional integral of f is somewhat *less* integrable than f in a neighborhood of ∞.

7. Periodic functions. Here G consists of the real numbers (mod 2π) with addition as the group operation. The general inequality is (1.4), which we write with $d\lambda(t) = K(t)\,dt$,

$$(7.1)\quad \int_0^{2\pi} \varphi\left\{L^{-1}\int_0^{2\pi} f(x+t)K(t)\,dt\right\}dx \leqq \int_0^{2\pi}\varphi(f(x))\,dx, \qquad L = \int_0^{2\pi} K(t)\,dt,$$

φ convex. Let us take, for example,

$$K(t) = \frac{1}{(n+1)\pi}\,\frac{\sin^2 \frac{1}{2}(n+1)t}{2\sin^2 \frac{1}{2}t},$$

the Fejér kernel. Then

$$\int_0^{2\pi} f(x+t)K(t)\,dt$$

is the nth arithmetic mean $\sigma_n(x)$ of the partial sums of the Fourier series of f, and $L = 1$. We therefore have

$$\int_0^{2\pi}\varphi(\sigma_n(x))\,dx \leqq \int_0^{2\pi}\varphi(f(x))\,dx$$

when $f(x) \geqq 0$, and consequently (as is well known)

$$\int_0^{2\pi}\varphi(|\sigma_n(x)|)\,dx \leqq \int_0^{2\pi}\varphi(|f(x)|)\,dx$$

when φ is both convex and increasing. When $\varphi(u) = u^p$, $p > 1$, this can be interpreted as saying that $|\sigma_n(x)|$ tends to be smaller than $|f(x)|$. On the other hand when $f(x) \geqq 0$ we have

$$\int_0^{2\pi}\{\sigma_n(x)\}^{-p}\,dx \leqq \int_0^{2\pi} f(x)^{-p}\,dx, \qquad p > 0,$$

and

$$\int_0^{2\pi}\{\sigma_n(x)\}^{p}\,dx \geqq \int_0^{2\pi} f(x)^{p}\,dx, \qquad 0 < p < 1;$$

either of these inequalities could be interpreted as saying that when $f(x)$ is positive, $\sigma_n(x)$ tends to be larger than $f(x)$.

By using $K(t) = (\sin\frac{1}{2}t)^{r-1}$, $r > 0$, we could obtain inequalities similar to those of § 6 for the integral

$$\int_0^{2\pi}\{\sin\tfrac{1}{2}(x-t)\}^{r-1}f(t)\,dt.$$

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UNIFORM APPROXIMATION OPERATORS GENERATED BY A POWER SERIES IDENTITY*

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Abstract. The classical Bernstein polynomials $B_n(f;x)$ may be regarded as transforms of the sequences $\{f(k/n)\}$ by the Euler summability matrix with parameter x. T. H. Gronwall has shown that the Euler matrix is generated by a certain formal power series identity. In this paper these ideas are combined to produce a wide class of approximation operators which generalize the Bernstein operators. A uniform convergence theorem, an order of convergence result and an asymptotic error formula are given.

1. Introduction. In 1912 Bernstein [1] introduced the polynomials $B_n(f;x)$, which for a given function f with domain [0, 1] are defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right). \tag{1.1}$$

Bernstein showed that $\lim_n B_n(f;x) = f(x)$ uniformly in x for each $f \in C[0,1]$. In recent years several authors [2], [3], [6], [7], [10], [11] have investigated classes of uniform approximation operators which contain the Bernstein polynomials.

In this paper we shall study a class of uniform approximation operators which generalize the Bernstein operators and are generated by a formal power series identity. We begin as in [2] by noting that (1.1) can be written

$$B_n(f;x) = \sum_{k=0}^{n} E_{nk}(x) f\left(\frac{k}{n}\right),$$

where $E_{nk}(x)$ is the Euler summability matrix with parameter x. For a given function $h(w)$ of a certain type which is analytic for $|w| < 1$ and satisfies $h(0) = 0$ and a fixed $\alpha > 0$, Gronwall [5] defined the $[h, (1-w)^{-\alpha}]$-transform of a sequence $\{s_n\}$ to be the sequence $\{U_n\}$ defined by the formal power series identity

$$(1-h(w))(1-w)^{-\alpha} \sum_{k=0}^{\infty} s_k[h(w)]^k = \sum_{n=0}^{\infty} \binom{n+\alpha-1}{n} U_n w^n. \tag{1.2}$$

Furthermore,

$$U_n = \sum_{k=0}^{n} A_{nk} s_k,$$

where

$$(1-h(w))(1-w)^{-\alpha}[h(w)]^k = \sum_{n=k}^{\infty} \binom{n+\alpha-1}{n} A_{nk} w^n. \tag{1.3}$$

In the particular case $\alpha = 1$ and

$$h(w) = \frac{xw}{1-(1-x)w},$$

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the transform (1.2) is the Euler summability transform and the matrix in (1.3) is the Euler matrix. We will study the operators $L_n(f; x)$ defined for $f \in C[0, 1]$ by $L_0(f; x) = f(0)$ and

$$L_n(f; x) = \sum_{k=0}^{n} A_{nk}(x) f\left(\frac{k}{n}\right), \qquad n > 0, \tag{1.4}$$

where the matrix $A_{nk}(x)$ is generated by the identity (1.3) and $h(w)$ is some function involving a parameter $x \in [0, 1]$.

2. A class of uniform approximation operators. Our first result shows that if the operators $L_n(f; x)$ of type (1.4) are to have the uniform approximation property, then $h(w)$ must have a very special form and α must equal 1.

THEOREM 2.1. *If the operators* (1.4) *have the uniform approximation property, then* $\alpha = 1$ *and*

$$h(w) = \frac{xw + Q(w)}{1 - (1 - x)w + Q(w)},$$

where $Q(w)$ is analytic in $|w| < 1$, $Q(0) = 0$, and $Q(w) \to 0$ as $w \to 1$ inside a Stolz angle at 1. Furthermore, the power series expansion of $Q(w)$ about $w = 0$ does not have all power series coefficients nonnegative unless $Q(w) \equiv 0$.

Proof. If the operators $L_n(f; x)$ have the uniform approximation property, then in particular for the functions $f(t) = t, t^2$ respectively we have

$$L_n(t^i; x) = x^i + \varepsilon_n^{(i)} \quad \text{for } i = 1, 2, \text{ where } \varepsilon_n^{(i)} \to 0. \tag{2.1}$$

Moreover, by (1.4) and (1.3), $U_0^{(i)} = L_0(t^i, x) = 0$ and

$$L_n(t^i, x) = \frac{U_n^{(i)}}{n^i} \qquad (n > 0), \tag{2.2}$$

where

$$(1 - h(w))(1 - w)^{-\alpha} \sum_{k=0}^{\infty} k^i [h(w)]^k = \sum_{n=0}^{\infty} \binom{n + \alpha - 1}{n} U_n^{(i)} w^n, \qquad i = 1, 2.$$

Summing the left-hand side we obtain

$$(1 - w)^{-\alpha} \frac{h(w)}{1 - h(w)} = \sum_{n=1}^{\infty} \binom{n + \alpha - 1}{n} U_n^{(1)} w^n \tag{2.3}$$

and

$$(1 - w)^{-\alpha} \left\{ 2\left[\frac{h(w)}{1 - h(w)}\right]^2 + \frac{h(w)}{1 - h(w)} \right\} = \sum_{n=1}^{\infty} \binom{n + \alpha - 1}{n} U_n^{(2)} w^n. \tag{2.4}$$

By (2.1) and (2.2) we may write

$$\begin{aligned} \sum_{n=1}^{\infty} \binom{n + \alpha - 1}{n} U_n^{(1)} w^n &= xw \sum_{n=1}^{\infty} n \binom{n + \alpha - 1}{n} w^{n-1} + \sum_{n=1}^{\infty} n \binom{n + \alpha - 1}{n} \varepsilon_n^{(1)} w^n \\ &= \alpha x w (1 - w)^{-\alpha - 1} + \sum_{n=1}^{\infty} n \binom{n + \alpha - 1}{n} \varepsilon_n^{(1)} w^n. \end{aligned}$$

Substituting this into (2.3) gives

$$(1 - w)^{-\alpha}\frac{h(w)}{1 - h(w)} = \alpha x w(1 - w)^{-\alpha - 1} + \sum_{n=1}^{\infty} n\binom{n + \alpha - 1}{n}\varepsilon_n^{(1)} w^n. \tag{2.5}$$

Hence we have

$$h(w) = \frac{\alpha x w + Q(w)}{1 - (1 - \alpha x)w + Q(w)},$$

where

$$Q(w) = (1 - w)^{\alpha + 1}\sum_{n=1}^{\infty} n\binom{n + \alpha - 1}{n}\varepsilon_n^{(1)} w^n.$$

Clearly $Q(0) = 0$ and $Q(w)$ is analytic for $|w| < 1$. We now show that $Q(w) \to 0$ as $w \to 1$ inside of a Stolz angle. Given $\varepsilon > 0$, suppose N is large enough so that $n > N$ implies $|\varepsilon_n^{(1)}| < \varepsilon$. Then

$$\begin{aligned}
|Q(w)| &\leqq |1 - w|^{\alpha + 1}\sum_{n=1}^{N} n\binom{n + \alpha - 1}{n}|\varepsilon_n^{(1)}|\,|w|^n \\
&\quad + \varepsilon|1 - w|^{\alpha + 1}\sum_{n=N+1}^{\infty} n\binom{n + \alpha - 1}{n}|w|^n \\
&\leqq |1 - w|^{\alpha + 1}\sum_{n=1}^{N} n\binom{n + \alpha - 1}{n}|\varepsilon_n^{(1)}|\,|w|^n + \alpha\varepsilon|w|\frac{|1 - w|^{\alpha + 1}}{(1 - |w|)^{\alpha + 1}}.
\end{aligned}$$

As $w \to 1$ inside of a Stolz angle the quantity $\alpha|w|(|1 - w|^{\alpha + 1}/(1 - |w|)^{\alpha + 1})$ remains bounded and the first term above approaches zero, hence $Q(w)$ has the properties asserted in the theorem. We now show that $\alpha = 1$. By (2.1) and (2.2) we may write

$$\begin{aligned}
\sum_{n=0}^{\infty}\binom{n + \alpha - 1}{n}U_n^{(2)} w^n &= x^2\sum_{n=1}^{\infty} n^2\binom{n + \alpha - 1}{n}w^n + \sum_{n=1}^{\infty} n^2\binom{n + \alpha - 1}{n}\varepsilon_n^{(2)} w^n \\
&= \alpha(\alpha + 1)x^2 w^2(1 - w)^{-\alpha - 2} + \alpha x^2 w(1 - w)^{-\alpha - 1} \qquad (2.6)\\
&\quad + \sum_{n=1}^{\infty} n^2\binom{n + \alpha - 1}{n}\varepsilon_n^{(2)} w^n.
\end{aligned}$$

It can be shown as above that

$$(1 - w)^{\alpha + 2}\sum_{n=1}^{\infty} n^2\binom{n + \alpha - 1}{n}\varepsilon_n^{(2)} w^n \to 0$$

as $w \to 1$ inside of a Stolz angle. By (2.5) we have

$$\begin{aligned}
\frac{h(w)}{1 - h(w)}(1 - w) &= \alpha x w + Q(w) \\
&\to \alpha x \quad \text{as } w \to 1 \text{ inside of a Stolz angle.}
\end{aligned}$$

Therefore,

$$2\left[\frac{h(w)}{1-h(w)}(1-w)\right]^2 + \frac{h(w)}{1-h(w)}(1-w)^2 \to 2\alpha^2x^2 \quad \text{as } w \to 1.$$

However, by (2.4) and (2.6) we obtain

$$\begin{aligned} &2\left[\frac{h(w)}{1-h(w)}(1-w)\right]^2 + \frac{h(w)}{1-h(w)}(1-w)^2 \\ &= \alpha(\alpha+1)x^2w^2 + \alpha x^2 w(1-w) + (1-w)^{\alpha+2}\sum_{n=1}^{\infty} n^2 \binom{n+\alpha-1}{n} \varepsilon_n^{(2)} w^n \\ &\to \alpha(\alpha+1)x^2 \quad \text{as } w \to 1 \text{ inside of a Stolz angle.} \end{aligned}$$

Therefore we must have $\alpha = 1$.

Finally, if $Q(w) = \sum_{n=0}^{\infty} q_n w^n$, then

$$\frac{h(w)}{1-h(w)}(1-w)^{-1} = \frac{xw + \sum_{n=0}^{\infty} q_n w^n}{(1-w)^2},$$

so that by (2.3),

$$L_n(t; x) = \frac{U_n^{(1)}}{n} = x + \frac{1}{n}\sum_{k=0}^{n} q_k(n+1-k).$$

We now observe that if $q_k \geqq 0$ for all k and some coefficient q_k is positive, then the second term above does not converge to zero as $n \to \infty$ and hence $\lim_n L_n(t; x) \neq x$. Therefore $Q(w)$ cannot have all power series coefficients non-negative unless $Q(w) \equiv 0$, which completes the proof.

It is clear from (1.4) that if $A_{nk}(x) \geqq 0$, then the operators $L_n(f; x)$ are monotone in the sense that $f(t) \geqq 0$ for $t \in [0, 1]$ implies that $L_n(f; x) \geqq 0$ for $x \in [0, 1]$. The remarkable theorem of Korovkin [8, p. 14] then asserts that these operators have the uniform approximation property if $L_n(t^i; x) \to x^i$ uniformly in x for $i = 0, 1, 2$.

In order to produce a class of uniform approximation operators we set $Q(w) = (x - \phi(x))w(w-1)$ or

$$h^{\phi}(w) = \frac{\phi(x)w + (x-\phi(x))w^2}{1-(1-\phi(x))w + (x-\phi(x))w^2}, \tag{2.7}$$

where $\phi(x)$ is a bounded function with domain $[0, 1]$. We then define the operators $L_n^{\phi}(f; x)$ by

$$L_n^{\phi}(f; x) = \sum_{k=0}^{n} A_{nk}^{\phi}(x) f\left(\frac{k}{n}\right), \tag{2.8}$$

where

$$(1-h^{\phi}(w))(1-w)^{-1}[h^{\phi}(w)]^k = \sum_{n=k}^{\infty} A_{nk}^{\phi}(x) w^n. \tag{2.9}$$

By considering the power series representation of (2.7) for $|w|$ sufficiently small and the identity (2.9), it is easy to see that the matrix entries $A_{nk}^{\phi}(x)$ are polynomials in x and $\phi(x)$ and hence so are the operators $L_n^{\phi}(f; x)$. We also note that $L_n^{\phi}(f; x) = B_n(f; x)$ if $\phi(x) \equiv x$.

THEOREM 2.2. *If* $((1 + \phi(x))/2)^2 \geqq x \geqq \phi(x) \geqq 0$, *then the operators* $L_n^{\phi}(f; x)$ *have the uniform approximation property.*

Proof. First we show that $A_{nk}^{\phi}(x) \geqq 0$. By (2.7) and (2.9) we have for $|w|$ sufficiently small,

$$\sum_{k=0}^{\infty} A_{nk}^{\phi}(x) w^n = \frac{[\phi(x)w + (x - \phi(x))w^2]^k}{[1 - (1 - \phi(x))w + (x - \phi(x))w^2]^{k+1}}.$$

Since $x \geqq \phi(x) \geqq 0$, it is therefore sufficient to show that

$$[1 - (1 - \phi(x))w + (x - \phi(x))w^2]^{-1} \tag{2.10}$$

has nonnegative power series coefficients. Clearly we may assume that $x > \phi(x)$ and then we have

$$1 - (1 - \phi(x))w + (x - \phi(x))w^2 = (x - \phi(x))e_1 e_2\left(1 - \frac{w}{e_1}\right)\left(1 - \frac{w}{e_2}\right), \tag{2.11}$$

where e_1 and e_2 are the roots of the quadratic in (2.11). Using the hypotheses of the theorem it can now be routinely verified that e_1 and e_2 are positive and hence (2.10) has nonnegative coefficients.

By substituting $s_k = 1$ in (1.2) we note that $L_n^{\phi}(1; x) = 1$ for all n. If we take $\alpha = 1$ and $h(w) = h^{\phi}(w)$ in (1.2) and set $s_k = k$ and $s_k = k^2$ respectively, we obtain as in (2.3) and (2.4),

$$L_n^{\phi}(t^i; x) = \frac{U_n^{(i)}}{n^i} \quad \text{for } i = 1, 2 \qquad (n > 0),$$

where

$$\sum_{n=0}^{\infty} U_n^{(1)} w^n = (1 - w)^{-1} \frac{h^{\phi}(w)}{1 - h^{\phi}(w)} = \frac{\phi(x)w + (x - \phi(x))w^2}{(1 - w)^2} \tag{2.12}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} U_n^{(2)} w^n &= (1 - w)^{-1}\left\{2\left[\frac{h^{\phi}(w)}{1 - h^{\phi}(w)}\right]^2 + \frac{h^{\phi}(w)}{1 - h^{\phi}(w)}\right\} \\ &= 2\frac{[\phi(x)w + (x - \phi(x))w^2]^2}{(1 - w)^3} + \frac{\phi(x)w + (x - \phi(x))w^2}{(1 - w)^2}. \end{aligned} \tag{2.13}$$

Straightforward calculations using (2.12) and (2.13) give

$$\frac{U_n^{(1)}}{n} = x - \frac{(x - \phi(x))}{n} \tag{2.14}$$

and

$$\frac{U_n^{(2)}}{n^2} = x^2 + \frac{x}{n}(1 + 4\phi(x) - 5x) + \frac{(x - \phi(x))}{n^2}(6x - 2\phi(x) - 1). \tag{2.15}$$

Therefore $L_n^\phi(t^i; x) \to x^i$ uniformly in x for $i = 0, 1, 2$ and the proof is completed by an application of Korovkin's theorem.

3. Order of convergence and asymptotic form of the remainder. As we pointed out prior to Theorem 2.2, the matrix entries $A_{nk}(x)$ are polynomials in x and $\phi(x)$. Hence in order for $L_n^\phi(f; x)$ to be a polynomial operator we must choose $\phi(x)$ to be a polynomial satisfying

$$0 \leqq \phi(x) \leqq x \leqq \frac{[1 + \phi(x)]^2}{4}.$$

A particularly simple class of such polynomials was pointed out to the authors by R. F. DeMar. We take $\phi(x) = x[1 - a(1 - x)^2]$, where $0 \leqq a \leqq 1/4$. When $a = 0$ the resulting operators $L_n^\phi(f; x)$ are just the Bernstein operators. We will now give an order of convergence result for these operators. We recall that the modulus of continuity of a real function $f(x)$ is the function $\omega(\delta)$ defined for $\delta > 0$ by

$$\omega(\delta) = \sup \{|f(x) - f(y)| : |x - y| < \delta\}.$$

THEOREM 3.1. *Suppose that* $\phi(x) = x(1 - a(1 - x)^2)$, *where* $0 \leqq a \leqq 1/4$. *Then*

$$|f(x) - L_n^\phi(f; x)| \leqq \frac{5}{4}\omega\left(\frac{\sqrt{4n - 2a(n - a - 2)}}{2n}\right).$$

Proof. The argument of Lorentz [9, p. 20] gives

$$|f(x) - L_n^\phi(f; x)| \leqq \omega(\delta)\left\{1 + \delta^{-2} \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 A_{nk}^\phi(x)\right\}$$

for each $\delta > 0$. Therefore if

$$q(n) \geqq \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 A_{nk}^\phi(x) = x^2 - 2xL_n^\phi(t; x) + L_n^\phi(t^2; x)$$

for each $x \in [0, 1]$, then

$$|f(x) - L_n^\phi(f; x)| \leqq \omega(\delta)\{1 + \delta^{-2}q(n)\}.$$

Setting $\delta = 2\sqrt{q(n)}$ we obtain

$$|f(x) - L_n^\phi(f; x)| \leqq \tfrac{5}{4}\omega(2\sqrt{q(n)}). \tag{3.1}$$

If $\phi(x) = x(1 - a(1 - x)^2)$, then we have by (2.14) and (2.15),

$$\begin{aligned} x^2 - 2xL_n^\phi(t; x) + L_n^\phi(t^2; x) &= x^2 - 2x\frac{U_n^{(1)}}{n} + \frac{U_n^{(2)}}{n^2} \\ &= [2x(x - \phi(x)) + x(1 + 4\phi(x) - 5x)]/n + (x - \phi(x))[6x - 2\phi(x) - 1]/n^2 \\ &\leqq [nx(1 - x) + (4 - 2n)ax^2(1 - x)^2 + 2a^2x^2(1 - x)^4]/n^2 \\ &\leqq [nx(1 - x) + (4 - 2n + 2a)ax^2(1 - x)^2]/n^2 \\ &\leqq [4n - 2a(n - a - 2)]/16n^2. \end{aligned}$$

Substituting this into (3.1) completes the proof.

We note that for $a = 0$ the bound given in Theorem 3.1 agrees with that given by Popoviciu for the Bernstein polynomials (see [9, p. 20]). It is also of interest to note that if $a > 0$, then for $n > 2$,

$$\frac{\sqrt{4n - 2a(n - a - 2)}}{2n} < \frac{1}{\sqrt{n}}$$

and hence the operators $L_n^\phi(f; x)$ have a more favorable order of convergence bound than the Bernstein polynomials if $\phi(x) = x(1 - a(1 - x)^2)$ $(a > 0)$.

Our final theorem extends Voronowskaja's asymptotic estimate of the remainder of the Bernstein polynomials (see [9, p. 22]) to the operators $L_n^\phi(f; x)$, where ϕ is any function satisfying the hypothesis of Theorem 2.2.

THEOREM 3.2. *Suppose that* $((1 + \phi(x))/2)^2 \geqq x \geqq \phi(x) \geqq 0$. *If f is bounded on* $[0, 1]$ *and has a second derivative at* $x \in [0, 1]$, *then*

$$\lim_n n[f(x) - L_n^\phi(f; x)] = \frac{-x(1 - x)}{2} f''(x) + (x - \phi(x)) \frac{d}{dt}[tf'(t)]_{t=x}.$$

Proof. By L'Hospital's rule we have

$$L_n^\phi(f; x) = f(x) + f'(x) L_n^\phi(t - x; x) + \frac{f''(x)}{2} L_n^\phi((t - x)^2; x) + \rho_n(x),$$

where

$$\rho_n(x) = L_n^\phi((t - x)^2 \varepsilon(t); x) \quad \text{and} \quad \varepsilon(t) \to 0 \text{ as } t \to x. \tag{3.2}$$

It follows from (2.14) and (2.15) that

$$\begin{aligned} f(x) - L_n^\phi(f; x) = f'(x) \frac{(x - \phi(x))}{n} - \frac{f''(x)}{2} \left[\frac{x(1 - x) - 2x(x - \phi(x))}{n} \right] \\ - \frac{f''(x)}{2} \frac{(6x - 2\phi(x) - 1)(x - \phi(x))}{n^2} - \rho_n(x). \end{aligned} \tag{3.3}$$

By (2.8) and (3.2) we have

$$|\rho_n(x)| \leqq \sum_{k=0}^n A_{nk}^\phi(x) \left[\frac{k}{n} - x \right]^2 \left| \varepsilon\left(\frac{k}{n}\right) \right|. \tag{3.4}$$

Given $\varepsilon > 0$, suppose that $\delta > 0$ is small enough so that $|k/n - x| \leqq \delta$ implies that $|\varepsilon(k/n)| < \varepsilon$. Denote by $\sum_1$, the sum of the terms in (3.4) extended over all k satisfying $|k/n - x| \leqq \delta$ and denote the complementary sum by $\sum_2$. Using (2.14) and (2.15) it is easy to see that $\sum_1 < \varepsilon x(1 - x)/n$ for all large n.

For a given n, we will denote by X_n a random variable with distribution $A_{nk}^\phi(x)$, i.e., $P\{X_n = k\} = A_{nk}^\phi(x)$. Then the mean μ_n of X_n is given by

$$\mu_n = nL_n^\phi(t; x) = nx - (x - \phi(x))$$

and the variance is

$$\begin{aligned}\sigma_n^2 &= \sum_{k=0}^{n} (k - \mu_n)^2 A_{nk}^{\phi}(x) \\ &= n^2 L_n^{\phi}(t^2; x) - 2n\mu_n L_n^{\phi}(t; x) + \mu_n^2 L_n^{\phi}(1; x) \\ &= O(n), \qquad\qquad \text{by (2.14) and (2.15).}\end{aligned}$$

For n sufficiently large ($n > N(\delta)$),

$$\left|\frac{k}{n} - x - \frac{x - \phi(x)}{n}\right| > 2\delta \quad \text{implies} \left|\frac{k}{n} - x\right| > \delta.$$

Therefore if M is an upper bound for $(k/n - x)^2|\varepsilon(k/n)|$, we have by use of Chebyshev's inequality [4, p. 219],

$$\begin{aligned}\sum\nolimits_2 &\leqq M \sum \left\{ A_{nk}^{\phi}(x) : \left|\frac{k}{n} - x\right| > \delta \right\} \\ &\leqq M \sum \left\{ A_{nk}^{\phi}(x) : \left|\frac{k}{n} - x + \frac{x - \phi(x)}{n}\right| > 2\delta \right\} \\ &= MP\{|X_n - \mu_n| > 2n\delta\} \leqq M \frac{\sigma_n^2}{4\delta^2 n^2} = o\left(\frac{1}{n}\right).\end{aligned}$$

It follows that $\rho_n(x) = o(1/n)$ and therefore by (3.3),

$$f(x) - L_n^{\phi}(f; x) = \frac{-x(1 - x)}{2n} f''(x) + \frac{(x - \phi(x))}{n}[f'(x) + xf''(x)] + o\left(\frac{1}{n}\right),$$

completing the proof.

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MULTIDIMENSIONAL STATIONARY PHASE AN ALTERNATIVE DERIVATION*

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Abstract. The method of multidimensional stationary phase is derived via a technique which makes strong use of integration by parts. The "diagonalization" of the matrix of second derivatives at the stationary point is carried out here in such a manner as to make all coefficients in the exponent ± 1. This modification of existing technique allows for the explicit calculation of the nth term of the asymptotic expansion in a closed form which involves the amplitude of the integrand in transformed coordinates. The first correction term in the multidimensional stationary phase formula is readily calculated from this result.

1. Introduction. The analysis of multidimensional Fourier integrals,

$$I(\lambda) = \int_{\mathscr{D}} g_0(\mathbf{x}) \exp\{i\lambda\phi(\mathbf{x})\}\, d\mathbf{x}, \qquad \mathbf{x} = (x_1, \cdots . x_n) \tag{1.1}$$

has been discussed by many authors, notably Focke [3], Jones and Kline [6], Lewis [7], Chako [1] and Jones [5]. A major result of this analysis is the multidimensional stationary phase formula on which all of the authors, including us, agree. Our derivation, however, differs from those in the literature. This alternative derivation allows for the calculation of new results.

Our analysis makes strong use of integration by parts (the divergence theorem), and thus the relation between the standard one-dimensional development and the multidimensional case becomes more transparent. We find, moreover, that the integration by parts procedure allows for easy identification of the critical points. This means of identification of critical points was previously used by Erdélyi [2] for the two-dimensional integral.

All derivations of the multidimensional stationary phase formula begin with a transformation that "diagonalizes" the matrix of second derivatives (the Hessian) of the phase function ϕ at the stationary point. The phase now is locally quadratic having only square terms, i.e., no cross-product terms. In previous derivations, the coefficients of the squares are the eigenvalues of the Hessian. In our derivation, we diagonalize so that these coefficients are ± 1. Furthermore, we do this via a transformation which makes this diagonalization valid in a neighborhood of the stationary point. This change allows us to write down the nth term of the asymptotic expansion in a closed form which involves the amplitude of the integrand in transformed coordinates. From this result, we are readily able to calculate the first correction term in the multidimensional stationary phase formula in terms of the original variables, thereby correcting a result in Chako [1]. Higher order corrections are still, admittedly, tedious to calculate.

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Because of the extensive background literature cited above, we shall minimize motivational discussion and primarily discuss those aspects of the development in which our result differs from previous results.

We remark that the same techniques are applicable to n-fold Laplace integrals, and thus we could recapture the results of Hsu [4].

2. Integration by parts and critical points. We first use integration by parts to recast (1.1) in another form when $\nabla\phi \neq \mathbf{0}$. This is accomplished in the following lemma.

LEMMA 2.1. *Suppose that in* $\bar{\mathscr{D}}$, (i) $\nabla\phi \neq \mathbf{0}$, (ii) ϕ *and* g_0 *are* M *times continuously differentiable.*[1] *Then*

$$\begin{aligned} I(\lambda) &= -\sum_{j=0}^{M-1}(-i\lambda)^{-(j+1)}\int_\Gamma (\mathbf{H}_j\cdot\mathbf{N})\exp\{i\lambda\phi\}\,d\Sigma \\ &\quad +(-i\lambda)^{-M}\int_{\mathscr{D}} g_M(\mathbf{x})\exp\{i\lambda\phi(\mathbf{x})\}\,d\mathbf{x}. \end{aligned} \tag{2.1}$$

Here Γ *denotes the* $(n-1)$*-dimensional boundary of* $\mathscr{D}$, $d\Sigma$ *denotes the differential element of content of the boundary, and the functions* $\mathbf{H}_j$ *and* g_j *are defined recursively by*

$$H_j = g_j\frac{\nabla\phi}{|\nabla\phi|^2}, \qquad g_{j+1} = \nabla\cdot\mathbf{H}_j, \qquad j = 0, 1, \cdots. \tag{2.2}$$

Remark. One sees in (2.2) the generalization of the recursion formula for the one-dimensional case. Also, in one dimension, the sum in (2.1) involves only the integrand evaluated at the endpoints of integration (boundary of $\mathscr{D}$).

Proof. In (1.1), set

$$\begin{aligned} g_0(\mathbf{x}_0)\exp\{i\lambda\phi\} &= (i\lambda)^{-1}\nabla\cdot\left\{\frac{g_0(\mathbf{x}_0)}{|\nabla\phi|^2}\nabla\phi\exp\{i\lambda\phi\}\right\} \\ &\quad -(i\lambda)^{-1}\nabla\cdot\left\{\frac{g_0(\mathbf{x}_0)}{|\nabla\phi|^2}\nabla\phi\right\}\exp\{i\lambda\phi\}. \end{aligned} \tag{2.3}$$

Upon substituting this into (1.1) and applying the divergence theorem to the first integral, we obtain (2.1) with M replaced by 1. We now apply the expansion procedure (2.3) to g_1 and integrate by parts to obtain (2.1) with M replaced by 2. Upon applying the expansion procedure (2.3) and integrating by parts M times, (2.1) is obtained. This completes the proof.

Under somewhat stronger conditions we have Lemma 2.2.

LEMMA 2.2. *Suppose that* (i) $\nabla\phi \neq \mathbf{0}$ *in* $\mathscr{D}$, (ii) ϕ *and* g_0 *are infinitely differentiable* (C^∞) *in* $\mathscr{D}$, *and* (iii) g_0 *vanishes* C^∞*-smoothly on* Γ. *Then*

$$I(\lambda) = o(\lambda^{-R}), \qquad \text{any } R. \tag{2.4}$$

Proof. Each of the boundary integrals in (2.1) is now zero, and the integration by parts process may be repeated an arbitrary number of times to yield (2.4). This completes the proof.

[1] Derivatives on the boundary are taken as their limits from the interior.

Remark. We conclude from Lemma 2.2 that the candidates for critical points of a multidimensional Fourier integral are

(i) points where $\nabla\phi = \mathbf{0}$, i.e., stationary points,
(ii) points where ϕ or g_0 fail to be C^∞,
(iii) the boundary of $\mathscr{D}$.

Through the use of *neutralizers*, we may isolate the various critical points of an integral. Indeed, we introduce the following definition.

DEFINITION. A function $\nu(\mathbf{x}, \mathbf{x}_0)$ is said to be a *neutralizer about* $\mathbf{x} = \mathbf{x}_0$ if

(i) there exists a neighborhood N_0 of $\mathbf{x}_0$ throughout which $\nu(\mathbf{x}, \mathbf{x}_0) \equiv 1$,
(ii) there exists a neighborhood N_1 of $\mathbf{x}_0$, containing N_0, outside of which $\nu(\mathbf{x}, \mathbf{x}_0) \equiv 0$,
(iii) in all of x-space, $\nu(\mathbf{x}, \mathbf{x}_0)$ is a C^∞ function with respect to its arguments and $0 \leqq \nu \leqq 1$.

Let us suppose that the integrand of I in (1.1) has isolated critical points of types (i) or (ii) in $\overline{\mathscr{D}}$ located at $\mathbf{x}_1, \cdots, \mathbf{x}_k$. Then we set

$$1 = \sum_{i=1}^{k} \nu_i(\mathbf{x}, \mathbf{x}_i) + \left[1 - \sum_{i=1}^{k} \nu_i(\mathbf{x}, \mathbf{x}_i)\right]. \tag{2.5}$$

Here we have taken the supports of the neutralizers to be disjoint from one another and, for interior critical points, disjoint from the boundary as well.

We substitute (2.5) into (1.1), thereby obtaining $k + 1$ integrals, the first k of which contain isolated critical points of types (i) and (ii), while the last has only a portion of the boundary as critical points. We could now discuss the various types of critical points and the contributions to the asymptotic expansion of $I(\lambda)$ arising from them. We remark that the neutralization process could certainly be extended to cover the case of interior critical curves, surfaces, hypersurfaces, etc., as well. We choose however to limit our considerations here to stationary critical points and refer the reader to Jones [5] for a detailed analysis of nonstationary critical points.

3. The multidimensional stationary phase formula. We consider now the integral

$$I_0(\lambda) = \int_{\mathscr{D}} g_0(\mathbf{x})\nu(\mathbf{x}, \mathbf{x}_0) \exp\{i\lambda\phi(\mathbf{x})\}\, d\mathbf{x} \tag{3.1}$$

in the case where ϕ has a simple stationary point at $\mathbf{x}_0$; i.e.,

$$\nabla\phi(\mathbf{x}_0) = \mathbf{0}, \tag{3.2}$$

but

$$\det A \neq 0, \qquad A = (\phi_{x_i x_j}), \qquad i, j = 1, \cdots, n. \tag{3.3}$$

The neutralizer $\nu(\mathbf{x}, \mathbf{x}_0)$ is chosen so that, in its support N_1, $\mathbf{x}_0$ is the *only* critical point of $I_0(\lambda)$. The positive eigenvalues of A are denoted by $\lambda_1, \cdots, \lambda_r$, the negative eigenvalues by $\lambda_{r+1}, \cdots, \lambda_n$. The signature of A (denoted by sig A) is given by

$$\operatorname{sig} A = 2r - n. \tag{3.4}$$

As is well known, there exists an orthogonal matrix Q which diagonalizes A. Furthermore, when one sets $(\mathbf{x} - \mathbf{x}_0)^T = QR\mathbf{z}^T$, with R the diagonal matrix

$$R = \operatorname{diag}\{|\lambda_i|^{-1/2}\}, \tag{3.5}$$

then

$$f(z) = \phi(\mathbf{x}(\mathbf{z})) - \phi(\mathbf{x}_0) \sim \frac{1}{2}\left\{\sum_{i=1}^{r} z_i^2 - \sum_{i=r+1}^{n} z_i^2\right\} \tag{3.6}$$

as $|\mathbf{z}| \to 0$. To make this approximate behavior hold throughout the effective domain of integration in (3.1), we introduce the second change of variables defined by

$$\xi_i = h_i(\mathbf{z}), \qquad i = 1, \cdots, n; \qquad h_i(\mathbf{z}) = z_i + o(|z|), \qquad |z| \to 0; \tag{3.7}$$

and

$$\sum_{i=1}^{r} h_i^2 - \sum_{i=r+1}^{n} h_i^2 = 2f. \tag{3.8}$$

Milnor [8, pp. 6–8] proves that such a nonsingular transformation exists in some neighborhood of the stationary point. This reference was cited for the authors by the referee.

Since, in N_1, $\nabla\phi$ vanishes only at $\mathbf{x} = \mathbf{x}_0$, the functions h_i can be chosen so that the Jacobian

$$J(\boldsymbol{\xi}) = \left|\det\left(\frac{\partial x_i}{\partial \xi_j}\right)\right|, \qquad \boldsymbol{\xi} = (\xi_1, \cdots, \xi_n), \tag{3.9}$$

is finite and nonzero throughout $\hat{N}_1$, the image of N_1 under (3.5), (3.7). Furthermore, one can readily check that

$$J(\mathbf{0}) = |\det A|^{-1/2} = \left(\prod_{j=1}^{n} |\lambda_j|\right)^{-1/2}. \tag{3.10}$$

In terms of $\boldsymbol{\xi}$, (3.1) becomes

$$I_0(\lambda) = \exp\{i\lambda\phi(\mathbf{x}_0)\} \int_{\hat{N}_1} G_0(\boldsymbol{\xi})\nu(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}. \tag{3.11}$$

Here

$$\boldsymbol{\rho} = (\xi_1, \cdots, \xi_r, -\xi_{r+1}, \cdots, -\xi_n), \tag{3.12}$$

$$G_0(\boldsymbol{\xi}) = g_0(\mathbf{x}(\boldsymbol{\xi}))J(\boldsymbol{\xi}), \tag{3.13}$$

and $\nu(\boldsymbol{\xi}, \mathbf{0})$ is a neutralizer with support in $\hat{N}_1$.

We set

$$G_0(\boldsymbol{\xi}) = G_0(\mathbf{0}) + \boldsymbol{\rho}\cdot\mathbf{H}_0, \tag{3.14}$$

with $\mathbf{H}_0$ chosen so that it is well-behaved throughout $\hat{N}_1$. For example, we may take the components of $\mathbf{H}_0$ to be

$$
\begin{aligned}
H_{0,1} &= \frac{1}{\xi_1}[G_0(\xi_1, \cdots, \xi_n) - G_0(0, \xi_2, \cdots, \xi_n)], \\
H_{0,2} &= \frac{1}{\xi_2}[G_0(0, \xi_2, \xi_3, \cdots, \xi_n) - G_0(0, 0, \xi_3, \cdots, \xi_n)], \\
H_{0,r+1} &= -\frac{1}{\xi_{r+1}}[G_0(0, \cdots, 0, \xi_{r+1}, \xi_{r+2}, \cdots, \xi_n) \\
&\qquad\qquad - G_0(0, \cdots, 0, \xi_{r+2}, \cdots, \xi_n)], \\
H_{0,n} &= -\frac{1}{\xi_n}[G_0(0, \cdots, 0, \xi_n) - G_0(\mathbf{0})].
\end{aligned} \tag{3.15}
$$

We shall see below that the ambiguity in $\mathbf{H}_0$ will not affect the asymptotic expansion we derive. We substitute (3.14) into (3.4) and obtain

$$I_0(\lambda) = \exp\{i\lambda\phi(\mathbf{x}_0)\}[I_0^{(2)}(\lambda)]. \tag{3.16}$$

Here

$$
\begin{aligned}
I_0^{(1)}(\lambda) &= G_0(\mathbf{0}) \int_{\hat{N}_1} v(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}, \\
I_0^{(2)}(\lambda) &= \int_{\hat{N}_1} (\boldsymbol{\rho}\cdot\mathbf{H}_0) v(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}.
\end{aligned} \tag{3.17}
$$

We first consider $I_0^{(2)}$. Application of the divergence theorem yields

$$
\begin{aligned}
I_0^{(2)}(\lambda) &= \frac{1}{i\lambda}\int_{N_1} [v\nabla_\xi\cdot\mathbf{H}_0 + \mathbf{H}_0\cdot\nabla_\xi v] \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}, \\
\nabla_\xi &= \left(\frac{\partial}{\partial\xi_1}, \cdots, \frac{\partial}{\partial\xi_n}\right).
\end{aligned} \tag{3.18}
$$

The support of $\mathbf{H}_0\cdot\nabla_\xi v$ is that annular region A on which v itself is nonconstant. In this region $\nabla\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2 \neq 0$. Thus, for the integral in (3.18) with $\mathbf{H}_0\cdot\nabla_\xi v$ as amplitude, Lemma 2.2 applies, and we conclude that

$$I_0^{(2)}(\lambda) = -\frac{1}{i\lambda}\int_{\hat{N}_1} v(\boldsymbol{\xi}, \mathbf{0}) G_1(\boldsymbol{\xi}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi} + o(\lambda^{-R}), \tag{3.19}$$

for all R. Here

$$G_1(\boldsymbol{\xi}) = \nabla_\xi\cdot\mathbf{H}_0. \tag{3.20}$$

Upon applying this procedure n times, we obtain

$$
\begin{aligned}
I(\lambda) &\sim \exp\{i\lambda\phi(\mathbf{x}_0)\} \\
&\quad\cdot \sum_{m=0}^{M-1} (-i\lambda)^{-m} G_m(\mathbf{0}) \int_{\hat{N}_1} v(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi} \\
&\qquad + (i\lambda)^{-M} \int_{\hat{N}_1} G_M(\boldsymbol{\xi}) v(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho}\cdot\boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}.
\end{aligned} \tag{3.21}
$$

The functions $G_m(\boldsymbol{\xi})$ are defined recursively by

$$G_m(\boldsymbol{\xi}) = G_m(\mathbf{0}) + \boldsymbol{\rho} \cdot \mathbf{H}_m(\boldsymbol{\xi}), \qquad m = 0, 1, \cdots . \tag{3.22}$$
$$G_{m+1}(\boldsymbol{\xi}) = \nabla_{\xi} \cdot \mathbf{H}_m(\boldsymbol{\xi}),$$

This result is simplified with the aid of the next lemma.

LEMMA 3.1. *Let*

$$K(\lambda) = \int_{\mathscr{D}} v(\boldsymbol{\xi}, \mathbf{0}) \exp\{i\lambda\boldsymbol{\rho} \cdot \boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}, \tag{3.23}$$

where $\boldsymbol{\rho}$ *is defined by* (3.12) *and* v *is any neutralizer about* $\boldsymbol{\xi} = \mathbf{0}$. *If the support of* v *is in* $\mathscr{D}$, *then*

$$K(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{n/2} \exp\left\{\frac{\pi i}{4}(2r - n)\right\} + o(\lambda^{-R}), \tag{3.24}$$

for all R.

Proof. Let $v_j(\xi_j, 0)$, $j = 1, \cdots, n$, be one-dimensional neutralizers such that the support of

$$\bar{v}(\boldsymbol{\xi}, \mathbf{0}) = \prod_{j=1}^{n} v_j(\xi_j, 0) \tag{3.25}$$

is completely inside the domain in which $v(\boldsymbol{\xi}, \mathbf{0}) \equiv 1$. Set

$$\begin{aligned} K(\lambda) = {} & \prod_{j=1}^{n} \int v_j(\xi_j, 0) \exp\{i\lambda\rho_j \cdot \xi_j/2\}\, \mathrm{d}\xi_j \\ & + \int (v - \bar{v}) \exp\{i\lambda\boldsymbol{\rho} \cdot \boldsymbol{\xi}/2\}\, d\boldsymbol{\xi}. \end{aligned} \tag{3.26}$$

By Lemma 2.2, the second integral is $o(\lambda^{-R})$ for all R. For the first integral, we can apply the one-dimensional theory to each factor, say $K_j(\lambda)$, to show that

$$K_j(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{1/2} \exp\left\{\frac{i\pi}{4} \operatorname{sig} \rho_j\right\} + o(\lambda^{-R}), \tag{3.27}$$

for all R. By multiplying these factors together and adding the estimate on the second integral in (3.26), we obtain (3.24). This completes the proof.

More surprisingly, the coefficients $G_m(\mathbf{0})$ in (3.21) can all be simply expressed in terms of $G_0(\boldsymbol{\xi})$. This is one of our main results and is proved in the following.

LEMMA 3.2. *For the functions* $G_m(\boldsymbol{\xi})$ *defined by* (3.22),

$$G_m(\mathbf{0}) = \frac{1}{2^m m!} \bar{\Delta}^m G_0(\mathbf{0}). \tag{3.28}$$

Here the operator $\bar{\Delta}$ *is defined by*

$$\bar{\Delta} = \frac{\partial^2}{\partial \xi_1^2} \cdots + \frac{\partial^2}{\partial \xi_r^2} - \frac{\partial^2}{\partial \xi_{r+1}^2} - \cdots - \frac{\partial^2}{\partial \xi_n^2}. \tag{3.29}$$

Proof. For any vector $\mathbf{F}$, we have

$$\bar{\Delta}(\boldsymbol{\rho} \cdot \mathbf{F}) = \boldsymbol{\rho} \cdot \bar{\Delta}\mathbf{F} + 2\nabla \cdot \mathbf{F}, \tag{3.30}$$

and by direct computation,

$$\bar{\Delta}^m(\boldsymbol{\rho} \cdot \mathbf{F})_{\boldsymbol{\xi}=\mathbf{0}} = 2m\bar{\Delta}^{m-1}(\nabla \cdot \mathbf{F})|_{\boldsymbol{\xi}=\mathbf{0}}. \tag{3.31}$$

We thus find, for any j,

$$\bar{\Delta}^m G_j(\mathbf{0}) = \bar{\Delta}^m \boldsymbol{\rho} \cdot \mathbf{H}_j|_{\boldsymbol{\xi}=\mathbf{0}} = 2m\bar{\Delta}^{m-1}\nabla \cdot \mathbf{H}_j|_{\boldsymbol{\xi}=\mathbf{0}}. \tag{3.32}$$

By starting with $j = 0$ and using (3.22) repeatedly to replace $\nabla \cdot \mathbf{H}_j$, we arrive at (3.28) after m applications of (3.22). This completes the proof.

By using the results of these two lemmas in (3.21), we obtain

$$I_0(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} \exp\left\{\frac{i\pi}{4}(2r - n) + i\lambda\phi(\mathbf{x}_0)\right\} \sum_{m=0}^{\infty} \left(\frac{i}{2\lambda}\right)^m \frac{\bar{\Delta}^m G_0(\mathbf{0})}{m!}. \tag{3.33}$$

By using (3.10) and (3.13), we can express $G_0(\mathbf{0})$ in terms of g_0 and ϕ. Thus, to leading order, we obtain the well-known multidimensional stationary phase formula

$$I_0(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{n/2} \frac{g_0(\mathbf{x}_0)}{\sqrt{|\det A|}} \exp\left\{\frac{i\pi}{4}(2r - n) + i\lambda\phi(\mathbf{x}_0)\right\}. \tag{3.34}$$

In order to find the first correction term to this expansion, we must use (3.5)–(3.8) along with (3.9) and (3.13) to express $\boldsymbol{\xi}$ derivatives of G_0 and ϕ. After a great deal of computation, we find that

$$\begin{aligned} \bar{\Delta}G_0(\mathbf{0}) = |A|^{-1/2}[&\phi_{x_s x_r x_q} B_{sq} B_{rp} (g_0)_{x_p} + \operatorname{tr}(CB) \\ &+ g_0\{\phi_{x_p x_q x_r}\phi_{x_s x_t x_u}(\tfrac{1}{4}B_{ps}B_{qr}B_{tu} + \tfrac{1}{6}B_{ps}B_{qt}B_{ru}) \\ &- \tfrac{1}{4}\phi_{x_p x_q x_r x_s} B_{pr} B_{qs}\}]|_{\mathbf{x}=\mathbf{x}_0}. \end{aligned} \tag{3.35}$$

Here we have used the summation convention where repeated indices are to be summed from 1 to n. A subscript x_p denotes differentiation with respect to x_p. The matrices B and C are defined by

$$B = (B_{pq}), \quad B_{pq}\phi_{x_q x_r}(\mathbf{x}_0) = \delta_{pr}, \quad C = (g_{0_{x_p x_q}}(\mathbf{x}_0)), \tag{3.36}$$

and tr denotes the trace of the matrix.

We remark that for the integral with real exponent,

$$I(\lambda) = \int_{\mathscr{D}} g_0(\mathbf{x}) \exp(-\lambda\phi(\mathbf{x}))\, d\mathbf{x}, \tag{3.37}$$

with $\mathbf{x}_0$ a simple absolute minimum of ϕ in $\mathscr{D}$, the same results obtain with very minor modification. Namely,

(i) replace the exponent in (3.33) and (3.34) by $-\lambda\phi(\mathbf{x}_0)$;
(ii) replace $(i/2\lambda)^m$ in (3.23) by $(2\lambda)^{-m}$;
(iii) replace $\bar{\Delta}$ in (3.33), (3.35) by $-\Delta$ (since all eigenvalues are now positive, but ϕ and B have been replaced by their negatives).

If the point $\mathbf{x}_0$ is located on the boundary Γ at a smooth point of Γ with a tangent hyperplane defined there, then it is fairly easy to show that (3.34) must be multiplied by $\frac{1}{2}$ and that the same factor must be introduced in the integral (3.37), above. In either case, the first correction term is difficult to obtain.

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THE CHARACTERIZATION OF THE CYLINDER FUNCTIONS BY A FUNCTIONAL EQUATION*

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Abstract. The following is a well-known identity involving the cylinder functions. If $x_{y,\theta} \equiv [x^2 + y^2 - 2xy\cos\theta]^{1/2}$, if $f(x) = (\mu x)^{-\nu} Z_\nu(\mu x)$ and if $H(y) = \pi^{-1/2}\Gamma(\nu + \frac{1}{2})(2/\mu y)^\nu J_\nu(\mu y)$, then

$$\frac{1}{2\pi}\int_0^{2\pi} f(x_{y,\theta})|\sin\theta|^{2\nu}\,d\theta = f(x)H(y), \qquad 0 \leqq y < x < \infty, \nu > -\tfrac{1}{2}.$$

It can be seen that the limit $\lim_{y\to 0^+} y^{-2}[H(0) - H(y)]$ exists.

In the present paper the author proves two theorems. One shows that if $\nu > -\frac{1}{2}$ and $f \in C(0, \infty)$, then there are no other real, nontrivial solutions of this functional equation for which the stated limit exists. His other theorem shows that provided $\nu \geqq 0$, then the condition $f \in C(0, \infty)$ will follow merely from the assumptions that the functional equation is meaningful (the left-hand side being a Lebesgue integral) and that H is continuous from the right at zero. He concludes by referring briefly to similar problems which have been considered by other authors.

1. Introduction. If Z_ν denotes any cylinder function of order $\nu > -\frac{1}{2}$, then the following identity holds (see [1, 7.7.2 (14)]):

$$\int_0^\pi (z_{\zeta,\theta})^{-\nu} Z_\nu(z_{\zeta,\theta})(\sin\theta)^{2\nu}\,d\theta = \frac{2\pi\Gamma(2\nu)}{(2z\zeta)^\nu\Gamma(\nu)} Z_\nu(z)J_\nu(\zeta), \tag{1.1}$$

where $z_{\zeta,\theta} \equiv [z^2 + \zeta^2 - 2z\zeta\cos\theta]^{1/2}$ and $0 \leqq |\zeta| < |z| < \infty$. From this it follows that the functional equation

$$\frac{1}{2\pi}\int_0^{2\pi} f(x_{y,\theta})|\sin\theta|^{2\nu}\,d\theta = f(x)H(y), \qquad 0 \leqq y < x < \infty, \tag{1.2}$$

is satisfied by

$$f(x) = (\mu x)^{-\nu} Z_\nu(\mu x), \qquad H(y) = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}}\left(\frac{2}{\mu y}\right)^\nu J_\nu(\mu y), \tag{1.3}$$

where μ denotes any nonzero complex number. Note that for this solution the limit

$$\lim_{y\to 0+} y^{-2}[H(0) - H(y)] \text{ exists.} \tag{1.4}$$

The value of this limit is

$$\frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu + 2)}\frac{\mu^2}{4}.$$

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The solution in (1.3) is meaningless if we put $\mu = 0$. However if we formally let $\mu \to 0$ in (1.3) we are led to expect that solutions of (1.2) will be

$$(1.5)\quad \begin{aligned} &f(x) = P + Qx^{-2\nu}, \qquad H(y) = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu + 1)}, \qquad \nu > -\tfrac{1}{2}, \quad \nu \neq 0, \\ &f(x) = P + Q \log x, \qquad H(y) = 1, \qquad \nu = 0, \end{aligned}$$

and in fact these are easily verified. The case $\nu = 0$ is elementary and the case $\nu \neq 0$ can be deduced from [1, 3.15.1 (18)]). In these cases the limit $\lim_{y \to 0+} y^{-2}[H(0) - H(y)]$ again exists and has the value zero.

It is our purpose to study the functional equation (1.2) subject to the side condition (1.4). We shall assume throughout that $\nu > -\frac{1}{2}$ and that all values are real with the single exception of the variable λ introduced in Theorem 1 below for which we assume that λ^2 is real.

For the functional equation (1.2) to be meaningful at all, some condition on f must be imposed to ensure the existence of the left-hand side. Clearly the most desirable hypothesis would be merely that this integral exists. It will be found that this hypothesis is sufficient if $\nu \geqq 0$, but for $-\frac{1}{2} < \nu < 0$ we shall assume continuity of f in $(0, \infty)$. Our results are embodied in the following two theorems.

THEOREM 1. *Suppose that* $\nu > -\frac{1}{2}$ *and that* $f \in C(0, \infty)$. *Then the only real nontrivial solutions of* (1.2) *for which*

$$\lim_{y \to 0+} y^{-2}[H(0) - H(y)] \quad \textit{exists} \quad \left(= \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu + 2)} \frac{\lambda^2}{4} \,(\textit{say}) \right)$$

are given by (1.3) *with* μ *put equal to* λ *in case* $\lambda \neq 0$ *and by* (1.5) *if* $\lambda = 0$.

THEOREM 2. *If* $\nu \geqq 0$ *and* (1.2) *is meaningful, the integral on the left being a Lebesgue integral and if* H *is continuous from the right at zero, then* $f \in C(0, \infty)$.

We shall make use of an operator A_ν defined as follows:

$$(1.6)\quad (A_\nu g)(x) = \lim_{h \to 0+} \frac{\sqrt{\pi}\,\Gamma(\nu + 2)}{\Gamma(\nu + \frac{1}{2})} \frac{2}{\pi h^2} \int_0^{2\pi} \{g(x_{h,\theta}) - g(x)\} |\sin \theta|^{2\nu} \, d\theta,$$

and certain properties of this operator must be obtained before proceeding. This will be done in § 2.

We conclude this section by mentioning how the present problem arose. In [2] we considered the uniqueness of representation of a function defined in (0, 1) by series of the form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n J_0(s_n x)$. To this end, the operator A_0 was introduced and was found to be effective largely due to the existence of the identity

$$\frac{1}{2\pi} \int_0^{2\pi} J_0(x_{y,\theta}) \, d\theta = J_0(x) J_0(y).$$

The question then arose of the further usefulness of the operator A_0 in connection with other series expansions, and this led to consideration of the functional equation obtained by putting $\nu = 0$ in (1.2). For the investigation carried out in [2] it was essential that the limit $\lim_{y \to 0+} y^{-2}[H(0) - H(y)]$ should exist. Hence so far as our original motivation is concerned the side condition (1.4) is no restriction at all. The condition (1.4) is equivalent to the hypothesis that $(A_\nu f)(a)$ should exist at any one point a for which $f(a) \neq 0$, and it is not obvious how this condition can be dropped or even relaxed.

2. The operator A_ν. The following two lemmas will be needed.

LEMMA 1. *Let* $g'' \in C(0, \infty)$. *Then* $A_\nu g$ *exists in* $(0, \infty)$ *and* $(A_\nu g)(a) = g''(a) + ((2\nu + 1)/a)g'(a)$.

COROLLARY. *If* $x > 0$ *then* $A_\nu(1) = 0$, $A_0(\log x) = 0$, $A_\nu(x^{-2\nu}) = 0$ *and* $A_\nu(x^2) = 4\nu + 4$.

LEMMA 2. *Let* $g \in C(a, b)$ *where* $a > 0$ *and suppose* $(A_\nu g)(x) = 0$ *for each* x *in* (a, b). *Then there are constants* P *and* Q *such that* $g(x) = P + Qx^{-2\nu}(\nu \neq 0)$ *or* $g(x) = P + Q \log x\ (\nu = 0)$.

Assuming for the moment that Lemma 1 has been proved, let us indicate how the proof of Lemma 2 proceeds. If we write

$$u_\nu(x) = \begin{cases} x^{-2\nu}, & \nu \neq 0, \\ \log x, & \nu = 0, \end{cases}$$

then the proof of Lemma 2 follows the same lines as the proof of Schwarz's theorem given in [3, p. 431]. We use the functions

$$\phi(x) = g(x) - g(a) - [g(b) - g(a)]\frac{u_\nu(x) - u_\nu(a)}{u_\nu(b) - u_\nu(a)}$$

$$\psi(x) = \phi(x) + \frac{\varepsilon}{4\nu + 4}\left\{x^2 + \frac{a^2(u_\nu(x) - u_\nu(b)) - b^2(u_\nu(x) - u_\nu(a))}{u_\nu(b) - u_\nu(a)}\right\}, \qquad \varepsilon > 0,$$

instead of the ϕ and ψ used there. The essential features of these functions are that $A_\nu\phi = 0$, $A_\nu\psi = \varepsilon$, whilst the expression in curly brackets is negative in (a, b).

Proof of Lemma 1. Write

$$G(h) = \int_0^{2\pi} g(a_{h,\theta})|\sin\theta|^{2\nu}\, d\theta, \qquad 0 < a < \infty.$$

If the limit exists, then

$$(A_\nu g)(a) = \frac{\sqrt{\pi}\,\Gamma(\nu + 2)}{\Gamma(\nu + \frac{1}{2})} \lim_{h \to 0+} \frac{2}{\pi h^2}[G(h) - G(0)].$$

We will write $[a^2 + h^2 - 2ah\cos\theta]^{1/2} = a + u(h, \theta)$ and we note that

$$|u(h, \theta)| \leqq h \quad \text{for } 0 \leqq \theta \leqq 2\pi. \tag{2.1}$$

Since $g' \in C(0, \infty)$, then $G'(h)$ exists and is given by

$$G'(h) = \int_0^{2\pi} g'(a + u)\frac{h - a\cos\theta}{a + u}|\sin\theta|^{2\nu}\, d\theta. \tag{2.2}$$

Now

$$\frac{2}{\pi h^2}[G(h) - G(0)] = \frac{G'(\xi h)}{\pi\xi h} \quad \text{for some } \xi \text{ in } 0 < \xi < 1,$$

so that, if the limit exists, then

$$(A_\nu g)(a) = \frac{\sqrt{\pi}\,\Gamma(\nu + 2)}{\Gamma(\nu + \frac{1}{2})} \lim_{h \to 0+} \frac{G'(h)}{\pi h}.$$

We write the integral in (2.2) as the sum of two integrals and integrate the latter one by parts. We get

$$\begin{aligned}
\frac{1}{\pi h} G'(h) &= \frac{1}{\pi} \int_0^{2\pi} \frac{g'(a + u)}{a + u} |\sin \theta|^{2\nu}\, d\theta \\
&\quad + \frac{a^2}{\pi} \int_0^{2\pi} |\sin \theta|^{2\nu + 2} \left\{ \frac{g''(a + u)}{(a + u)^2} - \frac{g'(a + u)}{(a + u)^3} \right\} d\theta \\
&\quad + \frac{2a\nu}{\pi h} \int_0^{2\pi} \frac{g'(a + u)}{a + u} |\sin \theta|^{2\nu} \cos \theta\, d\theta \\
&= I_1 + I_2 + I_3 \quad \text{(say).}
\end{aligned}$$

By (2.1), $u \to 0$ as $h \to 0$ uniformly with respect to θ in $[0, 2\pi]$. Since $g'(x)/x$ is continuous in $(0, \infty)$, then

$$\frac{g'(a + u)}{a + u} \to \frac{g'(a)}{a}$$

uniformly and so

$$I_1 \to \frac{1}{\pi} \frac{g'(a)}{a} \int_0^{2\pi} |\sin \theta|^{2\nu}\, d\theta = \frac{g'(a)}{a} \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)}.$$

In the same way we find that

$$I_2 \to \left\{ g''(a) - \frac{g'(a)}{a} \right\} \frac{2}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 2)}.$$

Now consider I_3. We can write

$$I_3 = \frac{2a\nu}{\pi h} \int_0^{2\pi} \left\{ \frac{g'(a + u)}{a + u} - \frac{g'(a)}{a} \right\} |\sin \theta|^{2\nu} \cos \theta\, \mathrm{d}\theta$$

since the latter integral here is zero. By an application of the mean value theorem it is easy to see that

$$\frac{1}{u} \left\{ \frac{g'(a + u)}{a + u} - \frac{g'(a)}{a} \right\} \to \left\{ \frac{g''(a)}{a} - \frac{g'(a)}{a^2} \right\}$$

uniformly in $[0, 2\pi]$. It is also easily seen that $u/h \to -\cos \theta$ uniformly in $[0, 2\pi]$. Hence as $h \to 0+$, then

$$\begin{aligned}
I_3 &\to \frac{2\nu}{\pi} \left\{ \frac{g'(a)}{a} - g''(a) \right\} \int_0^{2\pi} |\sin \theta|^{2\nu} \cos^2 \theta\, d\theta \\
&= \left\{ \frac{g'(a)}{a} - g''(a) \right\} \frac{4\nu}{\sqrt{\pi}} \left\{ \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} - \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\nu + 2)} \right\}.
\end{aligned}$$

Collecting these results we find that

$$(A_\nu g)(a) = \frac{\sqrt{\pi}\,\Gamma(\nu+2)}{\Gamma(\nu+\frac{1}{2})} \lim_{h\to 0+} \{I_1 + I_2 + I_3\} = g''(a) + \frac{2\nu+1}{a} g'(a),$$

which completes the proof of Lemma 1. The proof of the corollary is immediate.

3. Proof of the theorems. We now proceed to prove the first theorem.

Proof of Theorem 1. Throughout the proof we shall assume that λ is nonzero, but only minor alterations are needed to prove the case $\lambda = 0$. Put $y = 0$ in (1.2) and choose a value x for which $f(x) \neq 0$, and we find

$$H(0) = \frac{1}{2\pi}\int_0^{2\pi} |\sin\theta|^{2\nu}\, d\theta = \frac{1}{\sqrt{\pi}}\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}.$$

Next

$$\frac{1}{2\pi}\int_0^{2\pi} \{f(x_{y,\theta}) - f(x)\}|\sin\theta|^{2\nu}\, d\theta = f(x)\{H(y) - H(0)\}.$$

Multiplying this by $(4/y^2)(\sqrt{\pi}\,\Gamma(\nu+2)/\Gamma(\nu+\frac{1}{2}))$ and letting $y \to 0+$, we get

$$(A_\nu f)(x) = -\lambda^2 f(x).$$

Since $f \in C(0, \infty)$ we can define a function F in $(0, \infty)$ by

$$F(x) = \int_\delta^x u^{-2\nu-1}\int_\delta^u v^{2\nu+1} f(v)\, dv\, du, \qquad \delta > 0.$$

By Lemma 1, $A_\nu F$ exists in $(0, \infty)$ and $(A_\nu F)(x) = f(x)$. Hence $A_\nu(f + \lambda^2 F)(x) = 0$ when $x > 0$. By Lemma 2, $(f + \lambda^2 F)(x) = P + Qu_\nu(x)$ in any interval (a, b) with $a > 0$ and so in (a, ∞). As before we have written $u_\nu(x)$ to mean $u_\nu(x) = x^{-2\nu}$ $(\nu \neq 0)$, $u_\nu(x) = \log x$ $(\nu = 0)$. Accordingly,

$$f(x) = P + Qu_\nu(x) - \lambda^2\int_\delta^x u^{-2\nu-1}\int_\delta^u v^{2\nu+1} f(v)\, dv\, du, \qquad x > 0,$$

so that if $x > 0$, then $f'' \in C(0, \infty)$ and f satisfies

$$f''(x) + ((2\nu+1)/x)f'(x) + \lambda^2 f(x) = 0, \qquad x > 0.$$

Hence

$$f(x) = (\lambda x)^{-\nu} Z_\nu(\lambda x), \qquad x > 0, \quad \lambda^2 \text{ real}.$$

If this function f is now inserted in the left-hand side of (1.2), then the right-hand side $(\lambda x)^{-\nu} Z_\nu(\lambda x) H(y)$ is obtained. But in view of the known solution (1.3), this must equal

$$(\lambda x)^{-\nu} Z_\nu(\lambda x) \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}}\left(\frac{2}{\lambda y}\right)^\nu J_\nu(\lambda y).$$

Hence

$$H(y) = \frac{\Gamma(\nu + \frac{1}{2})}{\sqrt{\pi}}\left(\frac{2}{\lambda y}\right)^{\nu} J_{\nu}(\lambda y), \qquad y \geqq 0,$$

and the proof of Theorem 1 is complete.

Proof of Theorem 2. We assume that $\nu \geqq 0$ and that the Lebesgue integral of $f(x_{y,\theta})|\sin\theta|^{2\nu}$ exists over the θ interval $(0, 2\pi)$ and hence over the θ-interval $(0, \pi/2)$. Furthermore this integral is to exist for all x, y satisfying $0 \leqq y < x < \infty$.

Taking $0 < y < x$, we make the substitution $w = [x^2 + y^2 - 2xy\cos\theta]^{1/2}$ and obtain the existence of the integral

$$\int_{x-y}^{x+y} wf(w)[w^2 - (x-y)^2]^{\nu - 1/2}[(x+y)^2 - w^2]^{\nu - 1/2}\, dw.$$

This implies that $[w - (x - y)]^{\nu - 1/2} f(w) \in L(x - y, x)$. Since we can choose x and y so that $x - y$ is arbitrarily small and x is arbitrarily large, this means that

$$(w - \varepsilon)^{\nu - 1/2} f(w) \in L(\varepsilon, X) \quad \text{for all } \varepsilon > 0 \text{ and } X > 0.$$

Now it is easily proved that this is equivalent to the existence of the double Lebesgue integral of

$$[\sqrt{u^2 + v^2} - \varepsilon]^{\nu - 1/2} E(u, v) \quad \text{over the annulus } \varepsilon^2 \leqq u^2 + v^2 \leqq X^2,$$

where we have written $E(u, v) \equiv f(\sqrt{u^2 + v^2})$. Hence the double integral

$$\iint E(u, v)\, d\omega$$

will exist over any disc $S(x_0, b) \equiv \{(u, v) : (u - x_0)^2 + v^2 \leqq b^2\}$ provided that the origin is exterior to it; that is, provided that $0 < b < x_0$. If $\nu \geqq 0$ this, in turn, ensures the existence of the integral

$$\text{(3.1)} \qquad \iint_{S(x_0,b)} E(u, v) \frac{|v|^{2\nu}}{[(u - x_0)^2 + v^2]^{\nu}}\, d\omega, \qquad 0 < b < x_0.$$

Having established this, we turn to the proof that $f \in C(0, \infty)$. Since $H(0) > 0$ and since H is continuous from the right at zero, then $H(y)$ will be positive for all y in $0 \leqq y < b$ provided that b is sufficiently small. If x_0 denotes any number in $(0, \infty)$, let b be chosen and fixed so that both

$$H(y) > 0 \quad \text{in } [0, b) \quad \text{and} \quad 0 < b < x_0$$

hold. With this choice of b and x_0, consider the integral in (3.1). Putting $u = x_0 - y\cos\theta$, $v = -y\sin\theta$ and appealing to Fubini's theorem, we find that the integral in (3.1) is equal to the repeated integral

$$\int_0^b y\, dy \int_0^{2\pi} f([x_0^2 + y^2 - 2x_0 y\cos\theta]^{1/2})|\sin\theta|^{2\nu}\, d\theta.$$

By virtue of (1.2), this in turn must equal

$$f(x_0)\int_0^b yH(y)\,dy,$$

and by our choice of b the integral here is not zero.

Now if $\{x_n\}$ is an arbitrary sequence such that $x_n \to x_0$, then for all sufficiently large n we will have $0 < b < x_n$, and we can argue as before with x_n instead of x_0. Thus to show that f is continuous at x_0 it is sufficient to show that

$$\left|\iint_{S_n} Ef_n\,d\omega - \iint_{S_0} Ef_0\,d\omega\right| \to 0, \tag{3.2}$$

where we have written $S_n \equiv S(x_n, b)$ and $f_n \equiv |v|^{2\nu}[(u - x_n)^2 + v^2]^{-\nu}$. Now the left-hand side of (3.2) is dominated by

$$\iint_{S_n - S_0} |E| f_0\,d\omega + \iint_{S_0 - S_n} |E| f_0\,d\omega + \iint_{S_n} |E|\,|f_n - f_0|\,d\omega.$$

The first two integrals here tend to zero with n because meas $(S_n - S_0) \to 0$ and meas $(S_0 - S_n) \to 0$. For all sufficiently large n the third integral is dominated by

$$\iint_T |E|\,|f_n - f_0|\,d\omega, \tag{3.3}$$

where

$$T \equiv \{(u, v) : (u - x_0)^2 + v^2 \leqq \beta\}, \qquad 0 < b < \beta < x_0.$$

Clearly $T \supset S_n$ for all sufficiently large n, but the origin is exterior to T.

Since

$$\iint_T |E|\,|f_n|\,d\omega \leqq \iint_T |E|\,d\omega,$$

and since $f_n \to f_0$ almost everywhere, then the integral (3.3) tends to zero by the theorem of dominated convergence. This completes the proof of Theorem 2.

4. The product formula (1.1) can be obtained from Gegenbauer's addition theorem for the cylinder functions (see [1]) by treating the series there as an orthogonal expansion of Gegenbauer polynomials. To what extent the addition theorem itself characterizes the cylinder functions is a problem which has been considered by Al-Salam and Carlitz in [4]. A problem similar to the one considered in the present note originates from the product formula for the Gegenbauer polynomials themselves and this has been studied by Bingham in [5]. We note also that in [6], Koornwinder has proved a product formula for the Jacobi polynomials, but as far as we are aware, the question of whether this formula characterizes these polynomials is still open. In conclusion we mention that the solution provided by Theorem 1 to the functional equation obtained by taking $H(x) = f(x)$ in (1.2) has application to a problem considered—and solved—by Schwartz in [7] (see particularly his Theorem 4.1). Considerable space would need to be devoted to explaining the context of this problem, and so we merely refer the reader to Schwartz's paper.

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SOME IDENTITIES CONCERNING THE LAPLACIAN OF A FUNCTION SATISFYING MIXED BOUNDARY CONDITIONS*

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Abstract. The identity

$$\int_R (u_{x_ix_j}u_{x_ix_j}) = \int_R (\Delta u)^2 - \int_S (\mathrm{II} + 2\alpha E)((\nabla u)_S(\nabla u)_S) - (n-1)\alpha^2 \int_S Hu^2$$

for functions satisfying $u_n + \alpha u = 0$ on S, is established, where II is the second fundamental form of S, $E(X, X) = |X|^2$, and $(\cdot)_S$ indicates projection of a vector onto S. This identity is applied to obtain an isoperimetric characterization of eigenvalues for mixed boundary value problems and a maximum principle for an elliptic equation satisfying Cordes conditions.

Introduction. In what follows, we will derive an integral identity involving the second derivatives of a smooth function on a bounded n-dimensional region with piecewise smooth boundary, where the function in question satisfies either the Neumann or Robin boundary condition of potential theory. This identity is a companion to one derived by Talenti [1] for functions satisfying the Dirichlet boundary condition, and we will use it to establish isoperimetric inequalities characterizing the eigenvalues of the Laplacian relative to the Neumann and Robin conditions as was done by the author in [2] for the Dirichlet boundary condition. These inequalities in turn imply coercivity inequalities for the Laplacian which can be useful in dealing with semilinear equations. All of this can be put together in a straightforward manner to yield similar results for the case of mixed boundary conditions. The paper is concluded with an application to a mixed boundary value problem for an elliptic equation whose coefficients satisfy Cordes conditions.

1. Deriving of identities. The identity

$$\int_R (u_{x_ix_j}u_{x_ix_j}) = \int_R (\Delta u)^2 - (n-1)\int_S Hu_n^2 ,$$

which holds for $u \in W^2_{2,0}(R)$ (the closure in $W^2_2(R)$ of functions in $C^2(\bar{R})$ that vanish on S, the boundary of R), is well known [1], [3], [4]. The surface S is assumed piecewise smooth, H is the mean curvature of S, and the summation convention on repeated indices is employed. We will derive here analogous identities when u satisfies other boundary conditions on S.

The starting point of this derivation is the identity

$$\int_R (u_{x_ix_j}u_{x_ix_j}) = \int_R (\Delta u)^2 - \int_S (u_n \Delta u - u_{x_i}u_{x_in}),$$

which holds for $u \in C^2(\bar{R})$, and follows from two integrations by parts [3, Chap. 1]. The identities to be derived here follow from manipulation of the boundary integral when u satisfies the Neumann or Robin boundary condition on S.

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THEOREM 1. *Suppose that $u_n + \alpha u = 0$ on S (α a constant). Then*

$$\int_R (u_{x_i x_j} u_{x_i x_j}) = \int_R (\Delta u)^2 - \int_S (\mathrm{II} + 2\alpha E)((\nabla u)_S, (\nabla u)_S) + (n-1)\alpha^2 H u^2),$$

where II *is the second fundamental form of* S, $E(X, X) = |X|^2$, *and* $(\cdot)_S$ *indicates the projection of a vector onto* S.

Proof. We begin by observing that the boundary condition implies the identity

$$\int_R (u_{x_i x_j} u_{x_i x_j}) = \int_R (\Delta u)^2 + \int_S (\alpha u \, \Delta u + u_{x_i n} u_{x_i}).$$

In these integrals the indices are summed from 1 to n. We will carry out the proof in the case $n = 3$, other cases going through in the same way. The proof consists of appropriate manipulation of the above boundary integral. Suppose that we focus our attention on a point $P \in S$, and that S is given locally by a function $y_3 = w(y_1, y_2)$, where P is the origin of coordinates and the y_3-axis is directed along the exterior normal to S at P. The integrand can then be written

$$\alpha u \, \Delta u + u_{y_3} u_{y_3 n} + u_{y_\beta n} u_{y_\beta}, \tag{1}$$

with the repeated index summed from 1 to 2. Since $(-w_{y_1}, -w_{y_2}, 1)$ are direction numbers of the exterior normal to S, the boundary condition can be written

$$u_{y_3} - w_{y_1} u_{y_1} - w_{y_2} u_{y_2} + \alpha \sqrt{1 + w_{y_1}^2 + w_{y_2}^2}\, u = 0$$

and

$$u_{y_\beta n} = -u_{y_\beta y_1} w_{y_1} - u_{y_\beta y_2} w_{y_2} + u_{y_\beta y_3}, \qquad i = 1, 2.$$

Using the above form of the boundary condition, we find an expression for $u_{y_\beta n} u_{y_\beta}$ which reduces to

$$u_{y_\beta} w_{y_\beta y} - \alpha u_{y_\beta} u_{y_\beta}$$

when evaluated at P. If the y_1- and y_2-axes are chosen along the principle directions of S at P, the Hessian matrix of w at P becomes $-\operatorname{diag}(k_1, k_2)$, where k_1 and k_2 are the principle curvatures of S at P.

We turn now to the first term in (1). It is known ([8, Chap. 8]) that

$$\Delta u = \Delta_S u + (n-1) H u_n + u_{x_i x_j} n_i n_j, \tag{2}$$

where Δ_S is the Laplace–Beltrami operator for the surface S. Therefore the surface integral can be written as

$$\alpha \int_S u \Delta_S u - (n-1)\alpha^2 \int_S H u^2 - \int_S (u_{x_i x_j} n_i n_j u_n - u_{y_3} y_{y_3 n} - u_{y_\beta n} u_{y_\gamma}).$$

At P the first two terms in the last integral cancel. We observe that

$$\int_S u \Delta_S u = -\int_S |(\nabla u)_S|^2.$$

The proof is completed.

It should be remarked that (2) is derived in [8, Chap. 8] in the case $n = 3$. Using [9, §§ 43–50], the argument in [8, Chap. 8] can be extended to general n.

Each of these theorems holds for functions in $W_2^2(R)$ whose traces on ∂R satisfy the boundary conditions as is seen by taking limits of smooth functions.

A theorem which implies Theorem 1 in the special case $n = 2$ and $k_i \equiv 0$, i.e., R a polygon, has been proven by Grisvard [7]. It is worth noting that if $k_i \geqq \max\{0, -2\alpha\}$, the inequality

$$\int_R (u_{x_ix_j}u_{x_ix_j}) \leqq \int_R (\Delta u)^2$$

holds, and if R is a polyhedron and $u_n = 0$ on ∂R, equality holds here.

2. Some inequalities involving the Laplacian. In order to establish the main result of this section we need this preliminary result.

LEMMA. *Suppose that u satisfies $u_n + \alpha u = 0$ on S. Then*

$$\int_R (u^2 + |\nabla u|^2) + \alpha \int_S u^2 \leqq (1 + \mu)/\mu^2 \int_R (\Delta u)^2,$$

where μ is the first (positive) eigenvalue of $-\Delta$ subject to this boundary condition. (If $\alpha = 0$, we must require $\langle u, 1\rangle = \int_R u = 0$ since 0 is an eigenvalue.) Both sides are equal when u is an eigenfunction of $-\Delta$ corresponding to μ.

Proof. From

$$\int_R u\,\Delta u = -\int_R |\nabla u|^2 + \int_S uu_n$$

we get

$$\int_R |\nabla u|^2 \leqq \frac{\varepsilon}{2}\int_R u^2 + \frac{1}{2\varepsilon}\int_R (\Delta u)^2 - \alpha \int_S u^2, \qquad \varepsilon > 0.$$

From the variational characterization of μ,

$$\mu \int_R u^2 \leqq \int_R |\nabla u|^2 + \alpha \int_S u^2.$$

The last two inequalities imply

$$\varepsilon(2\mu - \varepsilon)\int_R u^2 \leqq \int_R (\Delta u)^2,$$

and then

$$\int_R (u^2 + |\nabla u|^2) + \int_S u^2 \leqq (1 + \mu)/\varepsilon(2\mu - \varepsilon)\int_R (\Delta u)^2.$$

The required inequality follows by minimizing with respect to ε. The equality when u is an eigenfunction corresponding to μ follows simply by substitution.

Suppose that the nonzero eigenvalues of $-\Delta$ with the boundary condition $u_n + \alpha u = 0$ are $\mu_1, \mu_2, \cdots$, enumerated in increasing magnitude. Then the conclusion of this lemma also holds if one restricts the discussion to functions u

such that $\langle u, u_1 \rangle = \cdots = \langle u, u_{n-1} \rangle = 0$, where $u_1, \cdots, u_{n-1}$ are eigenfunctions corresponding to $\mu_1, \cdots, \mu_{n-1}$, respectively, and $\mu = \mu_n$.

THEOREM 2. *Suppose the eigenfunction of* $-\Delta$ *subject to* $u_n + \alpha u = 0$ *corresponding to* μ *is an element of* $W_2^2(R)$. *Then*

$$\inf \frac{(\|u\|_2)^2 + \int_S \{\alpha u^2 + (\mathrm{II} + 2\alpha E)((\nabla u)_S, (\nabla u)_S) + (n-1)\alpha^2 H u^2\}}{\int_R (\Delta u)^2} = 1 + \mu^{-1} + \mu^{-2},$$

where the infimum is over functions in $W_2^2(R)$ *satisfying the boundary condition, and* $\|\cdot\|_2$ *is the norm in* $W_2^2(R)$.

Proof. We need only combine the lemma with the identity derived in paragraph one.

Verification of the hypothesis about the eigenfunctions belonging to $W_2^2(R)$ is intimately connected with regularity of ∂R and will be commented on below.

If u satisfies $u = 0$ on ∂R instead of $u_n + \alpha u = 0$, a similar theorem holds with the boundary integral replaced by

$$(n-1) \int_S (H u_n^2).$$

In the general case of mixed boundary conditions, say

$$\begin{aligned} u &= 0 && \text{on } S_1, \\ u_n &= 0 && \text{on } S_2, \\ u_n + \alpha u &= 0 && \text{on } S_3, \end{aligned}$$

and $R = S_1 + S_2 + S_3$, the boundary integral would be replaced by

$$(n-1) \int_{S_1} H u_n^2 + \int_{S_2} \mathrm{II}((\nabla u)_S, (\nabla u)_S) + \int_{S_3} (\alpha u^2 + (\mathrm{II} + 2\alpha E) \cdot ((\nabla u)_S, (\nabla u)_S) + (n-1)\alpha^2 H u^2$$

If $S_3 = 0$, and R is a polyhedron, the boundary integrals vanish. More generally, if $H \geqq 0$, on S_1, $k_i \geqq 0$ on S_2, and $k_i \geqq \max\{-2\alpha, 0\}$ on S_3 we have

$$(\|u\|_2)^2 = \int_R (u^2 + |\nabla u|^2 + u_{x_i x_j} u_{x_i x_j}) \leqq (1 + \mu^{-1} + \mu^{-2}) \int_R (\Delta u)^2,$$

where μ is the smallest eigenvalue corresponding to the above mixed boundary conditions.

All of the above can be applied to functions orthogonal to the first $n-1$ eigenfunctions, and a similar inequality is obtained with $\mu = \mu_n$.

Since $1 + \mu^{-1} + \mu^{-2}$ is a decreasing function of μ, $\mu > 0$, each of the above may be thought of as an isoperimetric inequality characterizing the eigenvalues of $-\Delta$ subject to the appropriate boundary conditions.

If Δ maps the subspace of $W_2^2(R)$ defined by the boundary condition(s) *onto* $L_2(R)$, the above results yield upper bounds on $\|\Delta^{-1}\|$. In particular, if $S_3 = \varnothing$ and R is a polyhedron,

$$\|\Delta^{-1}\| = (1 + \mu^{-1} + \mu^{-2})^{1/2},$$

and, more generally, if $H \geqq 0$ on S_1, $k_i \geqq 0$ on S_2, and $k_i \geqq \max\{0, -2\alpha\}$ on S_3,

$$\|\Delta^{-1}\| \leqq (1 + \mu^{-1} + \mu^{-2})^{1/2}.$$

Unfortunately, knowledge does not seem to be complete in this regard. For the Dirichlet boundary condition it is true (for all n) if and only if R is convex [5]. For $n = 2$, fairly complete results can be derived from [6]. In particular corners where the Dirichlet boundary condition is imposed on both incoming arcs, and those where the second or third boundary condition is imposed on each arc must have an opening not bigger than π, whereas a corner at which one arc has the Dirichlet boundary condition and the other either the second or third must have an opening not bigger than $\pi/2$.

3. A maximum principle for an equation of Cordes type. We conclude by showing how the results of paragraph one can be applied to mixed boundary value problems for the elliptic equation

$$Pu = a_{ij}u_{x_ix_j} = f,$$

where a_{ij} are bounded and measurable and satisfy a Cordes condition, that is, with the normalization

$$\sum_{i=1}^{n} a_{ii} = 1$$

we assume

$$\sum_{i,j=1}^{n} a_{ij}^2 < (n - 1 + \varepsilon)^{-1} \quad \text{for some } 0 < \varepsilon < 1.$$

P may be thought of as an operator mapping $W_2^2(R)$ into $L_2(R)$. In particular we will think of P as restricted to the subspace X of functions which satisfy the boundary conditions $u = 0$ on S_1, $u_n = 0$ on S_2, $u_n + \alpha u = 0$ on S_3. (In the case $S_1 = S_3 = \varnothing$, the further condition $\int_R u = 0$ is imposed.) We then have the following theorem.

THEOREM 3. *Suppose that the curvatures of S satisfy $H \geqq 0$ on S_1, $k_i \geqq 0$ on S_2, and $k_i \geqq \max\{0, -2\alpha\}$ on S_3. Then P is an isomorphism of X onto $L_2(R)$.*

Further, if $u \in X$ and $Lu \geqq 0$, then $u \leqq 0$ in R unless $S_1 = S_3 = \varnothing$ and u is a positive constant.

We need the following lemma, which is a consequence of the Cordes condition. It is proven in [1], where a result similar to the above was proved for $S = S_1$.

LEMMA. *If P_{ij} is a real, symmetric matrix, there is a positive constant A such that*

$$\sum_{i,j} P_{ij}^2 + A\sum_{i,j}\begin{vmatrix} P_{ii} & P_{ij} \\ P_{ij} & P_{jj}\end{vmatrix} \leqq B^2\Big(\sum_{i,j} a_{ij}P_{ij}\Big)^2,$$

where

$$B^2 = \varepsilon^{-2}(n - 1 + \varepsilon)((n - 1 + \varepsilon)^{1/2} + ((1 - \varepsilon)(n - 1))^{1/2})^2.$$

Proof. Set $P_{ij} = u_{x_i x_j}$, and integrate the above inequality. After observing that

$$\sum_{i,j} (u_{x_i x_i} u_{x_j x_j} - u_{x_i x_j} u_{x_i x_j}) = (\Delta u)^2 - u_{x_i x_j} u_{x_i x_j},$$

applying the identities of paragraph one, and invoking the hypotheses on the curvatures of S, we have

$$(*) \qquad \int_R (u_{x_i x_j} u_{x_i x_j}) \leqq B \int_R (Pu)^2.$$

Observe, now, that

$$\|u\|^2 = \int_R (u_{x_i x_j} u_{x_i x_j})$$

is a norm on X which is equivalent to the one used earlier. In fact, using the lemma of paragraph two, our fundamental identity, and the hypotheses on the boundary curvatures, we have

$$\int_R (u^2 + |\nabla u|^2 + |D^2 u|^2) \leqq (1 + \mu^{-1} + \mu^{-2}) \int_R |D^2 u|^2.$$

(We have assumed $\alpha > 0$ here.)

It now follows that P is an isomorphism of X onto $P(X)$, and our first conclusion will be verified if we can show that $P(X) = L_2(R)$. This and the second conclusion as well are demonstrated using a device introduced in [1]. We extend a_{ij} to R^n by defining it to be δ_{ij}/n outside of R, and introduce the functions

$$a_{ij}^{(m)}(x) = h^{-n} \int_{R^n} d\left(\frac{x-y}{h}\right) a_{ij}(y)\, dy,$$

where $h = 1/m$, and d is the usual "mollifier". Then it follows that $a_{ij}^{(m)} \in C^\infty$, they satisfy the Cordes condition, and that $a_{ij}^{(m)}$ converges to a_{ij} in the weak topology of L^∞. Observe that (*) holds for the operators $P^{(m)}$ as well as P.

For $P(X) = L_2(R)$ it suffices that $P(X)$ contain $C_0^\infty(R)$. Choose f in $C_0^\infty(R)$. By known theorems for elliptic equations with smooth coefficients there is $u^{(m)}$ in X such that $p^{(m)} u^{(m)} = f$. Since

$$\|u^{(m)}\|_2 \leqq B \|f\|_0,$$

the sequence $\{u^{(m)}\}$ contains a weakly convergent subsequence and it is easily shown that the limit of this subsequence is a solution of $Pu = f$.

Suppose now that $f \in L_2(R)$ is nonnegative, let $f^{(n)}$ be defined as above, and suppose that $u^{(m)} \in X$ is the solution of $P^{(m)} u^{(m)} = f^{(m)}$, and that $Pu = f$. Then, since

$$(P + P^m)(u^{(m)} - u) = (f^{(m)} - f) + (P - P^{(m)})u,$$

we have

$$\|u^{(m)} - u\|_2 \leqq 2B(\|f^{(m)} - f\|_0 + \|(P - P^{(m)})u\|_0).$$

The first term on the right tends to zero since $f^{(m)}$ approaches f in $L_2(R)$, and the second does because $a_{ij}^{(m)}$ converges weakly to a_{ij} in L^∞, and we deduce that $u^{(m)}$ converges strongly to u in X. Therefore, since each $u^{(m)} \leqq 0$ (unless it is a positive constant and $S_1 = S_3 = \varnothing$) the second assertion of the theorem follows.

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ON A FREE BOUNDARY PROBLEM, THE STARLIKE CASE*

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Abstract. Let $\mathscr{D}$ be a doubly connected region limited by the infinite point and a starlike boundary component Γ which does not reduce to a point. If λ is a given positive number, we show there exists a unique annulus $\omega_\lambda \subset \mathscr{D}$ having Γ as one boundary component and another boundary component γ_λ such that there is a harmonic function V in ω_λ satisfying $V \equiv 0$ on Γ, $V \equiv 1$ on γ_λ and $|\text{grad } V_\lambda| \equiv \lambda$ on γ_λ. We also show that γ_λ is starlike.

In this paper we generalize results proven by the author in [2]. We begin with a doubly connected region $\mathscr{D}$ limited by a compact boundary component Γ and the point at infinity. Suppose Γ is starlike, i.e., if $z \in \Gamma$, then the line segment $[0, z]$ contains no points in $\mathscr{D}$. If λ is a given positive constant, we are concerned with finding an annulus $\omega \subset \mathscr{D}$ having Γ as one boundary component and another boundary component γ, the "free boundary", such that there exists a harmonic function V in ω satisfying:

(a) $V = 0$ on Γ,

(b) $V = 1$ on γ,

(c) $|\text{grad } V| = \lambda$ on γ.

We will show that the above problem has a unique solution for fixed λ whose free boundary is also starlike. In [1], Beurling proved that the above problem has a unique solution when Γ is convex. Therefore, we give a generalization of his result in this paper.

Let $\mathscr{C}$ denote the family of all subannuli of $\mathscr{D}$ having Γ as one boundary component. If $\omega \in \mathscr{C}$, the harmonic function in ω with boundary values (a) and (b) will be denoted V_ω and referred to as the stream function of ω. We will make use of the following result which is proved in [1].

THEOREM (Beurling). *If there exists an annulus $\omega \in \mathscr{C}$ with free boundary γ such that*

$$|\text{grad } V_\omega| \leqq \lambda \text{ on } \gamma, \tag{1}$$

then there exists a solution $\Omega_\lambda \subset \omega$.

We now prove the following theorem.

THEOREM 1. *For fixed $\lambda > 0$, there exists a unique solution Ω_λ.*

Proof. We first prove existence. Consider the ring region

$$\Delta_{r,R} = \{z : r < |z| < R\},$$

where

$$\Gamma \subset \{z : |z| < r\},$$

and

$$\frac{1}{R(\log R - \log r)} \leqq \lambda.$$

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Let ω be the annulus in $\mathscr{C}$ whose free boundary is $|z| = R$. For $z \in \Delta_{r,R}$, the maximum modulus theorem gives

$$V_\omega(z) \geqq \frac{\log|z| - \log r}{\log R - \log r}. \tag{2}$$

Since equality holds in (2) when z lies on the free boundary of ω, we have

$$|\text{grad } V_\omega(z)| \leqq \frac{1}{R(\log R - \log r)} \leqq \lambda. \tag{3}$$

Hence, by Beurling's theorem, there exists a solution $\Omega_\lambda \subset \omega$.

We now prove uniqueness. It is shown in [1] that if there exist two solutions, then there exists a case where one solution is contained inside another solution. Suppose Ω'_λ and Ω''_λ are such a pair of solutions with $\Omega'_\lambda \subset \Omega''_\lambda$. Let γ'_λ and γ''_λ denote the respective free boundaries of Ω'_λ and Ω''_λ. There exists a largest positive number ρ_0 such that

$$A = \{z : \rho_0^{-1} z \in \gamma''_\lambda\}$$

lies inside γ'_λ. We note that $\rho_0 < 1$. Let

$$\Gamma' = \{z : \rho_0^{-1} z \in \Gamma\},$$

and let Ω be the annulus with boundary components Γ' and A. If V is the harmonic function in Ω such that $V = 0$ on Γ' and $V = 1$ on A, then

$$V(z) = V_{\Omega''_\lambda}(\rho_0^{-1} z). \tag{4}$$

By the maximum modulus principle, for $z \in \Omega \cap \Omega'_\lambda$ we have

$$V(z) \geqq V_{\Omega'_\lambda}(z). \tag{5}$$

However, there exists at least one point $z_0 \in A \cap \gamma'_\lambda$. From (5), we obtain

$$|\text{grad } V(z_0)| \leqq |\text{grad } V_{\Omega'_\lambda}(z_0)| = \lambda. \tag{6}$$

From (4) we obtain

$$\begin{aligned} |\text{grad } V(z_0)| &= \rho_0^{-1} |\text{grad } V_{\Omega''_\lambda}(\rho_0^{-1} z_0)| \\ &= \rho_0^{-1} \lambda \\ &> \lambda. \end{aligned} \tag{7}$$

This proves uniqueness.

Using techniques of [2], the following theorem may be proved. We omit the proof.

THEOREM 2. *If $\lambda_1 > \lambda_2$, then*

$$\Omega_{\lambda_1} \subset \Omega_{\lambda_2}. \tag{8}$$

Furthermore, if γ_λ is the free boundary of Ω_λ, then

$$\bigcup_{\lambda > 0} \gamma_\lambda = \mathscr{D}. \tag{9}$$

In [2] it is shown that if Γ is convex then γ_λ is convex. The following theorem shows that this "convex" may be replaced by "starlike".

THEOREM 3. *If* Γ *is starlike, then* γ_λ *is starlike.*

Proof. Let $\rho > 1$. Consider the annulus Ω whose boundary components are

$$\Gamma' = \{z : \rho^{-1}z \in \Gamma\},$$

$$\gamma'_\lambda = \{z : \rho^{-1}z \in \gamma_\lambda\}.$$

Let Ω' be the annulus in $\mathscr{C}$ whose free boundary is γ'_λ. By the maximum modulus theorem, we have

$$V_{\Omega'}(z) \geqq V_{\Omega_\lambda}(\rho^{-1}z). \tag{10}$$

Therefore, on the free boundary of Ω', we have

$$\begin{aligned} |\text{grad } V_{\Omega'}(z)| &\leqq \rho^{-1}|\text{grad } V_{\Omega_\lambda}(\rho^{-1}z)| \\ &= \rho^{-1}\lambda \\ &< \lambda. \end{aligned} \tag{11}$$

Hence, by Beurling's theorem, $\Omega' \supset \Omega_\lambda$ and γ_λ is starlike.

In [2], the author proved that as $\lambda \to 0$, the free boundaries γ_λ are in a certain sense asymptotic to a family of circles. We remark that similar reasoning applies in the starlike case which we consider in this paper. One could thus easily prove a theorem similar to Theorem 4 in [2].

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A STABLE MANIFOLD THEOREM FOR A SYSTEM OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract. We study the behavior of solutions of the perturbed system

$$z'(t) = Az(t) + B * z(t) + Hz(t), \qquad 0 \leqq t < \infty, \tag{N}$$

with initial condition $z(0) = z_0$, for $|z_0|$ sufficiently small; z is an n component vector, A is a real $n \times n$ matrix, B is a real $n \times n$ matrix of functions in $L^1(0, \infty)$, $B * z$ is the convolution, and the perturbation vector H of nonlinear functionals defined on appropriate Banach spaces represents higher order terms in z. The underlying hypothesis is that the equation

$$\det(sI - A - \hat{B}(s)) = 0, \qquad \operatorname{Re} s \geqq 0, \tag{E}$$

where $\hat{B}$ is the Laplace transform of B has N roots $\{s_1, s_2, \cdots, s_N\}$, $\operatorname{Re} s_j > 0$, $1 \leqq N < \infty$, where each root s_j has multiplicity m_j, $j = 1, \cdots, N$, and (E) has no roots s_j with $\operatorname{Re} s_j = 0$. The principal result is a stable manifold theorem for (N); the classical such theorem of H. Weyl for ordinary differential equations ($B \equiv 0$) is a special case.

1. Introduction. We consider the perturbed system of integro-differential equations

$$z'(t) = Az(t) + B * z(t) + Hz(t), \qquad 0 \leqq t < \infty, \tag{N}$$

with initial condition $z(0) = z_0$, where

$$B * z(t) = \int_0^t B(t - \xi)z(\xi)\, d\xi.$$

We shall assume throughout this paper that A is a real constant $n \times n$ matrix and that the real matrix $B(t) \in L^1(0, \infty)$; the perturbation vector H of nonlinear functionals satisfies $H0 = 0$ and represents higher order terms in z defined on certain natural Banach spaces. Let

$$\hat{B}(s) = \int_0^\infty \exp(-st)B(t)\, dt$$

denote the Laplace transform of B. It is well known (Grossman and Miller [4], Shea [12], Shea and Wainger [13]) that if the equation

$$\det(sI - A - \hat{B}(s)) = 0, \qquad \operatorname{Re} s \geqq 0, \tag{E}$$

where I is the identity matrix has no roots, then the resolvent kernel $R_L \in L^1(0, \infty)$; R_L is defined to be the solution of the linear problem, called the resolvent equation:

$$R_L'(t) = AR_L(t) + B * R_L(t), \qquad R_L(0) = I, \qquad 0 \leqq t < \infty.$$

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In this case the behavior of solutions of (N) with $|z_0|$ sufficiently small has been thoroughly studied for a variety of perturbations H. (See Grossman and Miller [3], Miller [6], [7], Nohel [9], [10], [11]).

In this paper we investigate the situation in which

(H_1) equation (E) has N roots $\{s_1, s_2, \cdots, s_N\}$, $\operatorname{Re} s_j > 0$, $1 \leqq N < \infty$, each root s_j has multiplicity m_j, $j = 1, \cdots, N$, and (E) has no roots s_j with $\operatorname{Re} s_j = 0$.

Our object is to study the behavior of solutions of the system (N) on $0 \leqq t < \infty$ for $|z_0|$ sufficiently small. This will be done in Theorems 1 and 2 below; these theorems will establish the existence of a stable manifold for equation (N) through the origin of $\mathbb{R}^n$. It may be remarked that assumption (H_1) and the Proposition in § 2 imply that the linear system arising from (N) by taking $H \equiv 0$ has an m-parameter family of solutions which tends to infinity as $t \to +\infty$, where $m = m_1 + m_2 + \cdots + m_N$ is the total multiplicity of all roots s_j of (E) with $\operatorname{Re} s_j > 0$. The classical stable manifold theorem for ordinary differential equations (see Coddington and Levinson [2, Chap. 13, Thm. 4.1]) follows from our results by taking $B(t) \equiv 0$ and H to be sufficiently smooth functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying assumption (H_2) below. The stable manifold for (N) is analysed in Corollary 1 below for the case that the roots s_j, $\operatorname{Re} s_j > 0$, of equation (E) are simple. A general result about the stable manifold for (N) is proved in Theorem 3. As will be seen in all these cases, the natural and interesting situation arises when the total multiplicity $m = m_1 + \cdots + m_N < n$. To prove Theorem 3 an extension of Theorems 1 and 2 is needed to systems more general than (N); this is done in Theorem 4.

2. Summary of results.

Define

$$F(s) = sI - A - \hat{B}(s). \tag{2.1}$$

Near each root s_j of E with $\operatorname{Re} s_j > 0$ the matrix $F^{-1}(s)$ has the Laurent series expansion

$$\sum_{k=0}^{m_j-1} P_{jk}(s - s_j)^{-k-1} + \sum_{k=0}^{\infty} L_{jk}(s - s_j)^k,$$

where P_{jk}, L_{jk} are constant $n \times n$ matrices. It may be noted that the resolvent kernel R_L is also completely determined by the relation $\hat{R}_L(s) = F^{-1}(s)$ as may be verified by taking Laplace transforms in the resolvent equation together with the initial condition $R_L(0) = I$. It will be convenient to decompose R_L in the following way. Define

$$W(t) = \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} P_{jk} \frac{t^k}{k!} \exp(s_j t), \qquad -\infty < t < \infty, \tag{2.2}$$

and define

$$R(t) = R_L(t) - W(t), \qquad 0 \leqq t < \infty, \tag{2.3}$$

or equivalently by its Laplace transform,

$$\hat{\hat{R}}(s) = F^{-1}(s) - \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} P_{jk}(s - s_j)^{-k-1}.$$

From the initial condition $R_L(0) = I$ and from (2.2), (2.3) one has

$$R(0) + W(0) = R(0) + \sum_{j=1}^{N} P_{j0} = I. \tag{2.4}$$

It may be noted further that while

$$\hat{W}(s) = \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} P_{jk}(s - s_j)^{-k-1},$$

the function W is well-defined on $-\infty < t < \infty$. The following result is known concerning the decomposition (2.3) of the resolvent kernel R_L and the representation of solutions of linear systems of integro-differential equations on $0 \leqq t < \infty$. Let $BC = \{f : [0, \infty) \text{ into } \mathbb{R}^n, f \text{ bounded and continuous}\}$ with uniform norm; let $BC_0 = \{f \in BC : \lim_{t\to\infty} f(t) = 0\}$.

PROPOSITION (for a proof see Miller [8, Thms. 1, 2, 3]). *Let* $B(t) \in L^1(0, \infty)$ *and let hypothesis* (H_1) *be satisfied; then* $R(t) \in C[0, \infty) \cap L^p(0, \infty)$, $1 \leqq p \leqq \infty$, *and* $\lim_{t\to\infty} R(t) = 0$. *Moreover, if* $f \in L^q(0, \infty)$, $1 \leqq q \leqq \infty$, *or* $f \in BC$ *or* $f \in BC_0$, *then the solution* $x(t)$ *of the linear initial value problem*

$$x'(t) = Ax(t) + B * x(t) + f(t), \qquad x(0) = x_0, \tag{L}$$

is given by (the variation of constants formula)

$$\begin{aligned} x(t) = {} & R(t)x_0 + W(t)x_0 + R * f(t) + \int_0^\infty W(t - \xi)f(\xi)\, d\xi \\ & - \int_t^\infty W(t - \xi)f(\xi)\, d\xi, \qquad 0 \leqq t < \infty. \end{aligned} \tag{2.5}$$

Remark. If one uses (2.2), elementary properties of Laplace transforms and the binomial theorem in (2.5), one obtains the following equivalent form of the solution (2.5) of (L):

$$\begin{aligned} x(t) = {} & R(t)x_0 + \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \frac{t^k}{k!} \exp(s_j t)\left[P_{jk}x_0 + \sum_{i=k}^{m_j-1} \frac{P_{ji}}{(i-k)!} \hat{f}^{(i-k)}(s_j)\right] \\ & + R * f(t) - \sum_{j=1}^{N} \sum_{k=0}^{m_j-1} \frac{P_{jk}}{k!} \int_t^\infty (t - \xi)^k \exp s_j(t - \xi) f(\xi)\, d\xi, \end{aligned} \tag{2.6}$$

where

$$\hat{f}^{(k)}(s_j) = \frac{d^k}{ds^k}\left[\int_0^\infty \exp(-st) f(t)\, dt\right]_{s=s_j}.$$

To state the main results let X with norm $\|\cdot\|$ denote any of the spaces BC, BC_0, $L^p(0, \infty)$, $1 \leqq p \leqq \infty$, of functions φ from $[0, \infty)$ into $\mathbb{R}^n$; let $|\cdot|$ denote any vector norm and the corresponding matrix norm. Concerning the perturbation functionals H in (N) assume

(H$_2$) The mapping $H: X \to X$ is continuous, $H0 = 0$ and for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|Hz_1 - Hz_2\| \leqq \varepsilon \|z_1 - z_2\|$ for all $z_1, z_2 \in X$ for which $\|z_j\| \leqq \delta$, $(j = 1, 2)$.

Define the operator K on X by the relation

$$K\varphi(t) = R(t)a + R * H\varphi(t) - \int_t^\infty W(t-\xi)H\varphi(\xi)\,ds, \qquad \varphi \in X, \quad 0 \leqq t < \infty, \tag{2.7}$$

where $a \in \mathbb{R}^n$ and where the matrices $R(t)$, $W(t)$ are defined in (2.2), (2.3) respectively. It may be noted that according to the Proposition, $R(t) \in X$; moreover from (2.2) and (H$_1$) there exist constants M, $\sigma > 0$ such that

$$|W(t)| \leqq M \exp(\sigma t), \qquad -\infty < t \leqq 0. \tag{2.8}$$

The first result concerns the Volterra equation

$$z(t) = Kz(t), \qquad 0 \leqq t < \infty. \tag{V}$$

THEOREM 1. *Let $B(t) \in L^1(0, \infty)$ and let hypotheses* (H$_1$), (H$_2$) *be satisfied. There exist constants $\delta_1, \delta_2 > 0$ such that if $|a| \leqq \delta_1$ the integral equation* (V) *has a unique solution $\psi(t, a) \in X$ and $\|\psi(\cdot, a)\| \leqq \delta_2$; moreover, $\psi(t, a)$ satisfies the condition*

$$\psi(0, a) = a - \sum_{j=1}^{N} \left[P_{j0}a + \sum_{k=0}^{m_j - 1} \frac{P_{jk}}{k!} \widehat{H\psi(\cdot, a)}^{(k)}(s_j) \right], \tag{2.9}$$

where

$$\widehat{H\psi(\cdot, a)}^{(k)}(s_j) = \frac{d^k}{ds^k} \left[\int_0^\infty \exp(-st) H\psi(t, a)\,dt \right]_{s = s_j}.$$

The proof of Theorem 1 is carried out in § 3 using a contraction mapping argument.

We next turn to the question of finding sufficient conditions in order that the solution $\psi(t, a)$ of (V) in *Theorem* 1 will satisfy the initial value problem (N) for some small $|a|$. It may be noted that (V) and (N) are not equivalent problems. It follows from (N), the *Proposition* and (2.6) that (N) is equivalent to the Volterra equation

$$\begin{aligned} z(t, z_0) = R(t)z_0 &+ \sum_{j=1}^{N} \sum_{k=0}^{m_j - 1} \frac{t^k}{k!} \exp(s_j t) \\ &\cdot \left[P_{jk} z_0 + \sum_{i=k}^{m_j - 1} \frac{P_{ji}}{(i-k)!} \widehat{Hz(\cdot, z_0)}^{(i-k)}(s_j) \right] + R * Hz(\cdot, z_0)(t) \\ &- \int_t^\infty W(t - \xi) Hz(\xi, z_0)\,d\xi, \qquad 0 \leqq t < \infty, \end{aligned} \tag{2.10}$$

where $z(t, z_0)$ is the solution of (N) satisfying the initial condition $z(0, z_0) = z_0$. The solution of equation (V) with $|a|$ sufficiently small is bounded in the sense of Theorem 1. The solution $z(t, z_0)$ of (N) is bounded and satisfies the integral equation (V) provided the coefficients of the terms $t^k \exp(s_j t)$ in (2.10) vanish for $k = 0, 1, \cdots, m_j - 1$, $j = 1, \cdots, N$. This is equivalent to the requirement that z_0 be chosen to satisfy the system of equations

$$P_{jk}z_0 + \sum_{i=k}^{m_j-1} \frac{P_{ji}}{(i-k)!} H(z(\cdot, z_0))^{(i-k)}(s_j) = 0, \tag{M}$$

$$j = 1, \cdots, N, \quad k = 0, 1, \cdots, m_j - 1.$$

By uniqueness of solutions of (V) for $|a|$ sufficiently small and by uniqueness of solutions of (N) for $|z_0|$ sufficiently small (implied by the Lipschitz condition in (H_2)) it follows that $z(t, z_0) = \psi(t, a)$ for some a, and it is bounded in the sense of Theorem 1. Moreover, if a in (V) is chosen to be a z_0 satisfying (M), then (2.9) becomes $\psi(0, z_0) = z_0$. We shall refer to equations (M) as defining the *stable manifold* for the system (N). This terminology is justified further by the following result.

THEOREM 2. *If $X = L^\infty(0, \infty)$ or BC, if the hypotheses of Theorem 1 are satisfied and if $y_0 \in \mathbb{R}^n$, $|y_0| \leqq \delta_1$, but y_0 is not on the manifold defined by* (M), *then the solution $z(t)$ of* (N), $z(0, y_0) = y_0$, *must leave the ball $\{z : |z| \leqq \delta_2\}$ in finite time.*

Theorem 2 is proved in § 4.

Theorems 1 and 2 generalize the familiar stable manifold theorem for ordinary differential equations (see Coddington and Levinson [2, Chap. 13, Thm. 4.1]). Hale [5] has obtained such a result for functional differential equations with finite time lags using strongly continuous semigroups and the Hille, Phillips, Yosida theorem to obtain an exponential dichotomy for solutions of the appropriate linearized equation. That method is not applicable here.

We also remark that Antosiewicz [1] has considered some abstract problems using fixed point methods which are related to theorems about stable manifolds. However, his technique does not yield our results.

We now consider several special cases.

Example 1. *The ODE case.* This arises from (N) and Theorems 1 and 2 by choosing $X = BC$ or BC_0, $B(t) \equiv 0$ and the functionals H to be continuous functions from $\mathbb{R}^n$ into $\mathbb{R}^n$ satisfying hypothesis (H_2); (E) is the characteristic equation for the matrix A and hypothesis (H_1) means that A has N eigenvalues $\lambda_1, \cdots, \lambda_N$ with positive real parts. If their multiplicities are $m_1, m_2, \cdots, m_N$ respectively, let $m = m_1 + m_2 + \cdots + m_N$. Let $m < n$; then in accordance with (H_1) the remaining $n - m$ eigenvalues of A have negative real parts. Without loss of generality we may assume that the matrix A is block diagonal:

$$A = \operatorname{diag}[A_1, A_2],$$

where the matrix A_1 of $n - m$ rows and columns has all eigenvalues with negative real parts and the $m \times m$ matrix A_2 has all eigenvalues with positive real parts. The resolvent kernel is $R_L(t) = \exp tA$; the kernels $R(t)$ and $W(t)$ of (2.2), (2.3) are given by

$$W(t) = \begin{bmatrix} 0 & 0 \\ 0 & \exp tA_2 \end{bmatrix}, \qquad R(t) = \begin{bmatrix} \exp tA_1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.11}$$

and the hypotheses of Theorems 1 and 2 are satisfied. The stable manifold is of the form

$$x_j = v_j(x_1, \cdots, x_{n-m}) = -\left(\int_0^\infty W(-\xi)H\psi(\xi, x)\, d\xi\right)_j \tag{2.12}$$

for $j = n - m + 1, \cdots, n$. This can be seen as follows. From (2.5) with f replaced by the functions $H(z)$, the special form of $R(t)$, $W(t)$ in (2.11) implies that (2.10) has the form

$$\begin{aligned} z(t, z_0) = R(t)z_0 + R * H(z(\cdot, z_0))(t) - \int_t^\infty W(t - \xi)H(z(\xi, z_0))\, d\xi \\ + W(t)\left[z_0 + \int_0^\infty W(-\xi)H(z(\xi, z_0))\, d\xi\right]. \end{aligned} \tag{2.13}$$

Using the special form of $W(t)$ in (2.11) again, equations (M) take the form

$$\left(z_0 + \int_0^\infty W(-\xi)H(z(\xi, z_0))\, d\xi\right)_j = 0, \qquad j = n - m + 1, \cdots, n. \tag{M^1}$$

If z_0 is a vector satisfying (M^1), the solution $z(t, z_0)$ of (N) satisfies (V) and by uniqueness $z(t, z_0)$ is $\psi(t, a)$ for some a. For any such bounded $z(t, z_0)$ the convergence of the integrals in (2.13) and in (M^1) follows from the hypothesis and from (2.11). From (V) and the special form of $R(t)$, $W(t)$ in (2.11), $\psi(t, a)$ satisfies

$$\psi_j(0, a) = \begin{cases} a_j & \text{if } j = 1, \cdots, n - m, \\ -\left(\int_0^\infty W(-\xi)H(\psi(\xi, a))\, d\xi\right)_j & \text{if } j = n - m + 1, \cdots, n; \end{cases}$$

it is clear that in view of (2.11) only the first $n - m$ components of a enter and a may be taken as a vector whose last m components are zero. Thus the initial values $x_j = \psi_j(0, a)$ satisfy the set of equations of the form (2.12). These define a manifold in x-space of dimension $n - m$ or codimension m.

Returning to the general case of (N) it is possible that the stable manifold (M) is the single point $\{z_0 = 0\}$; indeed, this is the case if, for example, $m = m_1 + \cdots + m_N \geqq n$ and $n = 1$. It is also clear that, in general, the equations in (M) are not independent. We now analyse the stable manifold (M) in another special case.

Example 2. If equation (E) has only simple roots s_j, $\operatorname{Re} s_j > 0$, $j = 1, \cdots, N$, the following result is true.

COROLLARY 1. *Let the hypothesis of Theorem* 1 *be satisfied and let the roots* s_j, $\operatorname{Re} s_j > 0$, $j = 1, \cdots, N$, *of equation* (E) *be simple. Then the stable manifold* (M) *for the system* (N) *is given by the relations*

$$\sum_{k=1}^{n} s_{1k}^{(j)}[z_{0k} + \widehat{H_k z(\cdot, z_0)}(s_j)] = 0, \qquad j = 1, \cdots, N, \tag{2.14}$$

where

$$\begin{aligned} \widehat{Hz(\cdot, z_0)}(s) &= (\widehat{H_1 z(\cdot, z_0)}, \cdots, \widehat{H_n z(\cdot, z_0)}) \\ &= \int_0^\infty \exp(-st)Hz(t, z_0)\, dt, \end{aligned}$$

and where $s_{ik}^{(j)}$ *are the elements of the matrix* Q_j^{-1} *and* Q_j *is the constant nonsingular* $n \times n$ *matrix such that* $Q_j^{-1}F(s_j)Q_j$ *is in Jordan canomical form* (5.3) *below,* $j = 1, \cdots, N$. *Moreover, if* $N\ (=m) < n$ *the stable manifold defined by* (2.14) *has codimension at most* N. *In particular, if* $N = 1$ *and* $n > 1$, *this stable manifold has codimension one.*

The smoothness of the stable manifold is determined by the smoothness of the perturbations as illustrated by the following result.

COROLLARY 2. *Let the hypothesis of Theorem* 1 *be satisfied and let*

$$Hz(t) = A_1 g_1(z(t)) + B_1 * g_2(z(\cdot))(t) + g_3(z(t))B_2 * g_4(z(\cdot))(t), \tag{2.15}$$

where A_1 *is a constant* $n \times n$ *matrix,* $B_1, B_2 \in L^1(0, \infty)$ *are given* $n \times n$ *matrices,* g_1, g_2, g_4 *are vector functions and* g_3 *is a scalar function, all in the class* $C^{(k)}$ (or analytic) *with respect to the components of* z, *in a region containing the origin* $z = 0$; *moreover,* $g_j(0) = 0, j = 1, \cdots, 4$, *and* $g_i'(z) = o(1)$ *as* $|z| \to 0$, $i = 1, 2$. *Then the degree of smoothness of the manifold defined by equation* (M) *assuming the form* (2.15) *is the same as the degree of smoothness of the perturbations.*

Corollaries 1 and 2 are proved in § 5.

When equation (E) has N roots s_j, $\operatorname{Re} s_j > 0$, which are not necessarily simple we have the following characterization of the stable manifold extending Corollary 1.

THEOREM 3. *Let* $X = BC$, *let the hypothesis of Theorem* 1 *be satisfied, and let* $0 < m = m_1 + m_2 + \cdots + m_N < n$. *Then the stable manifold for the system* (N) *defined by equations* (M) *has codimension at most* m.

This result is proved in § 6.

In the proof of Theorem 3 the following extension of Theorems 1 and 2 and of the associated stable manifold will be needed for an integro-differential system more general than (N) of the form

$$z'(t) = Az(t) + (B * z)(t) + h(z, z_0)(t), \qquad z(0) = z_0, \tag{2.16}$$

where A, B are as in (N) and where

$$h(z, z_0)(t) = g(z, z_0)(t) + \gamma(t)z_0 \tag{2.17}$$

in which g is a mapping from $BC \times \mathbb{R}^n$ into BC satisfying the hypothesis

(H$_3$) $g(0,0) = 0$ and for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\varphi_j \in BC$ and $\|\varphi_j\| \leqq \delta\ (j = 1, 2)$ and if $|z_0| \leqq \delta$, then $\|g(\varphi_1, z_0) - g(\varphi_2, z_0)\| \leqq \varepsilon\|\varphi_1 - \varphi_2\|$

and in which

(H$_4$) $\gamma(t)$ is an $n \times n$ matrix in $BC \cap L^1(0, \infty)$ and $\gamma(+\infty) = 0$.

In what follows it will be assumed that $X = BC$ or BC_0. Define the operator $\tilde{K}$ on X by the relation

$$\begin{aligned}(\tilde{K}\varphi)(t) = R(t)a &+ \int_0^t R(t-\xi)h(\varphi, a)(\xi)\,d\xi \\ &- \int_t^\infty W(t-\xi)h(\varphi, a)(\xi)\,d\xi, \qquad \varphi \in X, \quad 0 \leqq t < \infty,\end{aligned} \tag{2.18}$$

where the matrices R, W are defined by (2.2), (2.3) and $a \in \mathbb{R}_n$, $|a| \leqq \delta$.

THEOREM 4. *Let $B(t) \in L^1(0, \infty)$ and let hypotheses* (H_1), (H_3), (H_4) *be satisfied. There exist constants $\delta_1, \delta_2 > 0$ such that if $|a| \leqq \delta_1$, the integral equation*

$$(\tilde{\mathrm{V}}) \qquad z(t) = \tilde{K}z(t)$$

has a unique solution $\psi(t, a) \in X$ such that $\|\psi(\cdot, a)\| \leqq \delta_2$. The solution $\psi(t, a)$ will coincide with the solution $z(t, z_0)$ of the system (2.16) *for some $a \in \mathbb{R}^n$, $|a| \leqq \delta_1$ if $|z_0|$ is sufficiently small and if z_0 satisfies the stable manifold equations*

$$(\tilde{\mathrm{M}}) \qquad \left(P_{jk} + \sum_{i=k}^{m_j - 1} \frac{P_{ji}}{(i-k)!} \hat{\gamma}^{(i-k)}(s_j)\right) z_0 + \sum_{i=k}^{m_j - 1} \frac{P_{ji}}{(l-k!)} \widehat{h(z(\cdot, z_0), z_0)}^{(i-k)}(s_j) = 0,$$

$$j = 1, \cdots, N, \quad k = 0, 1, \cdots, m_{j-1}.$$

Moreover, if $y_0 \in \mathbb{R}^n$, $|y_0|$ small, but y_0 does not satisfy equations $(\tilde{\mathrm{M}})$, *then the solution $z(t, y_0)$ of* (2.16), *$z(0, y_0) = y_0$, must leave the ball $\{z : |z| \leqq \delta_2\}$ in finite time.*

The proof of Theorem 4 is an easy extension of the proofs of Theorems 1 and 2 and it will not be carried out in detail. It suffices to note from (2.17), (2.18), (H_3) and (H_4) that for $|a| \leqq \delta$ and $\|\varphi\| \leqq \delta$ one has

$$\|\tilde{K}\varphi\| \leqq \sup_{0 \leqq t < \infty} |R(t)|\,|a| + \left[\int_0^\infty |R(\xi)|\,d\xi + \int_{-\infty}^0 |W(\xi)|\,d\xi\right]\left[\varepsilon\|\varphi\| + \sup_{0 \leqq t < \infty} |\gamma(t)|\,|a|\right].$$

Thus for δ sufficiently small, it is easy to obtain an estimate like (3.5) in the proof of Theorem 1 and then establish that $\tilde{K}$ is a contraction on the appropriate ball S_2. Equations $(\tilde{\mathrm{M}})$ are obtained from (M) by replacing $Hz(\cdot, z_0)$ by $h(z(\cdot, z_0), z_0)$ and using (2.17) and $\gamma(t) \in L^1(0, \infty)$. To prove the last statement one follows the proof of Theorem 2 with Hz replaced by $h(z, z_0)$.

3. Proof of Theorem 1. Let $\|\cdot\|$ be the norm in X and let

$$M = \max\left(\sup_{0 \leqq t < \infty} |R(t)|,\ \|R\|,\ \|R\|_{L^1},\ \int_{-\infty}^0 |W(t)|\,dt\right).$$

Fix $\varepsilon > 0$ so that $\varepsilon M \leqq \frac{1}{4}$ and choose a corresponding $\delta > 0$ in accordance with (H_2). Let $0 < \delta_2 \leqq \delta$ and consider the ball

$$S_2 = \{\varphi \in X : \|\varphi\| \leqq \delta_2\}.$$

It will be shown that the mapping K defined by (2.7) is a contraction on S_2 for $|a|$ sufficiently small.

It is readily shown using $R(t) \in L^1(0, \infty)$ and inequality (2.8) that the mappings R and W defined respectively by

$$(3.1) \qquad Rf(t) = \int_0^t R(t - \xi) f(\xi)\,d\xi, \qquad 0 \leqq t < \infty, \quad f \in X,$$

$$(3.2) \qquad Wf(t) = \int_t^\infty W(t - \xi) f(\xi)\,d\xi, \qquad 0 \leqq t < \infty, \quad f \in X,$$

are continuous mappings from X into X. Note that for $X = L^p(0, \infty)$, $1 \leqq p < \infty$, one has the well-known inequality

$$\|Wf\|_{L_p} \leqq \int_{-\infty}^{0} |W(\xi)|\, d\xi \|f\|_{L_p}. \tag{3.4}$$

and similarly,

$$\|Wf\|_{L_p} \leqq \int_{-\infty}^{0} |W(\xi)|\, d\xi \|f\|_{L_p}. \tag{3.4}$$

If $X = BC_0$ or $BC_0 \cap L^p(0, \infty)$, one has from (3.1), (3.2) as well as (3.3), (3.4) that $\lim_{t\to\infty} Rf(t) = 0$ and $\lim_{t\to\infty} Wf(t) = 0$, whenever $f \in X$ and $\lim_{t\to\infty} f(t) = 0$.

Let $\delta_1 \leqq \min(\delta, \delta_2/(2M))$. It follows from (2.7), (2.8) and $R(t) \in L^1(0, \infty)$ that if $\varphi \in S_2$ and if X is any one of the spaces listed, then $K\varphi \in X$ and

$$\|K\varphi\| \leqq M|a| + \|H\varphi\| \left[\int_0^\infty |R(\xi)|\, d\xi + \int_{-\infty}^{0} |W(\xi)|\, d\xi \right];$$

here in case $X = L^p(0, \infty)$ use has been made of the Minkowski inequality and of (3.3), (3.4). Moreover, using the definitions of M, ε and δ_1 and of hypothesis (H_2), one obtains

$$\|K\varphi\| \leqq M\delta_1 + 2M\varepsilon \|\varphi\| \leqq \frac{\delta_2}{2} + \frac{\delta_2}{2} = \delta_2 \ (|a| \leqq \delta_1). \tag{3.5}$$

Hence $KS_2 \subset S_2$ provided $|a| \leqq \delta_1$.

Now let $\varphi_1, \varphi_2 \in S_2$. Then (2.7) and (H_2) imply

$$\begin{aligned} \|K\varphi_2 - K\varphi_1\| &\leqq \|H\varphi_2 - H\varphi_1\| \left[\int_0^\infty |R(\xi)|\, d\xi + \int_{-\infty}^{0} |W(\xi)|\, d\xi \right] \\ &\leqq 2\,\varepsilon M \|\varphi_2 - \varphi_1\| \leqq \tfrac{1}{2}\|\varphi_2 - \varphi_1\|. \end{aligned}$$

Thus K is a contraction on S_2 and the existence and uniqueness of the solution $\psi(t, a) \in X$ of the Volterra equation (V) follows from the contraction mapping principle. Returning to (V) and using the definitions (2.2), (2.3) yields the initial condition

$$\psi(0, a) = R(0)a - \sum_{j=1}^{N} \sum_{k=0}^{m_j - 1} \frac{P_{jk}}{k!} \int_0^\infty (-\xi)^k \exp(-\xi t) H\psi(\xi, a)\, d\xi.$$

The initial condition (2.9) results from this by using (2.4) as well as elementary properties of the Laplace transform. This completes the proof of Theorem 1.

4. Proof of Theorem 2. The proof is similar to the corresponding result in ordinary differential equations (see [2, p. 332]). Suppose the solution $z(t, y_0)$ of (N), $z(0, y_0) = y_0$, $|y_0| \leqq \delta_1$, $y_0 \notin (M)$, does not leave the ball $\{z : |z| \leqq \delta_2\}$ in finite time. The solution $z(t, y_0)$ of (N) satisfies the Volterra equation (2.10) with z_0 replaced by y_0 and $|z(t, y_0)| \leqq \delta_2$, $0 \leqq t < \infty$. Let

$$M = \max\left(\int_0^\infty |R(\xi)|\, d\xi,\ \int_{-\infty}^{0} |W(\xi)| d\xi,\ \sup_{0 \leqq t < \infty} |R(t)| \right),$$

$$K = \sup_{|z| \leqq \delta_2} |Hz|, \qquad K_1 = M(\delta_2 + 2K).$$

If $y_0 \notin (M)$ one has from (2.10) (with z_0 replaced by y_0)

$$\delta_2 \geqq |z(t, y_0)| \geqq \left| \sum_{j=1}^{N} \sum_{k=0}^{m_j - 1} \frac{t^k}{k!} \exp(s_j t) \cdot \left[P_{jk} y_0 + \sum_{i=k}^{m_j - 1} \frac{P_{ji}}{(i-k)!} \widehat{Hz(\cdot, y_0)}^{(i-k)}(s_j) \right] \right| - K_1, \qquad 0 \leqq t < \infty.$$

Since the exponential terms on the right-hand side have $\operatorname{Re} s_j > 0$ and since this exponential polynomial is not identically zero, this yields a contradiction and completes the proof of Theorem 2.

5. Proof of Corollaries 1 and 2.

(a) *Corollary* 1. Equations (M) for the stable manifold in the case of simple roots s_j, $\operatorname{Re} s_j > 0$, are

$$P_{j0}[z_0 + \widehat{Hz(0, z_0)}(s_j)] = 0, \qquad j = 1, \cdots, N. \tag{5.1}$$

Near each root s_j, $F^{-1}(s)$ has the Laurent expansion

$$F^{-1}(s) = P_{j0}(s - s_j)^{-1} + \sum_{k=0}^{\infty} L_{jk}(s - s_j)^k,$$

from which one readily obtains the relations

$$P_{j0} F(s_j) = F(s_j) P_{j0} = 0. \tag{5.2}$$

Since s_j is a simple zero of $\det F(s) = 0$, zero is a simple eigenvalue of the matrix $F(s_j)$. Hence there exists a nonsingular matrix Q_j such that

$$Q_j^{-1} F(s_j) Q_j = \begin{bmatrix} 0 & 0 & & & \\ & \lambda_2 & \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \delta_{n-1} \\ & & & & \lambda_n \end{bmatrix}, \tag{5.3}$$

where $\lambda_k \neq 0$, $k = 2, \cdots, n$ and $\delta_2, \cdots, \delta_{n-1}$ are either zero or one. From (5.2) one has

$$0 = Q_j^{-1} P_{j0} F(s_j) Q_j = Q_j^{-1} P_{j0} Q_j Q_j^{-1} F(s_j) Q_j.$$

Let $M_j = Q_j^{-1} P_{j0} Q_j = (m_{ik}^{(j)})$, $i, k = 1, \cdots, n$. It then follows from (5.2), (5.3) that

$$0 = Q_j^{-1} P_{j0} F(s_j) Q_j = \begin{bmatrix} 0 & \lambda_2 m_{12}^j & \delta_2 m_{12}^j + \lambda_3 m_{13}^j & \cdots & \delta_{n-1} m_{1,n-1}^j + \lambda_n m_{1n}^j \\ \cdot & \lambda_2 m_{22}^j & \delta_2 m_{22}^j + \lambda_3 m_{23}^j & \cdots & \delta_{n-1} m_{2,n-1}^j + \lambda_n m_{2n}^j \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \lambda_2 m_{n2}^j & \delta_2 m_{n2}^j + \lambda_3 m_{n3}^j & \cdots & \delta_{n-1} m_{n,n-1}^j + \lambda_n m_{nn}^j \end{bmatrix}.$$

Then $\lambda_k \neq 0$, $k = 2, \cdots, n$, readily yields

$$M_j = \begin{bmatrix} m_{11}^j & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ m_{n1}^j & 0 & \cdots & 0 \end{bmatrix},$$

A similar calculation using the second relation in (5.2) yields

$$M_j = \begin{bmatrix} m_{11}^j & 0 & \cdots & 0 \\ 0 & \cdot & & \cdot \\ \vdots & & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad m_{11}^j \neq 0. \tag{5.4}$$

Since $P_{j0} = Q_j M_j Q_j^{-1}$, (5.1) may be written as

$$M_j Q_j^{-1}[z_0 + \widehat{Hz(\cdot, z_0)}(s_j)] = 0, \qquad j = 1, \cdots, N, \tag{5.5}$$

which in view of the special form (5.4) means that "the first component of the vector $Q_j^{-1}[z_0 + \widehat{Hz(\cdot, z_0)}(s_j)] = 0$", and this written out gives the relations (2.14) for the stable manifold for $j = 1, \cdots, N$.

If $N = 1$, (2.14) is exactly one nontrivial relation. Hence if also $n = 1$, the stable manifold is the single point $\{z_0 = 0\}$. If $n > 1$, the stable manifold has codimension 1.

If $n > 1$ and $1 \leqq N < n$, consider the vectors

$$\begin{bmatrix} s_{11}^{(1)} \\ s_{12}^{(1)} \\ \cdot \\ \cdot \\ s_{1n}^{(1)} \end{bmatrix}, \quad \begin{bmatrix} s_{11}^{(2)} \\ s_{12}^{(2)} \\ \cdot \\ \cdot \\ s_{1n}^{(2)} \end{bmatrix}, \cdots, \begin{bmatrix} s_{11}^{(N)} \\ s_{12}^{(N)} \\ \cdot \\ \cdot \\ s_{1n}^{(N)} \end{bmatrix}. \tag{5.6}$$

The stable manifold has codimension N if and only if the N vectors (5.6) are linearly independent. Therefore, if $n \geqq 1$ and $N > 1$, the N vectors (5.6) contain a linearly independent subset which spans a subspace of dimension $1 \leqq k \leqq N$. If $k \geqq n$, the stable manifold consists of the single point $\{z_0 = 0\}$; if $1 \leqq k < n$, the stable manifold has codimension $k \leqq N$. This completes the proof of Corollary 1.

(b) *Corollary* 2. Here the stable manifold is defined by equations (M) with

$$\begin{aligned} \widehat{H(z(\cdot, z_0))}(s) &= A_1 \widehat{g_1(z(\cdot, z_0))}(s) + \hat{B}_1(s) \widehat{g_2(z(\cdot, z_0))(s)} \\ &\quad + \int_0^\infty e^{-s\xi}\left(\int_0^\infty e^{-su} g_3(z(\xi + u, z_0)B_2(u)\,du)\right) g_4(z(\xi, z_0))\,d\xi, \end{aligned} \tag{5.7}$$

where the last integral was obtained after an application of the Fubini theorem. Thus if the $g_j \in C^k$ or if the g_j are analytic near the origin, $j = 1, \cdots, 4$, the stable manifold defined by (2.14), (5.7) has the same type smoothness property. Use is made here of the differentiability (or analyticity) of solutions $\psi(\cdot, a)$ of equation (V) and of solutions $z(\cdot, z_0)$ of (N) with respect to a and z_0 respectively where the perturbation functionals H are replaced by the functions (2.15). Such results are well known for ordinary differential equations and essentially the same proofs carry over to the present situation. For a proof of the smoothness of the stable manifold in the ODE case the reader is referred to [2, Thm. 4,2, Chap. 13]. This completes the proof of Corollary 2.

6. Proof of Theorem 3. The result has already been established when each root s_j of (E) with $\operatorname{Re} s_j > 0, j = 1, \cdots, N$, has multiplicity $m_j = 1$ and if $m = m_1 + \cdots + m_N = N$; in particular the result is true if $m\ (=N) = 1$. We proceed by induction. Suppose that the theorem is true if $m = m_1 + \cdots + m_N = k$ and consider the case $m = k + 1$.

Without loss of generality we may assume that s_1 is at least a simple root of equation (E); this means that the matrix $F(s_1)$ defined by (2.1) has the Jordan canonical form

$$F(s_1) = \begin{bmatrix} 0 & 0 & & & \\ & \lambda_2 & \delta_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \delta_n \\ & & & & \lambda_n \end{bmatrix},$$

where $\delta_2, \cdots, \delta_{n-1}$ is zero or 1, the possibility $\lambda_j = 0, j \geqq 2$, is not excluded and where all the entries not shown explicitly are zero. Otherwise a linear change of coordinates applied to (N) can be used to achieve this without changing the form or character of (N).

Define the $n \times n$ matrix

$$C(t) = \operatorname{diag}[e^{s_1 t}, e^{-t}, \cdots, e^{-t}],$$

and note the formula (obtained by integration by parts)

$$\int_0^t C(t-s)z'(s)\,ds = z(t) - C(t)z(0) + \int_0^t C'(t-\xi)z(\xi)\,d\xi.$$

Let $z(t, z_0)$ be a solution of (N). If both sides of (N) are "convolution multiplied" by $C(t)$, it follows that the solution $z(t, z_0)$ satisfies the integral equation

$$z(t, z_0) = C(t)z_0 + C * Hz(\cdot, z_0)(t) + a * z(\cdot, z_0)(t), \tag{6.1}$$

where

$$a(t) = -C'(t) + C(t)A + C * B(t). \tag{6.2}$$

It follows from [8, Lemma 5] that $a(t), a'(t) \in L^1(0, \infty)$ and since $a(t)$ is continuous, also $\lim_{t\to\infty} a(t) = 0$.

Define $y = y(t, z_0)$ by the relation

$$y'(t, z_0) + y(t, z_0) = z(t, z_0), \qquad y(0, z_0) = z_0. \tag{6.3}$$

Thus

$$y(t, z_0) = e^{-t}z_0 + \int_0^t e^{-(t-\xi)}z(\xi, z_0)\,d\xi. \tag{6.4}$$

$$\begin{aligned} y'(t, z_0) = {} & A_1 y(t, z_0) + B_1 * y(\cdot, z_0)(t) \\ & + [C(t) - a(t)]z_0 + C * H[y(\cdot, z_0) + y'(\cdot, z_0)](t), \end{aligned} \tag{6.5}$$

where

$$A_1 = a(0) - I, \qquad B_1(t) = a(t) + a'(t). \tag{6.6}$$

LEMMA 6.1. *If* $z(t, z_0) \in BC$ *is a solution of* (6.1) *on* $0 \leqq t < \infty$, *then*

$$z_{01} + \widehat{H_1 z(\cdot, z_0)}(s_1) = 0, \tag{6.7}$$

where z_{01} *is the first component of* z_0 *and* $H_1 z$ *is the first component of* Hz.

Proof of Lemma 6.1. Since $z(t, z_0)$ is bounded and since $a(t) \in L^1(0, \infty)$, the last integral in (6.1) is bounded. Thus the quantity

$$C(t)z_0 + C * Hz(\cdot, z_0)(t) \tag{6.8}$$

is also bounded. From the definition of $C(t)$ the jth component of (6.8) is bounded for $j = 2, \cdots, n$. The first component of (6.8) is

$$\begin{aligned} &e^{s_1 t} z_{01} + \widehat{H_1 z(\cdot, z_0)}(s_1) \\ &= e^{s_1 t}\left[z_{01} + \int_0^\infty e^{-s_1 \xi} H_1 z(\xi, z_0)\, d\xi - \int_t^\infty e^{-s_1(t-\xi)} H_1 z(\xi, z_0)\, d\xi\right]. \end{aligned} \tag{6.9}$$

Since the last integral in (6.9) is clearly bounded, but $e^{s_1 t}$ is unbounded on $0 \leqq t < \infty$, (6.7) must hold. This completes the proof of Lemma 6.1.

It follows immediately from (6.4) that if $z(t, z_0)$ is bounded for $0 \leqq t < \infty$, then so is $y(t, z_0)$. The converse of this statement is also true provided z_0 and $z(t, z_0)$ satisfy (6.7); this will follow as a consequence of Lemma 6.2 below. Define the mapping $G: BC \times \mathbb{R}^n \to BC$ by the relation

$$G(\varphi, z_0)(t) = H\varphi(t, z_0). \tag{6.10}$$

It follows from hypothesis (H_2) and from Theorem 1 that a natural assumption for the functional G to satisfy is

$$\begin{gathered} G(0,0) = 0 \text{ and for every } \varepsilon > 0 \text{ there is a } \delta > 0 \text{ such that if} \\ \varphi_j \in BC \text{ and } \|\varphi_j\| \leqq \delta\ (j = 1, 2) \text{ and if } |z_0| \leqq \delta, \text{ then} \\ \|G(\varphi_1, z_0) - G(\varphi_2, z_0)\| \leqq \varepsilon \|\varphi_1 - \varphi_2\|. \end{gathered} \tag{6.11}$$

LEMMA 6.2. *Let* $h \in BC$ *be a given function and let the functional* G *satisfy* (6.11). *For* $|z_0|$ *and* $\|h\|$ *sufficiently small the system of integral equations*

$$\begin{aligned} w_1(t) &= -\int_t^\infty e^{s_1(t-\xi)} G_1(h + w, z_0)(\xi)\, d\xi, \\ w_k(t) &= \int_0^t e^{-(t-\xi)} G_k(h + w, z_0)(\xi)\, d\xi, \qquad k = 2, \cdots, n, \end{aligned} \tag{6.12}$$

where $G = [G_1, \cdots, G_n]$, $w = [w_1, \cdots, w_n]$, *has a unique solution* $w = w(t, z_0) \in BC$ *which may be written in the form*

$$w(t) = \tilde{G}(h, z_0)(t), \tag{6.13}$$

where the functional $\tilde{G}$ *satisfies* (6.11).

Proof of Lemma 6.2. If ε in (6.11) is regarded as a function of δ, then $\varepsilon(\delta) \to 0$ as $\delta \to 0^+$. Fix $\delta_0 > 0$ such that $0 < \varepsilon(\delta) \leqq \frac{1}{2}$, $0 \leqq \delta \leqq \delta_0$. Equations (6.12) have the form

$$w(t) = \bar{G}(h + w, z_0)(t), \tag{6.14}$$

where $\bar{G}$ denotes the operator defined by the right-hand side of (6.12) and $\bar{G}$ satisfies (6.11). Let $B_0 = \{w \in BC; \|w\| \leqq \delta_0/2\}$. Let $h \in BC$ such that $\|h\| \leqq \delta_0/2$ and let $z_0 \in \mathbb{R}^n$ such that $|z_0| \leqq \delta_0/2$. Then

$$\|\bar{G}(h + w_1 z_0)\| \leqq \tfrac{1}{2}\|w + h\| \leqq \tfrac{1}{2}(\|w\| + \|h\|) \leqq \delta_0/2.$$

for all $w \in B_0$ and all $h \in BC$, $\|h\| \leqq \delta_0/2$. Thus $\bar{G}$ maps B_0 into B_0. Moreover, if $v, w \in B$ and $h \in BC$, $\|h\| \leqq \delta_0/2$, $z_0 \in \mathbb{R}^n$, $|z_0| \leqq \delta_0/2$, then

$$\|\bar{G}(h + v, z_0) - \bar{G}(h + w, z_0)\| \leqq \tfrac{1}{2}\|v - w\|.$$

Thus $\bar{G}$ is a contraction on B_0 and thë system (6.12) has a unique solution as asserted.

To show that the solution $\tilde{G}(h, z_0)$ satisfies (6.11), let $h_j \in BC$, $\|h_j\| \leqq \sigma \leqq \delta_0/2$, let $|z_0| \leqq \sigma \leqq \delta_0/2$ and let $w^{(j)}$, $j = 1, 2$, be the corresponding solutions of (6.12). Then

$$w^{(j)} = \tilde{G}^{(j)}(h_j, z_0) = \bar{G}(h_j + w^{(j)}, z_0), \qquad j = 1, 2,$$

and therefore,

$$\begin{aligned} \|w^{(1)} - w^{(2)}\| &= \|\bar{G}(h_1 + w^{(1)}, z_0) - \bar{G}(h_2 + w^{(2)}, z_0)\| \\ &\leqq \varepsilon(\sigma)[\|h_1 - h_2\| + \|w^{(1)} - w^{(2)}\|] \\ &\leqq \tfrac{1}{2}\|w^{(1)} - w^{(2)}\| + \varepsilon(\sigma)\|h_1 - h_2\|. \end{aligned}$$

Thus $\|w^{(1)} - w^{(2)}\| \leqq 2\varepsilon(\sigma)\|h_1 - h_2\|$, and since $\varepsilon(\sigma) \to 0$ as $\sigma \to 0^+$, the solution $w = \tilde{G}(h, z_0)$ satisfies (6.11) as asserted. This completes the proof of Lemma 6.2.

To apply Lemma 6.2 to the system (6.5), let $[x]_k$ denote the kth component of x. Using Lemma 6.1 and (6.7) one has

$$\begin{aligned} y_1'(t, z_0) &= [A_1 y(t, z_0) + B_1 * y(\cdot, z_0)(t)]_1 - [a(t)z_0]_1 \\ &\quad - \int_t^\infty e^{s_1(t-\xi)} G_1(y + y', z_0)(\xi)\, d\xi. \end{aligned}$$

An easy calculation using (6.2), the definition of $C(t)$ and the canonical form $F(s_1)$ shows that

$$\begin{aligned} [a(t)z_0]_1 &= -e^{s_1 t}[F(s_1)z_0]_1 - \left[\int_t^\infty e^{s_1(t-\xi)} B(\xi)\, d\xi z_0\right]_1 \\ &= -\left[\int_t^\infty e^{s_1(t-\xi)} B(\xi)\, d\xi z_0\right]_1. \end{aligned}$$

Thus (6.5) becomes

$$
\begin{aligned}
y_1'(t, z_0) &= [A_1 y(t, z_0) + B_1 * y(\cdot, z_0)(t)]_1 + \left[\int_t^\infty e^{s_1(t-\xi)} B(\xi)\, d\xi z_0\right]_1 \\
&\quad - \int_t^\infty e^{s_1(t-\xi)} G_1(y + y', z_0)(\xi)\, d\xi, \\
y_k'(t, z) &= [A_1(t, z_0) + B_1 * y(\cdot, z_0)(t)]_k + [(C(t) - a(t))z_0]_k \\
&\quad + \int_0^t e^{-(t-\xi)} G_k(y + y', z_0)(\xi)\, d\xi, \qquad k = 2, \cdots, n.
\end{aligned} \tag{6.15}
$$

Define $f = \operatorname{col}(f_1, \cdots, f_n)$ by the relations

$$
\begin{aligned}
f_1(t) &= [A_1 y + B_1 * y]_1(t) - [a(t)z_0]_1, \\
f_k(t) &= [A_1 y + B_1 * y]_k(t) + [(C(t) - a(t))z_0]_k, \qquad k = 2, \cdots, n.
\end{aligned} \tag{6.16}
$$

Define $w = \operatorname{col}(w_1, \cdots, w_n)$ by the relations

$$
w_j(t) = y_j'(t) - f_j(t), \qquad j = 1, \cdots, n. \tag{6.17}
$$

Then equations (6.15) become

$$
\begin{aligned}
w_1(t) &= -\int_t^\infty e^{s_1(t-\xi)} G_1(y + f + w, z_0)(\xi)\, d\xi, \\
w_k(t) &= \int_0^t e^{-(t-\xi)} G_k(y + f + w, z_0)(\xi)\, d\xi, \qquad k = 2, \cdots, n,
\end{aligned} \tag{6.18}
$$

which is a system of the form (6.12) with $h = y + f$. Since $y \in BC$ and since $a(t)$ is continuous and tends to zero as $t \to \infty$, one has $f \in BC$ and $\|f\| \to 0$ as $\|y\| + |z_0| \to 0$. Thus Lemma 6.2, with $h = y + f$, applied to the system (6.18) shows that for $\|y\|$ and $|z_0|$ sufficiently small (6.18) has a unique solution w of the form

$$
w = y' - f = \tilde{G}(y + f, z_0),
$$

or equivalently

$$
y' = f + \tilde{G}(y + f, z_0), \tag{6.19}
$$

where $\tilde{G}$ satisfies (6.11). Note that (6.11) is the same as hypothesis (H_3). Writing out the system (6.19) using (6.16) yields

$$
\begin{aligned}
y'(t, z_0) &= A_1 y(t, z_0) + B_1 * y(\cdot, z_0)(t) \\
&\qquad + \tilde{G}(y + A_1 y + B_1 * y + \gamma(\cdot)z_0, z_0)(t) + \gamma(t)z_0, \\
y(0, z_0) &= t_0,
\end{aligned} \tag{6.20}
$$

where

$$
[\gamma(t)z_0]_1 = -[a(t)z_0]_1, \qquad [\gamma(t)z_0]_k = [(c(t) - a(t))z_0]_k,
$$

$k = 2, \cdots, n$. Thus $\gamma(t) \in BC \cap L^1(0, \infty)$ and $|\gamma(+\infty)| = 0$.

To see that the system (6.20) is of the right form to apply Theorem 4, define the mapping $h(\varphi, z_0)$ by the relation

$$h(\varphi, z_0)(t) = \tilde{G}(\varphi + A_1\varphi + B_1 * \varphi + \gamma(\cdot)z_0, z_0)(t) + \gamma(t)z_0 \tag{6.21}$$

for $\varphi \in BC$, where $\tilde{G}$ satisfies (6.11) for $|z_0|$, $\|\varphi\|$ sufficiently small and for $\gamma \in BC$. From (6.11) it follows that $h(\varphi, z_0)$ satisfies (H_3). Moreover for $\varepsilon > 0$ given and $\|\varphi\| \leqq \delta$, $|z_0| \leqq \delta$ one has

$$\begin{aligned} \|h(\varphi, z_0)\| &\leqq \|\tilde{G}(\varphi + A_1\varphi + B_1 * \varphi + \gamma(\cdot)z_0, z_0)\| + \|\gamma\|\,|z_0| \\ &\leqq \varepsilon\|\varphi\| + \|\gamma\|\delta. \end{aligned}$$

Thus $h(\varphi, z_0)$ satisfies the hypothesis of Theorem 4.

It is clear from (6.20), $y \in BC$ and $B_1 \in L^1(0, \infty)$ that $y' \in BC$; thus if (6.7) is satisfied and if $y \in BC$, it follows from (6.3) that $z(\cdot, z_0) \in BC$. Hence the stable manifold for the solution $z(\cdot, z_0)$ of (N) can be regarded as the intersection of the stable manifold of the solution $y(\cdot, z_0)$ of the system (6.20) and of the surface defined by (6.7). To complete the proof of Theorem 3 it therefore suffices to prove that the stable manifold of the solution $y(\cdot, z_0)$ of (6.20) has codimension at most k. This will be done using Theorem 4 and the induction hypothesis.

Using the definition of $C(t)$ and (6.2), (6.6) it is easy to compute

$$F_1(s) = sI - A_1 - \widehat{B_1(s)} = (s + 1)\widehat{C(s)}F(s),$$

and therefore

$$\det F_1(s) = (s - s_1)^{-1}(s + 1)^{-n+2} \det F(s). \tag{6.22}$$

Thus as in [8, Lemma 5] the multiplicity of the root $s = s_1$ of equation (E) has been reduced by one by means of the "convolution multiplication" of (N) by $C(t)$ and by the change of variable (6.1) which has transformed (N) to the system (6.20). In particular, from (6.22), note that s_1 will not be a zero of $\det F_1(s)$ in the case that s_1 is only a simple root of equation (E). Relation (6.22) also shows that the roots $s_2, \cdots, s_N$ of (E) are unaffected by the above procedure.

Returning to the induction proof assume that the total multiplicity of all the roots s_j, $\operatorname{Re} s_j > 0$, of the equation (E) is $k + 1$. Then the total multiplicity of the roots s_j, $\operatorname{Re} s_j > 0$, of the equation

$$\det F_1(s) = \det(sI - A_1 - \widehat{B_1(s)}) = 0, \qquad \operatorname{Re} s > 0, \tag{$\tilde{\mathrm{E}}$}$$

is k. Equation $(\tilde{\mathrm{E}})$ is the appropriate form of equation (E) with respect to the system (6.20). By the induction hypothesis and the theory of the stable manifold for the system (6.20) as extended in Theorem 4, it follows that the stable manifold of the solution $y(t, z_0)$ of (6.20) has codimension at most k. Hence that of the stable manifold of the solution $z(t, z_0)$ of the system (N) is at most $k + 1$. This completes the proof of Theorem 3.

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ASYMPTOTIC EXPANSIONS OF SOLUTIONS OF QUASI-LINEAR ELLIPTIC EQUATIONS WITH SMALL PARAMETER*

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Abstract. The Dirichlet problem for singularly perturbed elliptic equations of the type

$$\varepsilon^2 \sum a_{ij}(x, u, \varepsilon u_x)u_{x_i x_j} + \varepsilon \sum a_i(x, u, \varepsilon u_x)u_{x_i} - u + f(x, \varepsilon) = 0$$

is treated. An asymptotic series for the solution, containing boundary layer and interior terms is developed, and its uniform asymptotic validity proved under certain (not too restrictive) conditions on the coefficients and the boundary data.

1. Introduction. Let Ω be a bounded domain in E^n, Euclidean n-space, with boundary $\partial\Omega$. The boundary $\partial\Omega$ is a smooth $(n-1)$-dimensional manifold. Points x belonging to E^n are given by $x = (x_1, , \cdots, x_n)$. We denote $u_x = (u_{x_1}, \cdots, u_{x_n})$.

Consider, for each $\varepsilon > 0$, the Dirichlet problem for a quasi-linear elliptic equation

$$\begin{aligned}\varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x, u, \varepsilon u_x)u_{x_i x_j} &+ \varepsilon \sum_{i=1}^{n} a_i(x, u, \varepsilon u_x)u_{x_i} \\ &+ \varepsilon a(x, u, \varepsilon u_x) - u = -f(x, \varepsilon)\end{aligned} \tag{1.1a}$$

for $x \in \Omega$, and

$$u(x, \varepsilon) = g(x, \varepsilon) \quad \text{for } x \in \partial\Omega. \tag{1.1b}$$

The functions a_{ij}, a_i, f, and g have continuous partial derivatives of all orders with respect to the arguments x, u, u_x, and ε; and $\partial\Omega$ is infinitely smooth. These smoothness conditions are made for simplicity; only a finite degree of smoothness is actually needed in our proofs.

We assume that there exists a classical solution $u(x, \varepsilon)$ of (1.1) for ε in some range $0 < \varepsilon \leqq \varepsilon_0$, satisfying

$$|u(x, \varepsilon)| < M_0 \tag{1.2}$$

for some M_0 independent of ε and x. Such existence results, with such a priori bounds, may be found, for example, in [5]. In particular, under the assumption that $a(x, u, 0) \equiv 0$, it follows from Lemma 3.2 of this paper that hypothesis (1.2) will be satisfied.

The purpose of this paper is to obtain an asymptotic expansion of the solution which is valid as $\varepsilon \to 0+$. Under certain hypotheses, we shall obtain a uniformly valid expansion in the form

$$u(x, \varepsilon) \cong V(x, \varepsilon) + U(x, \varepsilon), \tag{1.3}$$

where $V(x, \varepsilon)$ and $U(x, \varepsilon)$ are inner and outer expansions respectively.

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Certain generalizations of these results are immediately suggested. For example, consider, in place of (1.1a), the equation

$$\varepsilon^2 \sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} + \varepsilon \sum_{i=1}^{n} a_iu_{x_i} + \varepsilon a(x,u,\varepsilon u_x) - h(x,u,\varepsilon) = 0. \tag{1.4}$$

If we assume that the equation $h(x, u, \varepsilon) = 0$ has only a single solution $u = f(x, \varepsilon)$, and that $h_u(x, f(x, \varepsilon), \varepsilon) > 0$, then we may express h in the form $h(x, u, \varepsilon) = (u - f \cdot (x, \varepsilon))k(x, u, \varepsilon)$ with $k > 0$. We may then divide (1.4) by k to obtain an equivalent equation of the form (1.1a), which may be treated by the methods of this paper.

A second, minor, generalization would allow a_{ij}, a_i and a to depend on ε as well.

The case when all the a_i are identically 0 involves enough simplifications to merit special consideration. Sections 2 and 3 are devoted to this case. The general case is then taken up in § 4.

Ladyženskaja and Ural'tseva [5] studied the question of existence and uniqueness of solutions of problems of the type given by (1.1). Tang [8] obtained L^2 and L^∞ convergence estimates (as $\varepsilon \to 0$) for the solution u_ε of some quasi-linear elliptic equations given in divergence form with small parameter ε.

An introduction to the literature of obtaining expansions of differential equations with small parameter is given in O'Malley [6]. Višik and Ljusternik [9] obtained expansions of solutions of linear elliptic equations with small parameter.

De Villiers, generalizing earlier work of Berger and Fraenkel, proved in [1] the validity of Višik–Ljusternik type expansions for equations of the form $\varepsilon^2\Delta u + f(x, u, \varepsilon) = 0$. Fife independently obtained such results for general second order semilinear elliptic problems [2] and for quasi-linear elliptic problems [3] of a certain type. In the present paper fewer restrictions are made on the higher order terms of the differential equation than in [1]–[3], but the reduced problem is of a more special type. Also our hypotheses on the boundary data are different from those in [1]–[3].

Murray in [7] used the type of asymptotic methods treated in this and the above papers to study singular perturbation problems in a cell biochemical context.

2. The formal expansion. Here we treat the special case of (1.1a),

$$\varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x,u,\varepsilon u_x)u_{x_ix_j} + \varepsilon a(x,u,\varepsilon u_x) - u = -f(x,\varepsilon). \tag{2.1}$$

In addition to the assumptions of smoothness, ellipticity, and existence of a solution made in § 1, we make two additional hypotheses—the first to be used in obtaining the formal expansion, and the second to be used in § 3 to establish its validity. For this purpose we define the function

$$B(x,v,\rho) \equiv \sum_{i,j=1}^{n} v_i(x)v_j(x)a_{ij}(x, f(x,0) + v, \rho\mathbf{v}) \tag{2.2}$$

for all $x \in \partial\Omega$, and all real v and ρ. Here $\mathbf{v} = (v_1, \cdots, v_n)$ is the inward-directed unit vector normal to $\partial\Omega$ at x. Also let $h(x, \varepsilon) \equiv g(x, \varepsilon) - f(x, \varepsilon)$.

Hypothesis I. The function B may be estimated below in the manner

$$B(x, v, \rho) \geqq q(x, v)s(|\rho|) \tag{2.3}$$

for some positive functions q and s satisfying

$$\int_0^{|h(x,0)|} \frac{t\,dt}{q(x,t)} < \int_0^\infty \rho s(\rho)\,d\rho$$

for all $x \in \partial\Omega$.

Hypothesis II. There exists a number $\kappa_1 > 0$ such that for all $x \in \partial\Omega$ and all real numbers v_1, v_2, and ρ,

$$\frac{h(x,0)(\partial B/\partial v)(x, v_1, \rho)}{B(x, v_2, \rho)} \leqq 1 - \kappa_1.$$

Assuming that an estimate of the form (2.3) holds for some positive q and s, both hypotheses will be satisfied if h is small enough. On the other hand, Hypothesis I places no restriction at all on h if B is bounded away from zero ($q = 1$; $s =$ const.; this is true if the operator in (2.1) is uniformly elliptic) or, more generally, if $B \geqq s(|\rho|)$ with $\int_0^\infty ts(t)\,dt = \infty$. Similarly, Hypothesis II is satisfied if $\partial B/\partial v$ is small enough, or if h and $\partial B/\partial v$ have opposite signs.

We begin by estimating a formal Taylor series expansion of the solution $u(x, \varepsilon)$ about $\varepsilon = 0$. To obtain the coefficients of the expansion, we differentiate (2.1) successively with respect to ε, and then set $\varepsilon = 0$ in the results. We thus obtain the system of equations

$$u_0 = f_0, \tag{2.4$_0$}$$

and in general for $r > 0$,

$$u_r = f_r + R_r(x, u_0, Du_0, D^2u_0, \cdots, u_{r-2}, Dr_{r-2}, D^2u_{r-2}), \tag{2.4$_r$}$$

where R_r are known functions. In this way, we obtain an expansion

$$U(x, \varepsilon) \sim \sum_{r=0}^\infty u_r\varepsilon^r. \tag{2.5}$$

We map a suitably small neighborhood N of a portion Σ of the boundary $\partial\Omega$ in a smooth invertible fashion into ζ-space, where $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$, so that Σ is mapped onto part of the hyperplane $\zeta_1 = 0$, and points in Ω are mapped onto points with $\zeta_1 > 0$. We denote the image of N by N'. For any given function of x, we shall denote by the same symbol the corresponding function of ζ resulting from this transformation; thus for example, we write simply $f(\zeta, \varepsilon)$ in place of $f(x(\zeta), \varepsilon)$.

In the new coordinate system, the equations (2.1), (1.1b) assume the following form:

$$\varepsilon^2 \sum_{i,j=1}^n b_{ij}(\zeta, u, \varepsilon u_\zeta)u_{\zeta_i\zeta_j} + \varepsilon b(\zeta, u, \zeta u_\zeta, \varepsilon) - u = -f(\zeta, \varepsilon); \tag{2.6}$$

$$u(0, \tilde{\zeta}, \varepsilon) = g(0, \tilde{\zeta}, \varepsilon), \tag{2.7}$$

where

$$\tilde{\zeta} = (\zeta_2, \cdots, \zeta_n), \quad \zeta = (\zeta_1, \tilde{\zeta}).$$

Thus, for example,

$$b_{11}(0, \tilde{\zeta}, f(0, \tilde{\zeta}, 0) + v, \rho\mathbf{v}) \equiv B(\tilde{\zeta}, v, \rho). \tag{2.8}$$

We stretch the normal coordinate by defining $t = \zeta_1/\varepsilon$, and seek a formal solution of (2.6), (2.7) in the form

$$u(\zeta, \varepsilon) = U(\varepsilon t, \tilde{\zeta}, \varepsilon) + V(t, \tilde{\zeta}, \varepsilon),$$

where

$$V = \sum_{r=0} v_r(t, \tilde{\zeta})\varepsilon^r.$$

Formally substituting this expression for u into (2.6), we obtain

$$b_{11}(w)V_{tt} - V + \varepsilon P = -f(\varepsilon t, \tilde{\zeta}, \varepsilon) + U(\varepsilon t, \tilde{\zeta}, \varepsilon), \tag{2.9}$$

where P is a function of ζ, ε, and derivatives of V of orders 0 to 2 involving at most one differentiation with respect to t, and where

$$(w) = (\varepsilon t, \tilde{\zeta}, U + V, \varepsilon U_{\zeta_1} + V_t, \varepsilon(U_{\tilde{\zeta}} + V_{\tilde{\zeta}})).$$

To obtain the terms v_r, we differentiate (2.9) successively with respect to ε, and set $\varepsilon = 0$ in the results. We obtain the system of equations

$$b_{11}(w_0)v_{0_{tt}} - v_0 = 0, \tag{2.10$_0$}$$

$$Lv_r \equiv b_{11}(w_0)v_{r_{tt}} + c_1(t, \tilde{\zeta})v_{r_t} + c(t, \tilde{\zeta})v_r = S_r(t, \tilde{\zeta}), \tag{2.10$_r$}$$

where

$$(w_0) \equiv (0, \tilde{\zeta}, u_0(0, \tilde{\zeta}) + v_0(t, \tilde{\zeta}), v_{0_t}(t, \tilde{\zeta}), 0),$$

$$c_1(t, \tilde{\zeta}) \equiv \frac{\partial b_{11}}{\partial u_{\zeta_1}}(w_0)v_{0_{tt}}(t, \tilde{\zeta}),$$

$$c(t, \tilde{\zeta}) \equiv b_{11u}(w_0)v_{0_{tt}} - 1,$$

and the functions S_r may be expressed in terms of the functions $v_0, \cdots, v_{r-1}$, and their derivatives, and known functions. They vanish when $v_0, \cdots, v_{r-1}$ are identically zero.

LEMMA 2.1. *There exists a unique monotone solution of* (2.10$_0$) *satisfying*

$$v_0(0, \tilde{\zeta}) = h_0(\tilde{\zeta}) \equiv g_0(0, \tilde{\zeta}) - u_0(0, \tilde{\zeta}); \qquad v_0(\infty, \tilde{\zeta}) \equiv 0. \tag{2.11}$$

It decays exponentially as $t \to \infty$.

Proof. If there exists a monotone solution, then we may write $W = v_{0_t}$ as a function of v_0 and $\tilde{\zeta}$. Thus (2.10$_0$) becomes

$$WW_{v_0} - \frac{v_0}{b_{11}(w_0)} = 0,$$

where $w_0 = (0, \tilde{\zeta}, u_0(0, \tilde{\zeta}) + v_0, W\mathbf{v})$ and the second condition (2.11) becomes $W(0, \tilde{\zeta}) = 0$. Any monotone solution will have the property that v_0 and h_0 have the same sign, and W has the opposite sign. We consider the case $v_0 > 0$ and $W < 0$; the other case may be treated similarly. We define $Q(v_0, \tilde{\zeta}) = W^2(v_0, \tilde{\zeta})$, so that the equation becomes

$$\frac{dQ}{dv_0} = H(\tilde{\zeta}, v_0, Q) \equiv 2v_0/b_{11}(0, \tilde{\zeta}, u_0(0, \tilde{\zeta}) + v_0, -\sqrt{Q}\mathbf{v});$$
$$Q(0, \tilde{\zeta}) = 0. \tag{2.12}$$

By [4, Lemma 5.2], the initial value problem (2·12) has a unique positive solution defined in some interval $v_0 \in [0, \alpha)$. To estimate α we do the following. By Hypothesis I and (2.8), we know that

$$H(\tilde{\zeta}, v_0, Q) \leqq 2v_0/q(\tilde{\zeta}, v_0)s(\sqrt{Q}).$$

Therefore a standard comparison argument shows that $Q(v_0, \tilde{\zeta}) \leqq R(v_0, \tilde{\zeta})$, where R satisfies

$$\frac{dR}{dv_0} = 2v_0/q(\tilde{\zeta}, v_0)s(\sqrt{R}); \qquad R(0, \tilde{\zeta}) = 0;$$

hence

$$\frac{1}{2}\int_0^R s(\sqrt{\tau})\, d\tau = \int_0^{v_0} \frac{v\, dv}{q(\tilde{\zeta}, v)}.$$

Setting $z = \sqrt{\tau}$, we have $\int_0^{\sqrt{R}} s(z)z\, dz = \int_0^{v_0} v\, dv/q(\tilde{\zeta}, v)$. By Hypothesis I, there exists a solution $R = R(v_0, \tilde{\zeta})$ defined for $0 \leqq v_0 \leqq h_0(\tilde{\zeta})$; hence $\alpha > h_0(\tilde{\zeta})$, and there exists a solution $Q(v_0, \tilde{\zeta})$ of (2.12) defined for the same range. Solving

$$\frac{dv_0}{dt} = -\sqrt{Q(\tilde{\zeta}, v_0)}; \qquad v_0(0, \tilde{\zeta}) = h_0(\tilde{\zeta}),$$

we obtain automatically that $v_0(\infty, \tilde{\zeta}) = 0$, and hence the desired monotone solution of (2.10_0). Its exponential decay follows by the same argument as in [1, Lemma 2.1].

LEMMA 2.2. *For each* $r \geqq 1$, *there exists a unique solution* $v_r(t, \tilde{\zeta})$ *of* (2.10_r) *satisfying*

$$v_r(0, \tilde{\zeta}) = h_r(\tilde{\zeta}) \equiv g_r(0, \tilde{\zeta}) - u_r(0, \tilde{\zeta}),$$
$$v_r(\infty, \tilde{\zeta}) \equiv 0,$$

with

$$v_r(t, \tilde{\zeta}) \leqq C_r e^{-\gamma_r t}.$$

Proof. Differentiating (2.10_0), we see that $\chi(t) = v_{0_t}$ is a solution of the homogeneous part of (2.10_r). The rest of the proof proceeds as in [2, Lemmas 2.2, 3.4].

We have thus obtained a formal expansion of $V(t, \tilde{\zeta}, \varepsilon)$ in the form

$$V(t, \tilde{\zeta}, \varepsilon) \sim \sum_{r=0}^{\infty} v_r(t, \tilde{\zeta})\varepsilon^r. \tag{2.13}$$

Let $X(x)$ be a smooth cut-off function such that

$$\begin{aligned} &0 \leqq X(x) \leqq 1, \qquad x \in \bar{\Omega}; \\ &X(x) \equiv 1 \quad \text{when } \operatorname{dist}(x, \partial\Omega) \leqq \mu/2. \\ &X(x) \equiv 0 \quad \text{when } \operatorname{dist}(x, \partial\Omega) \geqq \mu, \end{aligned} \tag{2.14}$$

where $\mu < \operatorname{dist}(\Sigma, \partial N)$.

We write the solution $u(x, \varepsilon)$ in the form

$$u(x, \varepsilon) \sim U(x, \varepsilon) + V(t, \tilde{\zeta}, \varepsilon)X(x). \tag{2.15}$$

This formal expansion has been constructed only in the neighborhood N of Σ. This was so that the differential operators involved could be written in terms of a local coordinate system $\tilde{\zeta}$ on Σ. However, this restriction to only a patch of $\partial\Omega$ is not necessary; in fact, we may interpret $\tilde{\zeta}$ as a symbol for any point on $\partial\Omega$, $\zeta_1(x)$ as distance from x to $\partial\Omega$, and (as before) $t = \zeta_1/\varepsilon$. We may then, in a unique way, write (2.1) as a formal power series in ε, the coefficients being differential operators in t and in $\tilde{\zeta}$, the latter type now being interpreted in the usual way as differential operators in the manifold $\partial\Omega$, no reference to a local coordinate system being necessary. With this understanding, our expansion V is now defined in a neighborhood of the complete boundary $\partial\Omega$.

3. Proof of the validity of the asymptotic expansion.

THEOREM 3.1. *Let $u(x, \varepsilon)$ be a $C^2(\Omega)$ solution of* (2.1), (1.1b) *satisfying* (1.2). *Let N be a fixed positive integer, and define*

$$S(x, \varepsilon) = u(x, \varepsilon) - u^N(x, \varepsilon), \tag{3.1}$$

where

$$u^N(x, \varepsilon) = \sum_{r=0}^{N} (u_r(x) + v_r(x, \varepsilon)X(x))\varepsilon^r, \tag{3.2}$$

and u_r, v_r and X are defined by (2.4), (2.10), *and* (2.14) *respectively. Then for some constant C independent of ε,*

$$|S(x, \varepsilon)|_{\bar{\Omega}} \leqq C\varepsilon^{N+1}. \tag{3.3}$$

Before we can prove the above theorem, we will need some preliminary results. By Ladyženskaja and Ural'tseva [5], we have the following.

LEMMA 3.2. *Let $u(x, \varepsilon)$ be a $C^2(\Omega)$ solution of*

$$\sum_{i,j=1}^{n} a_{ij}(x, u, u_x, \varepsilon)u_{x_i x_j} + A(x, u, u_x, \varepsilon) = 0 \quad \textit{in } \Omega, \tag{3.4}$$

and

$$u(x, \varepsilon) = \phi(x, \varepsilon) \quad \textit{for } x \in \partial\Omega.$$

Suppose the matrix (a_{ij}) *is positive definite for all arguments* x, u, u_x, *and* $\varepsilon > 0$. *Suppose*

$$A(x, u, 0, \varepsilon)u \leqq -b_1 u^2 + b_2 \quad \textit{for all } u \textit{ and } \varepsilon > 0; \tag{3.5}$$
$$b_1 > 0, \text{ and } b_2 \geqq 0.$$

Then

$$|u(x, \varepsilon)|_{\bar{\Omega}} \leqq \max \{\sqrt{b_2/b_1}, |\phi(x, \varepsilon)|_{\partial\Omega}\}. \tag{3.6}$$

Since u^N, as constructed, is bounded in maximum norm independently of ε, and since $S = u - u^N$, we obtain from (1.2) that for some $M_1 > 0$, possibly depending on N,

$$|S|_{\Omega} \leqq M_1. \tag{3.6 }$$

Proof of Theorem 3.1. Substituting $u(x, \varepsilon) = S(x, \varepsilon) + u^N(x, \varepsilon)$ into (2.1), we obtain the following equation in S:

$$\begin{aligned}\varepsilon^2 \sum_{i,j=1}^{n} & a_{ij}(x, u^N + S, \varepsilon(u_x^N + S_x))(u_{x_i x_j}^N + S_{x_i x_j}) \\ & + \varepsilon a(x, u^N + S, \varepsilon(u_x^N + S_x)) \\ & - u^N - S = -f,\end{aligned}$$

or

$$\varepsilon^2 \sum_{i,j=1}^{n} a_{ij} S_{x_i x_j} + A(x, S, S_x, \varepsilon) = 0,$$

where

$$\begin{aligned}A(x, S, 0, \varepsilon) &= \varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x, u^N + S, \varepsilon u_x^N) u_{x_i x_j}^N \\ &\quad + \varepsilon a(x, u^N + S, \varepsilon u_x^N) - u^N - S + f \\ &= \varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x, u^N, \varepsilon u_x^N) u_{x_i x_j}^N \\ &\quad + \varepsilon a(x, u^N, \varepsilon u_x^N) - u^N + f \\ &\quad + \varepsilon^2 \sum_{i,j=1}^{n} a_{iju}(x, u^N + \theta S, \varepsilon u_x^N) U_{x_i x_j}^N S \\ &\quad + \varepsilon a_u(x, u^N + \theta S, \varepsilon u_x^N) S - S \\ &= -A_1(x, S, \varepsilon) S + A_2(x, \varepsilon),\end{aligned}$$

where

$$A_1 = 1 - \left\{\varepsilon^2 \sum_{i,j=1}^{n} a_{iju}(x, u^N + \theta S, \varepsilon u_x^N)u_{x_i x_j}^N + \varepsilon a_u\right\},$$

$$A_2 = \varepsilon^2 \sum_{i,j=1}^{n} a_{ij}(x, u^N, \varepsilon u_x^N)u_{x_i x_j}^N + \varepsilon a - u^N + f.$$

By construction, we know all the terms $\varepsilon^2 u_{x_i x_j}^N$ are $O(\varepsilon)$ as $\varepsilon \to 0$ except those involving normal (ζ_1) derivatives of v_0. Pursuing this further, we find

$$A_1(x, S, \varepsilon) = 1 - b_{11u}(0, \tilde{\zeta}, u^N + \theta S, v_{0_t}\mathbf{v})v_{0_{tt}}(t, \tilde{\zeta}) + \varepsilon p(x, S, \varepsilon),$$

where $|p| < M_2$. From (2.10$_0$) we may replace $v_{0_{tt}}$ by b_0/b_{11}; and from Hypothesis II, we obtain

$$A_1 \geqq \kappa_1 - \varepsilon M_3 \geqq \tfrac{1}{2}\kappa_1 \quad \text{for small } \varepsilon. \tag{3.7}$$

LEMMA 3.3. *For some constant M_4 independent of ε,*

$$|A_2(x, \varepsilon)| \leqq M_4 \varepsilon^{N+1}. \tag{3.8}$$

Proof. This proof may be patterned after the proof of the analogous Theorem 3.6 of [2].

Combining this result with (3.7), we find

$$SA(x, S, 0, \varepsilon) \leqq -\frac{1}{2}\kappa_1 S^2 + M_4 \varepsilon^{N+1} S \leqq -\frac{1}{4}\kappa_1 S^2 + \frac{1}{\kappa_1}(M_4 \varepsilon^{N+1})^2;$$

hence from Lemma 3.2,

$$|S|_\Omega \leqq \max\left\{|S|_{\partial\Omega}, \frac{2M_4}{\kappa_1}\varepsilon^{N+1}\right\}. \tag{3.9}$$

Finally, by the construction of the u^N, we know they are within the order of ε^{N+1} of satisfying the correct boundary values: $|S|_{\partial\Omega} \leqq M_5 \varepsilon^{N+1}$. Combining this with (3.9), we have

$$|S|_\Omega \leqq C\varepsilon^{N+1},$$

which completes the proof.

4. The general case. Here we return to the more general equation (1.1a) and outline the changes in the above treatment necessary to account for the additional first order terms.

Hypotheses I and II are changed in the following manner. In addition to the function B defined by (2.2), we also define a function B_1 by

$$B_1(x, v, \rho) = \sum_{i=1}^{n} v_i(x)a_i(x, f(x, 0) + v, \rho\mathbf{v}).$$

We then require the following.

Hypothesis I$'$. The estimate

$$\frac{v + B_1(x, v, \rho)}{B(x, v, \rho)} \leqq \frac{Q(x, v)}{s(|\rho|)}$$

holds for positive functions Q and s satisfying

$$\int_0^{|h(x,0)|} Q(x,v)\,dv < \int_0^{\infty} \rho s(\rho)\,d\rho$$

for all $x \in \partial\Omega$.

Hypothesis II′. Let $R_0(x)$ be defined as the unique function satisfying

$$\int_0^{\sqrt{R_0}} \rho s(\rho)\,d\rho = \int_0^{|h(x,0)|} Q(x,v)\,dv, \qquad x \in \partial\Omega.$$

Then there exists a number $\kappa_1 > 0$ such that

$$\frac{\partial B}{\partial v}(x, v_1, \rho)\left(\frac{v - B_1(x,v,\rho)R}{B(x,v,\rho)}\right) + \frac{\partial B_1}{\partial v}(x, v_1, \rho)R \leqq 1 - \kappa_1$$

for all $x \in \partial\Omega$, all real numbers v_1, ρ, and all real numbers v, R satisfying

$$|v| \leqq |h(x,0)|, \qquad |R| \leqq R_0.$$

As before, these hypotheses are satisfied if h is small enough, or if $\partial B/\partial v$ and $\partial B_1/\partial v$ are small enough.

The construction of the asymptotic expansion and the proof of its validity are entirely analogous to those in the special case. For example, the expression on the left of (2.6) now includes terms $\varepsilon \sum b_i u_{\zeta_i}$; in place of (2.9), we have

$$b_{11}(w)V_{tt} + b_1(w)V_t - V + \varepsilon P' = -f + U;$$

and (2.10_0) becomes

$$b_{11}(w_0)v_{0_{tt}} + b_1(w_0)v_{0_t} - v_0 = 0.$$

Thus (2.12) is replaced by

$$\frac{dQ}{dv_0} = 2\left(\frac{v_0 + b_1(0, \tilde{\zeta}, u_0(0, \tilde{\zeta}) + v_0, -\sqrt{Q}\mathbf{v})}{b_{11}(\cdot\,\cdot)}\right); \qquad Q(0, \tilde{\zeta}) = 0.$$

Similar changes are needed in the other formulas; details will not be given.

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JACOBI POLYNOMIALS, III. AN ANALYTIC PROOF OF THE ADDITION FORMULA*

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Abstract. The addition formula for Jacobi polynomials is derived from the integral representation for the product $P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials. The proof uses integration by parts and some new differentiation formulas for Jacobi polynomials. Several formulas related to the addition formula are also discussed.

1. Introduction. This paper completes the analytic proof of the addition formula for Jacobi polynomials, which was initiated in [1] and [8]. In [1] Askey gave an elementary proof of the Laplace type integral representation for Jacobi polynomials. The author [8] derived from this formula the integral representation for the product $P_n^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials. The present paper contains the derivation of the addition formula from this product formula.

For the proof we need some new second order differential recurrence relations for Jacobi polynomials. These are obtained in § 2. The addition formula can be considered as an orthogonal expansion in terms of certain functions in two variables. It can be rewritten as an expansion in orthogonal polynomials in two variables. This is discussed in § 3. The addition formula is equivalent to a number of integration formulas which represent the respective terms of the orthogonal expansion. In § 4 these integration formulas are derived from the product formula which was proved in [8]. This is done by repeated integration by parts and by applying the differentiation formulas obtained in § 2. In a similar way the degenerate addition formula for Jacobi polynomials and a generalized addition formula for Bessel functions are obtained. Several related results are finally discussed in § 5.

Three different proofs of the addition formula for Jacobi polynomials have now been published. The first two proofs applied group theoretic methods. In [4], [5], [6] it was used that certain Jacobi polynomials are spherical functions on the homogeneous space $SU(q)/SU(q-1)$. The proof given in [7] was based on the interpretation of Jacobi polynomials as spherical harmonics. The present proof uses only analytic methods. A slightly different analytic proof by Gasper is unpublished (cf. § 5, Remark 2). The author can announce yet another proof of the addition formula which is rather short and involves a certain class of orthogonal polynomials in three variables.

Remark. In the following some elementary formulas for gamma, hypergeometric and Bessel functions and for orthogonal polynomials will be used without reference. For these formulas the reader is referred to the chapters 1, 2, 7 and 10, respectively, in Erdélyi [3].

2. Some new differentiation formulas for Jacobi polynomials. Let the hypergeometric function be defined by

$$ {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, \qquad |x| < 1. \tag{2.1}$$

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There are a number of well-known first order differential recurrence relations for hypergeometric functions (cf. [3, § 2.8, (20)–(27)] with $n = 1$). In this section some second order differential recurrence relations for hypergeometric functions and for Jacobi polynomials will be derived, which are probably new.

Replacement of x by x^2 in (2.1) and termwise differentiation gives

$$\left(\frac{d^2}{dx^2} + \frac{2c-1}{x}\frac{d}{dx}\right) {}_2F_1(a,b;c;x^2) = 4ab\, {}_2F_1(a+1,b+1;c;x^2). \tag{2.2}$$

Using the identity ${}_2F_1(a,b;c;x) = (1-x)^{c-a-b}\, {}_2F_1(c-a,c-b;c;x)$ we derive from (2.2) that

$$\begin{aligned}&\left(\frac{d^2}{dx^2} + \frac{2c-1}{x}\frac{d}{dx}\right)[(1-x^2)^{a+b-c+2}\, {}_2F_1(a+1,b+1;c;x^2)]\\ &\qquad = 4(c-a-1)(c-b-1)(1-x^2)^{a+b-c}{}_2F_1(a,b;c;x^2).\end{aligned} \tag{2.3}$$

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ can be expressed as hypergeometric functions by the formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(\beta+1)_n}{n!}\, {}_2F_1\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right).$$

Substituting this in (2.2) and (2.3) we obtain the pair of differential recurrence relations

$$\begin{aligned}&\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right)P_n^{(\alpha,\beta)}(2x^2-1)\\ &\qquad = 4(n+\alpha+\beta+1)(n+\beta)P_{n-1}^{(\alpha+2,\beta)}(2x^2-1),\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right)[(1-x^2)^{\alpha+2}P_{n-1}^{(\alpha+2,\beta)}(2x^2-1)]\\ &\qquad = 4n(n+\alpha+1)(1-x^2)^{\alpha}P_n^{(\alpha,\beta)}(2x^2-1).\end{aligned} \tag{2.5}$$

Repeated application of (2.5) gives a Rodrigues type formula

$$\begin{aligned}&2^{2n}n!(n+\alpha+1)_n(1-x^2)^{\alpha}P_n^{(\alpha,\beta)}(2x^2-1)\\ &\qquad = \left(\frac{d^2}{dx^2} + \frac{2\beta+1}{x}\frac{d}{dx}\right)^n(1-x^2)^{2n+\alpha}.\end{aligned} \tag{2.6}$$

If the variables x, y are expressed in the variables r, ϕ by $x = r\cos\phi$, $y = r\sin\phi$, then

$$\frac{\partial^2}{\partial r^2} + \frac{2\beta+1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \frac{2\beta\cot\phi}{r^2}\frac{\partial}{\partial\phi} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\beta}{y}\frac{\partial}{\partial y}.$$

Hence formula (2.6) can be rewritten as

$$\begin{aligned}&2^{2n}n!(n+\alpha+1)_n(1-x^2-y^2)^{\alpha}P_n^{(\alpha,\beta)}(2(x^2+y^2)-1)\\ &\qquad = (D_\beta)^n(1-x^2-y^2)^{2n+\alpha},\end{aligned} \tag{2.7}$$

where D_β denotes the partial differential operator

$$D_\beta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\beta}{y}\frac{\partial}{\partial y}. \tag{2.8}$$

Let the region $R = \{(x, y) | x^2 + y^2 < 1, y > 0\}$ denote the upper half unit disk.

LEMMA 2.1. *Let f be a C^∞-function on the closed unit disk $\{(x, y) | x^2 + y^2 \leqq 1\}$ such that $f(x, y) = f(x, -y)$. Then the same holds for $D_\beta f$. Furthermore, if $\alpha > -1$ and $\beta > -\frac{1}{2}$, then*

$$\begin{aligned} &\iint_R f(x, y) P_n^{(\alpha,\beta)}(2(x^2 + y^2) - 1)(1 - x^2 - y^2)^\alpha y^{2\beta}\, dx\, dy \\ &= \frac{1}{2^{2n} n!(n + \alpha + 1)_n} \iint_R ((D_\beta)^n f(x, y))(1 - x^2 - y^2)^{2n+\alpha} y^{2\beta}\, dx\, dy. \end{aligned} \tag{2.9}$$

Proof. It follows from (2.8) that $D_\beta f$ is a C^∞-function in x and y which is even in y. Let α be fixed and larger than -1. Both sides of (2.9) are well-defined and analytic in β if $\operatorname{Re} \beta > -\frac{1}{2}$. Since by (2.8),

$$D_\beta = y^{-2\beta}\left(\frac{\partial}{\partial x}\left(y^{2\beta}\frac{\partial}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{2\beta}\frac{\partial}{\partial y}\right)\right),$$

it follows by repeated integration by parts and by application of Gauss's theorem that for $k = 0, 1, \cdots, n - 1$ and $\beta > 0$ we have

$$\begin{aligned} &\iint_R ((D_\beta)^{n-k}(1 - x^2 - y^2)^{2n+\alpha})((D_\beta)^k f(x, y)) y^{2\beta}\, dx\, dy \\ &= \iint_R ((D_\beta)^{n-k-1}(1 - x^2 - y^2)^{2n+\alpha})((D_\beta)^{k+1} f(x, y)) y^{2\beta}\, dx\, dy. \end{aligned}$$

By these equalities and by (2.7), formula (2.9) is proved if $\alpha > -1$, $\beta > 0$. The case of general β follows by analytic continuation with respect to β. Q.E.D.

We mention two other second order differential recurrence formulas for Jacobi polynomials, although we do not need these formulas in the following sections. If the identity

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_n}{n!}\left(\frac{1 + x}{2}\right)^n {}_2F_1\left(-n, -n - \beta; \alpha + 1; \frac{x - 1}{x + 1}\right)$$

is substituted in (2.2) and (2.3), then we obtain the formulas

$$\begin{aligned} &\left(\frac{d^2}{dx^2} + \frac{2\alpha + 1}{x}\frac{d}{dx}\right)\left((1 + x^2)^n P_n^{(\alpha,\beta)}\left(\frac{1 - x^2}{1 + x^2}\right)\right) \\ &= -4(n + \alpha)(n + \beta)(1 + x^2)^{n-1} P_{n-1}^{(\alpha,\beta)}\left(\frac{1 - x^2}{1 + x^2}\right), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &\left(\frac{d^2}{dx^2} + \frac{2\alpha + 1}{x}\frac{d}{dx}\right)\left((1 + x^2)^{-n-\alpha-\beta} P_{n-1}^{(\alpha,\beta)}\left(\frac{1 - x^2}{1 + x^2}\right)\right) \\ &= -4n(n + \alpha + \beta)(1 + x^2)^{-n-\alpha-\beta-1} P_n^{(\alpha,\beta)}\left(\frac{1 - x^2}{1 + x^2}\right). \end{aligned} \tag{2.11}$$

Repeated application of (2.11) gives a Rodrigues type formula

$$(2.12)\qquad \begin{aligned}&(-1)^n 2^{2n} n!(\alpha+\beta+1)_n (1+x^2)^{-n-\alpha-\beta-1} P_n^{(\alpha,\beta)}\left(\frac{1-x^2}{1+x^2}\right)\\ &\quad = \left(\frac{d^2}{dx^2}+\frac{2\alpha+1}{x}\frac{d}{dx}\right)^n (1+x^2)^{-\alpha-\beta-1}.\end{aligned}$$

This formula is particularly nice, since for fixed α and β it expresses Jacobi polynomials $P_n^{(\alpha,\beta)}$, $n = 0, 1, 2, \cdots$, as functions which are obtained by n-fold application of a second order differential operator to an elementary function not depending on n. Tricomi obtained a simpler formula of this type for Gegenbauer polynomials C_n^λ, $n = 0, 1, 2, \cdots$, where λ is fixed. His formula [3, § 10.9 (37)] involves the first order operator d/dx. There does not exist a straightforward generalization of Tricomi's formula to general Jacobi polynomials, because they cannot be written as a solution of Truesdell's F-equation (cf. Truesdell [11], Miller [9, § 6.2]). However, formula (2.12) may be considered as a substitute.

3. A class of orthogonal polynomials in two variables. The addition formula for Gegenbauer polynomials (cf. [3, § 3.15.1 (19)] or (5.1)) can be considered as an expansion of the function $P_n^{(\alpha,\alpha)}(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,t)$ (x, y fixed, $\alpha > -\frac{1}{2}$) in terms of the orthogonal polynomials $P_k^{(\alpha-1/2,\alpha-1/2)}(t)$, $k = 0, 1, 2, \cdots$, i.e., with respect to the weight function $(1-t^2)^{\alpha-1/2}$ on the interval $(-1, 1)$.

Similarly, the addition formula for Jacobi polynomials (cf. [4, (3)] or (4.14)) can be considered as an orthogonal expansion of the function

$$P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 + \sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi - 1)$$

(x, y fixed and $\alpha > \beta > -\frac{1}{2}$) in terms of the functions $\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$, $k \geqq l \geqq 0$, defined by

$$(3.1)\qquad \mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi) = P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)\,r^{k-l}P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi),$$

which are orthogonal on the region $\{(r,\phi)|0 < r < 1, 0 < \phi < \pi\}$ with respect to the measure $(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi$. We shall prove that in terms of suitable coordinates the functions $\mathscr{P}_{k,l}^{(\alpha,\beta)}$ are orthogonal polynomials.

Let us define the functions $P_{n,k}^{(\alpha,\beta)}(u,v)$, $n \geqq k \geqq 0$, in terms of Jacobi polynomials by

$$(3.2)\qquad P_{n,k}^{(\alpha,\beta)}(u,v) = P_k^{(\alpha,\beta+n-k+1/2)}(2v-1)\,v^{(n-k)/2}P_{n-k}^{(\beta,\beta)}(v^{-1/2}u).$$

Since a Gegenbauer polynomial of degree n is even or odd according to whether n is even or odd it follows that $P_{n,k}^{(\alpha,\beta)}(u,v)$ is a polynomial in u and v. Comparing (3.1) and (3.2) we obtain that

$$(3.3)\qquad \mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi) = P_{k,l}^{(\alpha-\beta-1,\beta-1/2)}(r\cos\phi, r^2).$$

Let S denote the region $\{(u,v)|u^2 < v < 1\}$, bounded by the straight line $v = 1$ and by the parabola $v = u^2$ (cf. Fig. 1). The mapping $(x,y) \to (u,v)$ defined by $u = x$, $v = x^2 + y^2$ is a diffeomorphism from the upper half unit disk R onto the region S. If r, ϕ are polar coordinates on R such that $x = r\cos\phi$, $y = r\sin\phi$, then $u = r\cos\phi$, $v = r^2$ and $\partial(u,v)/\partial(r,\phi) = 2r^2\sin\phi$.

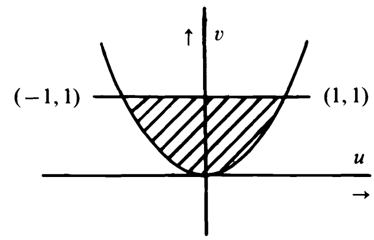

FIG. 1

THEOREM 3.1. *Let $\alpha, \beta > -1$. Then the polynomials $P_{n,k}^{(\alpha,\beta)}(u, v)$ satisfy the following properties*:

(i) $$P_{n,k}^{(\alpha,\beta)}(u, v) - \frac{(n + \alpha + \beta + 3/2)_k(n - k + 2\beta + 1)_{n-k}}{2^{n-k}k!(n-k)!}u^{n-k}v^k$$

is a polynomial of degree less than n;

(ii) $$\iint_S P_{n,k}^{(\alpha,\beta)}(u, v)P_{m,l}^{(\alpha,\beta)}(u, v)(1 - v)^\alpha(v - u^2)^\beta\, du\, dv = 0$$

if $(n, k) \neq (m, l)$.

Furthermore, conditions (i) *and* (ii) *define the polynomials $P_{n,k}^{(\alpha,\beta)}(u, v)$ uniquely.*

Proof. It is clear from (3.2) that for some constant c the polynomial $P_{n,k}^{(\alpha,\beta)}(u, v) - cu^{n-k}v^k$ has degree less than n. The value of c follows from [3, § 10.8, (5)].

To prove (ii) note that if $u = r\cos\phi$, $v = r^2$, then $(1 - v)^\alpha(v - u^2)^\beta\, du\, dv = 2(1 - r^2)^\alpha r^{2\beta+2}(\sin\phi)^{2\beta+1}\, dr\, d\phi$. Hence part (ii) follows by using (3.2) and the orthogonality relations for Jacobi polynomials. It is clear from (i) and (ii) that

(ii)′ $$\iint_S P_{n,k}^{(\alpha,\beta)}(u, v)q(u, v)(1 - v)^\alpha(v - u^2)^\beta\, du\, dv = 0$$

for each polynomial q of degree less than n.

Conditions (i) and (ii)′ uniquely determine the polynomials $P_{n,k}^{(\alpha,\beta)}(u, v)$. Q.E.D.

Since the region S is bounded, it follows that the polynomials $P_{n,k}^{(\alpha,\beta)}(u, v)$ form a complete orthogonal system on S with respect to the weight function $(1 - v)^\alpha (v - u^2)^\beta$. Hence the functions $\mathscr{P}_{k,1}^{(\alpha,\beta)}(r, \phi)$ form a complete orthogonal system with respect to the weight function $(1 - r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}$, $0 < r < 1$, $0 < \phi < \pi$.

The author is preparing a paper in which the orthogonal polynomials $P_{n,k}^{(\alpha,\beta)}(u, v)$ and the related classes of orthogonal polynomials inside the circle and inside the triangle are discussed in more detail.

4. The proof of the addition formula. It was pointed out in [8, § 5] that the integral representations for a Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, for the product $P_n^{(\alpha,\beta)}(x)$ $P_n^{(\alpha,\beta)}(y)$ of two Jacobi polynomials and for the product $J_\beta(x)\, J_\alpha(y)$ of two Bessel functions (cf. [8, (3.1), (3.7), (3.8)]) all have the form

(4.1) $$\int_0^1\int_0^\pi f(a^2r^2 + 2abr\cos\phi + b^2)\, dm_{\alpha,\beta}(r, \phi),$$

where f is a C^∞-function, a and b are positive real numbers, and $dm_{\alpha,\beta}(r,\phi)$, $\alpha > \beta > -\frac{1}{2}$, denotes the measure

$$dm_{\alpha,\beta}(r,\phi) = \mu_{\alpha,\beta}(1-r^2)^{\alpha-\beta-1}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi, \tag{4.2}$$

with the constant $\mu_{\alpha,\beta}$ such that

$$\int_0^1\int_0^\pi dm_{\alpha,\beta}(r,\phi) = 1, \quad \text{i.e., } \mu_{\alpha,\beta} = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}.$$

The functions $\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$, defined by (3.1), are orthogonal with respect to the measure $dm_{\alpha,\beta}(r,\phi)$. Hence the integral (4.1) can be considered as the first term of the orthogonal expansion of $f(a^2r^2+2abr\cos\phi+b^2)$ in terms of the functions $\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$. For the three cases mentioned above we shall derive this expansion in an explicit way and thus obtain three different addition formulas.

If f is a C^∞-function on $[0,\infty)$, then let $f^{(n)}$ denote the nth derivative of f and define the function $f_{k,l}^\beta$, $k \geqq l \geqq 0$, on $[0,\infty)$ by

$$f_{k,l}^\beta(t^2) = \left(\frac{d^2}{dt^2} + \frac{2(\beta+k-l)+1}{t}\frac{d}{dt}\right)^l f^{(k-l)}(t^2). \tag{4.3}$$

LEMMA 4.1. *Let $\alpha > \beta > -\frac{1}{2}$ and $k \geqq l \geqq 0$. Then for all C^∞-functions f and positive real numbers a, b there is the identity*

$$\begin{aligned}\int_0^1\int_0^\pi & f(a^2r^2+2abr\cos\phi+b^2)\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)\,dm_{\alpha,\beta}(r,\phi)\\ &= \frac{(\alpha-\beta)_l(\beta+\frac{1}{2})_{k-l}a^{k+l}b^{k-l}}{2^{2l}l!(k-l)!(\alpha+1)_{k+l}}\\ &\quad\cdot\int_0^1\int_0^\pi f_{k,l}^\beta(a^2r^2+2abr\cos\phi+b^2)\,dm_{\alpha+k+l,\beta+k-l}(r,\phi).\end{aligned} \tag{4.4}$$

Proof. The idea of the proof is to substitute Rodrigues' formula

$$(-1)^n2^nn!(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x) = \left(\frac{d}{dx}\right)^n[(1-x)^{n+\alpha}(1+x)^{n+\beta}] \tag{4.5}$$

and the Rodrigues type formula (2.6) in the explicit expression (3.1) for $\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)$ and then to perform repeated integration by parts. Let I be such that $\mu_{\alpha,\beta}I$ equals the left-hand side of (4.4). Then

$$\begin{aligned}I = \int_0^1 \Bigg[\int_0^\pi & f(a^2r^2+2abr\cos\phi+b^2)P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi)\\ &\cdot(\sin\phi)^{2\beta}\,d\phi\Bigg]P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)(1-r^2)^{\alpha-\beta-1}r^{2\beta+k-l+1}\,dr.\end{aligned}$$

By using (4.5) and by repeated integration by parts it follows that

$$I = \frac{(ab)^{k-l}}{(k-l)!}\int_0^1\int_0^\pi f^{(k-l)}(a^2r^2 + 2abr\cos\phi + b^2)$$

$$\cdot P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)(1-r^2)^{\alpha-\beta-1}(r\sin\phi)^{2(\beta+k-l)}r\,dr d\phi$$

$$= \frac{(ab)^{k-l}}{(k-l)!}\int\int_R f^{(k-l)}((ax+b)^2 + (ay)^2)$$

$$\cdot P_l^{(\alpha-\beta-1,\beta+k-l)}(2(x^2+y^2)-1)(1-x^2-y^2)^{\alpha-\beta-1}(y^2)^{\beta+k-l}\,dx\,dy,$$

where R denotes the upper half unit disk. Then, by Lemma 2.1,

$$I = \frac{(ab)^{k-l}}{2^{2l}l!(k-l)!(l+\alpha-\beta)_l}$$

$$\cdot\iint_R \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2(\beta+k-l)}{y}\frac{\partial}{\partial y}\right)^l f^{(k-l)}((ax+b)^2+(ay)^2)$$

$$\cdot(1-x^2-y^2)^{\alpha-\beta+2l-1}(y^2)^{\beta+k-l}\,dx\,dy.$$

Note that if $ax + b = t\cos\psi$, $ay = t\sin\psi$, then

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2(\beta+k-l)}{y}\frac{\partial}{\partial y}\right)^l f^{(k-l)}((ax+b)^2+(ay)^2)$$

$$= a^{2l}\left(\frac{\partial^2}{\partial t^2} + \frac{2(\beta+k-l)+1}{t}\frac{\partial}{\partial t}\right)^l f^{(k-l)}(t^2) = a^{2l}f_{k,l}^{\beta}(t^2).$$

Hence, by substituting $x = r\cos\phi$ and $y = r\sin\phi$ in the last expression for I, it follows that $\mu_{\alpha,\beta}I$ is equal to the right-hand side of (4.4). Q.E.D.

For the three choices of the function f in which we are interested the functions $f_{k,l}^{\beta}$ can easily be evaluated. We have

$$\begin{aligned}&f(t^2) = t^{2n},\\ (4.6)\qquad &f_{k,l}^{\beta}(t^2) = \frac{2^{2l}n!(n-l+\beta+1)_l}{(n-k)!}t^{2n-2k}, \qquad k \leqq n,\end{aligned}$$

$$\begin{aligned}&f(t^2) = P_n^{(\alpha,\beta)}(2t^2-1),\\ (4.7)\qquad &f_{k,l}^{\beta}(t^2) = 2^{2l}(n+\alpha+\beta+1)_k(n-l+\beta+1)_l P_{n-k}^{(\alpha+k+l,\beta+k-l)}(2t^2-1),\\ &\qquad\qquad k \leqq n,\end{aligned}$$

$$\begin{aligned}&f(t^2) = t^{-\beta}J_\beta(t),\\ (4.8)\qquad &f_{k,l}^{\beta}(t^2) = (-1)^k 2^{-k+l}t^{-\beta-k+l}J_{\beta+k-l}(t).\end{aligned}$$

In (4.6) and (4.7), $f_{k,l}^{\beta} = 0$ if $k > n$. Formula (4.6) is evident. Formula (4.7) follows from (2.4) and the formula $(d/dx)P_n^{(\alpha,\beta)}(2x-1) = (n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(2x-1)$.

To prove (4.8) we need the formulas $(d/dt)(t^{-\beta}J_\beta(t)) = -t^{-\beta}J_{\beta+1}(t)$ and $((d/dt)^2 + (2\beta+1)t^{-1}(d/dt))(t^{-\beta}J_\beta(t)) = -t^{-\beta}J_\beta(t)$.

Using (4.4), (4.6), (4.7), (4.8) and [8, (3.1), (3.7), (3.8)] we obtain

$$\begin{aligned}
&\int_0^1\int_0^\pi (\tfrac{1}{2}(x+1) + \tfrac{1}{2}(x-1)r^2 + \sqrt{x^2-1}\, r\cos\phi)^n \mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)\, dm_{\alpha,\beta}(r,\phi) \\
&\qquad = \frac{n!(\alpha-\beta)_l(n-l+\beta+1)_l(\beta+\frac{1}{2})_{k-l}}{2^k l!(k-l)!(\alpha+1)_{n+l}} \\
&\qquad\quad \cdot (x-1)^{(k+l)/2}(x+1)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\beta+k-l)}(x) \quad \text{if } k \leqq n,
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\int_0^1\int_0^\pi P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y) + \tfrac{1}{2}(1-x)(1-y)r^2 \\
&\qquad + \sqrt{1-x^2}\sqrt{1-y^2}\, r\cos\phi - 1)\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)\, dm_{\alpha,\beta}(r,\phi) \\
&\qquad = \frac{(n-k)!(n+\alpha+\beta+1)_k(\alpha-\beta)_l(n-l+\beta+1)_l(\beta+\frac{1}{2})_{k-l}}{2^{2k}l!(k-l)!(\alpha+1)_{n+l}} \\
&\qquad\quad \cdot (1-x)^{(k+l)/2}(1+x)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\beta+k-l)}(x) \\
&\qquad\quad \cdot (1-y)^{(k+l)/2}(1+y)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\beta+k-l)}(y) \quad \text{if } k \leqq n,
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
&\int_0^1\int_0^\pi (x^2 + y^2r^2 + 2xyr\cos\phi)^{-\beta/2} \\
&\qquad \cdot J_\beta((x^2+y^2r^2+2xyr\cos\phi)^{1/2})\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi)\, dm_{\alpha,\beta}(r,\phi) \\
&\qquad = \frac{2^\alpha\Gamma(\alpha+1)(-1)^k(\alpha-\beta)_l(\beta+\frac{1}{2})_{k-l}}{l!(k-l)!} x^{-\beta}J_{\beta+k-l}(x)y^{-\alpha}J_{\alpha+k-l}(y).
\end{aligned} \tag{4.11}$$

The left-hand sides of (4.9) and (4.10) are zero if $k > n$.

By using (3.1), (4.2) and [3, § 10.8 (4)] it follows that

$$\begin{aligned}
&\int_0^1\int_0^\pi (\mathscr{P}_{k,l}^{(\alpha,\beta)}(r,\phi))^2\, dm_{\alpha,\beta}(r,\phi) \\
&\qquad = \frac{(k+\alpha)((k-l)/2+\beta)(\beta+\frac{1}{2})_{k-l}(\beta+\frac{1}{2})_{k-l}(\beta+1)_k(\alpha-\beta)_l}{(k+l+\alpha)(k-l+\beta)(2\beta+1)_{k-l}(k-l)!(\alpha+1)_k l!}.
\end{aligned} \tag{4.12}$$

Hence the expansions corresponding to (4.9) and (4.10) are

$$\begin{aligned}
&(\tfrac{1}{2}(x+1) + \tfrac{1}{2}(x-1)r^2 + \sqrt{x^2-1}\, r\cos\phi)^n \\
&\quad = \sum_{k=0}^n\sum_{l=0}^k \frac{(k+l+\alpha)(k-l+\beta)(2\beta+1)_{k-l}(n-l+\beta+1)_l n!}{2^k(k+\alpha)((k-l)/2+\beta)(\beta+\frac{1}{2})_{k-l}(\beta+1)_k(\alpha+k+1)_{n-k+l}} \\
&\qquad \cdot (x-1)^{(k+l)/2}(x+1)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\ \beta+k-l)}(x) \\
&\qquad \cdot P_l^{(\alpha-\beta-1,\beta+k-l)}(2r^2-1)r^{k-l}P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi),
\end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&P_n^{(\alpha,\beta)}(\tfrac{1}{2}(1+x)(1+y)+\tfrac{1}{2}(1-x)(1-y)r^2\\
&\quad+\sqrt{1-x^2}\sqrt{1-y^2}\,r\cos\phi-1)=\sum_{k=0}^{n}\sum_{l=0}^{k}(k+l+\alpha)(k-l+\beta)\\
&\quad\cdot\frac{(n+\alpha+\beta+1)_k(2\beta+1)_{k-l}(n-l+\beta+1)_l(n-k)!}{2^{2k}(k+\alpha)((k-l)/2+\beta)(\beta+1)_k(\beta+\frac{1}{2})_{k-l}(k+\alpha+1)_{n-k+l}}\\
&\quad\cdot(1-x)^{(k+l)/2}(1+x)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\,\beta+k-l)}(x)\\
&\quad\cdot(1-y)^{(k+l)/2}(1+y)^{(k-l)/2}P_{n-k}^{(\alpha+k+l,\,\beta+k-l)}(y)\\
&\quad\cdot P_l^{(\alpha-\beta-1,\,\beta+k-l)}(2r^2-1)r^{k-l}P_{k-l}^{(\beta-1/2,\,\beta-1/2)}(\cos\phi).
\end{aligned}
\tag{4.14}
$$

Formula (4.14) is the addition formula for Jacobi polynomials (cf. [4, (3)]). We call (4.13) the degenerate addition formula for Jacobi polynomials (cf. § 5, Remark 4).

The formal expansion corresponding to (4.11) is

$$
\begin{aligned}
&(x^2+y^2r^2+2xyr\cos\phi)^{-\beta/2}J_\beta((x^2+y^2r^2+2xyr\cos\phi)^{1/2})\\
&\quad=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{2^\alpha\Gamma(\alpha+1)(-1)^k(k+l+\alpha)(k-l+\beta)(2\beta+1)_{k-l}(\alpha+1)_k}{(k+\alpha)((k-l)/2+\beta)(\beta+\frac{1}{2})_{k-l}(\beta+1)_k}\\
&\quad\cdot x^{-\beta}J_{\beta+k-l}(x)y^{-\alpha}J_{\alpha+k-l}(y)\\
&\quad\cdot P_l^{(\alpha-\beta-1,\,\beta+k-l)}(2r^2-1)r^{k-l}P_{k-l}^{(\beta-1/2,\,\beta-1/2)}(\cos\phi).
\end{aligned}
\tag{4.15}
$$

By rough asymptotic estimates it follows that in the right-hand side of (4.15) the term of index (k, l) is of order $(\Gamma(k-c))^{-1}$ if $k \to \infty$, where c is some real constant. This estimate is uniform in l, $0 \leqq l \leqq k$. Hence the series in (4.15) converges absolutely and the identity holds.

5. Discussion of the results. We conclude this paper with some remarks about the addition formulas (4.13), (4.14), (4.15). No proofs will be given in this section.

Remark 1. If both sides of (4.13) or (4.14) are differentiated once with respect to ϕ, then the same formula is obtained with n, α, β replaced by $n-1$, $\alpha+1$, $\beta+1$ respectively. If the partial differential operator

$$
\frac{\partial^2}{\partial r^2}+\frac{2\beta+1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+2\beta\frac{\cot\phi}{r^2}\frac{\partial}{\partial\phi}
$$

is applied on both sides of (4.13) or (4.14), then the same formula is obtained with n, α, β replaced by $n-1$, $\alpha+2$, β, respectively. The same is true for (4.15) except that the parameter n does not occur here. Both sides of (4.13) and (4.14) are rational functions in α and β. It follows that if these two formulas are known in one specific case (α_0, β_0), then they can be proved in the case of general (α, β) by repeated differentiation and by analytic continuation with respect to α and β.

Remark 2. Using the results in [8] Gasper obtained another analytic proof of the addition formula (4.14) (personal communication to the author). He first proved (4.9) by reducing the left-hand side of (4.9) to a multiple summation and by manipulating this sum, and next he derived (4.10) from (4.9) by using Bateman's formula [8, (2.19)].

Remark 3. If either $\alpha = \beta > -\frac{1}{2}$ or $\alpha > \beta = -\frac{1}{2}$, then (4.13), (4.14) and (4.15) degenerate to orthogonal expansions in terms of functions of one variable. For instance, putting $\alpha = \beta$ and $r = 1$ in (4.14) we obtain Gegenbauer's addition formula

$$\begin{aligned} &P_n^{(\alpha,\alpha)}(xy + \sqrt{1-x^2}\sqrt{1-y^2}\cos\phi) \\ (5.1)\qquad &= \sum_{k=0}^{n} \frac{(k+\alpha)(n+2\alpha+1)_k(2\alpha+1)_k(n-k)!}{2^{2k}(k/2+\alpha)(\alpha+\frac{1}{2})_k(\alpha+1)_n} \\ &\quad \cdot (1-x^2)^{k/2} P_{n-k}^{(\alpha+k,\alpha+k)}(x)(1-y^2)^{k/2} P_{n-k}^{(\alpha+k,\alpha+k)}(y) P_k^{(\alpha-1/2,\alpha-1/2)}(\cos\phi). \end{aligned}$$

The same formula with $n, x, y, \cos\phi$ replaced by $2n, ((1+x)/2)^{1/2}, ((1+y)/2)^{1/2}, r$, respectively, is obtained by putting $\beta = -\frac{1}{2}$ and $\phi = 0$ in (4.14) and by substituting the quadratic transformation formulas for Gegenbauer polynomials.

Remark 4. We call (4.13) the degenerate addition formula for Jacobi polynomials, since it can be derived from the addition formula (4.14) by dividing both sides of (4.14) by y^n and then letting $y \to \infty$.

Remark 5. The generalized addition formula (4.15) for Bessel functions is also a limit case of (4.14). The formula is obtained by dividing both sides of (4.14) by $P_n^{(\alpha,\beta)}(-1)$ and then letting $n \to \infty$, where the formulas [8, (3.9), (3.10)] are applied.

Remark 6. In Fig. 2 it is indicated how several related results concerning the addition formula for Jacobi polynomials follow from each other. Here an arrow denotes a direction of proof.

$$\begin{array}{llll} & \text{Laplace representation} & & \\ & \updownarrow \ \text{for } \alpha = \beta & & \\ \text{general} & \text{Laplace representation} & \leftrightarrow & \text{degenerate addition formula} \\ \alpha \text{ and } \beta & \updownarrow & & \updownarrow \\ & \text{product formula} & \leftrightarrow & \text{addition formula} \\ & & & \updownarrow \\ & & & \text{addition formula for any one specific } (\alpha, \beta) \end{array}$$

FIG. 2

In the approach used in [4], [5], [6] the author started at the bottom of Fig. 2 ($\alpha = 1, 2, \cdots$ and $\beta = 0$). In the approach used in the present series of papers we start at the top of Fig. 2.

Remark 7. The addition formula (4.14) in the case that $\beta = 0$ is also a special case of the addition formula for the so-called disk polynomials (cf. Šapiro [10, (1.20)] and Koornwinder [6, (5.4)]. In these two references the addition formula for disk polynomials was proved by group theoretic methods. An analytic proof of this formula might be given by using the methods of the present paper, starting from the product formula [5, (4.10)] for disk polynomials.

Remark 8. There is yet another limit case of the addition formula (4.14). Replacing the variables x, y, r in (4.14) by $2\alpha^{-1}x - 1, 2y - 1, \alpha^{-1/2}r$, respectively, letting $\alpha \to \infty$ and using that $\lim_{\alpha\to\infty} P_n^{(\alpha,\beta)}(2\alpha^{-1}x - 1) = (-1)^n L_n^{\beta}(x)$ and $\lim_{\alpha\to\infty} \alpha^{-n} P_n^{(\alpha,\beta)}(2x-1) = x^n/n!$, we obtain that

$$
\begin{aligned}
(5.2)\qquad & L_n^\beta(xy + (1-y)r^2 + 2\sqrt{xy(1-y)}\, r\cos\phi) \\
&= \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^{k+l}(k-l+\beta)(2\beta+1)_{k-l}(n-l+\beta+1)_l}{((k-l)/2+\beta)(\beta+1)_k(\beta+\frac{1}{2})_{k-l}} \\
&\quad \cdot x^{(k-l)/2} L_{n-k}^{\beta+k-l}(x) y^{n-(k+l)/2}(1-y)^{(k+l)/2} \\
&\quad \cdot L_l^{\beta+k-l}(r^2) r^{k-l} P_{k-l}^{(\beta-1/2,\beta-1/2)}(\cos\phi).
\end{aligned}
$$

This is a kind of addition formula for Laguerre polynomials $L_n^\beta(x)$. Integration of (5.2) gives

$$
\begin{aligned}
(5.3)\qquad L_n^\beta(x)y^n &= \frac{2}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})}\int_0^\infty\int_0^\pi L_n^\beta(xy+(1-y)r^2+2\sqrt{xy(1-y)}\,r\cos\phi) \\
&\quad \cdot e^{-r^2}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi, \qquad \beta > -\tfrac{1}{2}.
\end{aligned}
$$

Dividing both sides of (5.3) by y^n and letting $y \to \infty$ we finally obtain

$$
\begin{aligned}
(5.4)\qquad L_n^\beta(x) &= \frac{2(-1)^n}{\sqrt{\pi}\,\Gamma(\beta+\frac{1}{2})n!} \\
&\quad \cdot \int_0^\infty\int_0^\pi (x - r^2 + 2i\sqrt{x}\,r\cos\phi)^n e^{-r^2}r^{2\beta+1}(\sin\phi)^{2\beta}\,dr\,d\phi, \\
&\qquad\qquad \beta > -\tfrac{1}{2}.
\end{aligned}
$$

It was pointed out by Askey (personal communication) that the integral representation (5.4) can also be proved from the Laplace type integral representation for Gegenbauer polynomials by using Askey and Fitch [2, (3.29)].

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JACOBI POLYNOMIALS. IV: A FAMILY OF VARIATION DIMINISHING KERNELS*

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Abstract. A natural analogue of the de la Vallée Poussin kernel is considered for Jacobi polynomial series and is shown to be variation diminishing.

1. Introduction. Positive summability kernels have been extensively studied throughout this century. Fejér, whose results on $(C, 1)$ summability of Fourier series and $(C, 2)$ summability of Laplace series started this subject, gave the field another type of problem when he proved that the $(C, 3)$ means of even functions which are convex on $[0, \pi]$ are also convex [8]. The next important refinement was given by Pólya and Schoenberg [13] when they proved that the de la Vallée Poussin means of a Fourier series are variation diminishing. In particular, this includes an analogue of Fejér's convexity preserving theorem for de la Vallée Poussin means.

Fefér was the first to extend positive summability theorems to other orthogonal expansions when he proved that the $(C, 2)$ means of Laplace series are positive. Recently there has been interest in finding the smallest Césaro mean which is a positive operator for Jacobi polynomial series and for Hankel transforms [2]. Only partial results have been obtained for Jacobi series, but best possible results have been obtained for Hankel transforms.

The first attempt to define the analogue of the de la Vallée Poussin kernel for Legendre series was made by Kogbetliantz [10]. He had the right idea but, unfortunately, he was unaware of an even earlier formula of H. Bateman [5] which would have allowed him to say more about this method. This formula of Bateman has recently been rediscovered [3] and Bateman's second proof [6] has been analyzed [12] and used to give a simple proof of another important formula for Jacobi polynomials. H. Bavinck [7] had defined an analogue of the de la Vallée Poussin means for Jacobi series and he observed in a letter that Bateman's formula gave an explicit formula for the kernel associated with the summability method. In this paper an analogue of the Pólya–Schoenberg theorem is proved by use of this explicit formula.

Section 2 will contain background information on variation diminishing transformations, and the relevant facts about Jacobi polynomials will be given in § 3. Section 4 will contain the specific kernels and summability methods mentioned above, along with further comments, and the proof of the main theorem will be given in the last section.

2. Variation diminishing transformations. If $f(x)$ is a rational function defined on an open interval (a, b), $Z_{(a,b)}(f(x))$ will denote the number of zeros, counting multiplicity, of $f(x)$ in (a, b).

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Given a finite sequence of real numbers $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$, the number of variations of sign in the terms of the sequence will be represented by $v_{1 \leqq i \leqq n}(\alpha_i)$. A zero term in the sequence does not count as a variation; the sign must actually change from positive to negative or vice versa.

If $f(x)$ is a real-valued function on the interval $[a, b]$, to define the variation of $f(x)$ choose a finite sequence of values $x_1, x_2, \cdots, x_n$ such that

$$a < x_1 < x_2 < \cdots < x_n < b. \tag{2.1}$$

Then the variation of $f(x)$ on $[a, b]$, denoted by $v_{[a,b]}(f(x))$ is

$$\underset{[a,b]}{v}(f(x)) = \sup \underset{1 \leqq i \leqq n}{v}(f(x_i)),$$

where the supremum is taken over all finite sequences $x_1, x_2, \cdots, x_n$ which satisfy (2.1).

Finally, a real-valued kernel $K(x; y)$ defined on $[a, b] \times [a, b]$ is said to be variation diminishing if

$$\underset{[a,b]}{v}(Kf(x)) \leqq \underset{[a,b]}{v}(f(x))$$

for all real-valued integrable f, where

$$Kf(x) = \int_a^b K(x; y) f(y)\, dy.$$

If we are considering a sequence of kernels $K_1, K_2, \cdots, K_n$, then

$$K_n f(x) = \int_a^b K_n(x; y) f(y)\, dy.$$

See Karlin [9] for further information about variation diminishing transformations.

3. Jacobi polynomials. The Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, α, $\beta > -1$, are orthogonal with respect to the measure $\omega(x) = (1 - x)^\alpha (1 + x)^\beta$ on the interval $[-1, 1]$ and, as such, have no zero outside this interval.

A function $f(x)$ can be expanded on $[-1, 1]$ in a formal Jacobi series of the form

$$f(x) \sim \sum_{k=0}^{\infty} a_k h_k R_k^{(\alpha,\beta)}(x),$$

where

$$R_k^{(\alpha,\beta)}(x) = \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)} = {}_2F_1\left(-k, k + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\right),$$

$$a_k = \int_{-1}^{1} f(x) R_k^{(\alpha,\beta)}(x) \omega(x)\, dx,$$

and

$$h_k^{-1} = \int_{-1}^{1} [R_k^{(\alpha,\beta)}(x)]^2 \omega(x)\, dx,$$

providing all the a_k's exist and are finite. Notice that the polynomials are normalized to take the value one at $x = 1$; this minimizes the number of extraneous constants in the formulas we will be dealing with.

When $f(x)$ is expanded in this way, we will define the generalized translate of f to be

$$f(x;y) = \sum_{k=0}^{\infty} a_k h_k R_k^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(y). \tag{3.1}$$

This translate has the following familiar property. Let $f(x)$ be as before, $g(x)$ be expanded similarly with coefficients b_k, and $h_k = \int_{-1}^{1} f(x;y)g(y)\omega(y)\,dy$. Then, using the expansions for f and g and the orthogonality of the Jacobi polynomials, it can easily be shown that the expansion for h is

$$h(x) = \sum_{k=0}^{\infty} (a_k b_k) h_k R_k^{(\alpha,\beta)}(x). \tag{3.2}$$

See Askey and Wainger [1].

If we have a polynomial kernel $K_n(x)$, i.e., $K_n(x) = \sum_{k=0}^{n} c_k a_k R_k^{(\alpha,\beta)}(x)$, then (3.2) and the expansion for f indicate that

$$K_n f(x) = \int_{-1}^{1} K_n(x;y) f(y)\omega(y)\,dy = \sum_{k=0}^{n} (a_k c_k) h_k R_k^{(\alpha,\beta)}(x). \tag{3.3}$$

4. Family of kernels. Let us choose as our family of kernels the simplest possible set of positive bounded functions on $[-1, 1]$, namely, the polynomials $((1 + x)/2)^n$. However, because we want these kernels to reproduce constant functions, they must first be normalized so that their integrals over $[-1, 1]$ with respect to $\omega(x)$ equal 1. Thus, if

$$t_n = \left[\int_{-1}^{1} \left(\frac{1+x}{2}\right)^n \omega(x)\,dx\right]^{-1} = \frac{\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(n+\beta+1)\Gamma(\alpha+1)}, \tag{4.1}$$

we will consider

$$K_n(x) = t_n\left(\frac{1+x}{2}\right)^n = t_n \sum_{k=0}^{n} c_{k,n} h_k R_k^{(\alpha,\beta)}(x).$$

Using (3.1),

$$\begin{aligned} K_n(x;y) &= t_n \sum_{k=0}^{n} c_{k,n} h_k R_k^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(y) \\ &= t_n \left(\frac{x+y}{2}\right)^n R_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right). \end{aligned} \tag{4.2}$$

The simplification in (4.2) is due to Bateman [5]. Also see Koornwinder [11]. The result depends very strongly upon $c_{k,n}$ being the particular coefficients for $((1 + x)/2)^n$.

Because $|(1 + xy)/(x + y)| \geqq 1$ for $-1 \leqq x, y \leqq 1$, it is clear from (4.2) that $K_n(x; y)$ is nonnegative in the stated domain.

Also,

$$\begin{aligned} t_n c_{k,n} &= t_n \int_{-1}^{1} \left(\frac{1+x}{2}\right)^n R_k^{(\alpha,\beta)}(x)\omega(x)\,dx \\ &= \frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+1)}{\Gamma(n+k+\alpha+\beta+2)\Gamma(n-k+1)} \\ &\to n^{-k}n^k = 1 \quad \text{as } n \to \infty. \end{aligned}$$

Thus the kernels $K_n(x)$ provide us with a positive, finite summability method which is an approximate identity in the sense that the like coefficients in the orthogonal expansions of $K_n f(x)$ for $n = 0, 1, 2, \cdots$ (see (3.3)) converge to the coefficients in the expansion of $f(x)$.

One other kernel known to be positive for all $\alpha, \beta > -1$ is the Poisson kernel $\tilde{K}_r(x; y) = \sum_{k=0}^{\infty} r^k h_k R_k^{(\alpha,\beta)}(x) R_k^{(\alpha,\beta)}(y)$ which transforms

$$f(x) = \sum_{k=0}^{\infty} a_k h_k R_k^{(\alpha,\beta)}(x)$$

into

$$\tilde{K}_r f(x) = \int_{-1}^{1} \tilde{K}_r(x; y) f(y)\omega(y)\,dy = \sum_{k=0}^{\infty} r^k a_k h_k R_k^{(\alpha,\beta)}(x).$$

See Bailey [4].

Notice that, in the Poisson case, the multiplier coefficients r^n are very simple, but the associated generating kernel $\tilde{K}_r(x)$ is, in general, fairly complicated. On the other hand, our family of kernels, $t_n((1 + x)/2)^n$, is certainly simple enough, but the multiplier coefficients they generate,

$$\frac{\Gamma(n+\alpha+\beta+2)\Gamma(n+1)}{\Gamma(n+k+\alpha+\beta+2)\Gamma(n-k+1)},$$

are rather messy. This suggests that we can have either simple multiplier coefficients or simple positive generating functions, but not both.

Now expressing Jacobi polynomials as

$$P_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \frac{(n-k+\alpha+1)_k}{k!} \frac{(k+\beta+1)_{n-k}}{(n-k)!} (x+1)^k (x-1)^{n-k},$$

where $(a)_k = a(a + 1) \cdots (a + k - 1)$ is Pochhammer's shifted factorial, (4.2) can be rewritten as

$$\begin{aligned} K_n(x; y) &= \frac{t_n}{4^n P_n^{(\alpha,\beta)}(1)} (x+y)^n \sum_{k=0}^{n} \frac{(n-k+\alpha+1)_k}{k!} \frac{(k+\beta+1)_{n-k}}{(n-k)!} \\ &\quad \cdot \left(\frac{1+xy}{x+y} + 1\right)^k \left(\frac{1+xy}{x+y} - 1\right)^{n-k} \qquad (4.3) \\ &= \sum_{k=0}^{n} d_k (1+x)^k (1+y)^k (1-x)^{n-k} (1-y)^{n-k}, \end{aligned}$$

where

$$d_k = \frac{t_n}{4^n P_n^{(\alpha,\beta)}(1)} \frac{(n-k+\alpha+1)_k}{k!} \frac{(k+\beta+1)_{n-k}}{(n-k)!} > 0.$$

5. Main result.

THEOREM. *If f is an integrable function on $[-1,1]$, $K_n(x) = t_n((1+x)/2)^n$ (see (4.1)), and $K_n(x;y)$ and $K_nf(x)$ are defined as in (3.1) and (3.3) respectively, then*

$$\underset{[-1,1]}{v}(K_nf(x)) \leqq \underset{(-1,1)}{Z}(K_nf(x)) \leqq \underset{[-1,1]}{v}(f(x)).$$

Note that this is a stronger result than just variation diminishing. Not only is the variation of f bounded below by the variation of K_n acting on f, but it is also bounded below by the number of zeros of K_n acting on f, as in the Pólya–Schoenberg result.

Proof. Since $K_nf(x)$ is continuous, the first inequality is clear. For the second inequality, consider the integral

$$K_nf(x) = \int_{-1}^{1} K_n(x;y)f(y)\omega(y)\,dy.$$

We will approximate this integral with finite Riemann sums and show that the inequality holds for these sums.

So we must consider expressions of the form

$$S(x) = \sum_{\nu=1}^{m} c_\nu K_n(x;\xi_\nu), \tag{5.1}$$

where $-1 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$ and c_ν are constants.

Using (4.3) and factoring out $(1-x)^n$, we have

$$\frac{S(x)}{(1-x)^n} = \sum_{k=0}^{n} d_k \left(\sum_{\nu=1}^{m} c_\nu (1-\xi_\nu)^{n-k}(1+\xi_\nu)^k \right) \left(\frac{1+x}{1-x} \right)^k.$$

Let $z = (1+x)/(1-x)$ and note that $z \in (0,\infty)$ when $x \in (-1,1)$. Then

$$\begin{aligned} \underset{(-1,1)}{Z}(S(x)) &= \underset{(-1,1)}{Z}\left(\frac{S(x)}{(1-x)^n}\right) \\ &= \underset{(0,\infty)}{Z}\left(\sum_{k=0}^{n} d_k \left(\sum_{\nu=1}^{m} c_\nu (1-\xi_\nu)^{n-k}(1+\xi_\nu)^k\right) z^k\right) \\ &\leqq \underset{0\leqq k\leqq n}{v}\left(\sum_{\nu=1}^{m} c_\nu (1-\xi_\nu)^{n-k}(1+\xi_\nu)^k\right) \end{aligned} \tag{5.2}$$

(by Descartes' rule and $d_k > 0$).

Setting $\alpha_\nu = (1+\xi_\nu)/(1-\xi_\nu)$, the last expression becomes

$$\underset{0\leqq k\leqq n}{v}\left(\sum_{\nu=1}^{m} c_\nu (1-\xi_\nu)^n \alpha_\nu^k\right),$$

where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \infty$ and $(1-\xi_\nu) > 0$ for $1 \leqq \nu \leqq m$. In matrix form, this system becomes

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \alpha_1^2 & \alpha_2^2 & & \alpha_m^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^n & \alpha_2^n & \cdots & \alpha_m^n \end{pmatrix} \begin{pmatrix} c_1 (1 - \xi_1)^n \\ c_2 (1 - \xi_2)^n \\ \vdots \\ c_m (1 - \xi_m)^n \end{pmatrix}.$$

An induction argument then shows that all $k \times k$ determinants of the Vandermonde matrix on the left are nonzero and have the same sign. This is a sufficient condition for the matrix to be variation diminishing ([9, p. 219]); so

$$\begin{aligned} \underset{0 \leqq k \leqq n}{v} \left(\sum_{\nu=1}^{m} c_\nu (1 - \xi_\nu)^n \alpha_\nu^k \right) &\leqq \underset{0 \leqq \nu \leqq m}{v} (c_\nu (1 - \xi_\nu)^n) \\ &= \underset{0 \leqq \nu \leqq m}{v} (c_\nu), \qquad (1 - \xi_\nu > 0). \end{aligned} \tag{5.3}$$

Combining (5.1), (5.2) and (5.3), we see

$$\underset{(-1,1)}{Z} \left(\sum_{\nu=1}^{m} c_\nu K_n(x; \xi_\nu) \right) \leqq \underset{0 \leqq \nu \leqq m}{v} (c_\nu). \tag{5.4}$$

In particular choose c_ν to be

$$c_\nu = f(\xi_\nu)\omega(\xi_\nu)\delta_\nu,$$

where $f(x)$ satisfies the hypothesis, ξ_ν are chosen as in (5.1) and δ_ν is the length of the appropriate subinterval of $[-1, 1]$. Then (5.4) becomes

$$\begin{aligned} \underset{(-1,1)}{Z} \left(\sum_{\nu=1}^{m} K_n(x; \xi_\nu) f(\xi_\nu)\omega(\xi_\nu)\delta_\nu \right) &\leqq \underset{1 \leqq \nu \leqq m}{v} (f(\xi_\nu)\omega(\xi_\nu)\delta_\nu) \\ &= \underset{1 \leqq \nu \leqq m}{v} (f(\xi_\nu)), (\omega(\xi_\nu)\delta_\nu > 0) \\ &\leqq \underset{[-1,1]}{v} (f(x)). \end{aligned}$$

Since the sums on the left converge to $K_n f(x) = \int_{-1}^{1} K_n(x; y) f(y)\omega(y)\, dy$ as $m \to \infty$, we clearly have the second inequality of the theorem. □

Now recall from (4.1) that

$$K_n(x) = t_n \left(\frac{1 + x}{2} \right)^n = \frac{\Gamma(n + \alpha + \beta + 2)}{2^{\alpha+\beta+1}\Gamma(n + \beta + 1)\Gamma(\alpha + 1)} \left(\frac{1 + x}{2} \right)^n.$$

Let $\alpha = \beta = -\frac{1}{2}$ and $x = \cos\theta$. Then

$$\begin{aligned} K_n(\cos\theta) &= \left[\frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})} \left(\frac{1 + \cos\theta}{2} \right)^n \right] \frac{\Gamma(n + 1)2^{2n}}{n\Gamma(n)2^{2n}} \\ &= \frac{[\Gamma(n + 1)]^2 2^{2n} (\cos\theta/2)^{2n}}{\left[\dfrac{2^{2n-1}\Gamma(n)\Gamma(n + \frac{1}{2})}{\Gamma(\frac{1}{2})} \right] [\Gamma(\frac{1}{2})]^2 2n}. \end{aligned}$$

But the term in brackets is $\Gamma(2n)$, so

$$K_n(\cos\theta) = \frac{1}{[\Gamma(\frac{1}{2})]^2}\frac{[\Gamma(n+1)]^2}{\Gamma(2n+1)} 2\cos\left(\frac{\theta}{2}\right)^{2n} = \frac{1}{\pi}\frac{(n!)^2}{(2n)!} 2\cos\left(\frac{\theta}{2}\right)^{2n}.$$

This last expression is the de la Vallée Poussin kernel.

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ON THE POSITIVITY OF SOME ${}_1F_2$'S

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Abstract. By using asymptotic methods and fractional integration, it is shown that

$$ {}_1F_2\left(\begin{matrix} \lambda - a \\ \rho\lambda + b, \rho\lambda + c \end{matrix} \middle| \frac{-\mu^2}{4}\right) \geqq 0, \quad . \ \mu \text{ real}, $$

$0 \leqq a \leqq \lambda$, $0 \leqq b$, $1 \leqq 2c$, and either $2\rho \geqq 3$, $\lambda \geqq 1$ or $\rho \geqq 2$, $\lambda \geqq 0$. From this, it is deduced that for $x > 0$, $x^{2\lambda-2\rho\lambda-b}(1 + x^2)^{-\lambda}$ is completely monotonic for $b \geqq 0$ and either $2\rho \geqq 3$, $\lambda \geqq 1$, or $\rho \geqq 2$, $\lambda \geqq 0$. This extends the results of [3] and proves some conjectures of Askey [1].

1. Main result.

THEOREM 1. *If*

$$ (1.1) \qquad \begin{aligned} G(\rho, \lambda; \mu) &= {}_1F_2\left(\begin{matrix} \lambda \\ \rho\lambda, \rho\lambda + \frac{1}{2} \end{matrix} \middle| \frac{-\mu^2}{4}\right) & (\rho\lambda > 0) \\ &= 1 - \left(\frac{\mu^2}{2\rho}\right) {}_1F_2\left(\begin{matrix} 1 \\ 2, \frac{3}{2} \end{matrix} \middle| \frac{-\mu^2}{4}\right) & (\lambda = 0, \rho > 0), \end{aligned} $$

then

$$ (1.2) \qquad G(\rho, \lambda; \mu) > 0, \qquad \mu \text{ real}, $$

$2\rho = 3$, $1 < \lambda$; *or* $\rho = 2$, $0 < \lambda$.

Proof of Theorem 1. From hypergeometric function theory [4, p. 198],

$$ \begin{aligned} G(\rho, \lambda; \mu) = {} & \frac{\Gamma(2\rho\lambda)}{\Gamma(2\rho\lambda - 2\lambda)} \mu^{-2\lambda}\{1 + O(\mu^{-2})\} \\ & + \frac{\Gamma(2\rho\lambda)}{\Gamma(\lambda)} 2^{1-\lambda} \mu^{\lambda - 2\rho\lambda} \left\{ \cos\left[\mu + \frac{\pi}{2}(\lambda - 2\rho\lambda)\right] + O(\mu^{-1}) \right\}, \\ & \mu \to \infty, \qquad |\arg \mu| < \pi/2, \end{aligned} $$

and it follows that a necessary condition for the positivity of G is $2\rho \geqq 3$, $\lambda > 0$. Moreover, as $G(\rho, \lambda; 0) = 1$, it is sufficient to show that these "local" estimates overlap. First we extend the $\mu = 0$ estimate.

Let

$$ G(\rho, \lambda; \mu) = \sum_{k=0}^{\infty} \frac{(\lambda)_{2k}\mu^{4k}}{(2\rho\lambda)_{4k}(2k)!} C_k(\mu), \qquad (\sigma)_\omega \equiv \frac{\Gamma(\sigma + \omega)}{\Gamma(\sigma)}, $$

$$ C_k(\mu) = 1 - \frac{\mu^2(\lambda + 2k)}{(2k + 1)(4k + 2\rho\lambda)(4k + 2\rho\lambda + 1)}. $$

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Then

$$C_{k+1}(\mu) - C_k(\mu)$$
$$= \frac{\mu^2 \Delta_k}{(2k+1)(2k+3)(4k+2\rho\lambda)(4k+2\rho\lambda+1)(4k+2\rho\lambda+4)(4k+2\rho\lambda+5)},$$
$$\begin{aligned}\Delta_k &= (2k+\lambda)(2k+3)(4k+2\rho\lambda+4)(4k+2\rho\lambda+5) - (2k+\lambda+2)(2k+1)\\ &\quad \cdot (4k+2\rho\lambda)(4k+2\rho\lambda+1)\\ &= 128k^3 + (240 + 96\lambda + 64\rho\lambda)k^2 + (112 + 144\lambda + 64\rho\lambda + 64\rho\lambda^2)k\\ &\quad + [8\lambda^3\rho^2 + 4\lambda^2\rho(13 - 2\rho) + 4\lambda(15 - \rho)]\\ &\geqq 0, \qquad\qquad \lambda \geqq 0, \quad 0 \leqq 2\rho \leqq 13.\end{aligned}$$

This implies that if $C_k \geqq 0$, then $C_j > 0, j > k$, in the cases of interest. In particular, $0 \leqq \mu \leqq \sqrt{2\rho(1 + 2\rho\lambda)}$ implies that $C_0(\mu) \geqq 0$ and

$$G(\rho, \lambda; \mu) > 0, \qquad 0 \leqq \mu \leqq \sqrt{2\rho(1 + 2\rho\lambda)}.$$

We now develop the asymptotic estimates for $\mu \geqq 2$. A simple computation shows that

$$G(\rho, \lambda; \mu) = \frac{\Gamma(2\rho\lambda)}{2\pi i} \int_{-\infty}^{(0, \pm i\mu)^+} e^v v^{2\lambda - 2\rho\lambda}[v^2 + \mu^2]^{-\lambda}\, dv.$$

Deforming this loop contour, which is shown in Fig. 1 into the contours shown in Fig. 2, which includes branch cuts to make the binomial factors single-valued, we can write

$$G(\rho, \lambda; \mu) = G(\mu) = G^0(\mu) + G^+(\mu) + G^-(\mu),$$
$$G^0(\mu) = \frac{\Gamma(2\rho\lambda)}{2\pi i} \int_{-\infty}^{0+} e^v v^{2\lambda - 2\rho\lambda}(\mu^2 + v^2)^{-\lambda}\, dv,$$
$$G^+(\mu) = \frac{\Gamma(2\rho\lambda)}{2\pi i} \int_{-\infty}^{0+} e^{i\mu + t}(t + i\mu)^{2\lambda - 2\rho\lambda}(t + 2i\mu)^{-\lambda} t^{-\lambda}\, dt,$$

and $G^-(\mu)$ is the complex conjugate of $G^+(\mu)$. Each of these three functions has an asymptotic expansion for $\mu \to \infty$, and we must estimate them for $\mu \geqq 2$.

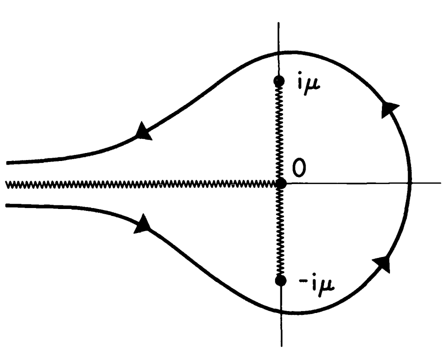

FIG. 1. *v-plane*

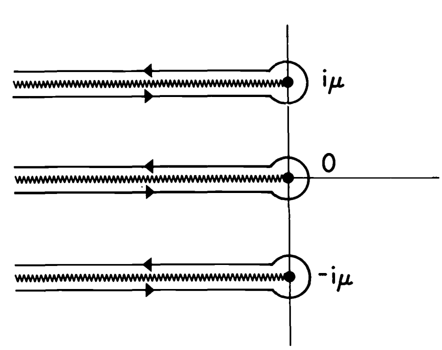

FIG. 2. *v-plane*

Let

$$\left(1+\frac{v^2}{\mu^2}\right)^{-\lambda} = \sum_{j=0}^{m-1} \frac{(\lambda)_j}{j!}\left(\frac{-v^2}{\mu^2}\right)^j + \frac{(\lambda)_m\left(\frac{-v^2}{\mu^2}\right)^m}{m!} r_m(v).$$

Then

$$G^0(\mu) = \frac{\Gamma(2\rho\lambda)\mu^{-2\lambda}}{\Gamma(2\rho\lambda - 2\lambda)}\left\{\sum_{j=0}^{m-1} \frac{(\lambda)_j(2\lambda+1-2\lambda\rho)_{2j}}{j!(-\mu^2)^j} + G_m^0(\mu)\right\},$$

$$\begin{aligned} G_m^0(\mu) &= \frac{\Gamma(2\rho\lambda - 2\lambda)(\lambda)_m(-1)^m}{m!2\pi i}\mu^{-2m}\int_{-\infty}^{0+} e^v v^{2\lambda-2\rho\lambda+2m} r_m(v)\,dv \\ &= \frac{(\lambda)_m(-1)^m\mu^{-2m}}{m!\Gamma(1+2\lambda-2\rho\lambda)}\int_0^\infty e^{-u}u^{2\lambda-2\rho\lambda+2m} r_m(u)\,du, \end{aligned}$$

provided $2m+1+2\lambda-2\rho\lambda>0$. Note that $r_m(v)$ takes on the same values at $v=|v|\,e^{i\pi}$ and $|v|\,e^{-i\pi}$.

From

$$r_m(v) = mT^{-m}\int_0^T \frac{(T-t)^{m-1}}{(1+t)^{m+\lambda}}\,dt, \qquad T = \frac{v^2}{\mu^2},$$

it follows that

$$(1+T)^{-m-\lambda} \leqq r_m(v) \leqq 1, \qquad T \geqq 0,$$

and

$$|G_m^0(\mu)| \leqq \frac{(\lambda)_m|(1+2\lambda-2\rho\lambda)_{2m}|}{m!}\mu^{-2m},$$

$$2m+1+2\lambda-2\rho\lambda>0.$$

For $\rho = 2$, we shall take $m=1$, while for $2\rho = 3$, we shall take $m=2$.

For $G^+(\mu)$, set

$$\begin{aligned} H(t) &= \left(1+\frac{t}{i\mu}\right)^{2\lambda-2\rho\lambda}\left(1+\frac{t}{2i\mu}\right)^{-\lambda} \\ &= \sum_{j=0}^{m-1} b_j\left(\frac{t}{i\mu}\right)^j + \left(\frac{t}{i\mu}\right)^m H_m(t), \qquad b_0=1,\quad 2b_1 = 3\lambda-4\rho\lambda, \\ G^+(\mu) &= \Gamma(2\rho\lambda)\mu^{\lambda-2\rho\lambda}\,e^{i\pi(\lambda-2\rho\lambda)/2}\frac{e^{i\mu}2^{-\lambda}}{2\pi i}\int_{-\infty}^{0+} e^t t^{-\lambda}H(t)\,dt, \\ &= \frac{\Gamma(2\rho\lambda)}{\Gamma(\lambda)}2^{-\lambda}\mu^{\lambda-2\rho\lambda}\,e^{i\pi(\lambda-2\rho\lambda)/2}\,e^{i\mu}\left\{\sum_{j=0}^{m-1}\frac{b_j(1-\lambda)_j}{(-i\mu)^j} + G_m^+(\mu)\right\}, \\ G_m^+(\mu) &= \frac{(i\mu)^{-m}\Gamma(\lambda)}{2\pi i}\int_{-\infty}^{0+} e^t t^{m-\lambda}H_m(t)\,dt. \end{aligned}$$

With the binomial factors defined as in Fig. 2, the factors $(t + i\mu)^{2\lambda - 2\rho\lambda}$ and $(t + 2i\mu)^{-\lambda}$ take the same values, respectively, at $t = |t|\, e^{i\pi}$ and $|t|\, e^{-i\pi}$. Hence we set

$$H_m(x\, e^{\pm i\pi}) = R_m(x), \qquad x \geqq 0.$$

Then

$$G_m^+(\mu) = \frac{(-i\mu)^{-m}}{\Gamma(1-\lambda)} \int_0^\infty e^{-x} x^{m-\lambda} R_m(x)\, dx, \qquad m + 1 > \lambda.$$

In particular, with $m = 0$,

$$G^+(2, \lambda; \mu) = \frac{\Gamma(4\lambda)}{\Gamma(\lambda)} 2^{-\lambda} \mu^{-3\lambda}\, e^{-3\pi i\lambda/2}\, e^{i\mu} G_0^+(\mu),$$

$$|G^+(2, \lambda; \mu)| \leqq \frac{\Gamma(4\lambda) 2^{-\lambda}}{\Gamma(\lambda)\Gamma(1-\lambda)} \mu^{-3\lambda} \int_0^\infty e^{-x} x^{-\lambda} \left(1 + \frac{x^2}{\mu^2}\right)^{-\lambda} \left(1 + \frac{x^2}{4\mu^2}\right)^{-\lambda/2} dx$$

$$\leqq \frac{\Gamma(4\lambda) 2^{-\lambda} \mu^{-3\lambda}}{\Gamma(\lambda)\Gamma(1-\lambda)} \int_0^\infty e^{-x} x^{-\lambda}\, dx$$

$$= \frac{\Gamma(4\lambda) 2^{-\lambda} \mu^{-3\lambda}}{\Gamma(\lambda)}, \qquad 0 < \lambda < 1.$$

When $2\rho = 3$, we estimate $H_2(t)$ as follows:

$$H(t) - 1 + \binom{3\lambda}{2}\left(\frac{t}{i\mu}\right) = \frac{\lambda}{4} \int_0^{-t/\mu} \left(v + \frac{t}{\mu}\right) \frac{[\lambda(9 - 4v^2 + 12iv) + 5 - 2v^2 + 6iv]}{(1 + iv)^{2+\lambda} \left(1 + \dfrac{iv}{2}\right)^{2+\lambda}} dv,$$

$$\left| H_2(t) \left(\frac{t}{\mu}\right)^2 \right| \leqq \frac{\lambda}{4} \int_0^{-t/\mu} \left| v + \frac{t}{\mu} \right| [2(2\lambda + 1)|v|^2 + (5 + 9\lambda) + 6(1 + 2\lambda)|v|] \cdot |dv|,$$

or

$$|H_2(t)| \leqq \frac{\lambda}{8} \left\{ (5 + 9\lambda) + \frac{2(1 + 2\lambda)|t|}{\mu} + \frac{(2\lambda + 1)|t|^2}{3\mu^2} \right\},$$

which leads to

$$|G_2^+(\tfrac{3}{2}, \lambda; \mu)|$$

$$\leqq \frac{\mu^{-2}\lambda}{8} \left\{ (5 + 9\lambda)|(1 - \lambda)_2| + \frac{2(1 + 2\lambda)|(1 - \lambda)_3|}{\mu} + \frac{(2\lambda + 1)|(1 - \lambda)_4|}{3\mu^2} \right\},$$

provided $0 < \lambda < 3$.

Now we derive our $G(2, \lambda; \mu)$ estimate for $\mu \geqq 2$, and $0 < \lambda < 1$. Clearly,

$$\begin{aligned} G(2, \lambda; \mu) &= \frac{\Gamma(4\lambda)}{\Gamma(2\lambda)} \mu^{-2\lambda}\{1 + G_1^0(\mu)\} + 2 \operatorname{Re}\{G^+(\mu)\} \\ &\geqq \frac{\Gamma(4\lambda)}{\Gamma(2\lambda)} \mu^{-2\lambda}\{1 - |G_1^0(\mu)|\} - 2|G^+(\mu)| \\ &\geqq \frac{\Gamma(4\lambda)}{\Gamma(2\lambda)} \mu^{-2\lambda} g(\lambda, \mu), \end{aligned}$$

$$\begin{aligned} g(\lambda, \mu) &= 1 - \frac{\lambda|(1 - 2\lambda)_2|}{\mu^2} - \frac{2^{1-\lambda}\Gamma(2\lambda)}{\Gamma(\lambda)\mu^\lambda}, \\ &= 1 - \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})}\left(\frac{2}{\mu}\right)^\lambda - \frac{2\lambda(1 - \lambda)|1 - 2\lambda|}{\mu^2}. \end{aligned}$$

To estimate $g(\lambda, \mu)$, consider the function

$$\begin{aligned} F(\lambda) &= 1 - \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\sigma^\lambda} \qquad (\sigma > 0, \lambda \geqq 0) \\ &= \frac{\sqrt{\sigma}}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-\sigma s} s^{-1/2}[1 - s^\lambda]\, ds. \end{aligned}$$

From the integral representation, it is clear that for σ fixed,

$$\frac{d^2}{d\lambda^2} F(\lambda) < 0,$$

implying that $F(\lambda)$ is concave down, or that the graph of $F(\lambda)$ lies above the straigh line connecting $F(0) = 0$ and $F(1) = 1 - (2\sigma)^{-1}$ for $0 < \lambda < 1$. Thus

$$F(\lambda) > \frac{\lambda(2\sigma - 1)}{2\sigma}, \qquad \sigma < 0, \quad 0 < \lambda < 1,$$

and

$$\begin{aligned} g(\lambda, \mu) &> \frac{\lambda(\mu - 1)}{\mu} - \frac{2\lambda(1 - \lambda)|1 - 2\lambda|}{\mu^2} \\ &\geqq \frac{\lambda(\mu - 2)(\mu + 1)}{\mu^2} > 0 \qquad (0 < \lambda \leqq \tfrac{1}{2}, \mu > 2) \\ &\geqq \frac{\lambda(4\mu^2 - 4\mu - 1)}{4\mu^2} > \frac{\lambda(\mu - 2)(\mu + 1)}{\mu^2} > 0 \quad (\tfrac{1}{2} \leqq \lambda < 1, \mu > 2) \end{aligned}$$

since

$$(1 - \lambda)|(1 - 2\lambda| \leqq \begin{cases} 1, & 0 \leqq \lambda \leqq \frac{1}{2}, \\ \frac{1}{8}, & \frac{1}{2} \leqq \lambda \leqq 1. \end{cases}$$

Clearly $g(\lambda, \mu) > 0$ implies $G(2, \lambda; \mu) > 0$.

Our $G(\frac{3}{2}, \lambda; \mu)$ estimate for $\mu \geqq \sqrt{12}$ and $1 < \lambda \leqq 2$ is more complicated. For convenience, let $\varepsilon = \lambda - 1$, $0 < \varepsilon \leqq 1$. Then, as before,

$$\begin{aligned}
&\frac{\Gamma(\lambda)}{\Gamma(3\lambda)}\mu^{2\lambda}G(\tfrac{3}{2}, \lambda; \mu) \\
&\quad = 1 + \frac{\varepsilon(1-\varepsilon^2)}{\mu^2} + G_2^0(\mu) + \frac{\cos(\mu - \lambda\pi)}{2^\varepsilon} - \frac{3\varepsilon(1+\varepsilon)}{2^{1+\varepsilon}\mu}\sin(\mu - \lambda\pi) \\
&\qquad + \operatorname{Re}\{2^{-\varepsilon}e^{i(\mu-\lambda\pi)}G_2^+(\mu)\} \\
&\quad \geqq 1 + 2^{-\varepsilon}\cos(\mu - \lambda\pi) - 2^{-1-\varepsilon}3\varepsilon(1+\varepsilon)\mu^{-1}\sin(\mu - \lambda\pi) \\
&\qquad + \frac{\varepsilon(1-\varepsilon^2)}{\mu^2}\left\{1 - \frac{(4-\varepsilon^2)(3-\varepsilon)}{2\mu^2}\right. \\
&\qquad \left. - 2^{-\varepsilon}\left[\frac{(14+9\varepsilon)}{8} + \frac{(3+2\varepsilon)(2-\varepsilon)}{4\mu} + \frac{(3+2\varepsilon)(2-\varepsilon)(3-\varepsilon)}{24\mu^2}\right]\right\}.
\end{aligned}$$

Denote this last expression by $G(\varepsilon, \mu)$. We need to find a positive lower bound for $G(\varepsilon, \mu)$ when $0 < \varepsilon \leqq 1$ or $1 < \lambda \leqq 2$, and $\mu \geqq \sqrt{12}$. We have

$$\begin{aligned}
G(\varepsilon, \mu) &\geqq 1 - 2^{-\varepsilon}\sqrt{1 + \frac{9\varepsilon^2(1+\varepsilon)^2}{4\mu^2}} + \frac{\varepsilon(1-\varepsilon^2)}{\mu^2} \\
&\quad \cdot \left\{1 - \frac{6}{\mu^2} - \frac{1}{2^{3+\varepsilon}}\left[23 + \frac{49}{4\mu} + \frac{6}{\mu^2}\right]\right\} \\
&\geqq 1 - 2^{-\varepsilon}[1 + (.375)\varepsilon^2] + \varepsilon(1-\varepsilon^2)[(.0416) - 2^{-\varepsilon}(.2817)] \\
&\geqq 1 + \varepsilon(1-\varepsilon^2)(.0416) + [1 + (.2817)\varepsilon(1-\varepsilon^2) \\
&\quad + (.375)\varepsilon^2][-1 + \varepsilon(\log 2) - \varepsilon^2 2^{-1}(\log 2)^2] \\
&\geqq (.453)\varepsilon - (.42)\varepsilon^2 + (.432)\varepsilon^3 - (.286)\varepsilon^4 + (.067)\varepsilon^5 \\
&> 0, \qquad\qquad 0 < \varepsilon \leqq 1, \quad \mu \geqq \sqrt{12}.
\end{aligned}$$

Note that $\log 2 = .6931 \cdots$. Thus,

$$G(\rho, \lambda; \mu) > 0, \qquad \mu \text{ real},$$

$2\rho = 3$, $1 < \lambda \leqq 2$; or $\rho = 2$, $0 < \lambda < 1$.

Next, from the Laplace transform

$$(1.3) \qquad \int_0^\infty e^{-tz}t^{2\sigma-1}\, {}_1F_2\left(\begin{matrix}A \\ B, C\end{matrix}\middle| \frac{-t^2}{4}\right) dt = \Gamma(2\sigma)z^{-2\sigma}\, {}_3F_2\left(\begin{matrix}A, \sigma, \sigma + \frac{1}{2} \\ B, C\end{matrix}\middle| \frac{-1}{z^2}\right),$$

$$\operatorname{Re}(\sigma) > 0, \quad \operatorname{Re}(z) > 0,$$

it follows that the function

$$\begin{aligned}
R(\rho, \lambda; x) &= x^{2\lambda - 2\rho\lambda}(1 + x^2)^{-\lambda} \\
&= \{\Gamma(2\rho\lambda)\}^{-1}\int_0^\infty e^{-tx}t^{2\rho\lambda - 1}G(\rho, \lambda; t)\, dt, \qquad x > 0,
\end{aligned}$$

is completely monotonic for $x > 0$ and either $2\rho = 3$, $1 < \lambda \leqq 2$, or $\rho = 2$,

$0 < \lambda < 1$. But as the product of two completely monotonic functions is again completely monotonic, and $R(\rho, \lambda; x)$ is multiplicative in λ, that is,

$$R(\rho, \lambda; x)R(\rho, \lambda'; x) = R(\rho, \lambda + \lambda'; x),$$

it is clear that $R(\rho, \lambda; x)$ is completely monotonic for $x > 0$ and either $2\rho = 3$, $1 < \lambda$ or $\rho = 2$, $0 < \lambda$. Theorem 1 then follows from Bernstein's theorem [6].

2. Corollaries and remarks.

COROLLARY 1.1. *If $x \geqq 0$, then*

$$\begin{aligned} I_1(\alpha; x) &= \int_0^x (x-t)^{\alpha+3/2} t^{\alpha+1} J_\alpha(t)\,dt \\ &\geqq 0, \qquad 2\alpha \geqq -1, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} I_2(\alpha; x) &= \int_0^x (x-t)^{2\alpha} t^{\alpha} J_\alpha(t)\,dt \\ &\geqq 0, \qquad 2\alpha > -1, \end{aligned} \tag{2.2}$$

where $J_\alpha(t)$ is the Bessel function of the first kind.

Proof. Expanding $J_\alpha(t)$ in powers of t and integrating term by term, one obtains

$$I_1(\alpha; x) = \frac{\Gamma(2\alpha+2)\Gamma(\alpha+\frac{5}{2})x^{3\alpha+7/2}}{2^\alpha\Gamma(\alpha+1)\Gamma(3\alpha+\frac{9}{2})} G\left(\frac{3}{2}, \frac{2\alpha+3}{2}; x\right),$$

$$I_2(\alpha; x) = \frac{2^{-1-3\alpha}\Gamma(\alpha+\frac{1}{2})x^{4\alpha+1}}{\Gamma(2\alpha+\frac{3}{2})} G\left(2, \frac{2\alpha+1}{2}; x\right).$$

COROLLARY 1.2.

$$0 \leqq G\left(\frac{3}{2}, 1; \mu\right) \leqq \frac{4}{\mu^2}, \qquad \mu \text{ real}. \tag{2.3}$$

Proof.

$$0 \leqq \lim_{\lambda \to 0+} G(2, \lambda; \mu) = 1 - \frac{\mu^2}{4} G\left(\frac{3}{2}, 1; \mu\right).$$

THEOREM 2.

$${}_1F_2\left(\begin{matrix}\lambda - a \\ \rho\lambda + b, \rho\lambda + c\end{matrix} \middle| \frac{-\mu^2}{4}\right) > 0, \qquad \mu \text{ real}, \tag{2.4}$$

$0 \leqq a < \lambda$, $0 \leqq b$, $1 \leqq 2c$, *and either* $2\rho \geqq 3$, $\lambda > 1$, *or* $\rho \geqq 2$, $\lambda > 0$.

Proof. Consider the beta transform,

$$\int_0^1 t^{2\alpha+1}(1-t^2)^\beta g(t)\,dt = \frac{\Gamma(\beta+1)\Gamma(\alpha+1)}{2\Gamma(\alpha+\beta+2)} {}_2F_3\left(\begin{matrix}A, \alpha+1 \\ B, C, \alpha+\beta+2\end{matrix} \middle| -z\right),$$

$$g(t) = {}_1F_2\left(\begin{matrix}A \\ B, C\end{matrix} \middle| -zt^2\right), \qquad \beta+1, \alpha+1 > 0.$$

With the conditions

$$K: 2\rho = 3, \lambda > 1 \quad \text{or} \quad \rho = 2, \lambda > 0$$

and

$$g_0(t) = {}_1F_2\left(\begin{matrix}\lambda \\ \rho\lambda, \rho\lambda + \frac{1}{2}\end{matrix}\middle| -zt^2\right) > 0, \qquad z \geqq 0,$$

$$\alpha + 1 = \rho\lambda, \qquad \beta + 1 = b > 0,$$

one obtains via the above transform,

$$g_1(t) = {}_1F_2\left(\begin{matrix}\lambda \\ \rho\lambda + \frac{1}{2}, \rho\lambda + b\end{matrix}\middle| -zt^2\right) > 0, \qquad z \geqq 0,$$

under the conditions K. Next, applying the Beta transform to $g_1(t)$ under the conditions K, and with

$$2\alpha + 1 = 2\rho\lambda, \quad \alpha + \beta + 2 = \rho\lambda + c, \quad c > \tfrac{1}{2},$$

one obtains

$$g_2(t) = {}_1F_2\left(\begin{matrix}\lambda \\ \rho\lambda + b, \rho\lambda + c\end{matrix}\middle| -zt^2\right) > 0, \qquad z \geqq 0,$$

under conditions K.

To relax the conditions K, we note that we can let ε be a nonnegative number and set

$$b = \lambda\varepsilon + b', \qquad b' > 0,$$

$$c = \lambda\varepsilon + c', \qquad c' > \tfrac{1}{2},$$

$$\rho' = \rho + \varepsilon.$$

Then our result for $g_2(t)$ implies that

$$g_3(t) = {}_1F_2\left(\begin{matrix}\lambda \\ \rho'\lambda + b', \rho'\lambda + c'\end{matrix}\middle| -zt^2\right) > 0, \qquad z \geqq 0,$$

$2\rho' \geqq 3$, $\lambda > 1$, $b' > 0$, $c' > \frac{1}{2}$, or $\rho' \geqq 2$, $\lambda > 0$, $b' > 0$, $c' > \frac{1}{2}$. Finally, if the beta transform is applied to $g_3(t)$, with $\alpha + 1 = \lambda - \beta - 1 = \lambda - a$, $0 < a < \lambda$, one obtains Theorem 2, except in the special cases when $a = 0$, $b = 0$ or $c = \frac{1}{2}$, which can be established directly.

COROLLARY 2.1. *For $x > 0$, $x^{2\lambda - 2\rho\lambda - b}(1 + x^2)^{-\lambda}$ is completely monotonic for $b \geqq 0$ and either $2\rho \geqq 3$, $\lambda > 1$, or $\rho \geqq 2$, $\lambda > 0$.*

Proof. In Theorem 2, let $c = b + \frac{1}{2}$, $a = 0$ and apply the Laplace transform (1.3) used in the proof of Theorem 1, with $\sigma = \rho\lambda + b$.

Remark 1. The above results partially answer some conjectures made by Askey [1].

Remark 2. A direct proof, not using asymptotic estimates, for the above results when $\rho \geqq 2$, has been given by Askey and Pollard [2], and the authors [3]. The methods used there are not applicable when $3 \leqq 2\rho < 4$.

Remark 3. If $3 < 2\rho < 4$, it is an open question whether $G(\rho, \lambda; \mu) > 0$ for some value of $\lambda < 1$, and if so, how small λ has to be.

Remark 4. Recently J. Steinig [5] proved that

$$ {}_1F_2\left[1; \frac{\omega - \nu + 3}{2}, \frac{\omega + \nu + 3}{2}; \frac{-x^2}{4}\right] > 0 \quad \text{for } x > 0$$

if $\omega = \frac{1}{2}$ and $|\nu| < \frac{1}{2}$, or if $\omega > \frac{1}{2}$ and $|\nu| \leqq \omega$, ν real. These results are only partially contained in Theorem 2 when $\lambda = 1$, $2\rho = 3$, $a = 0$, $2b = \omega - \nu$, $2c = \omega + \nu$.

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UNIFORM ASYMPTOTIC APPROXIMATION FOR VISCOUS FLUID FLOW DOWN AN INCLINED PLANE*

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Abstract. An asymptotic method is developed for the linearized Navier–Stokes equations governing the time-dependent motion of a viscous fluid flow down an inclined plane. A diffusion equation for the first order approximation of the fluid surface elevation in a perturbation scheme is derived and a critical Reynolds number is defined based upon the well-posedness of the equation. Under a set of sufficient conditions it is shown that the solution of the diffusion equation is a uniform asymptotic approximation to the generalized solution of the full equations for all time by means of various L_2 and pointwise estimates.

1. Introduction. The purpose of this paper is to develop an asymptotic method for the study of surface waves on a viscous fluid. The underlying ideas of this method center around the so-called long wave approximation, which has become an indispensable tool in dealing with problems of surface waves on ideal fluid [1]. Therefore, it should be of interest to extend such an approach to a viscous fluid system, and to give a rigorous justification of the asymptotic method.

In this paper we shall restrict ourselves to the linear time-dependent problem of a viscous fluid flow down an inclined plane. Based upon some of the ideas due to Ladyzhenskaya [2], Krein [3] and Kopachevskii [4], we first prove the uniqueness and existence of a generalized solution of the initial value problem posed by the linearized Navier–Stokes equations. Then by a formal asymptotic approximation we derive a two-dimensional diffusion equation as an asymptotic equation of the full equations. The solution method of this equation is well-known, but it is not well-posed if one of its coefficients changes sign. We make use of this property to define a critical Reynold number R_C by setting this coefficient equal to zero. Our main contribution is the following result. If the Reynolds number R of the flow is less than R_C and the initial data satisfy a compatibility condition at the free surface and a long wave condition, then the first order approximation obtained from the asymptotic expansion is shown to be an asymptotic approximation, uniform for all time, to the solution of the full equations. The uniformity in time of the asymptotic approximation is important in justifying the definition of the critical Reynolds number. Hence we not only show that the solution of the linearized equations in the long wave limit is stable for $R < R_C$, but also present a simple method to find an approximate solution to our initial value problem.

The linear instability of the flow down an inclined plane and similar problems were studied by Benjamin [5], Yih [6] and many others using a normal mode analysis under the long wave assumption and a critical Reynolds number was determined for each problem. Therefore, it is not coincidental that the critical Reynolds number we have defined is equivalent to the one obtained in [6]. Moreover, the theorems proved here may furnish an indirect justification for the

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normal mode approach used before. We also note that a discussion of formal results for nonlinear waves on viscous fluid may be found in [7].

In § 2, we formulate the problem and introduce some function spaces for later use. In § 3, we prove the uniqueness and existence theorem by use of the Galerkin method. The formal asymptotic expansion is carried out in § 4, and justified by various L_2 estimates in § 5, where pointwise estimates are also obtained as a consequence of Sobolev inequalities.

2. Formulation. The fluid domain Ω in our problem is a three-dimensional strip: $-1 < z < 0$ bounded by an upper plane $\Gamma: z = 0$ and a lower plane $S: z = -1$. We choose a moving coordinate with speed λ_0 in the direction of the x-axis (Fig. 1).

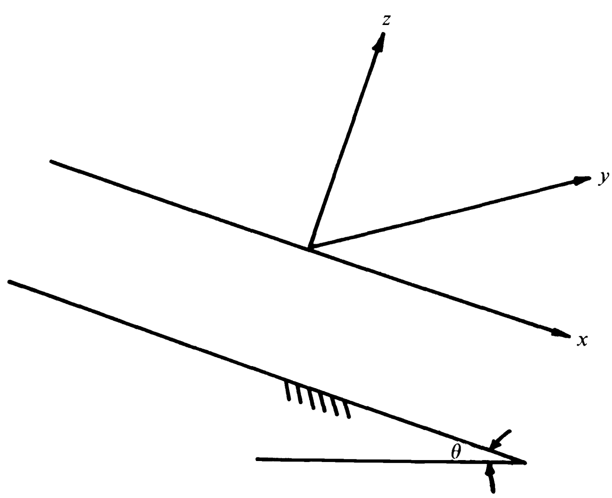

FIG. 1. *Viscous fluid flow down an inclined plane*

It is assumed that the undisturbed flow under gravity has only a velocity component in the x-direction given by [8]

$$u_0(z) = (aR/2)(1 - z^2) - \lambda_0, \tag{1}$$

and the linearized Navier–Stokes equations in this case are

$$\left.\begin{aligned} &V_t + u_0 V_x + V \cdot \nabla U = -\nabla p + R^{-1}\nabla^2 V, &(2)\\ &\nabla \cdot V = 0, &(3)\end{aligned}\right\} \text{ in } \Omega,$$

$$\left.\begin{aligned} &(R^{-1}T(V) - p)\cdot \bar{n}_3 = a\zeta\bar{n}_1 - b\zeta\bar{n}_3, &(4)\\ &\zeta_t + u_0\zeta_x - w = 0, &(5)\end{aligned}\right\} \text{ on } \Gamma,$$

$$V = 0 \quad \text{on } S, \tag{6}$$

$$V = V_0, \qquad \zeta = \zeta_0 \quad \text{at } t = 0. \tag{7}$$

Here a point is denoted by (x, y, z) or (x_1, x_2, x_3), $V = (v_1, v_2, v_3) = (u, v, w)$ is the velocity, $U = (u_0, 0, 0)$, p the pressure, ζ the surface elevation,

$$T(V) = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i},$$

$\bar{n}_i$ the unit vector of the coordinate system, R the Reynolds number, $a = \sin\theta$, $b = \cos\theta$, θ is the angle of inclination of the plane bottom and

$$T(V)\cdot\bar{n}_3 = \left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial v_3} + \frac{\partial v_3}{\partial x_2}, 2\frac{\partial v_3}{\partial x_3}\right).$$

We note that all the variables have been nondimensionalized by appropriate units.

In the following we introduce some function spaces pertaining to our problem. Let $J_s^\infty(\Omega)$ be the class of all solenoidal C^∞-vector functions with compact support in $\Omega \cup \Gamma$, and $J_s(\Omega)$ be the Hilbert space obtained by completing $J_s^\infty(\Omega)$ with the scalar product

$$(V, \Phi) = \int v_i\phi_i \, d\Omega,$$

where $V = (v_1, v_2, v_3)$, $\Phi = (\phi_1, \phi_2, \phi_3)$ and the summation convention has been adopted. For simplicity, we shall write a_i^2 for $a_i a_i$. The L_2-norm of a vector function V on Ω is defined by

$$\|V\|^2 = \int v_i^2 \, d\Omega.$$

It is known that the space $L_2(\Omega)$ of all square integrable vector functions has the orthogonal decomposition [4]

$$L_2(\Omega) = J(\Omega) \oplus G_\Gamma(\Omega).$$

Here $G_\Gamma(\Omega)$ consists of all vector functions ∇p, and p is a single-valued locally square integrable scalar function on Ω and possesses first order L_2-generalized derivatives such that if $\nabla p \in G_p(\Omega)$, then $p = 0$ on Γ.

Let $H(\Omega)$ be the Hilbert space obtained by completing $J_s^\infty(\Omega)$ with respect to the norm

$$\|V\|_H^2 = \int \left(\frac{\partial v_i}{\partial x_j}\right)^2 d\Omega.$$

Let $E(\Omega)$ be the completion of $J_s^\infty(\Omega)$ with respect to the energy norm [3]

$$\|V\|_E^2 = \frac{1}{2}\int \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 d\Omega,$$

and the scalar product in this case is

$$E(V, \Phi) = \frac{1}{2}\int \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\left(\frac{\partial \phi_i}{\partial x_j} + \frac{\partial \phi_j}{\partial x_i}\right) d\Omega.$$

That $\|V\|_E$ indeed defines a norm can be readily seen from Lemma 1 to be given later.

We also need the standard Sobolev spaces $W_2^m(\Omega)$, $m = 1, 2, \cdots$, and $W_2^m(\Gamma)$, which are respectively the completion of C^∞-vector functions defined on Ω and C^∞-scalar functions defined on Γ with respect to the norms $\|\cdot\|_{W_2^m(\Omega)}$ and $\|\cdot\|_{W_2^m(\Gamma)}$.

To put the Navier–Stokes equations in an abstract form, we introduce spaces of abstract functions of a real variable $t \in [0, T]$, where T is a positive number. By $L_2(T; X)$ we denote the class of functions $f:[0, T] \to X$ square integrable in norm over $[0, T]$, that is,

$$\int_0^T \|f(t)\|^2 \, dt < \infty,$$

where $\|\cdot\|$ is the norm of X. $L_2(T; X)$ is a Hilbert space with the scalar product

$$\int_0^T (f(t), g(t)) \, dt.$$

We denote by $C(T; X)$ the class of continuous functions with respect to the norm of X and by $C'(T; X)$ the class of continuously strongly differentiable functions. $C(T; X)$ and $C'(T; X)$ are Banach spaces under the norms

$$\|f(t)\|_{C(T;X)} = \sup_{t \in [0,T]} \|f(t)\|,$$

$$\|f(t)\|_{C'(T;X)} = \sup_{t \in [0,T]} \|f(t)\| + \sup_{t \in [0,T]} \|f'(t)\|,$$

where f' is the strong derivative of f.

We say $f \in L_2[T; X]$ has a generalized derivative $f_t \in L_2[T; X]$ if

$$\int_0^T (f_t(t), g(t)) \, dt = -\int_0^T (f(t), g_t(t)) \, dt$$

for all $g \in C'(T; X)$ such that $g(0) = g(T) = 0$. The generalized derivative so defined is unique and coincides with the strong derivative if the later exists.

In the lemmas given below, we shall establish some integral inequalities and properties pertaining to abstract functions.

LEMMA 1. *Norms $\|\cdot\|_E$ and $\|\cdot\|_H$ are equivalent. More precisely, for $V \in J_s^\infty(\Omega)$, the inequality*

$$\left(1 - \frac{1}{\sqrt{2}}\right) \|V\|_H^2 \leqq \|V\|_E^2 \leqq 2\|V\|_H^2,$$

holds.

Lemma 1 may be proved by use of the Fourier transform or Korn's inequality [4]. The details are omitted.

LEMMA 2. *For $V \in J_s^\infty(\Omega)$, we have*

$$\|V\| \leqq 2\sqrt{2}\, \|V\|_E, \tag{8}$$

$$\|V\|_\Gamma \leqq 2\sqrt{2}\, \|V\|_E, \tag{9}$$

where

$$\|V\|_\Gamma = \int (v_i)^2 \, d\Gamma.$$

Proof. Assume that $-1 < z < 0$. For $V \in J_s^\infty(\Omega)$, the inequality

$$\int_{x_3 = z} v_i^2\, dx_1\, dx_2 \leqq 2 \int_{-1}^{z} dx_3 \int \left| v_i \left(\frac{\partial v_i}{\partial x_3} + \frac{\partial v_3}{\partial x_i} \right) \right| dx_1\, dx_2, \tag{10}$$

holds. Hence

$$\|V\|^2 = \int_{-1}^{0} dz \int v_i^2\, dx_1\, dx_2 \leqq 2\sqrt{2}\, \|V\|\, \|V\|_E,$$

and (8) follows.

Now by setting z in (10) equal to zero, we obtain

$$\|V\|_\Gamma^2 = \int_{x_3 = 0} v_i^2\, dx_1\, dx_2 \leqq (2\sqrt{2})^2 \|V\|_E^2,$$

and (9) follows.

LEMMA 3.

(i) *Suppose that $f \in L_2(T; X)$ has a generalized derivative $f_t \in L_2(T; X)$. Then $f(t)$ is continuous in $[0, T]$ with respect to the norm of X. Furthermore,*

$$\int_0^t (f, f_t)\, ds = \tfrac{1}{2}(\|f(t)\|^2 - \|f(0)\|^2).$$

(ii) *If $f^n \in C'(t; X)$, $n = 1, 2, \cdots$, is a sequence of functions such that $\{f_t^n\}$ converges weakly in the space $L_2(T; X)$ to an element $f_t \in L_2(T; X)$, then $\{f^n\}$ converges weakly in $L_2(T, X)$ to an element $f \in L_2(T; X)$, and f_t is the generalized derivative of f.*

(iii) *If $f^n \in C'(T; X)$, $n = 1, 2, \cdots$, is a sequence of functions such that $f^n(0) = f_0 \in X$ for all n and $\{f_t^n\}$ converges weakly to f_t in $L_2(T; X)$, then $f(0) = f_0$.*

(iv) *If $f^n \in L_2(T; X_1)$, $n = 1, 2, \cdots$, is a sequence of functions converging weakly to f in $L_2(T; X_1)$, then $\{f^n\}$ also converges weakly to f in the space $L_2(T; X)$ if $X_1 \subset X$ and the norm of X_1 is stronger than the norm of X.*

Lemma 3 is a direct consequence of the definition of the generalized derivative of a function in $L_2(T; X)$ and the proof is omitted (see [9]).

The notion of a generalized solution follows the one given in [2], [4]. $\{V, \zeta\}$ is called a generalized solution of (2) to (7) if $V \in L_2(T; E(\Omega))$, $V_t \in L_2(T; E(\Omega))$, $\zeta \in L_2(T; L_2(\Gamma))$, $\zeta_t \in L_2(T; L_2(\Gamma))$, $\zeta_x \in L_2(T; L_2(\Gamma))$ and $\{V, \zeta\}$ satisfies (5), (7) and the following integral equality:

$$\begin{aligned} &\int_0^T dt \int (V_t + u_0 V_x + V \cdot \nabla U) \cdot \Phi\, d\Omega \\ &\quad = \int_0^T dt \int (a\zeta\phi_1 - b\zeta\phi_3)\, d\Gamma - \frac{1}{2R} \int_0^T dt \int T(V) \cdot T(\Phi)\, d\Omega \end{aligned} \tag{11}$$

for all $\Phi = (\phi_1, \phi_2, \phi_3) \in L_2(T; E(\Omega))$. It is easy to show that a classical solution is a generalized solution if this solution and its partial derivatives appearing in the differential equations also possess finite L_2-norms.

We say the initial data V_0, ζ_0 in (7) are compatible with the free surface condition (4) if there exists a $p_0 \in L_2(\Gamma)$ such that V_0, ζ_0 and p_0 satisfy (4). We also use the following standard notations:

$$\beta = (\beta_1, \beta_2), \qquad |\beta| = \beta_1 + \beta_2,$$

$$D^\beta = D_x^{\beta_1} D_y^{\beta_2}, \qquad D_x = \frac{\partial}{\partial x}, \quad D_y = \frac{\partial}{\partial y},$$

where β_1 and β_2 are nonnegative integers.

3. Uniqueness and existence theorem. In this section we shall prove the following theorem regarding the uniqueness and existence of a generalized solution to (2) to (7).

THEOREM 1. *If the initial data V_0 and ζ_0 are compatible with the free surface condition* (4), *and if V_0 and ζ_0 possess generalized derivatives with respect to x and y such that $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$ and $D^\beta \zeta_0$, $D^\beta \zeta_{0x}$, $D^\beta \zeta_{0xx} \in L_2(\Gamma)$ for any β in $0 \leqq |\beta| \leqq m$, where m is a nonnegative integer, then there exists a unique generalized solution $\{V, \zeta\}$ of* (2) *to* (7) *such that $D^\beta V \in C(T; E(\Omega))$, $D^\beta V_t \in L_2(T; E(\Omega))$, $D^\beta \zeta \in C(T; L_2(\Gamma))$ and $D^\beta \zeta_t \in L_2(T; L_2(\Gamma))$ for all $T > 0$ and any β in $0 \leqq |\beta| \leqq m$. Furthermore, for each β in $0 \leqq |\beta| \leqq m$, $\{D^\beta V, D^\beta \zeta\}$ is the generalized solution of* (2) *to* (7) *corresponding to the initial data $\{D^\beta V_0, D^\beta \zeta_0\}$.*

Proof. Let Φ in (11) be such that $\Phi(s) = V(s)$ for $0 \leqq s \leqq t$ and $\Phi(s) = 0$ for $t < s \leqq T$. By using (5), the Schwarz inequality, Lemma 2 and Lemma 3, we obtain from (11) an estimate as follows:

$$\|V(t)\|^2 + b\|\zeta(t)\|^2 \leqq \|V(0)\|^2 + b\|\zeta(0)\|^2 + M \int_0^t (\|V(s)\|^2 + b\|\zeta(s)\|^2)\, ds,$$

where M is a positive constant. If $\|V(0)\| = \|\zeta(0)\| = 0$, then $[dg(t)]/dt \leqq Mg(t)$, where $g(t) = \int_0^t (\|V(s)\|^2 + b\|\zeta(s)\|^2)\, ds$. Hence $g(t) \equiv 0$ and the uniqueness follows.

To prove the existence, we use the Galerkin method. Let $\Phi^1 = V_0$, $\Phi^k \in J_s^\infty(\Omega)$ for $k \geqq 2$ form an orthonormal basis in $J_s(\Omega)$. The differentiability assumption made on V_0 implies that $D^\beta \Phi^k \in E(\Omega) \cap W_2^2(\Omega)$ and $D^\beta \Phi^k \in W_2^1(\Gamma)$ on Γ for any β in $0 \leqq |\beta| \leqq m$ and any $k \geqq 1$. For each $n = 1, 2, \cdots$, let V^n and ζ^n be constructed as follows:

$$V^n(t) = \sum_{k=1}^n a_k^n(t) \Phi^k, \tag{12}$$

and V^n, ζ^n satisfy

$$\begin{aligned} &\sum_{|\beta|=0}^m \int (D^\beta V_t^n + u_0 D^\beta V_x^n + (D^\beta V^n) \cdot \nabla U) \cdot D^\beta \Phi^k \, d\Omega \\ &\quad = \sum_{|\beta|=0}^m \left\{ \int (a D^\beta \zeta^n D^\beta \Phi_1^k - b D^\beta \zeta^n D^\beta \phi_3^k)\, d\Gamma - \frac{1}{2R} \int T(D^\beta V^n) \cdot T(D^\beta \Phi^k)\, d\Omega \right\} \end{aligned} \tag{13}$$

for all $1 \leqq k \leqq n$,

$$\zeta_t^n + u_0 \zeta_x^n = v_3^n \quad \text{on } \Gamma, \tag{14}$$

$$V^n = V_0, \quad \zeta^n = \zeta_0 \quad \text{at } t = 0. \tag{15}$$

The solution ζ^n of (14), when expressed in terms of v_3^n, is given by

$$\zeta^n(t, x, y) = \zeta_0(x - yu_0(0), y) + \int_0^t v_3^n(s, x - (t - s)u_0(0), y, 0)\, ds. \tag{16}$$

Substituting (12) and (16) into (13), we obtain a system of integro-differential equations:

$$A_{kl}\frac{da_l^n}{dt} + B_{kl}a_l^n = \int_0^t C_{kl}(t - s)a_l^n(s)\, ds + d_k(t), \tag{17}$$

$$a_1^n(0) = 1, \qquad a_k^n(0) = 0 \quad \text{for } 2 \leqq k \leqq n, \tag{18}$$

where the matrices (A_{kl}), (B_{kl}), $(C_{kl}(t))$ and vector $(d_k(t))$ are given as follows:

$$A_{kl} = \sum_{|\beta|=0}^{m} \int D^\beta\Phi^l \cdot D^\beta\Phi^k\, d\Omega, \tag{19}$$

$$B_{kl} = \sum_{|\beta|=0}^{m} \left\{ \int (u_0 D^\beta\Phi_x^l + D^\beta\Phi^l \cdot \nabla U)D^\beta\Phi^k\, d\Omega + \frac{1}{2R}\int T(D^\beta\Phi^l)\cdot T(D^\beta\Phi^k)\, d\Omega \right\}, \tag{20}$$

$$\begin{aligned} C_{kl}(t) = \sum_{|\beta|=0}^{m} \int [&aD^\beta\phi_3^l(x - tu_0, y, 0)D^\beta\phi_1^k(x, y, 0) \\ &- bD^\beta\phi_3^l(x - tu_0, y, 0)D^\beta\phi_3^k(x, y, 0)]\, d\Gamma, \end{aligned} \tag{21}$$

$$\begin{aligned} d_k(t) = \sum_{|\beta|=0}^{m} \int [&aD^\beta\zeta_0(x - tu_0, y)D^\beta\phi_1^k(x, y, 0) \\ &- bD^\beta\zeta_0(x - tu_0, y)D^\beta\phi_3^k(x, y, 0)]\, d\Gamma, \end{aligned} \tag{22}$$

where u_0 in (21) and (22) are understood to be $u_0(0)$ which is a constant.

Since $D^\beta\zeta_0, D^\beta\zeta_{0x} \in L_2(\Gamma)$ and $D^\beta\phi_3$, $D^\beta\phi_{3x} \in L_2(\Gamma)$ on Γ for all β in $0 \leqq |\beta| \leqq m$, we can show that $C_{kl}(t)$ and $d_k(t)$ are continuously differentiable. It is also not difficult to show that the matrix (A_{kl}) is invertible. Therefore, the system (17) subject to (18) has a unique twice continuously differentiable solution $a_k^n(t)$, $k = 1, \cdots, n$. This further implies that (12) to (15) possess a unique solution $\{V^n, \zeta^n\}$ such that $D^\beta V^n(t) \in E(\Omega)$, $D^\beta\zeta^n(t) \in L_2(\Gamma)$, $D^\beta V^n \in C^2(T; W_2^2(\Omega))$, $D^\beta\zeta^n \in C^2(T; L_2(\Gamma))$ and $D^\beta\zeta_x^n \in C^1(T; L_2(\Gamma))$. The continuous differentiability of $D^\beta\zeta_x^n(t)$ and the twice continuous differentiability of $D^\beta\zeta^n(t)$ with respect to t in $L_2(\Gamma)$ follow from the fact that $\phi_3^k, \phi_{3x}^k \in L_2(\Gamma)$ on Γ and $\zeta_0, \zeta_{0x}, \zeta_{0xxx} \in L_2(\Gamma)$.

From (13) and (14), we easily obtain

$$\sum_{|\beta|=0}^{m} (\|D^\beta V^n(t)\|^2 + b\|D^\beta\zeta^n(t)\|^2) \leqq e^{Mt} \sum_{|\beta|=0}^{m} (\|D^\beta V_0\|^2 + b\|D^\beta\zeta_0\|^2), \tag{23}$$

$$\sum_{|\beta|=0}^{m} (\|D^\beta V_t(t)\|^2 + b\|D^\beta\zeta_t^n(t)\|^2) \leqq e^{Mt} \sum_{|\beta|=0}^{m} (\|D^\beta V_t^n(0)\|^2 + b\|D^\beta\zeta_t^n(0)\|^2), \tag{24}$$

$$\begin{aligned} \sum_{|\beta|=0}^{m} \int_0^T \|D^\beta V^n\|_E^2\, dt \leqq \sum_{|\beta|=0}^{m} \Bigg\{ &A_1 \int_0^T \|D^\beta V^n\|^2\, dt + A^2 \int_0^T \|D^\beta\zeta^n\|^2\, dt \\ &+ A_3\|D^\beta V_0\|^2 \Bigg\}, \end{aligned} \tag{25}$$

$$\sum_{|\beta|=0}^{m} \int_0^T \|D^\beta V_t^n\|_E^2\, dt \leqq \sum_{|\beta|=0}^{m} \left\{ A_1 \int_0^T \|D^\beta V_t^n\|^2\, dt + A_2 \int_0^T \|D^\beta \zeta_t^n\|^2\, dt + A_3 \|D^\beta V_t^n(0)\|^2 \right\}, \tag{26}$$

where M, A_1, A_2 and A_3 are positive constants independent of t and n. From the compatibility and differentiability assumptions made on V_0 and ζ_0, we see that $D^\beta V_0$ and $D^\beta \zeta_0$ are compatible with the free surface conditions (4) for any β in $0 \leqq |\beta| \leqq m$. Hence, one can show that for each β in $0 \leqq |\beta| \leqq m$, $\|D^\beta V_t^n(0)\|$ and $\|D^\beta \zeta_t^n(0)\|$ are uniformly bounded for all $n \geqq 1$. This implies the uniform boundedness of the right-hand sides of (23) to (26) for all $t \in [0, T]$ and all $n \geqq 1$. We therefore obtain that

$$\int_0^T \|D^\beta V^n\|_E^2\, dt, \qquad \int_0^T \|D^\beta V_t^n\|_E^2\, dt, \qquad \int_0^T \|D^\beta \zeta^n\|^2\, dt, \qquad \int_0^T \|D^\beta \zeta_t^n\|^2\, dt$$

are uniformly bounded for all β in $0 \leqq |\beta| \leqq m$ and all $n \geqq 1$. From Lemma 2 and the uniform boundedness of $D^\beta V^n$ in $L_2(T; E(\Omega))$, we also obtain the uniform boundedness of $D^\beta v_3^n$ on Γ in the space $L_2(T; L_2(\Gamma))$ for all β in $0 \leqq |\beta| \leqq m$ and all $n \geqq 1$. By the weak compactness in Hilbert spaces and Lemma 3, it follows that $\{V^n\}$ and $\{\zeta^n\}$ have respectively weakly convergent subsequences $\{V^{n_k}\}$ and $\{\zeta^{n_k}\}$ such that

$$D^\beta V^{n_k} \xrightarrow{w} D^\beta V, \quad D^\beta V_t^{n_k} \xrightarrow{w} D^\beta V_t \quad \text{in } L_2(T; E(\Omega)),$$

$$D^\beta v_3^{n_k} \xrightarrow{w} D^\beta v_3 \quad \text{in } L_2(T; L_2(\Gamma) \quad \text{on } \Gamma,$$

and

$$D^\beta \zeta^{n_k} \xrightarrow{w} D^\beta \zeta, \quad D^\beta \zeta_t^{n_k} \xrightarrow{w} D^\beta \zeta_t \quad \text{in } L_2(T; L_2(\Gamma)) \quad \text{as } n_k \to \infty$$

for any β in $0 \leqq |\beta| \leqq m$. Furthermore, from $D^\beta V_t \in L_2(T; E(\Omega))$, $D^\beta \zeta_t \in L_2(T; L_2(\Gamma))$ and Lemma 3, we have $D^\beta V \in C(T; E(\Omega))$ and $D^\beta \zeta \in C(T; L_2(\Gamma))$ for all β in $0 \leqq |\beta| \leqq m$.

By use of a similar argument as given in [2], [4], the weak limit $\{V, \zeta\}$, which possesses generalized derivatives with respect to x and y up to order m, can be shown to be a generalized solution of (2) to (7). Since if $T' < T''$, the generalized solution corresponding to the initial data $\{V_0, \zeta_0\}$ and the interval $[0, T']$ can be regarded as the restriction of the generalized solution corresponding to the same initial data and the interval $[0, T'']$, we conclude that for each nonnegative integer m, there exists a generalized solution $\{V, \zeta\}$ such that $D^\beta V \in C(T; E(\Omega))$, $D^\beta V_t \in L_2(T; E(\Omega))$, $D^\beta \zeta \in C(T; L_2(\Gamma))$ and $D^\beta \zeta_t \in L_2(T; L_2(\Gamma))$ for all $T > 0$ and all β in $0 \leqq |\beta| \leqq m$. This proves the first part of the theorem. To prove the last part, let β satisfy $0 \leqq |\beta| \leqq m$. Replacing Φ in (11) by $D^\beta \Phi$ and performing integration by parts with respect to x and y $|\beta|$ times, we see that $D^\beta V$ and $D^\beta \zeta$ satisfy (11). A similar argument shows that $D^\beta V$ and $D^\beta \zeta$ satisfy (5). Furthermore, the continuity of $D^\beta V(t)$ and $D^\beta \zeta(t)$ at $t = 0$ in $L_2(T; E(\Omega))$ and $L_2(T; L_2(\Gamma))$ respectively shows that $D^\beta V(0) = D^\beta V_0$ and $D^\beta \zeta(0) = D^\beta \zeta_0$. Therefore, $\{D^\beta V, D^\beta \zeta\}$ is the generalized solution corresponding to the initial data $\{D^\beta V_0, D^\beta \zeta_0\}$, and the proof of the theorem is completed.

COROLLARY. *If the initial data V_0 and ζ_0 are compatible with the free surface conditions* (4), *and if V_0 and ζ_0 possess generalized derivatives with respect to x and y of all orders such that $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$ and $D^\beta \zeta_0 \in L_2(\Gamma)$ for all β in $0 \leqq |\beta| < \infty$, then there exists a unique generalized solution $\{V, \zeta\}$ of* (2) *to* (7) *such that*

$$D^\beta V \in C(T; E(\Omega)), \quad D^\beta V_t \in L_2(T; E(\Omega)), \quad D^\beta \zeta \in C(T; L_2(\Gamma)) \quad \text{and} \quad D^\beta \zeta_t \in L_2(T; L_2(\Gamma))$$

for all $T > 0$ and any β. Furthermore, for each β, $\{D^\beta V, D^\beta \zeta\}$ is the generalized solution of (2) *to* (7) *corresponding to the initial data $\{D^\beta V_0, D^\beta \zeta_0\}$.*

Proof. For each $m = 0, 1, 2, \cdots$ such that $0 \leqq |\beta| \leqq m$, the initial data V_0, ζ_0 satisfy the condition of the above theorem, that is, $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$ and $D^\beta \zeta_0, D^\beta \zeta_{0x}, D^\beta \zeta_{0xx} \in L_2(\Gamma)$. Therefore, there exists a unique generalized solution $\{V^m, \zeta^m\}$ of (2) to (7) such that $D^\beta V^m \in C(T; E(\Omega))$, $D^\beta V_t^m \in L_2(t; E(\Omega))$, $D^\beta \zeta^m \in L_2(T; L_2(\Gamma))$ and $D^\beta \zeta_t^m \in L_2(T; L_2(\Gamma))$ for any $T > 0$ and all β in $0 \leqq |\beta| \leqq m$. Since all $\{V^m, \zeta^m\}$ are generalized solutions corresponding to the same initial data $\{V_0, \zeta_0\}$, by the uniqueness of the generalized solution, we see that all $\{V^m, \zeta^m\}$ are equal. Denote this unique generalized solution by $\{V, \zeta\}$. We then have $D^\beta V \in C(T; E(\Omega))$, $D^\beta V_t \in L_2(T; E(\Omega))$, $D^\beta \zeta \in C(T; L_2(\Gamma))$ and $D^\beta \zeta_t \in L_2(T; L_2(\Gamma))$ for all $T > 0$ and all β in $0 \leqq |\beta| < \infty$. Furthermore, integration by parts with respect to x and y and the continuity of $D^\beta V$ and $D^\beta \zeta$ at $t = 0$ show that $\{D^\beta V, D^\beta \zeta\}$ is the generalized solution of (2) to (7) corresponding to the initial data $\{D^\beta V_0, D^\beta \zeta_0\}$. This completes the proof.

4. Formal asymptotic expansion. Let α be a small positive parameter. Physically α may be considered as the ratio of the length scale in the x, y-direction to that in the z-direction. Suppose $\{V_0, \zeta_0\}$ has the property that $\partial/\partial x = O(\alpha)$ and $\partial/\partial y = O(\alpha)$. This means that $\partial^{k+l} f/\partial x^k \partial y^l = O(\alpha^{k+l})$, where f stands for ζ_0 or any component of V_0. Let $\{V, \zeta, p\}$ be the solution of (2) to (7) corresponding to the initial data $\{V_0, \zeta_0\}$. We assume that $\{V, \zeta, p\}$ also has the property $\partial/\partial x = O(\alpha)$, $\partial/\partial y = O(\alpha)$ and $\partial/\partial t = O(\alpha^2)$, and that $\{V, \zeta, p\}$ possesses asymptotic expansion of the form

$$\phi = \phi_1 + \phi_2 + \cdots. \tag{27}$$

In (27), it is assumed that $\phi_1 = O(1)$, $\phi_2 = O(\alpha)$ when ϕ stands for u, v, p and ζ, and $\phi = O(\alpha)$, $\phi_2 = O(\alpha^2)$ when ϕ stands for w. We further assume that R and θ are fixed. Substituting (27) into (2) to (7), and comparing orders of α, we obtain the following equations for the first order approximation:

$$\begin{array}{llll}
R^{-1}u_{1zz} = 0 & (-1 < z < 0), & R^{-1}v_{1zz} = 0 & (-1 < z < 0), \\
R^{-1}u_{1z} = a\zeta_1 & (z = 0), & R^{-1}v_{1z} = 0 & (z = 0), \\
u_1 = 0 & (z = -1). & v_1 = 0 & (z = -1). \\
p_{1z} = 0 & (-1 < z < 0), & u_{1x} + v_{1y} + w_{1z} = 0 & (-1 < z < 0), \\
p_1 = b\zeta_1 & (z = 0). & w_1 = 0 & (z = -1). \\
 & & u_0\zeta_{1x} = w_1 & (z = 0).
\end{array} \tag{28}$$

From (28), we obtain

$$u_1 = \bar{u}_1(z)\zeta_1(t, x, y), \qquad v_1 = 0, \qquad w_1 = \bar{w}_1(z)\zeta_{1x}, \qquad p_1 = \bar{p}_1(z)\zeta_1, \tag{29}$$

where $\bar{u}_1$, $\bar{w}_1$ and $\bar{p}_1$ are independent of α and given by

$$\bar{u}_1(z) = aR(1 + z), \qquad \bar{w}_1(z) = -(aR/2)(1 + z)^2, \qquad \bar{p}_1(z) = b. \tag{30}$$

The remaining equation $u_0\zeta_{1x} = w_1$ on $z = 0$ in (28) then implies that λ_0, called the critical speed, should assume the value

$$\lambda_0 = aR. \tag{31}$$

The equations for the second order approximation are

$$\begin{aligned}
&R^{-1}u_{2zz} = u_0u_{1x} + u_{0z}w_1 + p_{1x} && (-1 < z < 0),\\
&R^{-1}u_{2z} = a\zeta_2 && (z = 0),\\
&u_2 = 0 && (z = -1).\\
&R^{-1}v_{2zz} = p_{1y} \quad (-1 < z < 0), && p_{2z} = R^{-1}w_{1zz} \quad (-1 < z < 0),\\
&R^{-1}v_{2z} = 0 \quad (z = 0), && p_2 = b\zeta_2 + 2R^{-1}w_{1z} \quad (z = 0).\\
&v_2 = 0 \quad (z = -1).\\
&u_{2x} + v_{2y} + w_{2z} = 0 && (-1 < z < 0),\\
&w_2 = 0 && (z = -1).\\
&\zeta_{1t} = w_2 && (z = 0).
\end{aligned} \tag{32}$$

The solution for the second order approximation can be expressed as

$$\begin{aligned}
&u_2 = \bar{u}_2(z)\zeta_{1x} + \bar{u}_1(z)\zeta_2, \qquad v_2 = \bar{v}_2(z)\zeta_{1y},\\
&p_2 = \bar{p}_2(z)\zeta_{1x} + \bar{p}_1(z)\zeta_2, \qquad w_2 = \bar{w}_{21}(z)\zeta_{1xx} + \bar{w}_{22}(z)\zeta_{1yy} + \bar{w}_1(z)\zeta_{2x}.
\end{aligned} \tag{33}$$

$\bar{u}_2$, $\bar{v}_2$, $\bar{p}_2$, $\bar{w}_{21}$ and $\bar{w}_{22}$ are independent of α and are solutions of the following equations:

$$\begin{aligned}
&R^{-1}\bar{u}_{2zz} = u_0\bar{u}_1 + u_{0z}\bar{w}_1 + b && (-1 < z < 0),\\
&R^{-1}\bar{u}_{2z} = 0 && (z = 0),\\
&\bar{u}_2 = 0 && (z = -1).\\
&R^{-1}\bar{v}_{2zz} = b \quad (-1 < z < 0), && \bar{p}_{2z} = -a \quad (-1 < z < 0),\\
&R^{-1}\bar{v}_{2z} = 0 \quad (z = 0), && \bar{p}_2 = -2a \quad (z = 0).\\
&\bar{v}_2 = 0 \quad (z = -1).\\
&\bar{w}_{21z} = -\bar{u}_2 \quad (-1 < z < 0), && \bar{w}_{22z} = -\bar{v}_2 \quad (-1 < z < 0),\\
&\bar{w}_{21} = 0 \quad (z = -1). && \bar{w}_{22} = 0 \quad (z = 0).
\end{aligned} \tag{34}$$

The remaining equation $\zeta_{1t} = w_2$ on $z = 0$ in the second order approximation then gives the following linear evolution equation for $\zeta_1(t, x, y)$.

$$\zeta_{1t} = \mu_1 \zeta_{1xx} + \mu_2 \zeta_{1yy}, \tag{35}$$

where $\mu_1 = \bar{w}_{21}(0)$, $\mu_2 = \bar{w}_{22}(0)$. After some calculation, all the functions $\bar{u}_1, \cdots, \bar{p}_2$ are found to be polynomials in the variable z only, and the constants μ_1 and μ_2 are given as

$$\mu_1 = \bar{w}_{21}(0) = R\left(\frac{b}{3} - \frac{2}{15}a^2R^2\right), \qquad \mu_2 = \bar{w}_{22}(0) = \frac{Rb}{3}.$$

The evolution equation (35) is well-posed if $\mu_1 > 0$ and $\mu_2 > 0$. Since μ_2 is always positive, the condition $\mu_1 = 0$ then gives a criterion for the stability of the surface wave motion and defines a critical Reynolds number

$$R_C = \left(\frac{5b}{2a^2}\right)^{1/2} = \left(\frac{5 \cot \theta}{2 \sin \theta}\right)^{1/2}. \tag{36}$$

If the Reynolds number R is subcritical (i.e., $R < R_C$), then (35) coupled with the initial condition

$$\zeta_1 = \zeta_0 \quad \text{at } t = 0$$

gives a unique solution ζ_1. Hence we obtain a first order approximate solution $\{u_1, v_1, w_1, \zeta_1\}$ according to (29) and (35). It is noted that (u_1, v_1, w_1) cannot assume arbitrary value at $t = 0$. This is due to the fact that the above perturbation scheme is a singular one.

5. Uniform asymptotic approximation. Let $V_1 = (u_1, v_1, w_1)$ and ζ_1 be the first order approximation of the asymptotic expansion (27). According to (29) and (35), V_1 and ζ_1 are determined by

$$u_1 = \bar{u}_1 \zeta_1, \quad v_1 = 0, \quad w_1 = \bar{w}_1 \zeta_{1x}, \tag{37}$$

$$\zeta_{1t} = \mu_1 \zeta_{1xx} + \mu_2 \zeta_{1yy}, \tag{38}$$

$$\zeta_1 = \zeta_0 \quad \text{at } t = 0. \tag{39}$$

To prove that the first order approximate solution obtained in § 4 is a uniform asymptotic approximation to the generalized solution of (2) and (7), we make the following assumptions:

A1. $R < R_C$ and $0 < \theta < \pi/2$. For the case of $\theta = 0$, see Remark 2 at the end of this section.

A2. The initial data V_0 and ζ_0 satisfy the conditions given in the corollary to the existence theorem of § 2, that is, V_0 and ζ_0 possess generalized derivatives with respect to x and y of all orders such that $D^\beta V_0 \in E(\Omega) \cap W_2^2(\Omega)$, $D^\beta \zeta_0 \in L_2(\Gamma)$ for all β in $0 < |\beta| < \infty$, and V_0, ζ_0 are compatible with the free surface conditions (4).

A3. *Long wave condition*. There exist positive parameters α and ρ such that

$$
(40)\qquad
\begin{aligned}
&\|D^\beta\zeta_x(0)\| \leqq \alpha\rho^{|\beta|}, \qquad \|D^\beta\zeta_y(0)\| \leqq \alpha\rho^{|\beta|},\\
&\|D^\beta\zeta_{xx}(0)\| \leqq \alpha^2\rho^{|\beta|}, \qquad \|D^\beta\zeta_{yy}(0)\| \leqq \alpha^2\rho^{|\beta|},\\
&\|D^\beta(V(0) - V_1(0))\| \leqq \alpha\rho^{|\beta|},\\
&\|D^\beta(V_x(0) - V_{1x}(0))\| \leqq \alpha^2\rho^{|\beta|}, \qquad \|D^\beta(V_y(0) - V_{1y}(0))\| \leqq \alpha^2\rho^{|\beta|}
\end{aligned}
$$

for all β in $0 \leqq |\beta| < \infty$, where $V(0)$, $\zeta(0)$ and $V_1(0)$ denote the initial conditions for V, ζ and V_1, respectively, so we have $V(0) = V_0$, $\zeta(0) = \zeta_0$,

$$V_1(0) = (\bar{u}_1\zeta_0, 0, \bar{w}_1\zeta_{0x})$$

according to (37) and (38).

Remark 1. α corresponds to the perturbation parameter in § 4 and we shall always assume that $0 < \alpha < 1$. It is noted that each term in (40) is well-defined. Since $R < R_C$ by A1, we have $\mu_1 > 0$ according to (36), hence (38) is a well-posed heat equation. Since $D^\beta\zeta_0 \in L_2(\Gamma)$ according to A2, by means of the Poisson formula for heat equation, it is not difficult to see that $D^\beta\zeta_1(t)$ is continuous at $t = 0$ in the space $L_2(\Gamma)$ for all β. It follows that $D^\beta V_1(t)$ is continuous at $t = 0$ in the space $J_s(\Gamma)$ for all β since $\bar{u}_1$ and $\bar{w}_1$ are independent of x and y. Therefore, $D^\beta V_1(0)$ is well-defined and can be expressed as

$$D^\beta V_1(0) = (\bar{u}_1 D^\beta\zeta_0, 0, \bar{w}_1 D^\beta\zeta_{0x}).$$

It is also noted that conditions in A3 are not mutually independent.

Since V_0 and ζ_0 satisfy the assumptions in the corollary to the existence theorem of § 3, equations (2) to (7) have a unique generalized solution $\{V, \zeta\}$ corresponding to the initial data $\{V_0, \zeta_0\}$. This solution possesses generalized derivatives with respect to x and y of all orders, and for each β, $\{D^\beta V, D^\beta\zeta\}$ is the generalized solution of (2) to (7) corresponding to the initial data $\{D^\beta V_0, D^\beta\zeta_0\}$. Let

$$
(41)\qquad
\begin{aligned}
&V = (u, v, w) = V' + V'' + V^*,\\
&u = \bar{u}_1\zeta + \bar{u}_2\zeta_x + u^* = u' + u'' + u^*,\\
&v = 0 + \bar{v}_2\zeta_y + v^* = v' + v'' + v^*,\\
&w = \bar{w}_1\zeta_x + \bar{w}_{21}\zeta_{xx} + \bar{w}_{22}\zeta_{yy} + w^* = w' + w'' + w^*,
\end{aligned}
$$

where

$$
\begin{aligned}
&V' = (u', v', w') = (\bar{u}_1\zeta, 0, \bar{w}_1\zeta_x),\\
&V'' = (u'', v'', w'') = (\bar{u}_2\zeta_x, \bar{v}_2\zeta_y, \bar{w}_{21}\zeta_{xx} + \bar{w}_{22}\zeta_{yy}),\\
&V^* = (u^*, v^*, w^*).
\end{aligned}
$$

$\bar{u}_1, \bar{w}_1, \bar{u}_2, \bar{v}_2, \bar{w}_{21}$ and $\bar{w}_{22}$ are the same as defined in (30) and (34), and are given functions depending on z only. Since V and ζ are the generalized solutions of (2) to (7), substituting (41) into (5) and making use of the properties of $\bar{u}_1, \cdots, \bar{w}_{22}$, we obtain

$$\zeta_t = (\mu_1 \zeta_{xx} + \mu_2 \zeta_{yy}) = w^*, \tag{42}$$

$$\zeta = \zeta_0 \quad \text{at } t = 0. \tag{43}$$

If we substitute (41) in (11) and choose Φ in (11) in such a form that $\Phi(s) = 0$ for $t \leqq s \leqq T$, we obtain

$$\begin{aligned} \int_0^t ds \int [V'_t + V''_t + V^*_t + u_0(V'_x + V''_x + V^*_x) \\ + (V' + V'' + V^*) \cdot \nabla U] \cdot \Phi \, d\Omega \\ = \int_0^t ds \int (a\zeta\phi_1 - b\zeta\phi_3)\, d\Gamma - \frac{1}{2R}\int_0^t ds \int \Big[T(V') \\ + T(V'') + T(V^*) \Big] \cdot T(\Phi)\, d\Omega. \end{aligned} \tag{44}$$

Let

$$p' = \bar{p}_1 \zeta, \qquad p'' = \bar{p}_2 \zeta_x, \tag{45}$$

where $\bar{p}_1$ and $\bar{p}_2$ are defined in (30) and (34). Since ζ possesses generalized derivatives with respect to x and y of all order, and $\bar{u}_1, \bar{w}_1, \bar{u}_2, \bar{v}_2, \bar{w}_{21}, \bar{w}_{22}, \bar{p}_1$ and $\bar{p}_2$ are polynomials of z only, we see that $V', V'' \in W_2^2(\Omega)$ and $p', p'' \in W_2^1(\Omega)$, and integration by parts yields

$$\begin{aligned} -\frac{1}{2R}\int [T(V') + T(V'')] \cdot T(\Phi)\, d\Omega \\ = \int (-\nabla p' + R^{-1}\nabla^2 V' - \nabla p'' + R^{-1}\nabla^2 V'') \cdot \Phi \, d\Omega \\ - \int [(R^{-1}T(V') - p') \cdot \bar{n}_3 + (R^{-1}T(V'') - p'') \cdot \bar{n}_3] \cdot \Phi \, d\Gamma. \end{aligned}$$

Hence (44) can be written as

$$\begin{aligned} \int_0^t ds \int (V^*_t + u_0 V^*_x + V^* \cdot \nabla U) \cdot \Phi \, d\Omega \\ = \int_0^t ds \int G \cdot \Phi \, d\Gamma + \int_0^t ds \int F \cdot \Phi \, d\Omega - \frac{1}{2R}\int_0^t ds \int T(V^*) \cdot T(\Phi)\, d\Omega, \end{aligned} \tag{46}$$

where

$$G = -(R^{-1}T(V') - p') \cdot \bar{n}_3 - (R^{-1}T(V'') - p'') \cdot \bar{n}_3 + (\alpha\zeta\bar{n}_1 - b\zeta\bar{n}_3), \tag{47}$$

$$\begin{aligned} F = (-V'_t - u_0 V'_x - V' \cdot \nabla U - \nabla p' + R^{-1}\nabla^2 V') \\ + (V''_t - u_0 V''_x - V'' \cdot \nabla U - \nabla p'' + R^{-1}\nabla^2 V''). \end{aligned} \tag{48}$$

By (44) and (45), (47) and (48) can be expressed as follows:

$$G = G_0\zeta_{xy} + (G_1 + G_2D_x + G_3D_y)\zeta_{xx} + (G_4 + G_5D_x + G_6D_y)\zeta_{yy}, \tag{49}$$

$$\begin{aligned} F = {} & F_0\zeta_{xy} + (F_1 + F_2D_x + F_3D_y + F_4D_{xx} + F_5D_{yy})\zeta_{xx} \\ & + (F_6 + F_7D_x + F_8D_y + F_9D_{xx} + F_{10}D_{yy})\zeta_{yy} \\ & + (F_{11} + F_{12}D_x + F_{13}D_y + F_{14}D_{xx} + F_{15}D_{yy})\zeta_t, \end{aligned} \tag{50}$$

where $F_0, \cdots, F_{15}$ and $G_0, \cdots, G_6$ are vector functions of z related to the known functions $\bar{u}_1, \cdots, \bar{p}_2$ and u_0 only. We are now in a position to prove the following.

THEOREM 2. *There exists a positive number ρ_0 which depends only on the Reynolds number R and the angle of inclination θ. If the initial data V_0 and ζ_0 satisfy the assumptions* A1 *to* A3, *and if ρ in* A3 *satisfies $0 < \rho < \rho_0$, then the first order approximate solution $\{V_1, \zeta_1\}$, which is obtained from the perturbation scheme in* § 4, *is a uniform asymptotic approximation to the generalized solution $\{V, \zeta\}$ in the following sense: If we denote the L_2-norm $(\int u^2\, d\Omega)^{1/2}$ by $\|u\|$ and the L_2-norm $(\int \zeta^2\, d\Gamma)^{1/2}$ by $\|\zeta\|$, then*

$$\|u = u_1\| \leqq \alpha L_0, \quad \|v - v_1\| \leqq \alpha L_0, \quad \|w - w_1\| \leqq \alpha^2 L_0, \quad \|\zeta - \zeta_1\| \leqq \alpha L_0$$

for all $t \geqq 0$, where L_0 is a positive constant depending on R and θ only.

Since the proof of this theorem is fairly long, it will be divided into a series of lemmas. In the following many constants will appear in various estimates. Unless specified otherwise, A_i, B_i, C_i and L_i will denote positive constants depending upon R and θ only.

LEMMA 4. *There exist positive constants C_0, C_1 and C_2 such that*

$$\begin{aligned} \int_0^t \|D^\beta V^*\|_E^2\, ds \leqq C_0\|D^\beta V^*(0)\|^2 & + C_1(\|D^\beta\zeta_x(0)\|^2 + \|D^\beta\zeta_y(0)\|^2 \\ & + \|D^\beta\zeta_{xx}(0)\|^2 + \|D^\beta\zeta_{yy}(0)\|^2) \\ & + C_2\int_0^t (\|D^\beta V_x^*\|_E^2 + \|D^\beta V_y^*\|_E^2)\, ds \end{aligned} \tag{51}$$

for all β in $0 \leqq |\beta| < \infty$ and any $t \geqq 0$.

Proof. Let

$$\gamma = \sup_{V \in E(\Omega)} \frac{\|V\|}{\|V\|_E}, \qquad \delta = \sup_{V \in E(\Omega)} \frac{\|V\|_\Gamma}{\|V\|_E}.$$

By Lemma 2, $\gamma \leqq 2\sqrt{2}$, $\delta \leqq 2\sqrt{2}$. Replacing Φ in (46) by V^* and using Lemma 3 and the fact that $\int u_0 V_x^* V^*\, d\Omega = 0$, we obtain

$$\begin{aligned} \frac{1}{2}(\|V^*(t)\|^2 - \|V^*(0)\|^2) = & -\int_0^t ds \int u_{0_z} w^* u^*\, d\Omega + \int_0^t ds \int G \cdot V^*\, d\Gamma \\ & + \int_0^t ds \int F \cdot V^*\, d\Omega - R^{-1}\int_0^t \|V^*\|_E^2\, ds. \end{aligned} \tag{52}$$

From (1), (49), (50), Lemmas 1 and 2, Schwarz inequality, integration by parts, and noting that $\int |w^*|^2\, d\Omega \leqq 2\gamma^2(\|V_x^*\|_E^2 + \|V_y^*\|_E^2)$, we can obtain the following estimates:

$$\left|\int u_{0z} w^* u^* \, d\Omega\right| \leqq \sqrt{2}\, aR\gamma^2 \|V^*\|_E (\|V_x^*\|_E^2 + \|V_y^*\|_E^2)^{1/2}, \tag{53}$$

$$\left|\int F \cdot V^* \, d\Omega\right| \leqq A_1[Q(\zeta) + Q(\zeta_x) + Q(\zeta_y)]\|V^*\|_E, \tag{54}$$

$$\left|\int G \cdot V^* \, d\Omega\right| \leqq A_2[Q(\zeta) + Q(\zeta_x) + Q(\zeta_y)]\|V^*\|_E, \tag{55}$$

where

$$Q(\zeta) = (\mu_1^2\|\zeta_{xx}\|^2 + \mu_2^2\|\zeta_{yy}\|^2 + \|\zeta_t\|^2)^{1/2}. \tag{56}$$

By (53) to (55), the left side of (52) can be estimated as follows:

$$\begin{aligned}\|V^*(t)\|^2 - \|V^*(0)\|^2 \leqq{} & 2\sqrt{2}\, aR\gamma^2 \int_0^t (\|V_x^*\|_E^2 + \|V_y^*\|_E^2)^{1/2}\|V^*\|_E \, ds \\ & + 2A_3 \int_0^t [Q(\zeta) + Q(\zeta_x) + Q(\zeta_y)]\|V^*\|_E \, ds \\ & - 2R^{-1} \int_0^t \|V^*\|_E^2 \, ds \\ \leqq{} & A_4\left[\int_0^t (\|V_x^*\|_E^2 + \|V_y^*\|_E^2)\, ds\right] + A_5\left\{\int_0^t (|Q(\zeta)^2| \right. \\ & \left. + |Q(\zeta_x)|^2 + |Q(\zeta_y)|^2)\, ds\right\} - R^{-1}\int_0^t \|V^*\|_E^2 \, ds,\end{aligned} \tag{57}$$

where we have made use of the inequality $2ab \leqq \varepsilon a^2 + b^2/\varepsilon$, $\varepsilon > 0$. It follows that

$$\begin{aligned}\int_0^t \|V^*\|_E^2 \, ds \leqq{} & C_0\|V^*(0)\|^2 + C_3\int_0^t (|Q(\zeta)|^2 + |Q(\zeta_x)|^2 \\ & + |Q(\zeta_y)|^2)\, ds + C_4 \int_0^t (\|V_x^*\|_E^2 + \|V_y^*\|_E^2)\, ds.\end{aligned} \tag{58}$$

Now by (42), (56), Lemma 2, integration by parts and noting that $\nabla \cdot \bar{V}^* = 0$, we have

$$\begin{aligned}\int_0^t |Q(\zeta)|^2 \, ds \leqq{} & \mu_1\|\zeta_x(0)\|^2 + \mu_2\|\zeta_y(0)\|^2 + \int_0^t ds \int |w^*|^2 \, d\Gamma \\ \leqq{} & \mu_1\|\zeta_x(0)\|^2 + \mu_2\|\zeta_y(0)\|^2 + 2\gamma^2 \int_0^t (\|V_x^*\|_E^2 + \|V_y^*\|_E^2)\, ds.\end{aligned} \tag{59}$$

According to the corollary to the existence theorem of § 3, for each β in $0 \leqq |\beta| < \infty$, $D^\beta V$ and $D^\beta \zeta$ are the generalized solutions of the same equations (2) to (6) corresponding to the initial data $D^\beta V_0$ and $D^\beta \zeta_0$. Now, if we replace Φ in (46) by $D^{2\beta} V^*$ and perform integration by parts $|\beta|$ times with respect to x and y, we see that (52) still holds except that V^* is to be replaced by $D^\beta V^*$, $V^*(0)$

by $D^\beta V^*(0)$, G by $D^\beta G$ and F by $D^\beta F$. Since the given functions $G_0, \cdots, G_6$ and $F_0, \cdots, F_{15}$ are independent of x and y, we see that (49) and (50) are still satisfied by $D^\beta G$, $D^\beta F$ and $D^\beta \zeta$. Therefore every estimate made above also holds for $D^\beta V(t)$, $D^\beta \zeta^*(t)$, $D^\beta V^*(0)$ and $D^\beta \zeta^*(0)$, and, in particular, we have the following estimates which are similar to (58) and (59):

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds \leqq C_0\|D^\beta V^*(0)\|^2 + C_3 \int_0^t (|Q(D^\beta \zeta)|^2 + |Q(D^\beta \zeta_x)|^2 \\ + |Q(D^\beta \zeta_y)|^2)\, ds + C_4 \int_0^t (|D^\beta V_x^*\|_E^2 + \|D^\beta V_y^*\|_E^2)\, ds,\end{aligned} \tag{60}$$

and

$$\begin{aligned}\int_0^t |Q(D^\beta \zeta)|^2\, ds \leqq \mu_1\|D^\beta \zeta_x(0)\|^2 + \mu_2\|D^\beta \zeta_y(0)\|^2 \\ + 2\gamma^2 \int_0^t (\|D^\beta V_x^*\|_E^2 + \|D^\beta V_y^*\|_E^2)\, ds,\end{aligned} \tag{61}$$

for all β in $0 \leqq |\beta| < \infty$. Furthermore, by Lemma 2 we also obtain

$$\int |D^\beta w_x^*|^2\, d\Gamma \leqq \int |D^\beta V_x^*|^2\, d\Gamma \leqq \delta^2\|D^\beta V_x^*\|_E^2.$$

Hence, by replacing ζ in (59) by $D^\beta \zeta_x$, we have

$$\int_0^t |Q(D^\beta \zeta_x)|^2\, ds \leqq \mu_1\|D^\beta \zeta_{xx}(0)\|^2 + \mu_2\|D^\beta \zeta_{xy}(0)\|^2 + \delta^2 \int_0^t \|D^\beta V_x^*\|_E^2\, ds, \tag{62}$$

for all β in $0 \leqq |\beta| < \infty$. Similarly,

$$\int_0^t |Q(D^\beta \zeta_y)|^2\, ds \leqq \mu_1\|D^\beta \zeta_{xy}(0)\|^2 + \mu_2\|D^\beta \zeta_{yy}(0)\|^2 + \delta^2 \int_0^t \|D^\beta V_y^*\|_E^2\, ds, \tag{63}$$

for all β in $0 \leqq |\beta| < \infty$. It then follows from (60) to (63), integration by parts and $2ab \leqq a^2 + b^2$ that the following estimate holds:

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds \leqq C_0\|D^\beta V^*(0)\|^2 + C_1(\|D^\beta \zeta_x(0)\|^2 + \|D^\beta \zeta_y(0)\|^2 \\ + \|D^\beta \zeta_{xx}(0)\|^2 + \|D^\beta \zeta_{yy}(0)\|^2) \\ + C_2 \int_0^t (\|D^\beta V_x^*\|_E^2 + D^\beta\| V_y^*\|_E^2)\, ds\end{aligned} \tag{64}$$

for all β in $0 \leqq |\beta| < \infty$. This proves the lemma.

LEMMA 5. *There exists a positive number ρ_0 depending only on R and θ such that if ρ in the long wave condition* (40) *satisfies $0 < \rho < \rho_0$, then for any $t \geqq 0$, we have*

$$\int_0^t \|V^*\|_E^2\, ds \leqq \alpha^2 L_1^2/(1 - r), \tag{65}$$

where r is defined by $r = (\rho/\rho_0)^2$.

Proof. According to (40) and (41), we have

$$\begin{aligned}\|D^\beta V^*(0)\| &\leqq \|D^\beta(V(0) - V_1(0))\| + \|D^\beta V''(0)\| \\ &\leqq \alpha\rho^{|\beta|} + A_6(\|D^\beta\zeta_x(0)\| + \|D^\beta\zeta_y(0)\| \\ &\qquad + \|D^\beta\zeta_{xx}(0)\| + \|D^\beta\zeta_{yy}(0)\|) \leqq \alpha A_7\rho^{|\beta|}.\end{aligned} \tag{66}$$

By means of (40) and (66), (64) reduces to

$$\int_0^t \|D^\beta V^*\|_E^2\,ds \leqq \alpha^2 L_1^2\rho^{2|\beta|} + C_2\int_0^t (\|D^\beta V_x^*\|_E^2 + \|D^\beta V_y^*\|_E^2\,ds, \tag{67}$$

for any β in $0 < |\beta| < \infty$. If we multiply (67) by $C_2^{|\beta|}$, sum over β for all β in $0 \leqq |\beta| \leqq N - 1$, where N is a positive integer, we obtain that

$$\sum_{|\beta|=0}^{N-1} C_2^{|\beta|}\int_0^t \|D^\beta V^*\|_E^2\,ds \leqq \alpha^2 L_1^2 \sum_{|\beta|=0}^{N-1} (C_2\rho^2)^{|\beta|} + \sum_{|\beta|=1}^{N} C_2^{|\beta|}\int_0^t \|D^\beta V^*\|_E^2\,ds, \tag{68}$$

where

$$\|D^\beta V_x^*\|_E^2 + \|D^\beta V_y^*\|_E^2 = \sum_{|\sigma|=1} \|D^{\sigma+\beta}V^*\|_E^2.$$

Since

$$\sum_{|\beta|=0}^{N-1} (C_2\rho^2)^{|\beta|} \leqq \sum_{k=0}^{N-1} 2^k(C_2\rho^2)^k = \frac{1 - (2C_2\rho^2)^N}{1 - 2C_2\rho^2},$$

(68) becomes

$$\begin{aligned}\int_0^t \|V^*\|_E^2\,ds &\leqq \alpha^2 L^2[1 - (2C_2\rho^2)^N](1 - 2C_2\rho^2)^{-1} \\ &\qquad + C_2^N \sum_{|\beta|=N}\int_0^t \|D^\beta V^*\|_E^2\,ds,\end{aligned} \tag{69}$$

for all $N = 1, 2, \cdots$.

To make an estimate on the last term on the right side of (69), we first make estimates on the generalized solution $\{V, \zeta\}$ of (2) to (7). By (5) and (11), it is shown that

$$\|V(t)\|^2 + b\|\zeta(t)\|^2 \leqq e^{tB_1}(\|V(0)\|^2 + b\|\zeta(0)\|^2), \tag{70}$$

$$\int_0^t \|V\|_E^2\,ds \leqq 4R\,e^{tB_1}(\|V(0)\|^2 + b\|\zeta(0)\|^2). \tag{71}$$

Since $\{D^\beta V, D^\beta\zeta\}$ is the generalized solution corresponding to the initial condition $\{D^\beta V(0), D^\beta\zeta(0)\}$, we also have

$$\|D^\beta V(t)\|^2 + b\|D^\beta\zeta(t)\|^2 \leqq e^{tB_1}(\|D^\beta V(0)\|^2 + b\|D^\beta\zeta(0)\|^2), \tag{72}$$

$$\int_0^t \|D^\beta V\|_E^2\,ds \leqq 4R\,e^{tB_1}(\|D^\beta V(0)\|^2 + b\|D^\beta\zeta(0)\|^2). \tag{73}$$

From (41) and the definition of $\|\cdot\|_E$, we see that

$$\begin{aligned}\|D^\beta V^*\|_E &\leqq \|D^\beta V\|_E + \|D^\beta V'\|_E + \|D^\beta V''\|_E \\ &\leqq \|D^\beta V\|_E + A_8(\|D^\beta \zeta\| + \cdots + \|D^\beta \zeta_{yyy}\|),\end{aligned} \tag{74}$$

where $\|D^\beta \zeta\| + \cdots + \|D^\beta \zeta_{yyy}\|$ represents the sum of all derivatives of $D^\beta \zeta$ with respect to x and y up to order 3. Since derivatives of mixed type can be estimated in terms of derivatives with respect to one variable by means of integration by parts, it follows from (40) and (70) to (73) that, for $|\beta| \geqq 1$,

$$\begin{aligned}\int_0^t \|D^\beta V^*\|_E^2\, ds &\leqq A_9\, e^{tB_1}(\|D^\beta V(0)\|^2 + \|D^\beta V_{xxx}(0)\|^2 + \|D^\beta \zeta(0)\|^2 \\ &\quad + \cdots + \|D^\beta \zeta_{xxx}(0)\|^2 + \|D^\beta V_y(0)\|^2 \\ &\quad + \cdots + \|D^\beta V_{yyy}(0)\|^2 + \|D^\beta \zeta_y(0)\|^2 + \cdots + \|D^\beta \zeta_{yyy}(0)\|^2) \\ &\leqq A_{10}\, e^{tB_1}\alpha^2\rho^{2|\beta|}(\rho^{-2} + \cdots + \rho^6).\end{aligned} \tag{75}$$

Let

$$\rho_0 = (2C_2)^{-1/2},$$

where C_2 is the constant appearing in (67). It follows that ρ_0 depends on R and θ only. Suppose that the parameter ρ in (40) satisfies $0 < \rho < \rho_0$, and let

$$r = (\rho/\rho_0)^2.$$

From (69) and (75), it is obtained that

$$\begin{aligned}\int_0^t \|V^*\|_E^2\, ds &\leqq \alpha^2 L_1^2(1 - r^N)(1 - r)^{-1} \\ &\quad + A_{10} C_2^N \sum_{|\beta| = N} \alpha^2\, e^{tB_1}(\rho^{-2} + \cdots + \rho^6)\rho^{2|\beta|} \\ &= \alpha^2 L_1^2(1 - r)^N(1 - r)^{-1} + A_{10} r^N \alpha^2\, e^{tB}(\rho^{-2} + \cdots + \rho^6).\end{aligned}$$

Since $0 < r < 1$, N is arbitrary and A_{10} is independent of N, by letting $N \to \infty$, we obtain (65). This proves the lemma.

LEMMA 6. *If the assumption in Lemma 5 is met, then we have*

$$\|\zeta^*(t)\|,\ \|\zeta_x(t)\|,\ \|\zeta_y(t)\|,\ \|V^*(t)\| \leqq \alpha L_2,$$
$$\|\zeta_x^*(t)\|,\ \|\zeta_{xx}(t)\|,\ \|\zeta_{yy}(t)\|,\ \|w^*(t)\| \leqq \alpha^2 L_2,$$

for all $t \geqq 0$, *where* $\zeta^* = \zeta - \zeta_1$.

Proof. According to (38), (39), (42) and (43), ζ^* satisfies

$$\zeta_t^* - (\mu_1 \zeta_{xx}^* + \mu_2 \zeta_{yy}^*) = w^*, \tag{76}$$

$$\zeta^* = 0 \quad \text{at } t = 0. \tag{77}$$

If we multiply (76) by ζ^*, integrate over $[0, t] \times \Gamma$, and perform integrations by parts, we see that

$$
\begin{aligned}
(78)\qquad \frac{1}{2}\|\zeta^*(t)\|^2 + \int_0^t (\mu_1\|\zeta_x^*\|^2 + \mu_2\|\zeta_y\|^2)\,ds &= \int_0^t ds \int w^*\zeta^*\,d\Gamma \\
&\leqq \int_0^t (\mu_1\|\zeta_x^*\|^2 + \mu_2\|\zeta_y^*\|^2)\,ds + C_5\int_0^t \|V^*\|^2\,ds.
\end{aligned}
$$

From Lemma 2, (65) and (78), we obtain that

$$
(79)\qquad \|\zeta^*(t)\| \leqq \alpha L_1 C_6/(1-r)^{1/2} \quad \text{for all } t \geqq 0.
$$

Following a derivation similar to that of (78), we multiply (42) by $-\zeta_{xx}$, integrate over $[0, T] \times \Gamma$, and perform integration by parts to obtain

$$
\begin{aligned}
(80)\qquad \|\zeta_x(t)\|^2 - \|\zeta_x(0)\|^2 &\leqq 2C_5\int_0^t \|V_x^*\|^2\,ds \leqq 2C_5\int_0^t \|V^*\|_H^2\,ds \\
&\leqq C_7\int_0^t \|V^*\|_E^2\,ds,
\end{aligned}
$$

by Lemma 1. Therefore, by (40) and (65), we have

$$
(81)\qquad \|\zeta_x(t)\| \leqq \|\zeta_x(0)\| + C_8\alpha L_1/(1-r)^{1/2} \leqq \alpha C_9 \quad \text{for all } t \geqq 0.
$$

Similarly,

$$
(82)\qquad \|\zeta_y(t)\| \leqq \|\zeta_y(0)\| + \alpha L_1 C_8/(1-r)^{1/2} \leqq \alpha C_9 \quad \text{for all } t \geqq 0.
$$

Since V_x and ζ_x are the solution of the same problem corresponding to the initial data $V_x(0) = V_{0x}$ and $\zeta_x(0) = \zeta_{0x}$, (64) also holds if we replace V^*, $\zeta(0)$ and $V^*(0)$ by V_x^*, $\zeta_x(0)$ and $V_x^*(0)$ respectively. Hence

$$
\begin{aligned}
\int_0^t \|D^\beta V_x^*\|_E^2\,ds \leqq C_0\|D^\beta V_x^*(0)\| &+ C_1(\|D^\beta\zeta_{xx}(0)\|^2 + \|D^\beta\zeta_{xy}(0)\|^2 \\
&+ \|D^\beta\zeta_{xxx}(0)\|^2 + \|D^\beta\zeta_{xyy}(0)\|^2) \\
&+ C_2\int_0^t (\|D^\beta V_{xx}^*\|_E^2\,ds + \|D^\beta V_{xy}^*\|_E^2)\,ds.
\end{aligned}
$$

From (40), (41) and (76), we find that

$$
\begin{aligned}
\|D^\beta V_x^*(0)\| &\leqq \|D^\beta(V_x(0) - V_{1x}(0))\| + \|D^\beta V_x''(0)\| \\
&\leqq \alpha^2\rho^{|\beta|} + A_{11}(\|D^\beta\zeta_{xx}(0)\| + \|D^\beta\zeta_{xy}(0)\| + \|D^\beta\zeta_{xxx}(0)\| \\
&\qquad + \|D^\beta\zeta_{xyy}(0)\|) \leqq \alpha^2 A_{12}\rho^{|\beta|}.
\end{aligned}
$$

An estimate similar to (67) is then obtained as follows:

$$
\int_0^t \|D^\beta V_x^*\|_E^2\,ds \leqq \alpha^4 L_3^2\rho^{2|\beta|} + C_2\int_0^t (\|D^\beta V_{xx}^*\|_E^2 + \|D^\beta V_{xy}^*\|_E^2)\,ds.
$$

Since the last inequality is in the same form as (67) except that α in (67) is to be replaced by α^2 and V^* by V_x^*, using the same arguments to derive (65), we obtain

$$\int_0^t \|V_x^*\|_E^2\,ds \leqq \alpha^4 L_3^2/(1-r) \quad \text{for all } t \geqq 0. \tag{83}$$

Similarly,

$$\int_0^t \|V_y^*\|_E^2\,ds \leqq \alpha^4 L_3^2/(1-r) \quad \text{for all } t \geqq 0. \tag{84}$$

Furthermore, estimates similar to (80) can be obtained as follows:

$$\|\zeta_x^*(t)\|^2 \leqq C_7 \int_0^t \|V_x^*\|_E^2\,ds,$$

$$\|\zeta_{xx}(t)\|^2 - \|\zeta_{xx}(0)\|^2 \leqq C_{10} \int_0^t \|V_x^*\|_E^2\,ds,$$

$$\|\zeta_{yy}(t)\|^2 - \|\zeta_{yy}(0)\|^2 \leqq C_{10} \int_0^t \|V_y^*\|_E^2\,ds.$$

Hence from (40), (83) and (84), it follows that

$$\|\zeta_x^*(t)\| \leqq \alpha^2 C_{11} L_3/(1-r)^{1/2}, \tag{85}$$

$$\|\zeta_{xx}(t)\| \leqq \alpha^2 C_{12} L_3/(1-r)^{1/2}, \tag{86}$$

$$\|\zeta_{yy}(t)\| \leqq \alpha^2 C_{12} L_3/(1-r)^{1/2}, \tag{87}$$

for all $t \geqq 0$.

Estimates on $\|V^*(t)\|$ can be made as follows. By Schwarz inequality, Lemma 2, (52), (54) and (55), we arrive at

$$\begin{aligned} &\|V^*(t)\|^2 - \|V^*(0)\|^2 \\ &\leqq C_{13} \int_0^t \|V^*\|_E^2\,ds + A_{13} \int_0^t (|Q(\zeta)|^2 + |Q(\zeta_x)|^2 + |Q(\zeta_y)|^2)\,ds. \end{aligned} \tag{88}$$

Now by (40), (59), (66), (83), (84) and Lemma 2, we have

$$\begin{aligned} \int_0^t |Q(\zeta_x)|^2\,ds &\leqq \mu_1 \|\zeta_{xx}(0)\|^2 + \mu_2 \|\zeta_{xy}(0)\|^2 + \int_0^t ds \int |w_x^*|^2\,d\Gamma \\ &\leqq \alpha^2 \rho_0^2 C_{14} + 8 \int_0^t \|V_x^*\|_E^2\,ds \leqq \alpha^2 C_{15}, \end{aligned}$$

$$\int_0^t |Q(\zeta_y)|^2\,ds \leqq \alpha^2 C_{16}, \qquad \int_0^t |Q(\zeta)|^2 \leqq \alpha^2 C_{17}, \qquad \|V^*(0)\|^2 \leqq \alpha^2 A_7^2.$$

By the above estimates and (65), (88) then reduces to

$$\|V^*(t)\| \leqq \alpha C_{18}. \tag{89}$$

In exactly the same manner, we also have

$$\|V_x^*(t)\| \leqq \alpha^2 C_{19}, \tag{90}$$

$$\|V_y^*(t)\| \leqq \alpha^2 C_{20}. \tag{91}$$

Since

$$\|w^*(t)\|^2 = \int dx\, dy \int_{-1}^{0} dx_3 \left(\int_{-1}^{x_3} w_z^*\, dz\right)^2 \leqq \int (w_z^*)^2\, d\Omega \leqq 2(\|V_x^*\|^2 + \|V_y^*\|^2),$$

we obtain from (90) and (91) that

$$\|w^*(t)\| \leqq \alpha^2 C_{21} \quad \text{for all } t \geqq 0. \tag{92}$$

It follows from (79), (81), (82), (85) to (87), (89) and (92) that the proof of the lemma is completed.

Proof of Theorem 2. By Lemma 5, there exists a positive number ρ_0, which depends on R and θ only. If the initial data V_0 and ζ_0 satisfy the long wave condition (40) with $\rho < \rho_0$, then all the estimates in Lemma 6 hold. Let the L_2-norm $(\int u^2\, d\Omega)^{1/2}$ be denoted by $\|u\|$. From (41) and Lemma 6, we obtain

$$\|u - u_1\| \leqq \|\bar{u}_1\zeta^*\| + \|\bar{u}_2\zeta_x\| + \|u^*\| \leqq \alpha L_0,$$

$$\|v - v_1\| \leqq \|\bar{v}_{2y}\| + \|v^*\| \leqq \alpha L_0,$$

$$\|w - w_1\| \leqq \|\bar{w}_1\zeta_x^*\| + \|\bar{w}_{21}\zeta_{xx}\| + \|\bar{w}_{22}\zeta_{yy}\| + \|w^*\| \leqq \alpha^2 L_0,$$

and

$$\|\zeta - \zeta_1\| = \|\zeta^*\| \leqq \alpha L_0,$$

where $\bar{u}_1, \bar{u}_2, \bar{v}_2, \bar{w}_1, \bar{w}_{21}, \bar{w}_{22}$ are all bounded functions of z. Therefore, the first order approximation $\{V_1, \zeta_1\}$ is a uniform asymptotic approximation for all $t \geqq 0$ to the exact solution $\{V, \zeta\}$. Furthermore, if the perturbation parameter α in the asymptotic expansion is explicitly expressed, then we have the following:

$$u = u_1 + O(\alpha), \qquad v = O(\alpha), \qquad w = \alpha w_1 + O(\alpha^2), \qquad \zeta = \zeta_1 + O(\alpha),$$

for all $t \geqq 0$. This completes the proof of the theorem.

COROLLARY. *Theorem* 2 *also implies the following pointwise asymptotic approximation*

$$|D^\beta\zeta(t, x, y) - D^\beta\zeta_1(t, x, y)| \leqq \alpha M_0,$$

for all β in $0 \leqq |\beta| \leqq m$, where m is a nonnegative integer and M_0 is a positive constant depending only on R, θ and m.

Proof. The long wave condition (40) implies that

$$\|D^\beta D^\sigma\zeta_x(0)\| \leqq \alpha\rho^{|\sigma| + |\beta|} \leqq \alpha\rho_0^{|\sigma|}\rho^{|\beta|} = \alpha M\rho^{|\beta|},$$

$$\|D^\beta D^\sigma\zeta_y(0)\| \leqq \alpha M\rho^{|\beta|}, \qquad \|D^\beta D^\sigma(V(0) - V_1(0))\| \leqq \alpha M\rho^{|\beta|},$$

$$\|D^\beta D^\sigma\zeta_{xx}(0)\| \leqq \alpha^2 M\rho^{|\beta|}, \qquad \|D^\beta D^\sigma\zeta_{yy}(0)\| \leqq \alpha^2 M\rho^{|\beta|},$$

$$\|D^\beta D^\sigma(V_x(0) - V_{1x}(0))\| \leqq \alpha^2 M\rho^{|\beta|}, \qquad \|D^\beta D^\sigma(V_y(0) - V_{1y}(0))\| \leqq \alpha^2 M\rho^{|\beta|},$$

where $M = \rho_0^{|\sigma|}$, which depends on R, θ and $|\sigma|$ only. From the similarity between the above condition and the long wave condition (40), we see that all the estimates in Theorem 2 still hold if we replace V, ζ, V_1 and ζ_1 by $D^\sigma V$, $D^\sigma\zeta$, $D^\sigma V_1$ and $D^\sigma\zeta_1$, respectively. The positive constants acting as numerical upper bounds in those estimates now depend on R, θ and M. Therefore, according to Theorem 2, we obtain that

$$\|D^\sigma\zeta - D^\sigma\zeta_1\| = \|D^\sigma\zeta^*\| \leqq \alpha M_1,$$

where M_1 is a positive constant depending on R, θ and M only. According to the Sobolev inequality [10], for functions in $W_2^2(\Gamma)$, we have

$$\max_{\Gamma} |\zeta| \leqq C\|\zeta\|_{W_2^2(\Gamma)},$$

where C is a constant which does not depend on any other parameters. Hence, we obtain that, for any β in $0 \leqq |\beta| \leqq m$,

$$|D^\beta\zeta - D^\beta\zeta_1| \leqq C\|D^\beta(\zeta - \zeta_1)\|_{W_2^2(\Gamma)} = C\left(\sum_{|\sigma|=0}^{2} \|D^{\beta+\sigma}(\zeta - \zeta_1)\|^2\right)^{1/2} \leqq \alpha M_0,$$

where M_0 is a positive constant depending only on R, θ and $|m|$. This proves the corollary.

Remark 2. If $\theta = 0$, by starting from

$$u = \bar{u}_1\zeta_1 + \bar{u}_2\zeta_{1x} + u^*, \qquad v = \bar{v}_2\zeta_{1y} + v^*,$$
$$w = \bar{w}_1\zeta_{1x} + \bar{w}_{21}\zeta_{1xx} + \bar{w}_{22}\zeta_{1yy} + w^*, \qquad \zeta = \zeta_1 + \zeta^*,$$

and following the same ideas, it is easy to prove the following result.

If the initial data V_0 and ζ_0 satisfy the condition in Theorem 1 of § 3 for $m = 0$, that is, V_0 and ζ_0 are compatible with (4) and $V_0 \in E(\Omega) \cap W_2^2(\Omega)$, ζ_0, ζ_{0x}, $\zeta_{0xx} \in L_2(\Gamma)$, and if V_0 and ζ_0 have generalized derivatives with respect to x and y up to third order in $L_2(\Omega)$ and $L_2(\Gamma)$, respectively, such that

$$\|\zeta_x(0), \|\zeta_y(0)\|, \|V(0) - V_1(0)\| \leqq \alpha,$$
$$\|\zeta_{xx}(0)\|, \|\zeta_{yy}(0)\|, \|\zeta_{xxx}(0)\|,$$
$$\|\zeta_{yyy}(0)\|, \|V_x(0) - V_{1x}(0)\|, \|V_y(0) - V_{1y}(0)\| \leqq \alpha^2,$$

then for any $R > 0$, we have

$$\|u - u_1\| \leqq \alpha L_4, \quad \|v - v_1\| \leqq \alpha L_4, \quad \|w - w_1\| \leqq \alpha^2 L_4, \quad \|\zeta - \zeta_1\| \leqq \alpha L_4,$$

for all $t \leqq 0$.

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A THEORY OF FRACTIONAL INTEGRATION FOR GENERALIZED FUNCTIONS*

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Abstract. In this paper, we develop a theory of fractional integration for certain classes of generalized functions and give one simple application.

First, we introduce the appropriate spaces of testing-functions and generalized functions and state some of their basic properties. Next, we discuss the various operators of fractional integration including the Riemann–Liouville and Weyl fractional integrals and the Erdélyi–Kober operators. Use of analytic continuation enables us to obtain a precise description of the mapping properties of these operators relative to the testing-function spaces. We extend the operators to the generalized functions using adjoints and deduce the corresponding mapping properties using standard theorems. Finally, we solve a differential equation involving generalized functions using the previous theory.

The theory is much more general than that developed in Erdélyi and McBride [6].

1.1. Conventions. We begin by making certain conventions which will be adhered to throughout. Generalized functions will be denoted by letters such as f, g, etc., while testing-functions will be denoted by Greek letters such as ϕ, ψ, etc. The value assigned to a testing-function ϕ by a generalized function f will be denoted by (f, ϕ).

Our testing-functions will be complex-valued infinitely differentiable functions on the open interval $(0, \infty)$. The space of all such functions will be denoted by C^∞. For each p, $1 \leqq p < \infty$, L_p is the set of (measurable) functions ϕ for which

$$|\phi|_p = \left(\int_0^\infty |\phi(x)|^p \, dx\right)^{1/p} < \infty.$$

L^p will denote the set of equivalence classes of such functions which differ on a set of measure zero. L_∞ will denote the space of (measurable) functions ϕ for which

$$|\phi|_\infty = \text{essential supremum of } \phi \text{ over } (0, \infty)$$

is finite. L^∞ is the corresponding space of equivalence classes. The numbers p and q will always be related by

$$\frac{1}{p} + \frac{1}{q} = 1$$

and unless otherwise stated, $1 \leqq p \leqq \infty$.

For any $x \in (0, \infty)$ and complex number μ, x^μ means $\exp(\mu \log x)$ where $\log x$ is real.

Where any term is not defined explicitly, we shall use the terminology of Zemanian [16].

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1.2. Introduction. We shall be concerned with the following operators of fractional integration:

$$I^{\alpha}_{x^m}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_0^x (x^m - u^m)^{\alpha-1}u^{m-1}\phi(u)\,du, \tag{1.1}$$

$$K^{\alpha}_{x^m}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_x^{\infty} (u^m - x^m)^{\alpha-1}u^{m-1}\phi(u)\,du, \tag{1.2}$$

$$I^{\eta,\alpha}_{x^m}\phi(x) = x^{-m\eta-m\alpha}I^{\alpha}_{x^m}x^{m\eta}\phi(x), \tag{1.3}$$

$$K^{\eta,\alpha}_{x^m}\phi(x) = x^{m\eta}K^{\alpha}_{x^m}x^{-m\eta-m\alpha}\phi(x). \tag{1.4}$$

Here $m > 0$ is real, $\operatorname{Re}\alpha > 0$, η is a suitably restricted complex number and ϕ is defined on $(0, \infty)$. When $m = 1$, we obtain $I^{\alpha}_x\phi$ and $K^{\alpha}_x\phi$, which are respectively the Riemann–Liouville and Weyl integrals of order α of ϕ, while $I^{\eta,\alpha}_x$ and $K^{\eta,\alpha}_x$ are the Erdélyi–Kober operators [9].

Such operators arise in many situations, notably in connection with certain ordinary and partial differential equations (see, for instance, [3], [4] and [11]), integral transforms ([2], [10] and [14]) and dual and triple integral equations ([1] and [7]). On the other hand, the theory of generalized functions, or distributions, has led to great advances in the theory of differential equations ([8], [15]) and elsewhere. In this paper, we combine these two methods in developing a theory of fractional integration for a class of generalized functions.

It is possible to develop a theory for I^{α}_x and K^{α}_x based on the concept of the convolution of distributions [8], but this cannot be extended to the more general operators above. Instead, we pursue an approach based on adjoint operators. In [6], a space $\mathscr{I}$ of testing-functions was introduced such that (under suitable restrictions on the parameters) $K^{\eta,\alpha}_{x^m}$ is an automorphism of $\mathscr{I}$ and $I^{\eta,\alpha}_{x^m}$ is an automorphism of the generalized function space $\mathscr{I}'$. In this paper, we introduce classes $F'_{p,\mu}$ of generalized functions, relative to which the mapping properties of all four operators above can be obtained. The theory is much more general than that in [6] and also more flexible, since other operations such as differentiation and multiplication by arbitrary powers of x are easily handled.

In § 2, we study the spaces $F_{p,\mu}$ proceeding via the spaces $F_p \equiv F_{p,0}$. Certain simple operators are also discussed. The results are then extended to $F'_{p,\mu}$, and, in addition, we obtain a structure theorem for $F'_{p,\mu}$ in the case $p < \infty$.

Section 3 is devoted to a detailed study of the operators of fractional integration on $F_{p,\mu}$ and $F'_{p,\mu}$. It appears easier to obtain results for $I^{\eta,\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ first, deducing properties of $I^{\alpha}_{x^m}$ and $K^{\alpha}_{x^m}$, rather than to proceed in the opposite direction. The whole theory depends on the work of Kober in [9].

As indicated above, we would expect to obtain applications of the theory to generalized integral transforms (notably the Hankel transform) and to integral equations. These we hope to discuss in future papers, and we refer the interested reader to the author's thesis [13]. Here we content ourselves with just one application. In § 4, we discuss relations between fractional integration and the operator

$$L_\nu \equiv \frac{d^2}{dx^2} + \frac{2\nu+1}{x}\frac{d}{dx}. \tag{1.5}$$

Formulas are given for the solution of

$$L_\nu f = g,$$

where f and g are generalized functions. Again, the results are much more general than those in [6].

2.1. The testing-function spaces F_p. For each p, $1 \leqq p \leqq \infty$, we define F_p by

$$F_p = \left\{\phi : \phi \in C^\infty \text{ and } x^k \frac{d^k \phi}{dx^k} \in L_p\,(k = 0, 1, 2, \cdots)\right\}. \tag{2.1}$$

With the usual pointwise operations of addition and scalar multiplication, F_p becomes a complex linear space. For $\phi \in F_p$, $k = 0, 1, 2, \cdots$, define γ_k^p by

$$\gamma_k^p(\phi) = \left| x^k \frac{d^k \phi}{dx^k} \right|_p . \tag{2.2}$$

The collection

$$M_p = \{\gamma_k^p : k = 0, 1, 2, \cdots\} \tag{2.3}$$

is a countable multinorm and, with the topology generated by M_p, F_p becomes a countably multinormed space. We define convergent sequences and Cauchy (or fundamental) sequences as in [16, § 1.6]. As usual, every convergent sequence is a fundamental sequence, but the converse is also true, i.e., F_p is complete.

THEOREM 2.1. *For $1 \leqq p \leqq \infty$, F_p is a complete countably multinormed space (and hence a Fréchet space).*

Proof. Define an operator δ on F_p by

$$\begin{aligned} (\delta\phi)(x) &= x\frac{d\phi}{dx}, \\ \delta &\equiv x\frac{d}{dx}. \end{aligned} \tag{2.4}$$

Since $x^k(d^k\phi/dx^k) \in L_p$, $k = 0, 1, 2, \cdots$, $\Leftrightarrow \delta^k\phi \in L_p$, $k = 0, 1, 2, \cdots$, we may rewrite (2.1) as

$$F_p = \{\phi : \phi \in C^\infty \text{ and } \delta^k\phi \in L_p\,(k = 0, 1, 2, \cdots)\}. \tag{2.5}$$

The proof is completed by an argument analogous to that in [16, pp. 253–4] using Hölder's inequality rather than Schwarz's inequality at the appropriate stage.

It can be shown similarly that F_p is a testing-function space in the sense of [16, p. 39], and we will call the elements of F_p *testing-functions.*

We conclude this section with an easy lemma which will be used frequently.

LEMMA 2.2. *$\phi \in F_p \Rightarrow x^{1/p}\phi(x)$ is bounded on* $(0, \infty)$, $1 \leqq p \leqq \infty$.

Proof. It is sufficient to consider the case when $\phi(x)$ is real-valued. Suppose first that $1 \leqq p < \infty$. Choose a, b with $0 < a < b < \infty$. Integrating by parts, we have

$$\int_a^b x\phi'(x)\{\phi(x)\}^{p-1}\,dx = \frac{1}{p}[x\{\phi(x)\}^p]_a^b - \frac{1}{p}\int_a^b \{\phi(x)\}^p\,dx.$$

Now $\phi \in F_p \Rightarrow x\phi'(x) \in L_p$. Also $\{\phi(x)\}^{p-1} \in L_q$ so, by Hölder's inequality, the left-hand side is bounded as $a \to 0+$ or $b \to \infty$. Since the same is true of the integral on the right, the result follows in this case.

The case $p = \infty$ is trivial since then $x^{1/p}\phi(x) = \phi(x)$ is essentially bounded and continuous and hence bounded on $(0, \infty)$.

2.2. The generalized function spaces F'_p. A functional f on F_p is (sequentially) continuous if, whenever ϕ_n converges to ϕ in the topology of F_p, $(f, \phi_n) \to (f, \phi)$ as $n \to \infty$. F'_p will denote the complex linear space of continuous linear functionals on F_p with the usual operations of addition and scalar multiplication. We assign to F'_p the topology of weak (or pointwise) convergence. From Theorem 2.1 and also [16, Thm. 1.8-3] we have the following.

THEOREM 2.3. *F'_p is complete*, $1 \leqq p \leqq \infty$.

Any function $f \in L_q$ generates an element $\tilde{f} \in F'_p$ by means of the formula

$$(\tilde{f}, \phi) = \int_0^\infty f(x)\phi(x)\,dx, \qquad \phi \in F_p. \tag{2.6}$$

Generalized functions with an integral representation of this form will be called *regular*; those with no such representation will be called *singular*. An example of a singular element of F'_p is provided by δ_a, $a > 0$, defined by

$$(\delta_a, \phi) = \phi(a), \qquad \phi \in F_p.$$

We shall use regular functionals to motivate the definition of various operators on F'_p in the sequel.

It is interesting to compare the spaces F'_p with other spaces of generalized functions, in particular with $\mathscr{D}'$, the distributions on $(0, \infty)$, and $\mathscr{E}'$, the distributions on $(0, \infty)$ with compact support. For the theory of $\mathscr{D}(= \mathscr{D}(0, \infty))$, $\mathscr{D}'$, $\mathscr{E}(= \mathscr{E}(0, \infty))$ and $\mathscr{E}'$, see [16]. It is clear that for each p, $1 \leqq p \leqq \infty$,

$$\mathscr{D} \subset F_p \subset \mathscr{E},$$

both inclusions being strict. Further, since $\mathscr{D}$ is dense in $\mathscr{E}$, [16, p. 37], F_p is dense in $\mathscr{E}$. Also it can be shown that if $1 \leqq p < \infty$, $\mathscr{D}$ is dense in F_p; the proof, which is rather intricate, is omitted. However, $\mathscr{D}$ is *not* dense in F_∞; for instance, we cannot approximate a (nonzero) constant function in the F_∞-topology by functions with compact support.

Now suppose $1 \leqq p < \infty$. Let $\{\phi_n\}$ converge to ϕ in $\mathscr{D}$ (i.e., in the topology of $\mathscr{D}$). The supports of ϕ and ϕ_n, $n = 1, 2, \cdots$, are all contained in some closed interval $[a, b]$ with $0 < a < b < \infty$, so that

$$\gamma_k^p(\phi_n - \phi) = \left\{\int_a^b \left| x^k \frac{d^k}{dx^k}(\phi_n - \phi)\right|^p dx\right\}^{1/p}$$
$$\leqq b^k(b-a)^{1/p} \sup_{a \leqq x \leqq b} \left|\frac{d^k}{dx^k}(\phi_n - \phi)\right|$$
$$\to 0 \qquad \text{as } n \to \infty,$$

by definition of convergence in $\mathscr{D}$. Hence

$$\text{convergence in } \mathscr{D} \Rightarrow \text{convergence in } F_p, \qquad 1 \leqq p < \infty,$$

and hence, $F'_p \subset \mathscr{D}'$. We can also show that

$$\text{convergence in } \mathscr{D} \Rightarrow \text{convergence in } F_\infty,$$

but since $\mathscr{D}$ is not dense in F_∞, we cannot deduce that $F'_\infty \subset \mathscr{D}'$. In the other direction, however, we can show that for $1 \leqq p \leqq \infty$, $\mathscr{E}' \subset F'_p$. In summary, we have Theorem 2.4.

THEOREM 2.4. $\mathscr{E}' \subset F'_p$, $1 \leqq p \leqq \infty$, *and* $F'_p \subset \mathscr{D}'$, $1 \leqq p < \infty$.

After we have defined generalized differentiation below, we will be able to prove a structure theorem for the elements of F'_p, $p < \infty$.

2.3. The spaces $F_{p,\mu}$ and $F'_{p,\mu}$. In order to be able to consider certain operations such as multiplication by arbitrary powers of x and differentiation, we must introduce generalizations of the spaces F_p and F'_p. For any complex number μ and $1 \leqq p \leqq \infty$, we define $F_{p,\mu}$ by

$$F_{p,\mu} = \{\phi : x^{-\mu}\phi(x) \in F_p\}. \tag{2.7}$$

$F_{p,\mu}$ is given the topology generated by the multinorm

$$M_{p,\mu} = \{\gamma_k^{p,\mu} : k = 0, 1, 2, \cdots\}, \tag{2.8}$$

where, for $\phi \in F_{p,\mu}$,

$$\gamma_k^{p,\mu}(\phi) = \gamma_k^p(x^{-\mu}\phi), \tag{2.9}$$

where γ_k^p is given by (2.2). It follows that the mapping $\phi \to x^\mu\phi$ is an isomorphism of F_p onto $F_{p,\mu}$. From Theorems 2.1 and 2.3 we immediately have Theorem 2.5.

THEOREM 2.5. *For each complex number μ and $1 \leqq p \leqq \infty$, $F_{p,\mu}$ is a Fréchet space and $F'_{p,\mu}$ is complete.*

Note in passing that we will continue to write

$$F_p \equiv F_{p,0}. \tag{2.10}$$

For each complex number λ, we define the operator x^λ on $F_{p,\mu}$ by

$$(x^\lambda\phi)(x) = x^\lambda\phi(x), \qquad 0 < x < \infty. \tag{2.11}$$

No confusion should arise from using the same symbol for the function x^λ and the operation of multiplying by this function. We define δ' on $F_{p,\mu}$ by

$$(\delta'\phi)(x) = \frac{d}{dx}(x\phi), \tag{2.12}$$

while for $m > 0$ we shall write

$$D_m \equiv \frac{d}{dx^m}, \qquad D_1 = D.$$

Note that

$$\delta' = \delta + I,$$

where I is the identity operator and δ is defined by (2.4). It is easy to prove the next theorem.

THEOREM 2.6. *Let λ, μ be complex numbers and $1 \leqq p \leqq \infty$.*

(i) *x^λ is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu+\lambda}$ with inverse $x^{-\lambda}$.*

(ii) *δ, δ' are continuous linear mappings of $F_{p,\mu}$ into itself.*

(iii) *D_m is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-m}$.*

To define the corresponding operators on $F'_{p,\mu}$, we use adjoint operators. For $f \in F'_{p,\mu}$ we define $x^\lambda f$, δf, $\delta' f$ and $D_m f$ by

$$(x^\lambda f, \phi) = (f, x^\lambda \phi), \qquad \phi \in F_{p,\mu-\lambda}, \tag{2.13}$$

$$(\delta f, \phi) = (f, -\delta'\phi), \qquad \phi \in F_{p,\mu}, \tag{2.14}$$

$$(\delta' f, \phi) = (f, -\delta\phi), \qquad \phi \in F_{p,\mu}, \tag{2.15}$$

$$(D_m f, \phi) = \left(f, -\frac{1}{m} D x^{-m+1}\phi\right), \qquad \phi \in F_{p,\mu+m}. \tag{2.16}$$

The motivation for (2.14)–(2.16) is supplied by taking f to be a regular functional, $\phi \in \mathscr{D}$ and integrating by parts. Using Theorem 2.6 and [16, Thm. 1.10-1] we immediately obtain the following.

THEOREM 2.7. *Let λ, μ be complex numbers and $1 \leqq p \leqq \infty$.*

(i) *x^λ is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-\lambda}$ with inverse $x^{-\lambda}$.*

(ii) *δ, δ' are continuous linear mappings of $F'_{p,\mu}$ into itself.*

(iii) *D_m is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+m}$.*

We conclude this section with the following structure theorem.

THEOREM 2.8. *Let μ be any complex number and $1 \leqq p < \infty$. Any $f \in F'_{p,\mu}$ is of the form*

$$f = x^{-\mu} \sum_{k=0}^{r} x^k D^k \tilde{h}_k, \tag{2.17}$$

where r is a positive integer, $h_k \in L_q$, $k = 0, 1, \cdots, r$, and $\tilde{h}_k$ is defined as in (2.6).

Proof. The proof is analogous to a number of proofs in the literature and is omitted. (See, for instance, [15, pp. 272–274].)

3.1. The operators $I_{x^m}^{\eta;\alpha}$ on $F_{p,\mu}$. We are now ready to discuss the mapping properties of the operators (1.1)–(1.4) of fractional integration. In this section we study $I^{\alpha}_{x^m}$ and $I^{\eta,\alpha}_{x^m}$ and the corresponding results for $K^{\alpha}_{x^m}$ and $K^{\eta,\alpha}_{x^m}$ are given in the next section.

It is convenient to begin with $I^{\eta,\alpha}_{x^m}$ since (under suitable conditions) it maps L_p into L_p whereas $I^{\alpha}_{x^m}$ does not. The mapping properties of $I^{\alpha}_{x^m}$ can be derived using the relation

$$I^{\alpha}_{x^m}\phi(x) = x^{m\alpha}I^{0,\alpha}_{x^m}\phi(x). \tag{3.1}$$

This approach has the slight disadvantage that some of the more obvious results such as

$$I^{\alpha}_{x^m}\phi = \frac{d}{dx^m}I^{\alpha+1}_{x^m}\phi \tag{3.2}$$

appear much later than usual.

LEMMA 3.1. *For* $1 \leqq p \leqq \infty$, $\operatorname{Re}\alpha > 0$, $I^{\eta,\alpha}_{x^m}$ *is a continuous linear mapping of* L_p *into* L_p *provided* $m\operatorname{Re}\eta + m > 1/p$.

Proof. The result for $m = 1$ is proved by Kober in [9, Thm. 2]. The general result follows by a simple change of variable.

This leads to Theorem 3.2.

THEOREM 3.2. *For* $1 \leqq p \leqq \infty$, $\operatorname{Re}\alpha > 0$, $I^{\eta,\alpha}_{x^m}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu}$ *provided* $\operatorname{Re}(m\eta + \mu) + m > 1/p$.

Proof. Suppose first that $\mu = 0$. Since F_p is a subspace of L_p, Lemma 3.1 shows that $I^{\eta,\alpha}_{x^m}$ is a continuous linear mapping of F_p into L_p. From (1.1) and (1.3),

$$I^{\eta,\alpha}_{x^m}\phi(x) = \frac{m}{\Gamma(\alpha)}\int_0^1 (1 - t^m)^{\alpha-1}t^{m\eta+m-1}\phi(xt)\,dt. \tag{3.3}$$

Differentiating under the integral sign in (3.3) gives

$$\delta I^{\eta,\alpha}_{x^m}\phi = I^{\eta,\alpha}_{x^m}\,\delta\phi,$$

from which it follows by induction that for $k = 0, 1, 2, \cdots$,

$$x^k\frac{d^k}{dx^k}I^{\eta,\alpha}_{x^m}\phi = I^{\eta,\alpha}_{x^m}x^k\frac{d^k\phi}{dx^k}. \tag{3.4}$$

By Lemma 3.1, for some number M (depending only on η, α and m),

$$\gamma^p_k(I^{\eta,\alpha}_{x^m}\phi) \leqq M\gamma^p_k(\phi),$$

and the theorem is proved for $\mu = 0$.

The general result follows from the previous case using the relation

$$I^{\eta,\alpha}_{x^m}\phi(x) = x^{\mu}I^{\eta+(\mu/m),\alpha}_{x^m}x^{-\mu}\phi, \qquad \phi \in F_{p,\mu}$$

and Theorem 2.6 (i).

We shall, in fact, prove much more about $I^{\eta,\alpha}_{x^m}$ shortly. One result we shall need is

$$I^{\eta+\alpha,\beta}_{x^m}I^{\eta,\alpha}_{x^m}\phi = I^{\eta,\alpha+\beta}_{x^m}\phi, \qquad \phi \in F_{p,\mu}, \tag{3.5}$$

valid provided $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}(m\eta + \mu) + m > 1/p$. Theorem 3.2 involves restrictions on η and α. That the restriction $\operatorname{Re}(m\eta + \mu) + m > 1/p$ is necessary is seen by taking $p = \infty$, $\phi(x) = x^\mu$, whence

$$I_{x^m}^{\eta;\alpha}\phi(x) = \frac{\Gamma(\eta + (\mu/m) + 1)}{\Gamma(\alpha + \eta + (\mu/m) + 1)} x^\mu$$

which belongs to $F_{\infty,\mu}$ provided $\operatorname{Re}(\eta + (\mu/m) + 1) > 0$. On the other hand, we now proceed to remove the restriction $\operatorname{Re}\alpha > 0$ using analytic continuation. We make the following definition.

DEFINITION. Let V_1, V_2 be two countably multinormed spaces. Suppose that to each α in some domain D of the complex plane there corresponds a continuous linear mapping T_α from V_1 to V_2. We shall say that T_α is *analytic with respect to* α *in* D if there exists a continuous linear mapping $\partial T_\alpha/\partial\alpha$ of V_1 into V_2 such that, for each fixed $\phi \in V_1$,

$$\frac{1}{h}[T_{\alpha+h}\phi - T_\alpha\phi] - \frac{\partial T_\alpha}{\partial \alpha}\phi$$

converges to zero in the topology of V_2 as the (complex) increment $h \to 0$ in any manner.

It is easy to show that if $f(\alpha)$ is an analytic function of α in D (in the usual sense) and T_α is analytic in D (in the sense of the above definition), then the operator $f(\alpha)T_\alpha$ is analytic in D.

THEOREM 3.3. *On* $F_{p,\mu}$, $I_{x^m}^{\eta;\alpha}$ *is analytic with respect to* α *for* $\operatorname{Re}\alpha > 0$, *provided that* $\operatorname{Re}(m\eta + \mu) + m > 1/p$.

Proof. See [13].

Notes. 1. It is clear that, under the hypotheses of Theorem 3.3, $I_{x^m}^{\eta;\alpha}\phi(x)$ is, for each fixed x, an analytic function of α in the usual sense for $\operatorname{Re}\alpha > 0$.

2. A similar argument shows that, for each fixed α with $\operatorname{Re}\alpha > 0$, $I_{x^m}^{\eta;\alpha}$ is analytic on $F_{p,\mu}$ with respect to η in the half-plane $\operatorname{Re}\eta > (1/m)((1/p) - m - \operatorname{Re}\mu)$.

We shall be concerned with analytic continuation with respect to α. We require the following lemma which can be proved by straightforward differentiation.

LEMMA 3.4. *Let* $\operatorname{Re}\alpha > 0$, $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta + \mu) + m > 1/p$. *Then*

$$\delta I_{x^m}^{\eta;\alpha+1}\phi = I_{x^m}^{\eta;\alpha+1}\,\delta\phi = mI_{x^m}^{\eta;\alpha}\phi - (m\eta + m\alpha + m)I_{x^m}^{\eta;\alpha+1}\phi.$$

Rearranging the result of Lemma 3.4 gives

$$mI_{x^m}^{\eta;\alpha}\phi = (m\eta + m\alpha + m)I_{x^m}^{\eta;\alpha+1}\phi + I_{x^m}^{\eta;\alpha+1}\,\delta\phi. \tag{3.6}$$

By Theorem 3.3 and the remark following our definition above, the right-hand side is analytic with respect to α for $\operatorname{Re}\alpha > -1$. We use (3.6) to continue $I_{x^m}^{\eta;\alpha}$ analytically, in the first instance to $-1 < \operatorname{Re}\alpha \leqq 0$ and hence, step by step, to the whole complex α-plane.

Still assuming $\operatorname{Re}(m\eta + \mu) + m > 1/p$, we may put $\alpha = 0$ in (3.6) to obtain

$$I_{x^m}^{\eta;0}\phi = \phi. \tag{3.7}$$

We can now prove our first main result.

THEOREM 3.5. *Let* $\operatorname{Re}(m\eta + \mu) + m > 1/p$, $1 \leqq p \leqq \infty$.

(i) *For any complex number* α, $I_{x^m}^{\eta;\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into itself.*

(ii) *For fixed* η, $I_{x^m}^{\eta,\alpha}$ *is entire with respect to* α *on* $F_{p,\mu}$.

(iii) *If, in addition,* $\mathrm{Re}\,(m\eta + m\alpha + \mu) + m > 1/p$, $I_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and*

$$(I_{x^m}^{\eta,\alpha})^{-1} = I_{x^m}^{\eta+\alpha,-\alpha}.$$

Proof. Parts (i) and (ii) follow using Theorems 3.2 and 3.3 along with sufficiently many applications of formula (3.6). We now prove (iii).

By analytic continuation, (3.5) is valid provided only $\mathrm{Re}\,(m\eta + \mu) + m > 1/p$ and $\mathrm{Re}\,(m\eta + m\alpha + \mu) + m > 1/p$. (This second condition was redundant before with $\mathrm{Re}\,\alpha > 0$). In this case for $\phi \in F_{p,\mu}$,

$$I_{x^m}^{\eta+\alpha,-\alpha} I_{x^m}^{\eta,\alpha}\phi = I_{x^m}^{\eta,0}\phi = \phi$$

by (3.7) and

$$I_{x^m}^{\eta,\alpha} I_{x^m}^{\eta+\alpha,-\alpha}\phi = I_{x^m}^{\eta+\alpha,0}\phi = \phi$$

by (3.7). The result follows.

Finally in this section, we state the mapping properties of $I_{x^m}^{\alpha}$ For $\mathrm{Re}\,\alpha > 0$, we have, from (1.1) and (1.3),

$$I_{x^m}^{\alpha}\phi(x) = x^{m\alpha} I_{x^m}^{0,\alpha}\phi(x). \tag{3.8}$$

We use (3.8) to define $I_{x^m}^{\alpha}$ for all α, this definition coinciding with (1.1) for $\mathrm{Re}\,\alpha > 0$. By Theorems 2.6 (i) and 3.2, $I_{x^m}^{\alpha}$ is a contintinuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu+m\alpha}$ provided $\mathrm{Re}\,\mu + m > 1/p$. In this case also we can prove

$$I_{x^m}^{\alpha}\phi(x) = \frac{d}{dx^m} I_{x^m}^{\alpha+1}\phi(x). \tag{3.9}$$

If we had developed the theory of $I_{x^m}^{\alpha}$ without proceeding via $I_{x^m}^{\eta,\alpha}$, we would use (3.9) to continue $I_{x^m}^{\alpha}$ analytically from $\mathrm{Re}\,\alpha > 0$ to the whole complex α-plane.

Still assuming $\mathrm{Re}\,\mu + m > 1/p$, we have from (3.7) and (3.8) that, for $\phi \in F_{p,\mu}$,

$$I_{x^m}^{0}\,\phi = \phi.$$

It follows from (3.9) that, for $n = 0, 1, 2, \cdots$,

$$I_{x^m}^{-n}\phi = \left(\frac{d}{dx^m}\right)^n \phi, \tag{3.10}$$

as might be expected.

It can also be proved using (3.5) and (3.8) that for $\phi \in F_{p,\mu}$, $(1/p) - m - \mathrm{Re}\,\mu < \min(0, m\,\mathrm{Re}\,\alpha, m\,\mathrm{Re}\,\beta)$,

$$I_{x^m}^{\alpha}\, I_{x^m}^{\beta}\phi = I_{x^m}^{\alpha+\beta}\phi = I_{x^m}^{\beta} I_{x^m}^{\alpha}\phi. \tag{3.11}$$

This leads to Theorem 3.6.

THEOREM 3.6. $I_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$ *provided* $\mathrm{Re}\,\mu + m > 1/p$. $I_{x^m}^{0}$ *is the identity operator. If, in addition,* $\mathrm{Re}\,(\mu + m\alpha) + m > 1/p$, $I_{x^m}^{\alpha}$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(I_{x^m}^{\alpha})^{-1} = I_{x^m}^{-\alpha}.$$

Equation (3.11) enables us to write down an explicit expression for $I_{x^m}^{\alpha}$ for any α; if $\phi \in F_{p,\mu}$, $\operatorname{Re}\mu + m > 1/p$, $\operatorname{Re}\alpha + n > 0$, then

$$I_{x^m}^{\alpha}\phi(x) = \frac{m}{\Gamma(\alpha + n)}\left(\frac{d}{dx^m}\right)^n \int_0^x (x^m - u^m)^{\alpha+n-1}u^{m-1}\phi(u)\,du. \tag{3.12}$$

We mention also the second index law for the operators $I_{x^m}^{\alpha}$; if $\phi \in F_{p,\mu}$, $-\operatorname{Re}\mu - m + (1/p) < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$, then

$$x^{m\alpha}I_{x^m}^{\beta}x^{m\gamma}\phi(x) = I_{x^m}^{-\gamma}x^{-m\beta}I_{x^m}^{-\alpha}\phi(x). \tag{3.13}$$

We shall not prove (3.13) here, but defer the proof to a subsequent paper where the result arises naturally in connection with hypergeometric integral equations. Equations (3.11) and (3.13) have been studied in the case $m = 1$ by Love [12] for ordinary functions and by Erdélyi [5] for a class of generalized functions.

3.2. The operators $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$. We now consider the operators $K_{x^m}^{\eta,\alpha}$ on $F_{p,\mu}$. For $\operatorname{Re}\alpha > 0$, we have from (1.2) and (1.4) that

$$\begin{aligned} K_{x^m}^{\eta,\alpha}\phi(x) &= x^{m\eta}K_{x^m}^{\alpha}x^{-m\eta-m\alpha}\phi(x) \\ &= \frac{m}{\Gamma(\alpha)}\int_1^{\infty} (t^m - 1)^{\alpha-1}t^{-m\eta-m\alpha+m-1}\phi(xt)\,dt. \end{aligned} \tag{3.14}$$

We obtain the properties of $K_{x^m}^{\eta,\alpha}$ using arguments similar to those for $I_{x^m}^{\eta,\alpha}$. We shall mention only the salient points.

THEOREM 3.7. *Let* $\operatorname{Re}(m\eta - \mu) > -1/p$, $1 \leqq p \leqq \infty$.

(i) *For any complex number* α, $K_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into itself.*

(ii) *For fixed* η, $K_{x^m}^{\eta,\alpha}$ *is entire with respect to* α *on* $F_{p,\mu}$.

(iii) *If, in addition,* $\operatorname{Re}(m\eta + m\alpha - \mu) > -1/p$, $K_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F_{p,\mu}$ *and*

$$(K_{x^m}^{\eta,\alpha})^{-1} = K_{x^m}^{\eta+\alpha,-\alpha}.$$

Proof. (i) For $\operatorname{Re}\alpha > 0$, the result follows using a result of Kober [9] and differentiating under the integral sign in (3.14). We extend the definition of $K_{x^m}^{\eta,\alpha}$ to $\operatorname{Re}\alpha \leqq 0$ using the formula

$$mK_{x^m}^{\eta,\alpha}\phi(x) = (m\eta + m\alpha)K_{x^m}^{\eta,\alpha+1}\phi(x) - K_{x^m}^{\eta,\alpha+1}\,\delta\phi(x), \tag{3.15}$$

which is an analogue of (3.6) and is valid for $\phi \in F_{p,\mu}$ if $\operatorname{Re}(m\eta - \mu) > -1/p$. Use of (3.15) completes the proof of (i).

As regards (ii) and (iii), we proceed as for $I_{x^m}^{\eta,\alpha}$ using (3.15) and the additional results

$$K_{x^m}^{\eta,0}\phi = \phi, \tag{3.16}$$

valid for $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta - \mu) > -1/p$, and

$$K_{x^m}^{\eta,\alpha}K_{x^m}^{\eta+\alpha,\beta}\phi = K_{x^m}^{\eta,\alpha+\beta}\phi, \tag{3.17}$$

valid when $\phi \in F_{p,\mu}$, $\operatorname{Re}(m\eta - \mu) > -1/p$ and $\operatorname{Re}(m\eta + m\alpha - \mu) > -1/p$.

To obtain the results for $K_{x^m}^{\alpha}$, we note first that for $\operatorname{Re}\alpha > 0$,

$$K_{x^m}^{\alpha}\phi = K_{x^m}^{0,\alpha}x^{m\alpha}\phi \tag{3.18}$$

from (1.2) and (1.4). We use (3.18) to define $K_{x^m}^{\alpha}$ for all α. Using Theorem 3.7 gives the following.

THEOREM 3.8. *If* $\operatorname{Re}(\mu + m\alpha) < 1/p$, $K_{x^m}^{\alpha}$ *is a continuous linear mapping of* $F_{p,\mu}$ *into* $F_{p,\mu+m\alpha}$. *If also* $\operatorname{Re}\mu < 1/p$, $K_{x^m}^{\alpha}$ *is an isomorphism of* $F_{p,\mu}$ *onto* $F_{p,\mu+m\alpha}$ *and*

$$(K_{x^m}^{\alpha})^{-1} = K_{x^m}^{-\alpha}.$$

$K_{x^m}^{0}$ *is the identity operator on* $F_{p,\mu}$ *if* $\operatorname{Re}\mu < 1/p$.

Using (3.15), we can show that, for $\operatorname{Re}(\mu + m\alpha) < 1/p$,

$$K_{x^m}^{\alpha}\phi(x) = -K_{x^m}^{\alpha+1}\frac{d}{dx^m}\phi(x), \qquad \phi \in F_{p,\mu}, \tag{3.19}$$

from which it follows by induction that if $\operatorname{Re}\mu - mn < 1/p$, $n = 0, 1, 2, \cdots$,

$$K_{x^m}^{-n}\phi = \left(-\frac{d}{dx^m}\right)^n\phi. \tag{3.20}$$

The first index law for the operators $K_{x^m}^{\alpha}$ is

$$K_{x^m}^{\alpha}K_{x^m}^{\beta}\phi = K_{x^m}^{\alpha+\beta}\phi = K_{x^m}^{\beta}K_{x^m}^{\alpha}\phi, \tag{3.21}$$

valid when $(1/p) - \operatorname{Re}\mu > \max(m\operatorname{Re}\alpha, m\operatorname{Re}\beta, m\operatorname{Re}(\alpha+\beta))$. The second index law states that for $\phi \in F_{p,\mu}$, $\operatorname{Re}\mu - (1/p) < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$,

$$x^{m\gamma}K_{x^m}^{\beta}x^{m\alpha}\phi = K_{x^m}^{-\alpha}x^{-m\beta}K_{x^m}^{-\gamma}\phi. \tag{3.22}$$

For discussion of (3.21) and (3.22) we again refer the reader to [5] and [12].

3.3. The action of $I_{x^m}^{\eta,\alpha}$ and $K_{x^m}^{\eta,\alpha}$ on $F'_{p,\mu}$. We are now ready to develop the theory of fractional integration on the spaces $F'_{p,\mu}$ of generalized functions. As usual, our definitions are motivated by considering regular functionals.

Let $f \in F'_{p,\mu}$. From adjoint considerations we are led to define $I_{x^m}^{\eta,\alpha}f$, for $\operatorname{Re}\alpha > 0$, by

$$(I_{x^m}^{\eta,\alpha}f, \phi) = (f, K_{x^m}^{\eta+1-(1/m),\alpha}\phi), \tag{3.23}$$

where $\phi \in F_{p,\mu}$. However, the right-hand side is meaningful provided only $\operatorname{Re}(m\eta - \mu) + m > 1/q$ by Theorem 3.7; in this case, we can remove the restriction $\operatorname{Re}\alpha > 0$ and use (3.23) to define $I_{x^m}^{\eta,\alpha}f$ for all complex α.

THEOREM 3.9. *Let* $1 \leqq p \leqq \infty$ *and let* α *be any complex number.*

(i) $I_{x^m}^{\eta,\alpha}$ *is a continuous linear mapping of* $F'_{p,\mu}$ *into* $F'_{p,\mu}$ *provided that* $\operatorname{Re}(m\eta - \mu) + m > 1/q$.

(ii) *If, in addition,* $\operatorname{Re}(m\eta + m\alpha - \mu) + m > 1/q$, $I_{x^m}^{\eta,\alpha}$ *is an automorphism of* $F'_{p,\mu}$ *and*

$$(I_{x^m}^{\eta,\alpha})^{-1} = I_{x^m}^{\eta+\alpha,-\alpha}.$$

Proof. By Theorem 3.7 (i), $K_{x^m}^{\eta+1-(1/m),\alpha}$ is a continuous linear mapping of $F_{p,\mu}$ into itself provided $\operatorname{Re} m(\eta + 1 - (1/m)) - \mu > -1/p$, i.e., $\operatorname{Re}(m\eta - \mu)$

$+ m > 1/q$. Part (i) now follows by [16, Thm. 1.10-1]. Part (ii) follows similarly using Theorem 3.7 (iii) in conjunction with [16, Thm. 1.10-2].

Using (3.17) and (3.23) we see that if $f \in F'_{p,\mu}$,

$$I_{x^m}^{\eta+\alpha,\beta} I_{x^m}^{\eta,\alpha} f = I_{x^m}^{\eta,\alpha+\beta} f \tag{3.24}$$

provided $\operatorname{Re}(m\eta - \mu) + m > 1/q$, $\operatorname{Re}(m\eta + m\alpha - \mu) + m > 1/q$, while from (3.16),

$$I_{x^m}^{\eta,0} f = f \tag{3.25}$$

provided $\operatorname{Re}(m\eta - \mu) + m > 1/q$. Equations (3.24) and (3.25) are analogous to (3.5) and (3.7), respectively.

It is now clear that to obtain results for $F'_{p,\mu}$ from the corresponding results for $F_{p,\mu}$ (e.g., to obtain Theorem 3.9 from Theorem 3.5) we interchange μ and $-\mu$, p and q in the restrictions on the parameters. This trend, which continues below, is to be expected from consideration of Hölder's inequality. If $\phi \in F_{p,\mu}$, $\int_0^\infty f(x)\phi(x)\,dx$ will converge if $f(x) = x^{-\mu} g(x)$ with $g \in L_q$ and, in particular, if $f \in F_{q,-\mu}$.

We note in passing that for fixed $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu}$ and $\operatorname{Re}(m\eta - \mu) + m > 1/q$,

$$(I_{x^m}^{\eta,\alpha} f, \phi)$$

is an entire function of α by virtue of Theorem 3.7 (ii). However, this will not be needed here.

Proceeding as before, we are led to define $K_{x^m}^{\eta,\alpha}$ for any α and any $f \in F'_{p,\mu}$ by

$$(K_{x^m}^{\eta,\alpha} f, \phi) = (f, I_{x^m}^{\eta-1+(1/m),\alpha} \phi), \qquad \phi \in F_{p,\mu}. \tag{3.26}$$

Using Theorem 3.5 we obtain the following result analogous to Theorem 3.7.

THEOREM 3.10. *For $1 \leqq p \leqq \infty$ and any complex α, $K_{x^m}^{\eta,\alpha}$ is a continuous linear mapping of $F'_{p,\mu}$ into itself provided* $\operatorname{Re}(m\eta + \mu) > -1/q$. *If, in addition,* $\operatorname{Re}(m\eta + m\alpha + \mu) > -1/q$, *$K_{x^m}^{\eta,\alpha}$ is an automorphism of $F'_{p,\mu}$ and*

$$(K_{x^m}^{\eta,\alpha})^{-1} = K_{x^m}^{\eta+\alpha,-\alpha}.$$

For $f \in F'_{p,\mu}$, we have analogues of (3.16) and (3.17).

$$K_{x^m}^{\eta,0} f = f \tag{3.27}$$

for $\operatorname{Re}(m\eta + \mu) > -1/q$; if in addition, $\operatorname{Re}(m\eta + m\alpha + \mu) > -1/q$,

$$K_{x^m}^{\eta,\alpha} K_{x^m}^{\eta+\alpha,\beta} f = K_{x^m}^{\eta,\alpha+\beta} f. \tag{3.28}$$

Finally we discuss the properties of $I_{x^m}^{\alpha}$ and $K_{x^m}^{\alpha}$ on $F'_{p,\mu}$.

Let $f \in F'_{p,\mu}$. As before, from adjoint considerations, we are led to define $I_{x^m}^{\alpha} f$ for any complex α by

$$(I_{x^m}^{\alpha} f, \phi) = (f, x^{m-1} K_{x^m}^{\alpha} x^{-m+1} \phi). \tag{3.29}$$

The right-hand side is meaningful provided only that $\phi \in F_{p,\mu-m\alpha}$ and $m - \operatorname{Re}\mu > 1/q$ by Theorem 3.8. Similarly, for $\phi \in F_{p,\mu-m\alpha}$, $f \in F'_{p,\mu}$, we define $K_{x^m}^{\alpha} f$ by

$$(K_{x^m}^{\alpha} f, \phi) = (f, x^{m-1} I_{x^m}^{\alpha} x^{-m+1} \phi). \tag{3.30}$$

Use of Theorems 3.6 and 3.8 proves the next theorem.

THEOREM 3.11. (i) *I^{α}_{xm} is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided $m - \operatorname{Re}\mu > 1/q$. I^{0}_{xm} is the identity operator. If, in addition, $m + \operatorname{Re}(m\alpha - \mu) > 1/q$, I^{α}_{xm} is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and*

$$(I^{\alpha}_{xm})^{-1} = I^{-\alpha}_{xm}.$$

(ii) *K^{α}_{xm} is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu-m\alpha}$ provided $\operatorname{Re}(m\alpha - \mu) < 1/q$. If, in addition, $-\operatorname{Re}\mu < 1/q$, K^{α}_{xm} is an isomorphism of $F'_{p,\mu}$ onto $F'_{p,\mu-m\alpha}$ and*

$$(K^{\alpha}_{xm})^{-1} = K^{-\alpha}_{xm}.$$

K^{0}_{xm} is the identity operator on $F'_{p,\mu}$ if $-\operatorname{Re}\mu < 1/q$.

For $f \in F'_{p,\mu}$, we have the following index laws analogous to (3.11), (3.13), (3.21) and (3.22).

$$I^{\alpha}_{xm}I^{\beta}_{xm}f = I^{\alpha+\beta}_{xm}f = I^{\beta}_{xm}I^{\alpha}_{xm}f \tag{3.31}$$

provided $(1/q) - m + \operatorname{Re}\mu < \min(0, m\operatorname{Re}\alpha, m\operatorname{Re}\beta)$.

$$x^{m\alpha}I^{\beta}_{xm}x^{m\gamma}f = I^{-\gamma}_{xm}x^{-m\beta}I^{-\alpha}_{xm}f \tag{3.32}$$

provided $(1/q) - m + \operatorname{Re}\mu < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$.

$$K^{\alpha}_{xm}K^{\beta}_{xm}f = K^{\alpha+\beta}_{xm}f = K^{\beta}_{xm}K^{\alpha}_{xm}f \tag{3.33}$$

provided $(1/q) + \operatorname{Re}\mu > \max(m\operatorname{Re}\alpha, m\operatorname{Re}\beta, m\operatorname{Re}(\alpha + \beta))$.

$$x^{m\gamma}K^{\beta}_{xm}x^{m\alpha}f = K^{-\alpha}_{xm}x^{-m\beta}K^{-\gamma}_{xm}f \tag{3.34}$$

provided $-(1/q) - \operatorname{Re}\mu < \min(0, m\operatorname{Re}\gamma)$, $\alpha + \beta + \gamma = 0$.

4.1. The operators L_ν. For any suitable function ϕ and any complex number ν, we define the differential operator L_ν by

$$(L_\nu\phi)(x) = \frac{d^2\phi}{dx^2} + \frac{2\nu + 1}{x}\frac{d\phi}{dx}. \tag{4.1}$$

In this section, we consider connections between L_ν and operators of fractional integration which have been discussed for ordinary functions by Erdélyi in [3], and one of which has been established for the class $\mathscr{S}'$ of generalized functions by Erdélyi and McBride in [6].

It is immediate from Theorem 2.6 that for all complex numbers μ and ν and for $1 \leqq p \leqq \infty$, L_ν is a continuous linear mapping of $F_{p,\mu}$ into $F_{p,\mu-2}$. For $f \in F'_{p,\mu}$, we define $L_\nu f$ by

$$(L_\nu f, \phi) = (f, xL_{-\nu}x^{-1}\phi), \qquad \phi \in F_{p,\mu+2}. \tag{4.2}$$

The motivation for (4.2) is supplied by taking f to be a regular functional generated by a C^2 function, taking $\phi \in \mathscr{D}$ and integrating by parts. Using [16, Thm. 1.10-1], we immediately deduce Theorem 4.1.

THEOREM 4.1. *For any complex numbers μ and ν and for $1 \leqq p \leqq \infty$, L_ν is a continuous linear mapping of $F'_{p,\mu}$ into $F'_{p,\mu+2}$.*

As regards connections with fractional integration, we have Theorem 4.2.

THEOREM 4.2. *Let* $\phi \in F_{p,\mu}$, $1 \leqq p \leqq \infty$.

(i) *If* $\operatorname{Re}(2\nu + \mu) > 1/p$,

$$I_{x^2}^{\nu,\alpha} L_\nu \phi = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} \phi. \tag{4.3}$$

(ii) *If* $\operatorname{Re}(2\nu - \mu) > -1/p$,

$$L_{-\nu} K_{x^2}^{\nu,\alpha} \phi = K_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} \phi. \tag{4.4}$$

Proof. To prove (i) we can proceed as in [6, § 6], or use (3.15). The proof of (ii) is similar, so we shall omit the details.

(4.3) and (4.4) give perhaps the neatest relations between L_ν and fractional integration operators on $F_{p,\mu}$. We now give the corresponding results for $F'_{p,\mu}$.

THEOREM 4.3. *Let* $f \in F'_{p,\mu}$, $1 \leqq p \leqq \infty$.

(i) *If* $\operatorname{Re}(2\nu - \mu) > 1/q$,

$$I_{x^2}^{\nu,\alpha} L_\nu f = L_{\nu+\alpha} I_{x^2}^{\nu,\alpha} f. \tag{4.5}$$

(ii) If $\operatorname{Re}(2\nu + \mu) > -1/q$,

$$L_{-\nu} K_{x^2}^{\nu,\alpha} f = K_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} f. \tag{4.6}$$

Proof. (i) For $f \in F'_{p,\mu}$, $\phi \in F_{p,\mu+2}$, (3.23) and (4.2) give

$$\begin{aligned}(I_{x^2}^{\nu,\alpha} L_\nu f, \phi) &= (f, x L_{-\nu} x^{-1} K_{x^2}^{\nu+(1/2),\alpha} \phi) \\ &= (f, x L_{-\nu} K_{x^2}^{\nu,\alpha} x^{-1} \phi)\end{aligned}$$

and similarly

$$(I_{x^2}^{\nu,\alpha} L_{\nu+\alpha} f, \phi) = (f, x K_{x^2}^{\nu,\alpha} L_{-\nu-\alpha} x^{-1} \phi).$$

The result now follows from Theorem 4.2 (ii) with μ, ϕ replaced by $\mu + 1$ and $x^{-1}\phi$, respectively. Part (ii) follows similarly from Theorem 4.2 (i).

4.2. Solution of $L_\nu \phi = \psi$. Suppose $\psi \in F_{p,\mu-2}$ is given. We wish to find $\phi \in F_{p,\mu}$ such that $L_\nu \phi = \psi$, i.e.,

$$\begin{aligned}&\frac{d^2\phi}{dx^2} + \frac{2\nu + 1}{x}\frac{d\phi}{dx} = \psi \\ &\Rightarrow \frac{d}{dx}\left(x^{2\nu+1}\frac{d\phi}{dx}\right) = x^{2\nu+1}\psi.\end{aligned} \tag{4.7}$$

The problem then reduces to inverting $D = d/dx$, and for this we fall back on Theorems 3.6 and 3.8, which tell us that D is an isomorphism of $F_{p,\mu}$ onto $F_{p,\mu-1}$ provided $\operatorname{Re}\mu \neq 1/p$ and

$$D^{-1} = \begin{cases} I_x^1, & \operatorname{Re}\mu > 1/p, \\ -K_x^1, & \operatorname{Re}\mu < 1/p. \end{cases}$$

As regards the case $\operatorname{Re}\mu = 1/p$, take $p = \infty$ so that $\mu = 0$. Then $D\phi = 0$ for *every* constant function $\phi \in F_\infty$, and clearly D is not invertible in this case. Using (4.7) we easily obtain the following theorem.

THEOREM 4.4. *For each* $\psi \in F_{p,\mu-2}$, *the equation* $L_\nu \phi = \psi$ *has a unique solution* $\phi \in F_{p,\mu}$ *provided* $\mathrm{Re}\,(2\nu+\mu) \neq 1/p$ *and* $\mathrm{Re}\,\mu \neq 1/p$.

(i) *If* $\mathrm{Re}\,(2\nu+\mu) > 1/p$, $\mathrm{Re}\,\mu > 1/p$,

$$\phi = I_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1}\psi.$$

(ii) *If* $\mathrm{Re}\,(2\nu+\mu) > 1/p$, $\mathrm{Re}\,\mu < 1/p$,

$$\phi = -K_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1}\psi.$$

(iii) *If* $\mathrm{Re}\,(2\nu+\mu) < 1/p$, $\mathrm{Re}\,\mu > 1/p$,

$$\phi = -I_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1}\psi.$$

(iv) *If* $\mathrm{Re}\,(2\nu+\mu) < 1/p$, $\mathrm{Re}\,\mu < 1/p$,

$$\phi = K_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1}\psi.$$

The results are particularly simple when $\nu = -\frac{1}{2}$ (in which case (ii) is redundant) and $\nu = 0$ (when (ii) and (iii) are redundant). We can use these special cases in conjunction with Theorem 4.2 to derive alternative expressions for the solution ϕ in the general case. For instance, for the case $\nu = -\frac{1}{2}$, we obtain Theorem 4.5.

THEOREM 4.5. *If* $\mathrm{Re}\,(2\nu+\mu) \neq 1/p$ and $\mathrm{Re}\,\mu \neq 1/p$, *the* (*unique*) *solution* $\phi \in F_{p,\mu}$ *of* $L_\nu\phi = \psi$, $\psi \in F_{p,\mu-2}$, *is given as follows*:

(i) *If* $\mathrm{Re}\,(2\nu+\mu) > 1/p$, $\mathrm{Re}\,(-1+\mu) > 1/p$

$$\phi = I_{x^2}^{-1/2,\nu+1/2} I_x^2 I_{x^2}^{\nu,-\nu-1/2}\psi,$$

(ii) *If* $\mathrm{Re}\,(2\nu+\mu) < 1/p$, $\mathrm{Re}\,(-1+\mu) < 1/p$

$$\phi = -K_{x^2}^{1/2,-\nu-1/2} I_x^1 K_x^1 K_{x^2}^{-\nu,\nu+1/2}\psi, \qquad 1/p < \mathrm{Re}\,\mu < (1/p)+1,$$

$$\phi = K_{x^2}^{1/2,-\nu-1/2} K_x^2 K_{x^2}^{-\nu,\nu+1/2}\psi, \qquad \mathrm{Re}\,\mu < 1/p.$$

Proof.

(i) Since $\mathrm{Re}\,(2\nu+\mu) > 1/p$, we may take $\alpha = -\nu-\frac{1}{2}$ in (4.3) to get

$$I_{x^2}^{\nu,-\nu-1/2} L_\nu \phi = L_{-1/2} I_{x^2}^{\nu,-\nu-1/2}\phi$$

$$\Rightarrow \psi = L_\nu\phi = I_{x^2}^{-1/2,\nu+1/2} L_{-1/2} I_{x^2}^{\nu,-\nu-1/2}\phi$$

using Theorem 3.5 (since $\mathrm{Re}\,\mu + 1 > 1/p$). Provided $(L_{-1/2})^{-1}$ exists, we may now invert obtaining

$$\phi = L_\nu^{-1}\psi = I_x^{-1/2,\nu+1/2}(L_{-1/2})^{-1} I_{x^2}^{\nu,-\nu-1/2}\psi,$$

and we use Theorem 4.4 (i) to substitute for $(L_{-1/2})^{-1}$.

Part (ii) is proved similarly using (4.4).

There are other possible expressions for ϕ, but we shall not list them here.

4.3. Solution of $L_\nu f = g$. Suppose now that $g \in F'_{p,\mu+2}$ is given. We have to find $f \in F'_{p,\mu}$ such that $L_\nu f = g$. To obtain the solution, we can either imitate the methods of § 4.2 or else take adjoints.

THEOREM 4.6. *For each* $g \in F'_{p,\mu+2}$, *the equation* $L_\nu f = g$ *has a unique solution* $f \in F'_{p,\mu}$ *provided* $\operatorname{Re}(2\nu - \mu) \neq 1/q$ *and* $\operatorname{Re}\mu \neq 1/q$.

(i) *If* $\operatorname{Re}(2\nu - \mu) > 1/q$, $-\operatorname{Re}\mu > 1/q$,

$$f = I_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1} g.$$

(ii) *If* $\operatorname{Re}(2\nu - \mu) > 1/q$, $-\operatorname{Re}\mu < 1/q$,

$$f = -K_x^1 x^{-2\nu-1} I_x^1 x^{2\nu+1} g.$$

(iii) *If* $\operatorname{Re}(2\nu - \mu) < 1/q$, $-\operatorname{Re}\mu > 1/q$,

$$f = -I_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1} g.$$

(iv) *If* $\operatorname{Re}(2\nu - \mu) < 1/q$, $-\operatorname{Re}\mu < 1/q$,

$$f = K_x^1 x^{-2\nu-1} K_x^1 x^{2\nu+1} g.$$

As an illustration consider (i). Under the given conditions, $xL_{-\nu}x^{-1}$ is invertible on $F_{p,\mu+2}$, and from Theorem 4.4 (iv), if $\psi \in F_{p,\mu}$,

$$xL_{-\nu}x^{-1}\phi = \psi \Leftrightarrow \phi = xK_x^1 x^{2\nu-1} K_x^1 x^{-2\nu}\psi.$$

Hence taking adjoints, where $g \in F'_{p,\mu+2}$, we see that

$$L_\nu f = g \Leftrightarrow f = x^{-2\nu} I_x^1 x^{2\nu-1} I_x^1 x g.$$

This is a perfectly acceptable expression for the solution, but to obtain the form in (i) we use the index laws (3.31) and (3.32). Indeed,

$$\begin{aligned} f = x^{-2\nu} I_x^1 x^{2\nu-1} I_x^1 x g &= I_x^{1-2\nu} x^{-1} I_x^{2\nu} I_x^1 x g \\ &= I_x^1 (I_x^{-2\nu} x^{-1} I_x^{2\nu+1}) x g = I_x^1 (x^{-2\nu-1} I_x^1 x^{2\nu}) x g, \end{aligned}$$

from which (i) follows; the above steps are all valid under the given conditions. Parts (ii)–(iv) are similar.

Again other equivalent solution formulas can be obtained if required via (4.5) and (4.6).

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ERRATA: CONNECTION FORMULAS FOR ASYMPTOTIC SOLUTIONS OF SECOND ORDER TURNING POINTS IN UNBOUNDED DOMAINS

ANTHONY LEUNG†

The correction on line 32 of p. 98 for the matrix $N(\varepsilon)$ is the most important one. The rest are misprints or arithmetic errors in computations leading to the correction for $N(\varepsilon)$.

On p. 95, the entries of the lower left-hand corners of the 2×2 matrices displayed on lines 3, 16 and 25 should be

$$``\varepsilon\frac{dq_1}{dt} + q_2\left(t^2 + \frac{a_0\varepsilon^2}{2}\right)".$$

On p. 96, the formula for $a_1(\mu)$ on line 19 should be "$a_1(\mu) = -\exp[+2\pi i \cdot (\frac{1}{4}\mu - \frac{1}{4})]$",

On p. 97, the formulas for $z_2(t, \varepsilon)$, $v_{11}(t, \varepsilon)$ and $v_{21}(t, \varepsilon)$ on lines 15, 21, and 24 respectively should be changed to

$$``z_2(t, \varepsilon) = \frac{\Gamma(\frac{1}{2})^2}{\Gamma(\frac{1}{2})^2 - 2\pi}\frac{\varepsilon^{1/4}}{2^{1/4}t^{1/2}}\left[\left(\frac{-e^{3\pi i/4}\sqrt{2\pi}}{\Gamma(\frac{1}{2})} + O(\varepsilon\log\varepsilon)\right)(e^{-t^2/(2\varepsilon)}) - (e^{\pi i/4} + O(\varepsilon\log\varepsilon))\,e^{t^2/(2\varepsilon)}\right]",$$

$$``v_{11}(t, \varepsilon) = -\varepsilon^{1/4}2^{-1/4}t^{-1/2}[e^{-t^2/(2\varepsilon)}(\sqrt{2} + O(\varepsilon\log\varepsilon)) - e^{t^2/(2\varepsilon)}(e^{\pi i/2} + O(\varepsilon\log\varepsilon))]",$$

$$``v_{21}(t, \varepsilon) = -\varepsilon^{1/4}2^{-1/4}t^{-1/2}\cdot t[e^{-t^2/(2\varepsilon)}(-\sqrt{2} + O(\varepsilon\log\varepsilon)) - e^{t^2/(2\varepsilon)}(e^{\pi i/2} + O(\varepsilon\log\varepsilon))]".$$

On p. 98, line 3 and 28, "$-\sqrt{2}i + O(\varepsilon\log\varepsilon)$" should read "$-\sqrt{2} + O(\varepsilon\log\varepsilon)$". On line 12, "$C(\varepsilon) = 2^{1/2}e^{-\pi i/2}\ldots$" should read "$C(\varepsilon) = 2^{1/4}e^{-\pi i/2}\ldots$". On line 20, the entry on the lower right-hand corner of the 2×2 matrix should be "$2\sqrt{2}i + O(\varepsilon\log\varepsilon) - e^{t^2/\varepsilon}O(\varepsilon\log\varepsilon)$".

On p. 98, line 32, the (2, 2)th entry of the matrix $N(\varepsilon)$ should be "$-\sqrt{2}i + c_{22}(\varepsilon) + O(\varepsilon\log\varepsilon)$".

On p. 101, line 8, "$\sqrt{2}$" should read "$-\sqrt{2}i$".

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A NONLINEAR BOUNDARY VALUE PROBLEM ON AN UNBOUNDED INTERVAL*

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Abstract. A constructive existence and uniqueness result is obtained for small values of the positive parameter ε for the problem (1.1), (1.2). The multivariable method is used to provide a candidate u_0 about which the problem is perturbed, and the method of successive approximation is then used to obtain existence and uniqueness in a functional ball centered at u_0. It is shown that u_0 provides a uniformly valid approximation to the resulting solution on the full interval $1 \leqq r \leqq \infty$ for small ε. The problem (1.1), (1.2) has been interpreted as a scalar model of the stationary incompressible flow of a viscous Navier-Stokes fluid past a sphere, with prescribed constant velocity at infinity and zero velocity on the surface of the sphere. For this scalar model the results obtained here provide a first step towards a verification of the analogue of the long standing conjecture concerning the nature of a stationary viscous flow at large Reynolds number.

1. Introduction. We consider the boundary value problem consisting of the differential equation

$$\varepsilon\left(\frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr}\right) + u\frac{du}{dr} = 0, \qquad r > 1, \tag{1.1}$$

and the boundary conditions

$$u(1, \varepsilon) = \alpha, \qquad u(\infty, \varepsilon) = 1 \tag{1.2}$$

for the function $u = u(r, \varepsilon)$, where n denotes a given, fixed positive integer (the case $n = 1$ is trivial, and hence uninteresting), ε is a small positive parameter, and α is a fixed given constant which satisfies the condition

$$\alpha + 1 > 0. \tag{1.3}$$

We could replace the second boundary condition of (1.2) with the more general condition

$$u(\infty, \varepsilon) = \beta \tag{1.4}$$

for a given fixed constant β. However, the method of calculation used below in § 2 to obtain equation (2.4) can be used to show in this case that the condition $\beta > 0$ is a necessary condition for the existence of solutions to the present problem, and then a fixed similarity transformation (cf. Von Mises and Friedrichs (1971)) can be used to reduce the more general condition (1.4) back to the case $\beta = 1$. Hence there is no loss in taking $\beta = 1$, and this we shall do as in (1.2). (On the other hand if we were to retain (1.4), then the condition (1.3) would be replaced with the condition $\alpha + \beta > 0$.)

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Such two-point boundary value problems on *bounded* intervals for equations of the form

$$\varepsilon\frac{d^2u}{dr^2} + F\left(r, u, \frac{du}{dr}, \varepsilon\right) = 0$$

have been studied by several authors including Coddington and Levinson (1952), Wasow (1956), Willett (1966), Erdélyi (1968), O'Malley (1968), and others. We indicate in this paper by illustration how certain of these earlier results on bounded intervals can be extended to handle similar problems of interest on *unbounded* intervals. The problem (1.1), (1.2) is of independent interest in its own right, and for this reason and for the sake of simplicity we only consider this special problem here.

The expression $d^2u/dr^2 + [(n-1)/r]\,du/dr$ which appears in (1.1) is the spherically symmetric part of the n-dimension Laplacian of u. Hence (1.1) may be interpreted as a mathematical model for the equilibrium temperature distribution in a homogeneous medium which fills out the region $r = |x| > 1$, with nonlinear heat loss represented by the term $u\,du/dr$. The boundary conditions (1.2) specify the values of the temperature on the surface of the unit sphere in n-space and at infinity. This interpretation of (1.1), (1.2) is due to Lagerstrom and Casten (1972) who used matching techniques to study this problem for *large* values of the parameter ε in the case $\alpha = 0$.

The problem (1.1), (1.2) with $\alpha = 0$ has also been interpreted by Lagerstrom (1961) as a scalar model of the stationary incompressible flow of a viscous Navier–Stokes fluid past a sphere, with prescribed constant velocity at infinity and zero velocity on the surface of the sphere. In this case the parameter ε corresponds to the kinematic viscosity of the fluid, and Lagerstrom used matching techniques to study the problem for large ε (i.e., small Reynolds number). This problem has been considered further by Bush (1971), again for large ε.

In the case of the stationary incompressible flow of a viscous Navier–Stokes fluid past an obstacle, it is widely expected that uniqueness *fails* for certain values of ε due to turbulence. Uniqueness is known to hold for *large* ε in this case as a consequence of the work of Finn (1959; 1965) and Finn and Smith (1967), but the case of *small* ε as studied in the present paper for (1.1) remains open for the stationary incompressible Navier–Stokes equations. (See Chapter 4 of Meyer (1971) for a discussion of this problem.) Hence we consider the uniqueness question to be of some importance for (1.1), (1.2). We also consider that it is important whenever possible to use constructive methods which lead to quantitative results. For example, if existence and uniqueness hold for all small values of ε, then we wish to obtain quantitative information regarding the admissible size of ε, and we also wish to obtain both qualitative and quantitative information about the resulting solutions.

The existence of solutions of (1.1), (1.2) can be proved nonconstructively with the Schauder fixed point theorem directly or with the subfunction/superfunction theory of Jackson (1968) which in this case is again based on the Schauder fixed point theorem. However, from our standpoint such an existence result is not satisfactory since it is in principle nonconstructive, and the difficult question of uniqueness would still remain to be studied (using, perhaps, the topological degree theory of Leray). Moreover there would also remain the important question of the qualitative and quantitative behavior of the solutions with respect to their dependence on r and ε.

In our study of (1.1), (1.2) we shall use a direct, elementary approach based on the multivariable expansion method (cf. Smith (1975)). It would seem that the multivariable expansion method provides an ideal, natural approach in the present case since it provides an elementary, direct, unified approach which gives at once a constructive existence and uniqueness result along with the additional quantitative and qualitative information of interest.

Specifically, we shall obtain a constructive existence and uniqueness result for (1.1), (1.2) for (small) values of ε which satisfy

$$(1.5) \qquad \varepsilon < \begin{cases} \dfrac{(1+\alpha)^6}{8(n-1)[(1+\alpha)^4 + 16(1-\alpha^2)/n + 32(1-\alpha)^2]} & \text{if } \alpha \leqq 1, \\[2ex] \dfrac{1}{2(n-1) + (n-1)(\alpha-1)[(1+\alpha)^2/n + 2(\alpha^2-1)]} & \text{if } \alpha \geqq 1, \end{cases}$$

and at the same time we shall also obtain an approximation to the solution function u which is easy to interpret for small ε. We prove that the resulting approximation is uniformly valid (for small ε) on the full interval $1 \leqq r \leqq \infty$, and we obtain detailed qualitative and quantitative information about the structure of the solution.

We shall be able to conclude directly from our analysis that the flow function u obtained undergoes a rapid variation in the Prandtl boundary layer region near $r = 1$, whereas outside this boundary layer region the flow function is well-approximated by the smooth solution ($u_0 = 1$) of the reduced "Euler flow problem"

$$u_0 \frac{du_0}{dr} = 0, \qquad u_0(\infty) = 1.$$

Hence for the scalar model (1.1), (1.2) we provide a first step towards a verification of the analogue of the long standing conjecture concerning the nature of a stationary flow of a viscous incompressible fluid at large Reynolds number. In the case of the Navier–Stokes equations the conjecture remains open: not even the first step has been taken.

As a direct by-product of our analysis we obtain a constructive existence result for the nonlinear Dirichlet problem

$$\varepsilon\,\Delta u + u\frac{\partial u}{\partial r} = 0 \quad \text{in } r = |x| > 1,$$

$$u|_{|x|=1} = \alpha, \qquad u|_{x\to\infty} = 1$$

for small ε, where Δ denotes the Laplace operator in n-space. Moreover we obtain an easily interpretable, uniformly valid approximation to the resulting solution u, from which we conclude that the solution exhibits the same boundary layer behavior described above. The boundary layer region which adjoins the surface $|x| = 1$ in n-space has a thickness of order ε, for any $n = 2, 3, 4 \cdots$.

We mention finally that our existence and uniqueness condition (1.5) requires ε to be relatively small in those cases in which the given boundary value α (at $r = 1$) departs significantly (within the range specified by (1.3)) from the given boundary value $\beta = 1$ at infinity. Indeed, ε must be small [of order $(1 + \alpha)^6$] if α is close to the extreme value -1, and ε must also be small [of order α^{-3}] if α is large. On the other hand, when the boundary value α is close to the value at infinity, $\beta = 1$, then our existence, uniqueness, and approximation results remain valid for relatively large values of ε approaching the value $\varepsilon = 1/[2(n-1)]$. (If $\alpha = \beta = 1$, then (2.4) shows that the trivial solution $u \equiv 1$ is the unique L_1-solution of (1.1), (1.2) for *any* $\varepsilon > 0$.)

2. Qualitative properties of the solutions. It is convenient in this section to work with functions in $L_1(1, \infty)$ (which vanish at infinity), and for this purpose we rewrite (1.1), (1.2) in terms of the function w given as

$$w = 1 - u. \tag{2.1}$$

The function w is then found to satisfy the conditions

$$\varepsilon\left(\frac{d^2w}{dr^2} + \frac{n-1}{r}\frac{dw}{dr}\right) + \frac{dw}{dr} = w\frac{dw}{dr}, \qquad r > 1, \tag{2.2}$$

and

$$w(1, \varepsilon) = 1 - \alpha, \qquad w(\infty, \varepsilon) = 0. \tag{2.3}$$

The equation (2.2) can be integrated twice with the boundary conditions (2.3) to yield the nonlinear integral equation

$$w(r, \varepsilon) = \frac{(1-\alpha)\displaystyle\int_r^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon} - \frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/ \sigma^{n-1}\right] d\sigma}{\displaystyle\int_1^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon} - \frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/ \sigma^{n-1}\right] d\sigma} \qquad \text{for } r \geqq 1. \tag{2.4}$$

Conversely, any solution of the latter nonlinear integral equation will be a smooth solution of (2.2), (2.3).

We conclude directly from (2.4) that any solution w of (2.2), (2.3) must satisfy the following bounds (which can also be obtained from the maximum principle):

$$\begin{aligned} &0 \leqq w(r, \varepsilon) \leqq 1 - \alpha, \quad \frac{dw}{dr}(r, \varepsilon) < 0 \quad &&\text{if } \alpha < 1, \\ &w(r, \varepsilon) = 0 &&\text{if } \alpha = 1, \\ &1 - \alpha \leqq w(r, \varepsilon) \leqq 0, \quad \frac{dw}{dr}(r, \varepsilon) > 0 &&\text{if } \alpha > 1 \end{aligned} \tag{2.5}$$

for all $r \geqq 1$. The following estimates, which are useful for large r, follow now from

(2.4) and (2.5):

$$
(2.6)\quad
\begin{aligned}
&0 \leqq w(r,\varepsilon) \leqq \frac{\varepsilon(1-\alpha)\exp\left(-\dfrac{r}{\varepsilon}\right)}{r^{n-1}\displaystyle\int_1^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon}-\frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/\sigma^{n-1}\right]d\sigma} && \text{if } \alpha < 1,\\
&0 \leqq -w(r,\varepsilon) \leqq \frac{\varepsilon(\alpha-1)\exp\left(-\dfrac{r}{\varepsilon}\right)\exp\left(-\dfrac{1}{\varepsilon}\displaystyle\int_1^\infty w\right)}{r^{n-1}\displaystyle\int_1^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon}-\frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/\sigma^{n-1}\right]d\sigma} && \text{if } \alpha > 1,
\end{aligned}
$$

and

$$
(2.7)\quad
\begin{aligned}
&0 \leqq -\frac{dw}{dr}(r,\varepsilon) \leqq \frac{(1-\alpha)\exp\left(-\dfrac{r}{\varepsilon}\right)}{r^{n-1}\displaystyle\int_1^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon}-\frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/\sigma^{n-1}\right]d\sigma} && \text{if } \alpha < 1,\\
&0 \leqq \frac{dw}{dr}(r,\varepsilon) \leqq \frac{(\alpha-1)\exp\left(-\dfrac{r}{\varepsilon}\right)\exp\left(-\dfrac{1}{\varepsilon}\displaystyle\int_1^\infty w\right)}{r^{n-1}\displaystyle\int_1^\infty \left[\exp\left(-\frac{\sigma}{\varepsilon}-\frac{1}{\varepsilon}\int_\sigma^\infty w\right)\Big/\sigma^{n-1}\right]d\sigma} && \text{if } \alpha > 1.
\end{aligned}
$$

In particular, every $L_1(1,\infty)$-solution must vanish exponentially at infinity (for any fixed $\varepsilon > 0$).

3. A formal two-variable approximation. The nonlinear term $w\,dw/dr$ which appears in (2.2) will be negligible at infinity compared to the linear term dw/dr appearing in (2.2) (see (2.6) and (2.7)), and so we are led to consider the following related *linear* problem:

$$
\varepsilon\left(\frac{d^2W}{dr^2}+\frac{n-1}{r}\frac{dW}{dr}\right)+\frac{dW}{dr}=0, \qquad r>1,
$$
$$
W(1,\varepsilon)=1-\alpha, \qquad W(\infty,\varepsilon)=0,
$$

which is obtained from (2.2) and (2.3) by omitting the nonlinear term in (2.2). The unique solution W of this linear problem can be given explicitly up to quadratures, and then repeated integrations by parts can be used to obtain the result

$$
(3.1)\qquad W(r,\varepsilon)=e^{-(r-1)/\varepsilon}\left[\sum_{k=0}^{N} A_k(r)\varepsilon^k + O(\varepsilon^{N+1})\right]
$$

for suitable functions $A_k = A_k(r)$ which we need not (and shall not) give here. The functions A_k depend only on r (and n), and not on ε. (For example, $A_0(r) = (1-\alpha)/r^{n-1}$, while $A_1(r) = (1-\alpha)(n-1)(r-1)/r^n$.)

The corresponding linear problem associated with (1.1), (1.2) is obtained by setting $U = 1 - W$. It satisfies the differential equation

$$
(3.2)\qquad \varepsilon\left(\frac{d^2U}{dr^2}+\frac{n-1}{r}\frac{dU}{dr}\right)+\frac{dU}{dr}=0, \qquad r>1,
$$

whose solution satisfies

$$U(r,\varepsilon) = 1 - e^{-(r-1)/\varepsilon}\left[\sum_{k=0}^{N} A_k(r)\varepsilon^k + O(\varepsilon^{N+1})\right]. \tag{3.3}$$

In particular we see that the solution U of (3.2) depends in an essential way (for small ε) on the two variables r and $(r-1)/\varepsilon$.

We now turn to the solution u of the original nonlinear problem (1.1), (1.2). By analogy with (3.3) we seek to represent the solution u with an appropriate asymptotic expansion of the form

$$u(r,\varepsilon) \sim \sum_{k=0}^{\infty} u_k(r,\rho)\varepsilon^k, \qquad \rho = \frac{r-1}{\varepsilon}, \tag{3.4}$$

for suitable functions $u_k = u_k(r,\rho)$ of two independent variables r and ρ. We shall be mainly interested in obtaining the leading term u_0 in the expansion (3.4), but for the moment it will be convenient to work with the full expansion.

If we insert (3.4) into the differential equation (1.1), we obtain formally the result

$$\begin{aligned}&\frac{u_{0,\rho\rho} + u_0u_{0,\rho}}{\varepsilon} + \left[u_{1,\rho\rho} + (u_0u_1)_{,\rho} + 2u_{0,r\rho} + \frac{n-1}{r}u_{0,\rho} + u_0u_{0,r}\right]\\ &+ \sum_{k=1}^{\infty}\left\{u_{k+1,\rho\rho} + (u_0u_{k+1})_{,\rho} + \left(2u_{k,r\rho} + \frac{n-1}{r}u_{k,\rho} + u_0u_{k,r}\right)\right.\\ &\left.+\left(u_{k-1,rr} + \frac{n-1}{r}u_{k-1,r}\right) + \sum_{l=0}^{k-1} u_{k-l}(u_{l,r} + u_{l+1,\rho})\right\}\varepsilon^k = 0,\end{aligned}$$

and this equation will hold automatically if we impose the following conditions:

$$u_{0,\rho\rho} + u_0u_{0,\rho} = 0, \tag{3.5}$$

$$u_{1,\rho\rho} + (u_0u_1)_{,\rho} + (2u_{0,r\rho} + ((n-1)/r)u_{0,\rho} + u_0u_{0,r}) = 0, \tag{3.6}$$

and

$$\begin{aligned}&u_{k+1,\rho\rho} + (u_0u_{k+1})_{,\rho} + (2u_{k,r\rho} + ((n-1)/r)u_{k,\rho} + u_0u_{k,r})\\ &+ (u_{k-1,rr} + ((n-1)/r)u_{k-1,r}) + \sum_{l=0}^{k-1} u_{k-l}(u_{l,r} + u_{l+1,\rho}) = 0\end{aligned} \tag{3.7}$$

for $k = 1, 2, \cdots$.

Similarly, from (3.4) and (1.2) we find formally the boundary relations (note that $\rho = 0$ when $r = 1$, and $\rho = \infty$ when $r = \infty$)

$$\sum_{k=0}^{\infty} u_k(1,0)\varepsilon^k = \alpha$$

and

$$\sum_{k=0}^{\infty} u_k(\infty,\infty)\varepsilon^k = 1,$$

and these relations will hold automatically if we impose the requirements

$$u_k(1,0) = \begin{cases} \alpha & \text{for } k = 0, \\ 0 & \text{for } k = 1, 2, \cdots, \end{cases} \tag{3.8}$$

and

$$u_k(\infty,\infty) = \begin{cases} 1 & \text{for } k = 0, \\ 0 & \text{for } k = 1, 2, \cdots. \end{cases} \tag{3.9}$$

The general solution of (3.5) can be given (for each fixed r) as

$$u_0(r,\rho) = A_0(r)\frac{1 - B_0(r)\,e^{-\rho A_0(r)}}{1 + B_0(r)\,e^{-\rho A_0(r)}}, \tag{3.10}$$

where the "constants" of integration A_0 and B_0 may still depend on the variable r. To completely specify u_0, we impose the Lindstedt–Poincaré condition [cf. Smith (1975)] that A_0 and B_0 be chosen to eliminate the dominant terms contributed by u_0 to the next function u_1 of the expansion (3.4).

The function u_1 is obtained by inserting (3.10) back into (3.6) and then integrating the resulting equation (3.6) with respect to ρ (for each fixed r). In this way we find the result

$$\begin{aligned}
&[1 + B_0(r)\,e^{-\rho A_0(r)}]^2 u_1(r,\rho) = -A_0'(r)\rho \\
&\quad + 2A_0(r)\{[B_0'(r) + ((n-1)/r)B_0(r)] - A_0'(r)B_0(r)\rho\}\rho\, e^{-\rho A_0(r)} \\
&\quad + B_0(r)^2[A_0'(r)\rho - 2(n-1)/r]\,e^{-2\rho A_0(r)} \\
&\quad + A_1(r)\,e^{-\rho A_0(r)} + B_1(r)[1 + 2A_0(r)B_0(r)\rho\, e^{-\rho A_0(r)} \\
&\quad - B_0(r)^2\, e^{-2\rho A_0(r)}],
\end{aligned} \tag{3.11}$$

where $A_1(r)$ and $B_1(r)$ are again constants of integration.

In view of (2.1), (2.6), (3.4), and (3.10), we anticipate that $A_0(r)$ will be positive (at least for large r) so that for large ρ the dominant term contributed by u_0 to the right-hand side of (3.11) is expected to be the leading term $-\rho A_0'(r)$. Since this will otherwise be large with ρ, we impose the condition $A_0'(r) = 0$, and obtain

$$A_0(r) = a \quad \text{for } r \geqq 1,$$

for some suitable positive constant a. Substituting into (3.11), we now find for large ρ that the dominant term contributed by u_0 to the right-hand side will be the term involving $\rho e^{-a\rho}$ unless there holds

$$B_0'(r) + ((n-1)/r)B_0(r) = 0.$$

Hence we also impose this last condition and find the result

$$B_0(r) = b/r^{n-1}$$

for some suitable constant of integration b. Thus, u_0 is determined up to the constants a and b. These remaining constants can be determined by imposing the appropriate boundary conditions for u_0 given by (3.8) and (3.9). In this way we find the results $a = 1$ and $b = (1-\alpha)/(1+\alpha)$, so that u_0 is completely determined as

$$u_0(r,\rho) = \frac{(1+\alpha)r^{n-1} - (1-\alpha)\,e^{-\rho}}{(1+\alpha)r^{n-1} + (1-\alpha)\,e^{-\rho}}, \tag{3.12}$$

while u_1 is now given by the relation

$$\begin{aligned}[(1+\alpha)r^{n-1} + (1-\alpha)\,e^{-\rho}]u_1(r,\rho) = {} & -\frac{2(n-1)(1-\alpha)^2\,e^{-2\rho}}{r} \\ & + A_1(r)(1+\alpha)^2 r^{2n-2}\,e^{-\rho} + B_1(r)[(1+\alpha)^2 r^{2n-2} \\ & + 2(1-\alpha)(1+\alpha)r^{n-1}\rho\,e^{-\rho} - (1-\alpha)^2\,e^{-2\rho}].\end{aligned} \tag{3.13}$$

The functions $A_1(r)$ and $B_1(r)$ in (3.13) can be chosen so as to eliminate the dominant terms contributed by u_1 to the next function u_2 in (3.4). The procedure can be continued recursively so as to generate the successive terms appearing in (3.4). However, in practice, the actual calculations become successively (and rapidly) more complicated and we shall be content here with only the leading term u_0 given by (3.12).

Note that the condition $\alpha + 1 > 0$ (see (1.3)) is necessary if the function u_0 given by (3.12) is to provide an acceptable approximation to a solution u of (1.1), (1.2). Indeed, if $\alpha + 1 = 0$, then $u_0 = -1$ and $du_0/dr = 0$, whereas (2.1) and (2.5) imply that $du/dr > 0$. Similarly, if $\alpha + 1 < 0$, then u_0 has a singularity at a certain point $r_1 > 1$, and $du_0/dr < 0$ whereas again $du/dr > 0$.

4. Existence, uniqueness, and behavior of solutions. In this section we shall obtain a constructive existence and uniqueness result for the original boundary value problem (1.1), (1.2) by perturbing that problem about the leading term u_0 of the two-variable expansion (3.4). (We assume throughout that (1.3) holds.) To this end we introduce a function $v = v(r,\varepsilon)$ by the relation

$$u(r,\varepsilon) = u_0(r,\rho) + v(r,\varepsilon), \qquad \rho = (r-1)/\varepsilon, \tag{4.1}$$

where u is to satisfy (1.1) and (1.2), and where u_0 is given by (3.12). The boundary conditions (1.2) along with (3.12) imply that

$$v(1,\varepsilon) = v(\infty,\varepsilon) = 0, \tag{4.2}$$

while the differential equation (1.1) along with (3.12) imply

$$\varepsilon\left(\frac{d^2v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr}\right) + \frac{d}{dr}\left(u_0 v + \frac{1}{2}v^2\right) + f(r,\varepsilon) = 0, \tag{4.3}$$

where u_0 is evaluated at $\rho = (r-1)/\varepsilon$ and where the forcing term f is given as

$$\begin{aligned} f(r,\varepsilon) &= 2u_{0,r\rho} + \frac{n-1}{r}u_{0,\rho} + u_0 u_{0,r} + \varepsilon\left[u_{0,rr} + \frac{n-1}{r}u_{0,r}\right] \\ &= \frac{4(1+\alpha)(1-\alpha)^2(n-1)r^{n-2}\exp[-2(r-1)/\varepsilon]}{\{(1+\alpha)r^{n-1} + (1-\alpha)\exp[-(r-1)/\varepsilon]\}^3} \\ &\quad + \frac{\begin{aligned}&2\varepsilon(1-\alpha^2)(n-1)r^{n-3}\exp[-(r-1)/\varepsilon] \\ &\qquad \cdot\{-(1+\alpha)r^{n-1} + (2n-3)(1-\alpha)\exp[-(r-1)/\varepsilon]\}\end{aligned}}{\{(1+\alpha)r^{n-1} + (1-\alpha)\exp[-(r-1)/\varepsilon]\}^3}.\end{aligned} \tag{4.4}$$

For small ε this forcing term f is *small* as compared with the forcing terms which would result from using other u_0 than that given by (3.12). (See Lemma 2 and Lemma 3 below.)

It is useful to observe that any possible $L_1(1, \infty)$-solution of (4.2), (4.3) must vanish exponentially at infinity along with its first derivative. Indeed any such solution function v can be represented with (2.1), (3.12) and (4.1) in the form

$$v(r, \varepsilon) = \frac{2(1 - \alpha)e^{-\rho}}{(1 + \alpha)r^{n-1} + (1 - \alpha)e^{-\rho}} - (1 - \alpha)w(r, \varepsilon)$$

for a suitable function w which satisfies the estimates (2.6) and (2.7). Hence we can integrate (4.3) twice and impose the given boundary conditions to find the nonlinear integral equation

$$v = Lv + Nv + F \quad \text{for } r \geqq 1, \tag{4.5}$$

where the function F is defined for $r \geqq 1$ by the formula

$$F(r, \varepsilon) = \frac{1}{\varepsilon}\int_1^r \exp\left(-\frac{1}{\varepsilon}\int_\sigma^r \left[u_0 + \frac{\varepsilon(n-1)}{\tau}\right] d\tau\right)\left(\int_\sigma^\infty f(\nu, \varepsilon)\, d\nu\right) d\sigma, \tag{4.6}$$

and where the expressions Lv and Nv are defined by

$$Lv(r) = (n-1)\int_1^r \exp\left(-\frac{1}{\varepsilon}\int_\sigma^r \left[u_0 + \frac{\varepsilon(n-1)}{\tau}\right] d\tau\right)\left(\int_\sigma^\infty \frac{v(\nu)}{\nu^2}\, d\nu\right) d\sigma \tag{4.7}$$

and

$$Nv(r) = -\frac{1}{2\varepsilon}\int_1^r \exp\left(-\frac{1}{\varepsilon}\int_\sigma^r \left[u_0 + \frac{\varepsilon(n-1)}{\tau}\right] d\tau\right) v(\sigma)^2\, d\sigma \tag{4.8}$$

for any L_1-function v. The nonlinear integral equation (4.5) is equivalent (in a suitable class of functions) to the given boundary value problem (4.2), (4.3).

We shall consider (4.5) in the normed vector space $\mathscr{V}_\varepsilon$ consisting of all continuous functions v with finite norm $\|v\|_\varepsilon$, where we take the norm to be defined by the formula

$$\|v\|_\varepsilon = \sup_{r \geqq 1} |v(r)|\, e^{(r-1)/\varepsilon} \tag{4.9}$$

for any suitable function v. The operators L and N defined by (4.7) and (4.8) will map the space $\mathscr{V}_\varepsilon$ into itself, and for small ε we shall prove that the operator $L + N + F$ is a contraction operator of a certain ball into itself. Hence (4.5) can be solved by successive approximation to yield a unique solution in this ball centered at the zero vector in $\mathscr{V}_\varepsilon$. We shall also prove that the resulting solution is actually unique in a certain larger ball. The solution function v will decay exponentially at infinity, with $|v(r)| \leqq \|v\|_\varepsilon \exp[-(r-1)/\varepsilon]$. These results will follow directly from the following lemmas, Lemma 1 through Lemma 7.

LEMMA 1. *The function u_0 given by* (3.12) *satisfies the inequality*

$$\exp\left(-\frac{1}{\varepsilon}\int_\sigma^r\left[u_0\left(\tau,\frac{\tau-1}{\varepsilon}\right)+\frac{(n-1)\varepsilon}{\tau}\right]d\tau\right)\leqq \mathscr{u}_1\, e^{-(r-\sigma)/\varepsilon}$$

for $1\leqq\sigma\leqq r$, $\varepsilon>0$, *where* $\mathscr{u}_1=4/(1+\alpha)^2$ *if* $\alpha\leqq 1$ *and* $\mathscr{u}_1=1$ *if* $\alpha\geqq 1$.

Proof. From (3.12) and (1.3) we find

$$-\frac{1}{\varepsilon}\int_\sigma^r\left[u_0+\frac{(n-1)\varepsilon}{\tau}\right]d\tau\leqq-\frac{1}{\varepsilon}\int_\sigma^r u_0\,d\tau$$

$$=-\frac{(r-\sigma)}{\varepsilon}+\frac{2(1-\alpha)}{\varepsilon}\int_\sigma^r\frac{e^{-(\tau-1)/\varepsilon}\,d\tau}{(1+\alpha)\tau^{n-1}+(1-\alpha)\,e^{-(\tau-1)/\varepsilon}},$$

from which the stated result follows upon exponentiation since the last term on the right-hand side here is nonpositive if $\alpha\geqq 1$, while if $|\alpha|\leqq 1$ there holds

$$\frac{2(1-\alpha)}{\varepsilon}\int_\sigma^r\frac{e^{-(\tau-1)/\varepsilon}\,d\tau}{(1+\alpha)\tau^{n-1}+(1-\alpha)\,e^{-(\tau-1)/\varepsilon}}\leqq\frac{2(1-\alpha)}{\varepsilon}\int_\sigma^r\frac{e^{-(\tau-1)/\varepsilon}\,d\tau}{1+\alpha+(1-\alpha)\,e^{-(\tau-1)/\varepsilon}}$$

$$=\log\left[\frac{1+\alpha+(1-\alpha)\,e^{-(\sigma-1)/\varepsilon}}{1+\alpha+(1-\alpha)\,e^{-(r-1)/\varepsilon}}\right]^2\leqq\log\left[\frac{4}{(1+\alpha)^2}\right].$$

LEMMA 2. *The function f given by* (4.4) *satisfies the estimate*

$$\left|\int_r^\infty f(\sigma,\varepsilon)\,d\sigma\right|\leqq\mathscr{u}_2\left\{\frac{\varepsilon^2(n-1)\,e^{-(r-1)/\varepsilon}|1-\alpha|}{r^{n+1}}\right.$$
$$\left.+\frac{\varepsilon(n-1)(1-\alpha)^2\,e^{-2(r-1)/\varepsilon}}{(1+\alpha)r^{2n-1}}\left[1+\varepsilon\frac{(2n-3)}{2}\right]\right\}$$

for $r\geqq 1$, $\varepsilon>0$, $n\geqq 2$, *and* $\alpha+1>0$, *where*

$$\mathscr{u}_2=\begin{cases}2/(1+\alpha) & \text{if } \alpha\leqq 1,\\ (1+\alpha)^2/4 & \text{if } \alpha\geqq 1.\end{cases}$$

Proof. For $\sigma\geqq 1$ there holds $(1+\alpha)\sigma^{n-1}+(1-\alpha)\exp[-(\sigma-1)/\varepsilon]\geqq\mathscr{u}_0\sigma^{n-1}$, with $\mathscr{u}_0=1+\alpha$ if $\alpha\leqq 1$ and $\mathscr{u}_0=2$ if $\alpha\geqq 1$. Hence we find from (4.4) the result

$$\left|\int_r^\infty f(\sigma,\varepsilon)\,d\sigma\right|\leqq\frac{4(1+\alpha)(1-\alpha)^2(n-1)}{\mathscr{u}_0^3}\int_r^\infty\frac{\exp[-2(\sigma-1)/\varepsilon]}{\sigma^{2n-1}}\,d\sigma$$
$$+\frac{2\varepsilon(1+\alpha)^2(n-1)}{\mathscr{u}_0^3}|1-\alpha|\int_r^\infty\frac{\exp[-(\sigma-1)/\varepsilon]\,d\sigma}{\sigma^{n+1}}$$
$$+\frac{2\varepsilon(1+\alpha)(1-\alpha)^2(n-1)(2n-3)}{\mathscr{u}_0^3}\int_r^\infty\frac{\exp[-2(\sigma-1)/\varepsilon]}{\sigma^{2n}}\,d\sigma$$

$$\leqq \frac{4(1+\alpha)(1-\alpha)^2(n-1)}{\mathscr{U}_0^3 r^{2n-1}} \int_r^\infty e^{-2(\sigma-1)/\varepsilon}\, d\sigma$$
$$+ \frac{2\varepsilon(1+\alpha)(n-1)}{\mathscr{U}_0^3 r^{n+1}}$$
$$\cdot \left[|1-\alpha|(1+\alpha) + (1-\alpha)^2 \frac{(2n-3)}{r^{n-1}} \right] \int_r^\infty e^{-(\sigma-1)/\varepsilon}\, d\sigma$$

and the desired result then follows easily upon evaluating the integrals.

LEMMA 3. *The function F given by* (4.6) *satisfies the inequality*

$$|F(r,\varepsilon)| \leqq \varepsilon \mathscr{U}_3 \exp\left[-(r-1)/\varepsilon\right]$$

for $r \geqq 1$ and $\varepsilon > 0$, where $\mathscr{U}_3$ is given as $\mathscr{U}_3 = (n-1)|1-\alpha|\mathscr{U}_1\mathscr{U}_2\{(1/n) + |1-\alpha|(1+\alpha)^{-1}[1+\varepsilon(2n-3)/2]\}$ with $\mathscr{U}_1$ and $\mathscr{U}_2$ as in Lemmas 1 *and* 2.

Proof. This result follows directly from (4.6) and Lemma 1 and Lemma 2. We omit the details.

LEMMA 4. *The mapping L defined by* (4.7) *satisfies the inequality $\|Lv\|_\varepsilon \leqq m\varepsilon\mathscr{U}_1(n-1)\|v\|_\varepsilon \leqq \varepsilon\mathscr{U}_1(n-1)\|v\|_\varepsilon$ for any function v in the space $\mathscr{V}_\varepsilon$ with* (*here and below*) *the norm of* (4.9), *and*

$$0 < m = \frac{1}{\varepsilon}\int_1^\infty e^{(\sigma-1)/\varepsilon} \int_\sigma^\infty \frac{\exp\left[-(\nu-1)/\varepsilon\right]}{\nu^2}\, d\nu\, d\sigma < 1.$$

Proof. From (4.7), Lemma 1, and (4.9) we find that

$$|Lv(r)|e^{(r-1)/\varepsilon} \leqq (n-1)\mathscr{U}_1 \int_1^r e^{(\sigma-1)/\varepsilon} \int_\sigma^\infty |v(\nu)|\, e^{(\nu-1)/\varepsilon} \frac{\exp\left[-(\nu-1)/\varepsilon\right]}{\nu^2}\, d\nu\, d\sigma$$
$$\leqq (n-1)\mathscr{U}_1 \|v\|_\varepsilon \int_1^\infty e^{(\sigma-1)/\varepsilon} \int_\sigma^\infty \frac{\exp\left[-(\nu-1)/\varepsilon\right]}{\nu^2}\, d\nu\, d\sigma$$
$$= (n-1)\mathscr{U}_1 \|v\|_\varepsilon \varepsilon m,$$

where

$$0 < m < \frac{1}{\varepsilon}\int_1^\infty \frac{\exp\left[(\sigma-1)/\varepsilon\right]}{\sigma^2} \int_\sigma^\infty e^{-(\nu-1)/\varepsilon}\, d\nu\, d\sigma = 1.$$

LEMMA 5. *The mapping N defined by* (4.8) *satisfies the inequality $\|Nv\|_\varepsilon \leqq \mathscr{U}_1\|v\|_\varepsilon^2/2$ for any function v in $\mathscr{V}_\varepsilon$.*

Proof. From (4.8), Lemma 1, and (4.9) we find the result

$$|Nv(r)|\, e^{(r-1)/\varepsilon} \leqq \frac{\mathscr{U}_1}{2\varepsilon}\int_1^r e^{(\sigma-1)/\varepsilon} v(\sigma)^2\, d\sigma$$
$$\leqq \frac{\mathscr{U}_1\|v\|_\varepsilon^2}{2\varepsilon}\int_1^r e^{-(\sigma-1)/\varepsilon}\, d\sigma \leqq \frac{\mathscr{U}_1\|v\|_\varepsilon^2}{2}.$$

LEMMA 6. *For any positive number ε let $M = M_\varepsilon$ be the mapping defined on $\mathscr{V}_\varepsilon$ by*

$$Mv = Lv + Nv + F \tag{4.10}$$

(*cf.* (4.5)). *Let k be any fixed number with*

$$0 < k < 2/u_1, \tag{4.11}$$

and let $\varepsilon_0 = \varepsilon_0(k) = \min(\delta, 2/(2n-3))$, *where*

$$\delta = \frac{k(2 - u_1 k)}{2(n-1)u_1\{k + |1-\alpha|u_2[(1/n) + 2|1-\alpha|(1+\alpha)^{-1}]\}}. \tag{4.12}$$

Then $M = M_\varepsilon$ *maps the ball* $\{\|v\|_\varepsilon \leqq k\}$ *into itself, uniformly for all* ε *in the interval* $0 < \varepsilon \leqq \varepsilon_0$.

Proof. We first note that δ, and hence also ε_0, is positive for any k satisfying (4.11). It follows now directly from (4.10), Lemma 3, Lemma 4, and Lemma 5 that there holds

$$\|Mv\|_\varepsilon \leqq \frac{u_1 k^2}{2} + \varepsilon_0(n-1)u_1\left\{k + |1-\alpha|u_2\left[\frac{1}{n} + |1-\alpha|(1+\alpha)^{-1}\right.\right.$$
$$\left.\left.\cdot\left(1 + \frac{\varepsilon_0(2n-3)}{2}\right)\right]\right\}$$

for $\|v\|_\varepsilon \leqq k$ and $0 < \varepsilon \leqq \varepsilon_0$. Since $\varepsilon_0(2n-3)/2 \leqq 1$ and $\varepsilon_0 \leqq \delta$, we find from this last inequality and (4.12) the desired result

$$\|Mv\|_\varepsilon \leqq \frac{u_1 k^2}{2} + \delta(n-1)u_1\left\{k + |1-\alpha|u_2\left[\frac{1}{n} + |1-\alpha|(1+\alpha)^{-1}\right.\right.$$
$$\left.\left.\cdot\left(1 + \frac{\varepsilon_0(2n-3)}{2}\right)\right]\right\} = k.$$

LEMMA 7. *The mapping M given by* (4.10) *satisfies the inequality* $\|Mv_1 - Mv_2\|_\varepsilon \leqq u_1[k + m(n-1)\varepsilon]\|v_1 - v_2\|_\varepsilon$ *for any two functions v_1 and v_2 in the ball* $\{\|v\|_\varepsilon \leqq k\}$, *where the quantity m is defined in Lemma* 4.

Proof. Lemma 4 and (4.7) yield the inequality $\|Lv_1 - Lv_2\|_\varepsilon \leqq m\varepsilon u_1(n-1)\|v_1 - v_2\|_\varepsilon$, while (4.8) implies the result

$$Nv_1(r) - Nv_2(r) = -\frac{1}{2\varepsilon}\int_1^r \exp\left(-\frac{1}{\varepsilon}\int_\sigma^r\left[u_0 + \frac{\varepsilon(n-1)}{\tau}\right]d\tau\right)$$
$$\cdot [v_1(\sigma) - v_2(\sigma)][v_1(\sigma) + v_2(\sigma)]\,d\sigma$$

from which we find with Lemma 1 and (4.9) the result $\|Nv_1 - Nv_2\|_\varepsilon \leqq u_1(\|v_1 + v_2\|_\varepsilon\|v_1 - v_2\|_\varepsilon)/2$. Hence we find

$$\|Mv_1 - Mv_2\|_\varepsilon \leqq \|Lv_1 - Lv_2\|_\varepsilon + \|Nv_1 - Nv_2\|_\varepsilon$$
$$\leqq u_1[m\varepsilon(n-1) + (\|v_1 + v_2\|_\varepsilon/2)]\|v_1 - v_2\|_\varepsilon,$$

and the stated result follows immediately from this inequality since

$$\|v_1 + v_2\|_\varepsilon \leqq 2k.$$

As a consequence of the previous lemmas it can now be shown that the integral equation (4.5) can be solved for small ε by successive approximation. For definiteness we shall restrict ε to the interval

$$0 < \varepsilon \leqq \varepsilon_1 = \{2(n-1)u_1 + 4(n-1)u_1^2|1-\alpha|u_2[(1/n) + 2|1-\alpha|(1+\alpha)^{-1}]\}^{-1}. \tag{4.13}$$

For example if $\alpha = 0$, there will hold $\varepsilon_1 \approx 0.003$ if $n = 2$, while $\varepsilon_1 \approx 0.0017$ if $n = 3$.

One can show directly that ε_1 satisfies the inequality $\varepsilon_1 \leqq \varepsilon_0(1/(2u_1))$, where $\varepsilon_0(k)$ is defined in Lemma 6. Since $k = 1/(2u_1)$ satisfies (4.11), it follows by Lemma 6 that the operator M maps the ball $\{\|v\|_\varepsilon \leqq 1/(2u_1)\}$ into itself, uniformly for all ε with $0 < \varepsilon \leqq \varepsilon_1$. Moreover, Lemma 7 implies that there holds

$$\|Mv - Mw\|_\varepsilon \leqq \gamma\|v - w\|_\varepsilon, \qquad \gamma = \tfrac{1}{2} + mu_1(n-1)\varepsilon, \tag{4.14}$$

for any two functions v and w in the stated ball. From (4.13) it follows that $mu_1(n-1)\varepsilon_1 \leqq m/2 < 1/2$ since $0 < m < 1$. Hence γ is less than one, uniformly for all ε $(0 < \varepsilon \leqq \varepsilon_1)$, and the mapping M is a contraction mapping.

We conclude directly now from the fixed-point theorem for contraction operators [cf. Apostol (1962, pp. 483–486)] that *the equation $v = Lv + Nv + F$ has precisely one solution $v = v^*$ in the ball $\{\|v\|_\varepsilon \leqq 1/(2u_1)\}$ for any fixed ε satisfying $0 < \varepsilon \leqq \varepsilon_1$. Moreover this solution v^* can be obtained as the uniform limit of the sequence $\{v_j : j = 0, 1, 2, \cdots\}$ defined recursively as*

$$\begin{aligned} &v_0 = 0, \\ &v_{j+1} = Mv_j = Lv_j + Nv_j + F \quad \text{for } j = 0, 1, 2, \cdots, \end{aligned} \tag{4.15}$$

where v_j satisfies the inequalities

$$\begin{aligned} \|v_j\|_\varepsilon &\leqq \sum_{\nu=0}^{j-1} \|v_{\nu+1} - v_\nu\|_\varepsilon \leqq \|F\|_\varepsilon \sum_{\nu=0}^{j-1} \gamma^\nu \\ &= \|F\|_\varepsilon \frac{1-\gamma^j}{1-\gamma} \quad \text{for } j = 1, 2, 3, \cdots. \end{aligned}$$

If we let j tend toward infinity, the limit function v^* satisfies the inequality $\|v^*\|_\varepsilon \leqq \|F\|_\varepsilon/(1-\gamma)$, which implies in turn with Lemma 3, (4.9) and (4.14) the result

$$\|v^*\|_\varepsilon \leqq \frac{2\varepsilon(n-1)|1-\alpha|u_1u_2\{(1/n) + |1-\alpha|(1+\alpha)^{-1}[1 + \varepsilon(2n-3)/2]\}}{1 - 2u_1(n-1)\varepsilon}, \tag{4.16}$$

where the denominator is positive for $0 < \varepsilon \leqq \varepsilon_1$. This last estimate (4.16) shows that the solution v^* is actually uniformly small, of order ε, as ε tends toward zero.

We remark also that the solution v^*, which is already known to be unique in the ball $\{\|v\|_\varepsilon \leqq 1/(2u_1)\}$, is actually unique in a larger ball $\{\|v\|_\varepsilon < R\}$ with radius R which approaches the value $2/u_1$ as ε tends toward zero. The precise expression for R follows by a direct calculation from (4.16) and the proof of Lemma 7.

Moreover the results of our previous calculations can be shown to remain valid even if we replace the previous norm $\|\cdot\|_\varepsilon$ with the usual supremum norm

$$\|v\| = \sup_{r \geqq 1} |v(r)|,$$

so that *the solution v^* is actually unique in the larger class of functions v in the ball of radius $1/(2\mathscr{u}_1)$ centered at the origin in the vector space $L_\infty(1, \infty)$*. We leave the verification of this last result to the reader. (Again, the radius $1/(2\mathscr{u}_1)$ can be replaced with a larger value R which approaches $2/\mathscr{u}_1$ as ε tends toward zero.)

Finally we can use (4.1) to translate these results for v back into analogous results for the original boundary value problem (1.1), (1.2). In this way we find that we have proved the following result.

THEOREM. *For any ε in the interval given by* (4.13) *the boundary value problem* (1.1), (1.2), (1.3) *has a solution $u^* = u^*(r, \varepsilon)$ which satisfies the uniform estimate* (*for* $1 \leqq r < \infty$)

$$|u^*(r, \varepsilon) - u_0(r, \varepsilon)| \leqq \|v^*\|_\varepsilon \, e^{-(r-1)/\varepsilon} \tag{4.17}$$

with

$$u_0(r, \varepsilon) = \frac{(1 + \alpha)r^{n-1} - (1 - \alpha)\, e^{-(r-1)/\varepsilon}}{(1 + \alpha)r^{n-1} + (1 - \alpha)\, e^{-(r-1)/\varepsilon}}$$

and where $\|v^*\|_\varepsilon$ satisfies the bound (4.16) (*and in particular $\|v^*\|_\varepsilon$ vanishes of order ε as ε tends toward zero*). *Moreover this solution u^* can be obtained as the limit of the uniformly converging sequence of functions $\{u_j : j = 0, 1, 2, \cdots\}$ defined as $u_j(r, \varepsilon) = u_0(r, \varepsilon) + v_j(r, \varepsilon)$, where the functions v_j are defined recursively by* (4.15), (4.6), (4.7), *and* (4.8). *The solution u^* is unique in the class of bounded integrable functions $u = u(r, \varepsilon)$ which satisfy the inequality*

$$\sup_{r \geqq 1} |u(r, \varepsilon) - u_0(r, \varepsilon)| < \frac{1}{(2\mathscr{u}_1)} = \begin{cases} (1 + \alpha)^2/8 & \text{if } \alpha \leqq 1, \\ 1/2 & \text{if } \alpha \geqq 1. \end{cases}$$

(The constant $1/(2\mathscr{u}_1)$ on the right-hand side here can be replaced with the related, larger quantity R mentioned above, where R approaches the value $2/\mathscr{u}_1$ as ε vanishes.)

The inequalities (4.17) and (4.16) verify that the leading term u_0 of the two-variable expansion of § 3 actually provides a uniformly valid approximation to the solution u^* for small ε. In particular (4.17) and (4.16) imply that the solution $u^* = u^*(r, \varepsilon)$ satisfies the result

$$\lim_{\varepsilon \to 0} u^*(r, \varepsilon) = 1 \tag{4.18}$$

for any fixed $r > 1$, so that the boundary condition at infinity is retained while the boundary condition at $r = 1$ is lost in the limit as ε vanishes. The limiting function given by (4.18) for $r > 1$ is the unique smooth solution of the reduced equation

$$u\frac{du}{dr} = 0 \quad \text{for } r > 1$$

which also satisfies the given boundary condition at $r = \infty$,

$$\lim_{r \to \infty} u(r) = 1.$$

Finally we mention that it is possible to improve upon the approximation given by (4.17) by going to higher terms in the two-variable expansion (3.4).

However, the leading term u_0 has sufficed in providing the existence, uniqueness, and behavior (both qualitative and quantitative) of solutions of the original problem, and indeed the leading term u_0 has been entirely adequate for our purposes.

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PERTURBATIONS IN A CLASS OF NONLINEAR ABSTRACT EQUATIONS*

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Abstract. Let V be a real reflexive Banach space and denote the dual of V by V'. Let $\{B_\varepsilon : 0 < \varepsilon \leqq \varepsilon_0\}$ be a family of (possibly) nonlinear operators from V into V' and $\{A_\varepsilon : 0 < \varepsilon \leqq \varepsilon_0\} \subset \mathscr{L}(V, V')$, and let Λ be an unbounded linear operator in V'. Consider the equation $\Lambda A_\varepsilon u_\varepsilon + B_\varepsilon u_\varepsilon = f_\varepsilon \in V'$. It is shown that $\{u_\varepsilon\}$ converges in a certain sense to a solution of $\Lambda u + Bu = f$ provided $A_\varepsilon \to 1$, $B_\varepsilon \to B$ and $f_\varepsilon \to f$ in some appropriate sense, and provided certain other conditions on the operators involved are satisfied. This result is shown to apply to certain nonlinear evolution equations. Two examples are discussed: the first concerns a nonlinear, pseudoparabolic partial differential equation with a small parameter; the second concerns a nonlinear, degenerate parabolic equation with a small parameter.

1. Introduction. Let H be a Hilbert space over R and V a reflexive Banach space with $V \subset H$ algebraically and topologically such that V is dense in H. If V' denotes the dual of V we have the usual inclusions

$$V \subset H \subset V'.$$

Let B_ε $(0 < \varepsilon \leqq 1$ for example) be a nonlinear operator from V into V' and $A_\varepsilon \in \mathscr{L}(V, V')$. Let Λ be a linear operator in H such that $-\Lambda$ generates a continuous semigroup of contractions on H. We consider the equation

$$\Lambda A_\varepsilon u_\varepsilon + B_\varepsilon u_\varepsilon = f_\varepsilon. \tag{1.1}$$

The purpose of this paper is to study the limiting behavior of u_ε $(\varepsilon \to 0_+)$ and to show that u_ε converges to a solution of the equation

$$\Lambda u + Bu = f \tag{1.2}$$

provided $f_\varepsilon \to f$, $B_\varepsilon \to B$ and $A_\varepsilon \to 1$ in some appropriate sense.

The hypotheses we impose to insure the solvability of (1.1) and (1.2) are those of Bardos–Brezis [2] and many of the ideas from that paper play an important role here, although the question of convergence of u_ε was not considered there. If B_ε is *linear* and $\Lambda = d/dt$, the question of convergence of u_ε to u and also its rate of convergence has been more or less settled in [10], [11]. As for the case of nonlinear B_ε, some special cases of equation (1.3) below have been considered by Davis [6] vis-à-vis limiting behavior of solutions.

There are three additional sections. In § 2 we prove a general convergence theorem for solutions of (1.1). In § 3 we apply the results of § 2 to the case where

$$V = L^p(0, T; \mathscr{V}), \qquad H = L^2(0, T; \mathscr{H}), \qquad p \geqq 2,$$

$\Lambda = d/dt$ and where A_ε arises from a positive, self-adjoint operator $\mathscr{A}_\varepsilon \in \mathscr{L}(\mathscr{V}, \mathscr{V}')$; $\mathscr{V}$ is a reflexive Banach space and $\mathscr{H}$ a Hilbert space with properties analogous to those of V and H. In § 4 we apply the results of § 3 to two examples. The first is an

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equation of "pseudoparabolic" type with a small parameter:

$$
\begin{aligned}
(1.3)\qquad &\frac{\partial}{\partial t}\left[u_\varepsilon + \varepsilon \sum_{|i|,|j|\le m} (-1)^i D^i(a_{ij}D^j u_\varepsilon)\right] \\
&\quad + (-1)^m \sum_{i=1}^{N} \frac{\partial^m}{\partial x_i^m}\left(\left|\frac{\partial^m u_\varepsilon}{\partial x_i^m}\right|^{p-2} \frac{\partial^m u_\varepsilon}{\partial x_i^m}\right) = f_\varepsilon \qquad \left(D^i = \frac{\partial^{|i|}}{\partial x_1^{i_1} \cdots \partial x_N^{i_N}}\right).
\end{aligned}
$$

Particular cases of equations of this type ($p \neq 2$) have been studied in [6]. The second example concerns a degenerate parabolic equation with a small parameter:

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\varepsilon}{p-1} \sum_{|i|,|j|\le m} (-1)^{|i|} D^i(a_{ij} D^j |u_\varepsilon|^{p-2} u_\varepsilon) + \frac{1}{p-1}|u_\varepsilon|^{p-2}u_\varepsilon = f.$$

Equations of this type have been studied by Dubinsky [7], Raviart [13] and others. For each of these equations we shall show that u_ε converges (in a sense to be made precise) to a solution of the corresponding unperturbed equation.

2. The convergence theorem. Let V and H be as in § 1. We denote by $(\cdot, \cdot)$ and $|\cdot|$, respectively, the scalar product and norm in H. The norm in V is denoted by $|\cdot|_V$. If $x \in V$ and $f \in V'$, we also use (f, x) to denote their scalar product in the duality between V and V'; (f, x) coincides with their scalar product in H whenever $f \in H$.

We denote by $\mathscr{G}(H)$ the class of operators Λ such that $-\Lambda$ generates a strongly continuous semigroup of contractions in H. If $\Lambda \in \mathscr{G}(H)$ and $G(h)$ is the contraction semigroup generated by $-\Lambda$, it is known that Λ^*, the adjoint of Λ, belongs to $\mathscr{G}(H)$ and that $-\Lambda^*$ is the generator of the semigroup $G^*(h)$. We assume

$$(2.1)\qquad \Lambda \in \mathscr{G}(H),$$

(2.2) $G(h)$ and $G^*(h)$ induce strongly continuous semigroups of operators on V.

Then Λ is *V-regular* in the sense of Bardos–Brezis [2]. Since V is reflexive, the dual semigroups (which act in V') of $G(h)$ and $G^*(h)$ are $G^*(h)$ and $G(h)$ respectively (without abuse of notation; these duals coincide with the adjoints of $G(h)$ and $G^*(h)$ on H) and their generators are $-\Lambda^*$ and $-\Lambda$, the duals of $-\Lambda$ and $-\Lambda^*$. The domain of the generator $-\Lambda$ of the semigroup $G(h)$ in the spaces V, H and V' is denoted by $D(\Lambda; V)$, $D(\Lambda; H)$ and $D(\Lambda; V')$ respectively.

Let B be an operator from V into V'. We assume

(2.3) B is of type M, bounded and coercive,

that is,

(i) *Type M*: for every filter $\{u_i\} \subset V$ such that u_i converges weakly to u in V, Bu_i converges weakly to f in V' and $\limsup (Bu_i, u_i) \leqq (f, u)$, we have $Bu = f$.

(ii) *Bounded*: B maps bounded sets in V into bounded sets in V'.

(iii) *Coercive*: $$\lim_{|v|_V \to \infty} \frac{(B(v + v_0), v)}{|v|_V} = \infty, \quad \forall v_0 \in V.$$

It is known that every bounded, monotone, hemicontinuous operator from V into V' is of type M [12, Prop. 2.2.5].

Assuming (2.1)–(2.3) the problem

$$\Lambda u + Bu = f, \qquad u \in V \cap D(\Lambda, V'), \tag{2.4}$$

has, for each $f \in V'$, at least one solution and uniqueness holds if, in addition, B is strictly monotone [2]. The same is true for the problem

$$\Lambda A_\varepsilon u_\varepsilon + B_\varepsilon u_\varepsilon = f_\varepsilon, \qquad u_\varepsilon \in V, \qquad A_\varepsilon u_\varepsilon \in D(\Lambda, V'), \tag{2.5}$$

provided B_ε is of type M, bounded and coercive, $f_\varepsilon \in V'$ and

$$A_\varepsilon \in \mathscr{L}(V, V') \text{ satisfies } (G(h)A_\varepsilon u, u) \leqq (A_\varepsilon u, u), \qquad h > 0, \quad u \in V. \tag{2.6}$$

To treat the question of convergence of the family $\{u_\varepsilon\}$ we first of all suppose

(2.7) B_ε is of type M and the family $\{B_\varepsilon\}$ is uniformly bounded and uniformly coercive:

(i) *Uniformly bounded*: If S is a bounded set in V then $\bigcup_{\varepsilon>0} B_\varepsilon S$ is a bounded set in V'.

(ii) *Uniformly coercive*:

$$\lim_{|v|_V \to \infty} \inf_\varepsilon \frac{(B_\varepsilon(v + v_0), v)}{|v|_V} = \infty, \quad \forall v_0 \in V.$$

We further assume that $B_\varepsilon \to B$ and A_ε converges to the identity in the following way.

(2.8) For every filter $\{u_i\} \subset V$ $(i \to 0)$ such that $u_i \to u$ weakly in V, $B_i u_i \to f$ weakly in V' and

$$\limsup\, (B_i u_i, u_i) \leqq (f, u),$$

we have

$$Bu = f.$$

(2.9) For every filter $\{v_i\} \subset V$ $(i \to 0)$ such that $v_i \to v$ weakly in V we have $A_i v_i \to v$ weakly in V'.

(2.10) For every filter $\{v_i\} \subset V(i \to 0)$ such that $A_i v_i \in D(\Lambda, V')$, $v_i \to v \in D(\Lambda, V')$ weakly in V and $\Lambda A_i v_i \to \Lambda v$ weakly in V' we have

$$(\Lambda v, v) \leqq \liminf\, (\Lambda A_i v_i, v_i).$$

THEOREM 2.1. *Assume*

(i) Λ *satisfies* (2.1) *and* (2.2).
(ii) B *satisfies* (2.3) *and is strictly monotone.*
(iii) A_ε *satisfies* (2.6), (2.9) *and* (2.10).
(iv) B_ε *satisfies* (2.7) *and* (2.8).
(v) $f_\varepsilon \to f$ *strongly in* V'.

Let u be the unique solution of (2.4) *and, for each ε, let u_ε be a solution of* (2.5). *Then $u_\varepsilon \to u$ weakly in V, $\Lambda A_\varepsilon u_\varepsilon \to \Lambda u$ weakly in V' and*

$$\limsup\,(B_\varepsilon u_\varepsilon - Bu, u_\varepsilon - u) \leqq 0. \tag{2.11}$$

Before proving this theorem, we give two results dealing with hypotheses (2.7) and (2.8).

PROPOSITION 2.1. *Suppose B is of type M and A is a bounded mapping from V into V'. Then $B_\varepsilon = B + \varepsilon A$ satisfies* (2.8). *If in addition B is monotone and A is monotone and hemicontinuous, then B_ε is of type M.*

Proof. If $u_\eta \to u$ weakly in V and $Bu_\eta + \eta Au_\eta \to f$ weakly in V', then $\eta Au_\eta \to 0$ strongly in V' and so $Bu_\eta \to f$ weakly in V'. If also

$$\limsup\,(Bu_\eta + \eta Au_\eta, u_\eta) \leqq (f, u),$$

then $\limsup\,(Bu_\eta, u_\eta) \leqq (f, u)$, hence $Bu = f$ since B is of type M.

Suppose also that B is monotone and A is monotone and hemicontinuous. Then $B + A$ is of type M. In fact, suppose $u_i \to u$ weakly in V, $Bu_i + Au_i \to f$ weakly in V' and $\limsup\,(Bu_i + Au_i, u_i) \leqq (f, u)$. We may suppose $Au_i \to g$ weakly in V'. We have $(Bu_i, u_i) \leqq (Bu_i + Au_i, u_i) - (Au, u_i - u) - (Au_i, u)$ and so

$$\limsup\,(Bu_i, u_i) \leqq (f - g, u).$$

Hence $Bu = f - g$ and $Bu_i \to Bu$ weakly in V'.

Let $v \in V$. We have

$$\begin{aligned}(Bu_i + Au_i - Av, u_i - v) &\geqq (Bu_i, u_i - v)\\ &\geqq (Bu_i - Bu, u - v) + (Bu, u_i - v),\end{aligned}$$

hence,

$$\liminf\,(Bu_i + Au_i - Av, u_i - v) \geqq (Bu, u - v).$$

Therefore,

$$\begin{aligned}(Av, u - v) &\leqq \liminf\,(Bu_i + Au_i, u_i) - (f, v) - (Bu, u - v)\\ &= (f - Bu, u - v), \quad \forall v \in V.\end{aligned}$$

It follows that $Au = f - Bu$ since A is monotone and hemicontinuous. Thus $B + A$ is of type M.

PROPOSITION 2.2. *Let $\{B_\varepsilon\}$ be a family of monotone mappings of V into V'. Let B be a monotone, hemicontinuous mapping of V into V' and suppose $B_\varepsilon \to B$ in the following sense: for every $v \in V$ there is a filter $\{v_\varepsilon\} \subset V$ such that $v_\varepsilon \to v$ strongly in V and $B_\varepsilon v_\varepsilon \to Bv$ strongly in V'.*

Then (2.8) *is satisfied.*

Proof. Let $u_i \to u$ weakly in V, $B_i u_i \to f$ weakly in V' and $\limsup\,(B_i u_i, u_i)$

$\to Bv$ strongly in V'. We have

$$\begin{aligned}(B_iu_i - Bv, u_i - v) = (B_iu_i &- B_iv_i, u_i - v_i)\\ &+ (B_iu_i - B_iv_i, v_i - v)\\ &+ (B_iv_i - Bv, u_i - v).\end{aligned}$$

Hence

$$\liminf (B_iu_i - Bv, u_i - v) \geqq 0$$

so that

$$\begin{aligned}(Bv, u - v) &\leqq \liminf (B_iu_i, u_i) - (f, v)\\ &= (f, u - v), \quad \forall v \in V.\end{aligned}$$

Therefore $Bu = f$ since B is monotone and hemicontinuous.

Proof of Theorem 2.1. Since $(\Lambda A_\varepsilon u_\varepsilon, u_\varepsilon) \geqq 0$,

$$(B_\varepsilon u_\varepsilon, u_\varepsilon) \leqq (f_\varepsilon, u_\varepsilon) \leqq |f_\varepsilon|_{V'}|u_\varepsilon|_V.$$

Since $\{B_\varepsilon\}$ is uniformly coercive and uniformly bounded, it follows that $\{u_\varepsilon\}$ is bounded in V, $\{B_\varepsilon u_\varepsilon\}$ is bounded in V' and thus for some ultrafilter $\{u_\eta\}$ we have

$$\begin{aligned} u_\eta &\to w \quad \text{weakly in } V,\\ B_\eta u_\eta &\to \xi \quad \text{weakly in } V',\\ A_\eta u_\eta &\to w \quad \text{weakly in } V'.\end{aligned}$$

Multiply (2.5) by $v \in D(\Lambda^*, V)$ to obtain

$$(A_\eta u_\eta, \Lambda^* v) + (B_\eta u_\eta, v) = (f_\eta, v). \tag{2.12}$$

(2.12) yields, upon passage to the limit,

$$(w, \Lambda^* v) + (\xi, v) = (f, v), \quad \forall v \in D(\Lambda^*, V).$$

It follows that $w \in D(\Lambda, V')$ and

$$\begin{aligned}\Lambda w = f - \xi &= \operatorname*{weak\,lim}_{(V')} (f_\eta - B_\eta u_\eta)\\ &= \operatorname*{weak\,lim}_{(V')} \Lambda A_\eta u_\eta.\end{aligned} \tag{2.13}$$

We wish to show that $\xi = Bw$. It will then follow from (2.13), by virtue of the uniqueness of solution of (2.4), that $w = u$, $u_\varepsilon \to u$ weakly in V and that $\Lambda A_\varepsilon u_\varepsilon \to \Lambda u$ weakly in V'.

We have from (2.5),

$$(B_\eta u_\eta, u_\eta) = (B_\eta u_\eta, w) + (f_\eta, u_\eta - w) + (\Lambda A_\eta u_\eta, w) - (\Lambda A_\eta u_\eta, u_\eta).$$

Therefore,

$$\limsup (B_\eta u_\eta, u_\eta) \leqq (\xi, w) + (\Lambda w, w) - \liminf (\Lambda A_\eta u_\eta, u_\eta) \leqq (\xi, w)$$

because of (2.10), and hence $Bw = \xi$ in view of (2.8).

To prove (2.11) we note the identity

$$(B_\varepsilon u_\varepsilon - Bu, u_\varepsilon - u) = (f_\varepsilon - f, u_\varepsilon - u) + (\Lambda u, u_\varepsilon - u) + (\Lambda A_\varepsilon u_\varepsilon, u) - (\Lambda A_\varepsilon u_\varepsilon, u_\varepsilon).$$

Therefore,

$$\limsup\, (B_\varepsilon u_\varepsilon - Bu, u_\varepsilon - u) \leqq (\Lambda u, u) - \liminf\, (\Lambda A_\varepsilon u_\varepsilon, u_\varepsilon) \leqq 0.$$

Remark. If B_ε and B are as in Propositions 2.1 or 2.2, it follows from Theorem 2.1 that

$$(B_\varepsilon u_\varepsilon - Bu, u_\varepsilon - u) \to 0. \tag{2.14}$$

In particular, if $B_\varepsilon \equiv B$ is the duality map from V into V', then (2.14) implies $u_\varepsilon \to u$ *strongly* in V [4, p. 33].

3. Application to a class of nonlinear evolution equations. Let H be a real Hilbert space, $\mathscr{V}$ a reflexive Banach space with $\mathscr{V} \subset \mathscr{H}$ algebraically and topologically and $\mathscr{V}$ dense in $\mathscr{H}$. Let $0 < T < \infty$, $p \geqq 2$ and set

$$V = L^p(0, T; \mathscr{V}), \qquad H = L^2(0, T; \mathscr{H}).$$

Then $V \subset H$ algebraically and topologically and V is dense in H. Identifying H with its dual, we have the usual inclusions

$$V = L^p(0, T; \mathscr{V}) \subset H = L^2(0, T; \mathscr{H}) \subset V' = L^{p'}(0, T; \mathscr{V}'),$$

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

We shall use $(\cdot, \cdot)$ to designate both the scalar product in $\mathscr{H}$ and in H as well as the scalar product in the duality between $\mathscr{V}$ and $\mathscr{V}'$ and in the duality between V and V'. The particular meaning intended will be clear from context.

Set

$$\Lambda = \frac{d}{dt}$$

with

$$D(\Lambda, H) = \left\{ u \in H : \frac{du}{dt} \in H, u(0) = 0 \right\}.$$

Then Λ satisfies (2.1) and (2.2) and

$$[G(h)u](t) = \begin{cases} 0, & 0 < t < h, \\ u(t - h), & h < t < T. \end{cases}$$

Let $\mathscr{A}_\varepsilon \in \mathscr{L}(\mathscr{V}, \mathscr{V}')$ be positive and self-adjoint:

$$(\mathscr{A}_\varepsilon v, v) \geqq 0, \qquad \forall v \in \mathscr{V}, \tag{3.1}$$

$$(\mathscr{A}_\varepsilon u, v) = (\mathscr{A}_\varepsilon v, u), \qquad \forall u, v \in \mathscr{V}. \tag{3.2}$$

An unbounded operator $\mathring{\mathscr{A}}_\varepsilon$ in $\mathscr{H}$ may be defined as follows:

$$D(\mathring{\mathscr{A}}_\varepsilon; \mathscr{H}) = \{u \in \mathscr{V} : \mathscr{A}_\varepsilon u \in \mathscr{H}\}$$

and

$$\mathring{\mathscr{A}}_\varepsilon u = \mathscr{A}_\varepsilon u, \qquad \forall u \in D(\mathring{\mathscr{A}}_\varepsilon; \mathscr{H}).$$

We suppose

$$D = \bigcap_{\varepsilon > 0} D(\mathring{\mathscr{A}}_\varepsilon; \mathscr{H}) \quad \text{is dense in } \mathscr{V}. \tag{3.3}$$

It follows from (3.3) that $\mathring{\mathscr{A}}_\varepsilon$ is closable. Let $\bar{\mathring{\mathscr{A}}}_\varepsilon$ denote its closure and assume

$$\bar{\mathring{\mathscr{A}}}_\varepsilon \in \mathscr{G}(\mathscr{H}). \tag{3.4}$$

We may identify $\bar{\mathring{\mathscr{A}}}_\varepsilon$ and $\mathscr{A}_\varepsilon$ without abuse of notation: it is clear that $\mathscr{A}_\varepsilon$ and $\bar{\mathring{\mathscr{A}}}_\varepsilon$ coincide on $\mathscr{V} \cap D(\bar{\mathring{\mathscr{A}}}_\varepsilon; \mathscr{H})$.

By virtue of (3.4) the square root $\mathscr{A}_\varepsilon^{1/2}$ may be defined [8] with domain $D(\mathscr{A}_\varepsilon^{1/2}; \mathscr{H})$ and [2, Prop. V.1]

$$\mathscr{V} \subset D(\mathscr{A}_\varepsilon^{1/2}; \mathscr{H}).$$

We now define $A_\varepsilon \in \mathscr{L}(V, V')$ by

$$(A_\varepsilon u)(t) = \mathscr{A}_\varepsilon(u(t)), \quad \forall u \in V.$$

It is clear that A_ε satisfies properties analogous to those of $\mathscr{A}_\varepsilon$. In addition, hypothesis (2.6) is satisfied [2, p. 386]. Therefore the problems

$$\begin{aligned} &\frac{du}{dt} + Bu = f, \qquad u(0) = 0, \\ &u \in L^p(0, T; \mathscr{V}), \quad \frac{du}{dt} \in L^{p'}(0, T; \mathscr{V}') \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\frac{d}{dt} A_\varepsilon u_\varepsilon + B_\varepsilon u_\varepsilon = f_\varepsilon, \qquad (A_\varepsilon u_\varepsilon)(0) = 0, \\ &u_\varepsilon \in L^p(0, T; \mathscr{V}), \qquad \frac{d}{dt} A_\varepsilon u_\varepsilon \in L^{p'}(0, T; \mathscr{V}') \end{aligned} \tag{3.6}$$

each have solutions provided $f, f_\varepsilon \in L^{p'}(0, T; \mathscr{V}')$ and B, B_ε are of type M, bounded and coercive from $L^p(0, T; \mathscr{V})$ into $L^{p'}(0, T; \mathscr{V}')$.

Remark. It follows from (3.5) that $u \in C(0, T; \mathscr{H})$. From (3.6) we conclude that $A_\varepsilon u_\varepsilon \in C(0, T, \mathscr{V}')$, hence $(A_\varepsilon u_\varepsilon)(0)$ has a sense. The stronger continuity result $A_\varepsilon^{1/2} u_\varepsilon \in C(0, T; \mathscr{H})$, $A_\varepsilon^{1/2} u_\varepsilon(0) = 0$ was proved in [2, Thm. V.2].

THEOREM 3.1. *Assume $\mathscr{A}_\varepsilon$ satisfies* (3.1)–(3.4) *and in addition*

$$\mathscr{A}_\varepsilon v \to v \quad \textit{strongly in } \mathscr{H}, \quad \forall v \in D, \tag{3.7}$$

$$\sup |\mathscr{A}_\varepsilon|_{\mathscr{L}(\mathscr{V}, \mathscr{V}')} < +\infty. \tag{3.8}$$

Suppose B is an operator from $L^p(0, T; \mathscr{V})$ into $L^{p'}(0, T; \mathscr{V}')$, strictly monotone, of type M, bounded and coercive, and that B_ε satisfies (2.7) *and* (2.8). *If $f_\varepsilon \to f$ strongly in $L^{p'}(0, T; \mathscr{V}')$, then*

$$u_\varepsilon \to u \quad \text{weakly in } L^p(0, T; \mathscr{V}), \tag{3.9}$$

$$\frac{d}{dt} A_\varepsilon u_\varepsilon \to \frac{du}{dt} \quad \text{weakly in } L^{p'}(0, T; \mathscr{V}'), \tag{3.10}$$

$$\mathscr{A}_\varepsilon^{1/2} u_\varepsilon(t) \to u(t) \quad \text{weakly in } \mathscr{H}, \qquad 0 \leqq t \leqq T, \tag{3.11}$$

$$\limsup\,(B_\varepsilon u_\varepsilon - Bu, u_\varepsilon - u) \leqq 0. \tag{3.12}$$

The proof utilizes Theorem 2.1 and the following lemma.

LEMMA 3.1. *Assume $\mathscr{A}_\varepsilon$ satisfies* (3.1)–(3.4), (3.6) *and* (3.7). *Then*

(i) *$\mathscr{A}_\varepsilon v \to v$ strongly in $\mathscr{V}'$, $\forall v \in \mathscr{V}$.*

(ii) *$\mathscr{A}_\varepsilon^{1/2}|_{\mathscr{V}} \in \mathscr{L}(\mathscr{V}, \mathscr{H})$ and $\sup |\mathscr{A}_\varepsilon^{1/2}|_{\mathscr{L}(\mathscr{V},\mathscr{H})} < \infty$.*

(iii) *$\mathscr{A}_\varepsilon^{1/2} v \to v$ strongly in $\mathscr{H}$, $\forall v \in \mathscr{V}$.*

Proof.

(i) If $v \in \mathscr{V}$ there is a sequence $\{v_n\} \subset D$ such that $v_n \to v$ strongly in $\mathscr{V}$. Therefore,

$$\begin{aligned} |\mathscr{A}_\varepsilon v - v|_{\mathscr{V}'} &\leqq |\mathscr{A}_\varepsilon(v - v_n)|_{\mathscr{V}'} + |\mathscr{A}_\varepsilon v_n - v_n|_{\mathscr{V}'} + |v_n - v|_{\mathscr{V}'} \\ &\leqq \text{const.}\,(|v - v_n|_{\mathscr{V}} + |\mathscr{A}_\varepsilon v_n - v_n|_{\mathscr{H}}), \\ \limsup |\mathscr{A}_\varepsilon v - v|_{\mathscr{V}'} &\leqq \text{const.}\,|v - v_n|_{\mathscr{V}}, \qquad n = 1, 2, \cdots. \end{aligned}$$

(ii) If $v \in \mathscr{V}$ we have

$$|\mathscr{A}_\varepsilon^{1/2} v|^2 = (\mathscr{A}_\varepsilon v, v) \leqq |\mathscr{A}_\varepsilon|_{\mathscr{L}(\mathscr{V},\mathscr{V}')} |u|_{\mathscr{V}}^2.$$

(iii) Suppose $v \in D$. Then (see [9])

$$\mathscr{A}_\varepsilon^{1/2} v = \frac{1}{\pi} \int_0^\infty \lambda^{-1/2} (\mathscr{A}_\varepsilon + \lambda)^{-1} \mathscr{A}_\varepsilon v \, d\lambda,$$

$$\mathscr{A}_\varepsilon^{1/2} v - v = \frac{1}{\pi} \int_0^\infty \lambda^{1/2} (\lambda + 1)^{-1} (\mathscr{A}_\varepsilon + \lambda)^{-1} (\mathscr{A}_\varepsilon v - v) \, d\lambda.$$

Since $|(\mathscr{A}_\varepsilon + \lambda)^{-1}|_{\mathscr{L}(\mathscr{H},\mathscr{H})} \leqq 1/\lambda$, we have

$$|\mathscr{A}_\varepsilon^{1/2} v - v| \leqq |\mathscr{A}_\varepsilon v - v|, \qquad v \in D.$$

Thus (iii) is proved for $v \in D$. The general case is now easily proved with the aid of (ii) and (3.3).

Proof of Theorem 3.1. We first verify hypothesis (2.9). Thus let $\{v_i\} \subset L^p(0, T; \mathscr{V})$, $v_i \to v$ weakly in $L^p(0, T; \mathscr{V})$. Since

$$(A_i v_i, w) = (A_i w, v_i), \quad \forall w \in V,$$

it suffices to show that $A_i w \to w$ strongly in V', $\forall w \in V$. We have

$$|A_i w - w|_{V'} = \left[\int_0^T |\mathscr{A}_i w(t) - w(t)|_{\mathscr{V}'}^{p'} \, dt \right]^{1/p'}. \tag{3.13}$$

By (3.8) and Lemma 3.1,

$$|\mathscr{A}_i w(t) - w(t)|_{\mathscr{V}'}^{p'} \leqq \text{const.}\, |w(t)|_{\mathscr{V}}^{p'},$$
$$|\mathscr{A}_i w(t) - w(t)|_{\mathscr{V}'} \to 0, \qquad 0 < t < T.$$

Thus by the dominated convergence theorem the limit in (3.13) may be taken inside the integral and so $A_i w \to w$ strongly in V'.

We next verify hypothesis (2.10). Suppose therefore that $\{v_i\} \subset V, \cdot A_i v_i \in D(\Lambda, V'), v_i \to v \in D(\Lambda, V')$ weakly in V and $\Lambda A_i v_i \to \Lambda v$ weakly in V'. Let $w \in \mathscr{V}$ and X_t be the characteristic function of $[0, t]$. Then

$$(3.14)\qquad \begin{aligned}(\mathscr{A}_i^{1/2} v_i(t), \mathscr{A}_i^{1/2} w) &= \int_0^T \left(\frac{d}{ds}\mathscr{A}_i v_i(s), X_t(s) w\right) ds \\ &= (\Lambda A_i v_i, X_t w) \to (\Lambda v, X_t w) \\ &= (v(t), w), \qquad 0 \leqq t \leqq T.\end{aligned}$$

In addition,

$$(3.15)\qquad \begin{aligned}|\mathscr{A}_i^{1/2} v_i(t)|^2 &= 2\int_0^T \left(\frac{d}{ds}\mathscr{A}_i v_i(s), X_t(s) v_i(s)\right) ds \\ &\leqq 2|\Lambda A_i v_i|_{V'} |v_i|_V < \infty.\end{aligned}$$

Since $\mathscr{A}_i^{1/2} w \to w$ strongly in $\mathscr{H}$, $\forall w \in \mathscr{V}$, it follows from (3.14) and (3.15) that

$$(\mathscr{A}_i^{1/2} v_i(t), w) \to (v(t), w), \quad \forall w \in \mathscr{V}, \quad 0 \leqq t \leqq T.$$

But $\mathscr{V}$ is dense in $\mathscr{H}$ and therefore we may conclude that

$$\mathscr{A}_i^{1/2} v_i(t) \to v(t) \quad \text{weakly in } \mathscr{H}, \qquad 0 \leqq t \leqq T.$$

Therefore,

$$\begin{aligned}(\Lambda v, v) = \tfrac{1}{2}|v(T)|^2 &\leqq \tfrac{1}{2}\liminf |\mathscr{A}_i^{1/2} v_i(T)|^2 \\ &= \liminf (\Lambda A_i v_i, v_i)\end{aligned}$$

as was to be proved.

(3.9), (3.10) and (3.12) now follow immediately from Theorem 2.1, and (3.11) follows from (3.9) and (3.10) by the same argument just given for the verification of (2.10).

4. Examples.

Example 1. *Nonlinear pseudoparabolic equations with a small parameter.* Let Ω be a bounded open set in R^N with smooth boundary, m a nonnegative integer and $p \geqq 2$. Set

$$\mathscr{V} = W_0^{m,p}(\Omega), \qquad \mathscr{H} = L^2(\Omega).$$

The norm in $\mathscr{V}$ is written $|\cdot|_{m,p}$. $\mathscr{V}'$ may be identified with the space $W^{-m,p'}(\Omega)$. Define a linear operator $\mathscr{A} \in \mathscr{L}(\mathscr{V}, \mathscr{V}')$ by

$$(4.1)\qquad \mathscr{A}u = \sum_{|i|,|j| \leqq m} (-1)^{|i|} D^i(a_{ij}(x) D^j u), \qquad a_{ij} = a_{ji},$$

where $a_{ij} \in C^{|i|}(\bar{\Omega})$, and set

$$\mathscr{A}_\varepsilon = I + \varepsilon\mathscr{A},$$

where I is the injection of $\mathscr{V}$ into $\mathscr{V}'$. For $u, v \in \mathscr{V}$, we have

$$(\mathscr{A}u, v) = \sum_{|i|,|j| \leqq m} \int_\Omega a_{ij} D^j u D^i v \, dx \equiv a(u, v)$$

and we assume

$$a(v, v) \geqq \alpha |v|_{m,2}^2, \quad \forall v \in W_0^{m,2}(\Omega), \tag{4.2}$$

where $\alpha > 0$ is a constant. It then follows that

$$D(\mathring{\mathscr{A}}_\varepsilon; \mathscr{H}) = W^{2m,2}(\Omega) \cap W_0^{m,p}(\Omega),$$
$$D(\mathscr{A}_\varepsilon; \mathscr{H}) = W^{2m,2}(\Omega) \cap W_0^{m,2}(\Omega),$$
$$D(\mathscr{A}_\varepsilon^{1/2}; \mathscr{H}) = W_0^{m,2}(\Omega)$$

and that $\mathring{\mathscr{A}}_\varepsilon \in \mathscr{G}(\mathscr{H})$. Clearly $\mathscr{A}_\varepsilon v \to v$ strongly in $\mathscr{H}$ for each $v \in D(\mathscr{A}_\varepsilon; \mathscr{H})$ and

$$|(\mathscr{A}_\varepsilon u, v)| \leqq |(u, v)_{L^2(\Omega)}| + \varepsilon \cdot \text{const.}\, |u|_{m,2} |v|_{m,2}, \quad \forall u, v \in V.$$

Therefore,

$$|\mathscr{A}_\varepsilon|_{\mathscr{L}(\mathscr{V}, \mathscr{V}')} \leqq \text{const.}\,(1 + \varepsilon).$$

Let $B: L^p(0, T; W_0^{m,p}(\Omega)) \to L^{p'}(0, T; W^{-m,p'}(\Omega))$ be defined by

$$Bv = (-1)^m \sum_{i=1}^N \frac{\partial^m}{\partial x_i^m} \left(\left| \frac{\partial^m v}{\partial x_i^m} \right|^{p-2} \frac{\partial^m v}{\partial x_i^m} \right).$$

B is strictly monotone, hemicontinuous, bounded and coercive (Brezis–Sibony [3]). We may therefore conclude, by virtue of Theorem 3.1, that for each $f_\varepsilon \in L^{p'}(0, T; W^{-m,p'}(\Omega))$ the unique solution $u_\varepsilon \in L^p(0, T; W_0^{m,p}(\Omega))$ of the problem

$$\begin{gathered} \frac{\partial}{\partial t}\left(u_\varepsilon + \varepsilon \sum_{|i|,|j| \leqq m} (-1)^{|i|} D^i (a_{ij} D^j u) \right) \\ + (-1)^m \sum_{i=1}^N \frac{\partial^m}{\partial x_i^m} \left(\left| \frac{\partial^m u_\varepsilon}{\partial x_i^m} \right|^{p-2} \frac{\partial^m u_\varepsilon}{\partial x_i^m} \right) = f_\varepsilon \quad \text{on } \Omega \times (0, T], \\ (1 + \varepsilon\mathscr{A})^{1/2} u_\varepsilon(x, 0) = 0 \quad \text{on } \Omega \end{gathered} \tag{4.3$_\varepsilon$}$$

satisfies the following: If $f_\varepsilon \to f$ strongly in $L^{p'}(0, T; W^{-m,p'}(\Omega))$, then

$$u_\varepsilon \to u \quad \text{weakly in } L^p(0, T; W_0^{m,p}(\Omega)),$$

$$\frac{\partial}{\partial t}(I + \varepsilon\mathscr{A}) u_\varepsilon \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^{p'}(0, T; W^{-m,p'}(\Omega)),$$

$$(1 + \varepsilon\mathscr{A})^{1/2} u_\varepsilon(t) \to u(t) \quad \text{weakly in } L^2(\Omega), \qquad 0 \leqq t \leqq T,$$

$$\sum_{i=1}^N \int_0^T \int_\Omega \left(\left| \frac{\partial^m u_\varepsilon}{\partial x_i^m} \right|^{p-2} \frac{\partial^m u_\varepsilon}{\partial x_i^m} - \left| \frac{\partial^m u}{\partial x_i^m} \right|^{p-2} \frac{\partial^m u}{\partial x_i^m} \right) \left(\frac{\partial^m u_\varepsilon}{\partial x_i^m} - \frac{\partial^m u}{\partial x_i^m} \right) dx \, dt \to 0,$$

where u is the unique solution of $(4.3)_0$ with vanishing initial data.

Remark. If $m = 1$, the operator B is the duality map of $L^p(0, T; W_0^{m,p}(\Omega))$ onto $L^{p'}(0, T; W^{-m,p'}(\Omega))$ with respect to the function $\phi(r) = r^{p-1}$. We may therefore conclude from the last convergence statement that

$$u_\varepsilon \to u \quad \text{strongly in } L^p(0, T; W_0^{1,p}(\Omega))$$

provided $m = 1$.

In the linear case equation $(4.3)_\varepsilon$ is sometimes called "pseudoparabolic" (cf. [5], [14]) since boundary value problems which are well-posed for parabolic equations are generally well-posed for $(4.3)_\varepsilon$ and solutions of the two classes of equations generally have the same qualitative (e.g., regularity) properties.

Example 2. Degenerate parabolic equations with a small parameter. Let Ω be a bounded open set in R^N with smooth boundary and $\mathscr{A}$ be the differential operator (4.1) and satisfying (4.2). Then $I + \varepsilon\mathscr{A}$ is a bijection of $W^{2m,q}(\Omega) \cap W_0^{m,q}(\Omega)$ onto $L^q(\Omega)$ for each $q \in (1, \infty)$ and

$$|(I + \varepsilon\mathscr{A})^{-1}u|_{L^q(\Omega)} \leqq C_q|u|_{L^q(\Omega)}, \quad \forall u \in L^q(\Omega),$$

where we may choose $C_2 = 1$ (cf. Agmon [1]). In addition, $(1 + \varepsilon\mathscr{A})^{-1}u \to u$ strongly in $L^q(\Omega)$ for each $u \in L^q(\Omega)$ (Hille–Phillips [8, Lemma 12.2.1]). Let $p \geqq 2$ and set

$$\mathscr{V} = L^p(\Omega), \qquad \mathscr{H} = L^2(\Omega)$$

and

$$\mathscr{A}_\varepsilon u = (I + \varepsilon\mathscr{A})^{-1}u, \quad Bu = \frac{1}{p-1}|u|^{p-2}u, \quad \forall u \in L^p(\Omega).$$

$\mathscr{A}_\varepsilon$ is a linear, bounded, positive, self-adjoint operator on $\mathscr{H}$ and B is the duality map of $L^p(0, T; L^p(\Omega))$ onto $L^{p'}(0, T; L^{p'}(\Omega))$ relative to the function $\phi(r) = r^{p-1}/(p-1)$. Theorem 3.1 applies and we may conclude that for each $f \in L^{p'}(0, T; L^{p'}(\Omega))$ the unique solution $u_\varepsilon \in L^p(0, T; L^p(\Omega))$ of the problem

$$(4.4)_\varepsilon \quad \begin{aligned} &\frac{\partial}{\partial t}(I + \varepsilon\mathscr{A})^{-1}u_\varepsilon + \frac{1}{p-1}|u_\varepsilon|^{p-2}u_\varepsilon = (I + \varepsilon\mathscr{A})^{-1}f \quad \text{on } \Omega \times (0, T], \\ &(1 + \varepsilon\mathscr{A})^{-1/2}u_\varepsilon(x, 0) = 0 \quad \text{on } \Omega, \end{aligned}$$

satisfies the following:

$$u_\varepsilon \to u \quad \textit{strongly} \text{ in } L^p(0, T; L^p(\Omega)),$$

$$\frac{\partial}{\partial t}(I + \varepsilon\mathscr{A})^{-1}u_\varepsilon \to \frac{\partial u}{\partial t} \quad \text{weakly in } L^{p'}(0, T; L^{p'}(\Omega)),$$

$$(I + \varepsilon\mathscr{A})^{-1/2}u_\varepsilon(t) \to u(t) \quad \text{weakly in } L^2(\Omega), \qquad 0 \leqq t \leqq T,$$

where u is the unique solution of $(4.4)_0$ with vanishing initial data. As in [2, p. 393] we observe that $(4.4)_\varepsilon$ indicates that u_ε is a weak solution of the degenerate parabolic equation

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\varepsilon}{p-1} \sum_{|i|,|j| \leqq m} (-1)^{|i|} D^i(a_{ij} D^j(|u_\varepsilon|^{p-2} u_\varepsilon)) + \frac{1}{p-1} |u_\varepsilon|^{p-2} u_\varepsilon = f \quad \text{in } \Omega \times (0, T]$$

satisfying the Dirichlet boundary conditions

$$\frac{\partial^j u}{\partial n^j} = 0 \quad \text{on } \partial\Omega \times (0, T], \qquad j = 0, 1, \cdots, m-1,$$

and the initial condition

$$u_\varepsilon(x, 0) = 0 \quad \text{on } \Omega.$$

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A DISTRIBUTIONAL APPROACH TO DUAL INTEGRAL EQUATIONS OF TITCHMARSH TYPE*

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Abstract. The existence of solutions to dual integral equations of Titchmarsh type is rigorously proved using the theory of generalized functions. Generalizations of the relevant classical operators to certain spaces of generalized functions are effected, and a generalized dual relation problem is formulated and solved. A regularity theorem which essentially solves the classical problem is then proved. Finally, under rather severe restrictions, a uniqueness result is also obtained.

1. Introduction. The subject of this work is existence and uniqueness questions for a class of dual relations on the positive real line which involve Hankel transforms. These equations, sometimes referred to as equations of Titchmarsh type, have kernels that are Bessel functions of the first kind whose orders are real numbers not less than $-\frac{1}{2}$. There have been numerous investigations of these relations (for a detailed survey see Sneddon [6]) most of which have been purely formal manipulations and dealt with special cases. Two exceptions are the doctoral dissertation of I. W. Busbridge [1] and the work of Erdélyi and Sneddon (Erdélyi and Sneddon [4], Sneddon [6]). Busbridge solves rigorously the special case in which the orders of the Bessel functions are equal and the equation over the infinite interval is homogeneous, whereas, Erdélyi and Sneddon present a formal solution of the general problem using operators of fractional integration. It is the intent of this paper to combine the approach of Erdélyi and Sneddon with the theory of generalized functions to effect a rigorous treatment of the general problem.

In the next section the classical problem is stated and the relevant operators are defined. Section 3 contains a formulation of the generalized problem and the introduction of the appropriate topological spaces. A theorem guaranteeing the existence of solutions to the general problem is proved in § 4, and a regularity theorem which essentially solves the classical problem is proved in § 5. Section 6 contains an example, and the final section is concerned with a special case for which a uniqueness result is obtained. The notation and terminology used for the classical operators is consistent with Sneddon [6] and for the generalized functions with Zemanian [8].

2. The classical problem. The relevant operators are introduced in the following definitions. For any real number μ, J_μ will denote the μth ordered Bessel function of the first kind, and $L^1(a, b)$ will denote the set of all equivalence classes of functions Lebesgue integrable on the interval (a, b).

DEFINITION 2.1. The *ordinary Hankel transform*, $\mathscr{H}_\mu\{f(t); x\}$, is defined by the equation

$$\mathscr{H}_\mu\{f(t); x\} \equiv \int_0^\infty (xt)^{1/2} J_\mu(xt) f(t)\, dt, \tag{2.1}$$

valid for any real number μ whenever the integral exists.

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DEFINITION 2.2. For any real numbers η and α, the *modified Hankel transform* $S_{\eta,\alpha}\{f(t);x\}$ is defined by the equation

$$S_{\eta,\alpha}\{f(t);x\} \equiv 2^{\alpha}x^{-\alpha}\int_0^\infty t^{1-\alpha}f(t)J_{2\eta+\alpha}(xt)\,dt, \tag{2.2}$$

whenever the integral converges.

From (2.1) and (2.2) it is clear that

$$S_{\eta,\alpha}\{f(t);x\} = 2^{\alpha}x^{-\alpha-1/2}\mathscr{H}_{2\eta+\alpha}\{t^{1/2-\alpha}f(t);x\}. \tag{2.3}$$

DEFINITION 2.3. If η and α are real numbers and $\alpha > 0$, then the *fractional integrals* $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined by the equations

$$I_{\eta,\alpha}\{f(t);x\} = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-t^2)^{\alpha-1}t^{2\eta+1}f(t)\,dt, \tag{2.4}$$

$$K_{\eta,\alpha}\{f(t);x\} = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (t^2-x^2)^{\alpha-1}t^{-2\eta-2\alpha+1}f(t)\,dt, \tag{2.5}$$

where $\Gamma(\alpha)$ denotes the gamma function.

If $\alpha = 0$, $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined to be the identity operators, that is, $I_{\eta,0}(f) = K_{\eta,0}(f) = f$.

DEFINITION 2.4. When $\alpha < 0$ the *fractional derivatives* $I_{\eta,\alpha}(f)$ and $K_{\eta,\alpha}(f)$ are defined to be functions $F(x)$ and $G(x)$, respectively, satisfying the Abel integral equations

$$f(t) = I_{\eta+\alpha,-\alpha}\{F(x);t\},$$
$$f(t) = K_{\eta+\alpha,-\alpha}\{G(x);t\}.$$

These operators of fractional integration are slight variations, due to Sneddon [6], of those introduced by Erdélyi and Kober (Erdélyi and Kober [3], Erdélyi [2], Kober [5]). However, it is a simple matter to extend their results to these operators. The pertinent properties are summarized below.

(a) If $\alpha > 0, \eta > -\frac{1}{2}$ and $f(t) \in L^1(0,\infty)$, then both $I_{\eta,\alpha}\{f(t);x\}$ and $K_{\eta,\alpha}\{f(t);x\}$ are also in the space $L^1(0,\infty)$.

(b) If $\alpha < 0, \eta + \alpha > -\frac{1}{2}$ and $f(t) \in L^1(0,\infty)$, then the fractional derivatives $I_{\eta,\alpha}(f)$ and $K_{\eta,\alpha}(f)$ are uniquely determined in the space $L^1(0,\infty)$ whenever they exist. We shall also have need of the trivial identities

$$x^{2\delta}I_{\eta,\alpha}\{f(t);x\} = I_{\eta-\delta,\alpha}\{t^{2\delta}f(t);x\},$$
$$x^{2\delta}K_{\eta,\alpha}\{f(t);x\} = K_{\eta+\delta,\alpha}\{t^{2\delta}f(t);x\}.$$

The classical problem involving dual relations is that of finding a function $\psi(t)$ such that

$$\int_0^\infty t^{-2\alpha}\psi(t)J_\mu(tx)\,dt = F(x), \qquad 0 < x < 1, \tag{2.6}$$

$$\int_0^\infty t^{-2\beta}\psi(t)J_\nu(tx)\,dt = G(x), \qquad 1 < x < \infty, \tag{2.7}$$

where F and G are prescribed functions and α, β, μ and ν are real numbers. By making the substitutions $\psi(t) = A(t)t$, $f_1(x) = 2^{2\alpha}x^{-2\alpha}F(x)$ and $g_2(x) = 2^{2\beta}x^{-2\beta}G(x)$, it is seen easily that (2.6) and (2.7) are equivalent to the operator equations

$$S_{1/2\mu-\alpha,2\alpha}\{A(t);x\} = f_1(x), \qquad 0 < x < 1, \tag{2.8}$$

$$S_{1/2\nu-\beta,2\beta}\{A(t);x\} = g_2(x), \qquad 1 < x < \infty. \tag{2.9}$$

A formal solution to (2.8) and (2.9) has been known for several years (see Sneddon [6]). The technique involves using the Erdélyi–Kober operators to factor the modified Hankel transform. More precisely, let

$$\lambda = 1/2(\mu + \nu) - (\alpha - \beta), \tag{2.10}$$

and let $H(x)$ by the Heaviside step function equal to 1 for $x > 0$, and vanishing for $x < 0$. Then using the relations

$$I_{\eta+\alpha,\beta}S_{\eta,\alpha} = S_{\eta,\alpha+\beta}, \tag{2.11}$$

$$K_{\eta,\alpha}S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}, \tag{2.12}$$

first discussed by Erdélyi and Kober [3], we may deduce from (2.8), (2.9), (2.11) and (2.12) that

$$S_{1/2\mu-\alpha,\lambda-\mu+2\alpha}\{A(t);x\} = h(x), \qquad 0 < x < \infty, \tag{2.13}$$

where

$$h(x) = H(1 - x)I_{1/2\mu+\alpha,\lambda-\mu}\{f_1(t);x\} + H(x - 1)K_{1/2\mu-\alpha,\nu-\lambda}\{g_2(t);x\}. \tag{2.14}$$

Therefore, providing the Fredholm equation (2.13) is invertible, we obtain the solution

$$A(t) = S^{-1}_{1/2\mu-\alpha,\lambda-\mu+2\alpha}\{h(x);t\}. \tag{2.15}$$

Using the obvious fact that

$$S^{-1}_{\eta,\alpha} = S_{\eta+\alpha,-\alpha},$$

(2.15) is seen to be equivalent to

$$A(t) = S_{1/2\mu+\beta,\lambda-\nu-2\beta}\{h(x);t\}. \tag{2.16}$$

These operations are purely formal. A rigorous proof that (2.16) satisfies (2.8) and (2.9) is given in § 4 and § 5 within the framework of the generalized function spaces discussed in the next section.

3. The generalized problem. The spaces presented here were developed by Zemanian [8] to treat the ordinary Hankel transform, and for a detailed discussion, the reader is referred to that reference.

DEFINITION 3.1. For each real number μ, let H_μ be the set of all functions $\phi(x)$ which are defined on $0 < x < \infty$, complex-valued, infinitely differentiable and such that for each pair of nonnegative integers m and k the quantity

$$\gamma^{\mu}_{m,k}(\phi) \equiv \sup_{0<x<\infty} |x^m(x^{-1}D)^k[x^{-\mu-1/2}\phi(x)]|$$

is finite.

The following facts due to Zemanian [8] are stated without proof.

(a) Each $\gamma^{\mu}_{m,k}$ is a seminorm on H_μ.

(b) The collection $\{\gamma^{\mu}_{m,k}\}^{\infty}_{m,k=0}$ is a multinorm on H_μ.

(c) With the topology generated by the multinorm, H_μ is a Fréchet space.

(d) Each $\phi(x) \in H_\mu$ is of rapid descent at infinity, that is, for any $N > 0$

$$\lim_{x\to\infty} x^N\phi(x) = 0.$$

We are now permitted to make the following definition.

DEFINITION 3.2. Let H'_μ denote the dual space of H_μ with the weak topology.

Remark. It is easy to see that if $f(x)$ is locally integrable on $0 < x < \infty$, of slow growth at infinity (that is, $f(x) = O(x^n)$ as $x \to \infty$ for some real number n) and if $x^{\mu+1/2}f(x) \in L^1(0, 1)$, then $f(x)$ generates a regular generalized function in H'_μ by $\langle f, \phi\rangle = \int_0^\infty f(x)\phi(x)\,dx$ for each $\phi(x) \in H_\mu$. From this, we easily obtain that $L^1(0, \infty) \subset H'_\mu$, when $\mu \geqq -\frac{1}{2}$. Use will also be made of the trivial observation that the map $\phi(x) \to x^n\phi(x)$ defines an isomorphism from H_μ onto $H_{\mu+n}$ for any real n. The next theorem, due to Zemanian [8], is the fundamental result.

THEOREM 3.3. *For $\mu \geqq -\frac{1}{2}$, the ordinary Hankel transform, $\mathscr{H}_\mu$, is an automorphism on H_μ, with $\mathscr{H}_\mu^{-1} = \mathscr{H}_\mu$.*

The generalized Hankel transform is defined as the adjoint of $\mathscr{H}_\mu$, that is, for each $\phi \in H_\mu$ and $f \in H'_\mu$,

$$\langle \mathscr{H}'_\mu f, \phi\rangle = \langle f, \mathscr{H}_\mu\phi\rangle.$$

Then clearly $\mathscr{H}'_\mu$ is an automorphism on H'_μ with $\mathscr{H}'^{-1}_\mu = \mathscr{H}'_\mu$.

The modified Hankel transform is related to the space H_μ by the following.

THEOREM 3.4. *If $2\eta + \alpha \geqq -\frac{1}{2}$, then $S_{\eta,\alpha}$ is an isomorphism from $H_{2\eta+2\alpha-1/2}$ onto $H_{2\eta-1/2}$ with $S^{-1}_{\eta,\alpha} = S_{\eta+\alpha,-\alpha}$.*

Proof. The proof is straightforward from (2.3) and Theorem 3.3.

DEFINITION 3.5. Let $2\eta + \alpha \geqq -1/2$. Then $S'_{\eta,\alpha}$ is defined to be the *adjoint* of $S_{\eta,\alpha}$, that is, $S'_{\eta,\alpha}$ is that mapping from $H'_{2\eta-1/2}$ onto $H'_{2\eta+2\alpha-1/2}$ defined for each $\phi \in H_{2\eta+2\alpha-1/2}$ and $f \in H'_{2\eta-1/2}$ by the equation

$$\langle S'_{\eta,\alpha}f, \phi\rangle = \langle f, S_{\eta,\alpha}\phi\rangle.$$

Clearly $S'_{\eta,\alpha}$ is an isomorphism with $S'^{-1}_{\eta,\alpha} = S'_{\eta+\alpha,-\alpha}$. Within this framework we may now formulate the generalization of (2.8) and (2.9). In the general problem, f_1 and g_2 are measurable, and we seek a regular generalized function $A(t)$ such that $S'_{1/2\mu-\alpha,2\alpha}\{A(t); x\}$ and $S'_{1/2\nu-\beta,2\beta}\{A(t); x\}$ are also regular generalized functions and such that

$$S'_{1/2\mu-\alpha,2\alpha}\{A(t); x\} = f_1(x), \qquad 0 < x < 1, \tag{3.1}$$

$$S'_{1/2\nu-\beta,2\beta}\{A(t); x\} = g_2(x), \qquad 1 < x < \infty, \tag{3.2}$$

where in (3.1) and (3.2) we have equality almost everywhere. The requirement that $A(t)$ be a regular generalized function is necessary since the domains of $S'_{1/2\mu-\alpha,2\alpha}$ and $S'_{1/2\nu-\beta,2\beta}$ are $H'_{\mu-2\alpha-1/2}$ and $H'_{\nu-2\beta-1/2}$, respectively. In general they are not the same, and it is not at all clear how a singular generalized function can be in $H'_{\mu-2\alpha-1/2} \cap H'_{\nu-2\beta-1/2}$. In the next section this problem is shown to possess a solution.

4. Existence theorem.

THEOREM 4.1. *If* 1. $f_1(x)$ *and* $g_2(x)$ *are measurable functions on* $0 < x < 1$ *and* $1 < x < \infty$, *respectively,*

2. $x^{\nu+2\beta}h(x) \in L^1(0, \infty)$, *where* $h(x)$ *is given by*

$$h(x) = H(1 - x)I_{1/2\mu+\alpha-1/2,\lambda-\mu}\{f_1(t); x\} + H(x - 1)K_{1/2\mu-\alpha+1/2,\nu-\lambda}\{g_2(t); x\},$$

3. $\lambda \geqq -1/2$, 4. $\mu \geqq -1/2$, 5. $\nu \geqq 1/2$, *and either*

6. $\lambda = \mu$ *or*

7. $\mu - \lambda > 0$ *and* $x^{\nu+2\beta-2\delta}h(x) \in L^1(0, \infty)$ *for arbitrarily small* $\delta > 0$ *or*

8. (a) $\mu - \lambda < 0$,
(b) $2\mu - \lambda > -3/2$ *and*
(c) *the Abel equation*

$$h(t)t^{\mu+2\alpha-\delta} = I_{1/2\delta-1/2,\lambda-\mu}\{H(x); t\}$$

possesses a solution $H(x) \in L^1(9, \infty)$, *where* $\delta > 0$ *is arbitrarily small and either*

9. $\lambda - \nu = 0$ *or*

10. $\lambda - \nu > 0$ *or*

11. (a) $\lambda - \nu < 0$ *and*
(b) *the Abel equation*

$$h(t)t^{\nu+2\beta} = K_{\lambda+1/2,\nu-\lambda}\{G(x); t\}$$

has a solution $G(x) \in L^1(0, \infty)$,

then

(i)

$$A(t) = tS_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x}; t\right\} \tag{4.1}$$

is a regular generalized function in $H'_{\mu-2\alpha-1/2}$ *and* $H'_{\nu-2\beta-1/2}$,

(ii) $S'_{1/2\mu-\alpha,2\alpha}(A)$ *is a regular generalized function in* $H'_{\mu+2\alpha-1/2}$ *which equals* $f_1(x)$ *for almost all* $x \in (0, 1)$,

(iii) $S'_{1/2\nu-\beta,2\beta}(A)$ *is a regular generalized function in* $H'_{\nu+2\beta-1/2}$ *which equals* $g_2(x)$ *for almost all* $x \in (1, \infty)$.

Proof. Since the Erdélyi–Kober operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are defined differently for $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$, it is necessary to distinguish special cases in Theorem 4.1, and therefore, the proof is presented in a series of lemmas. The first lemma verifies (i).

LEMMA 4.2. *If assumptions* 1, 2 *and* 3 *in Theorem* 4.1 *hold, then* (i) *is valid.*

Proof. Expanding $A(t)$ we obtain

$$
\begin{aligned}
A(t) &= tS_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x};t\right\} \\
&= 2^{\lambda-\nu-2\beta}t^{\nu+2\beta-\lambda+1/2}\int_0^\infty \sqrt{xt}J_\lambda(xt)x^{\nu+2\beta-\lambda-1/2}h(x)\,dx \qquad (4.2)\\
&= 2^{\lambda-\nu-2\beta}t^{\nu+2\beta+1}\int_0^\infty \frac{J_\lambda(xt)}{(xt)^\lambda}x^{\nu+2\beta}h(x)\,dx.
\end{aligned}
$$

On using assumptions 2, 3 and the fact that $J_\lambda(x)/x^\lambda$ is bounded for $0 < x < \infty$, whenever $\lambda \geqq -\frac{1}{2}$, it is clear that (4.2) defines a continuous function of slow growth at infinity.

The lemma is completed by demonstrating that

$$\int_0^1 |t^{\nu-2\beta}A(t)|\,dt < \infty, \tag{4.3}$$

and

$$\int_0^1 |t^{\mu-2\alpha}A(t)|\,dt < \infty, \tag{4.4}$$

and appealing to the remark immediately following Definition 3.2.

Expanding (4.3) we obtain

$$
\begin{aligned}
&\int_0^1 \left|t^{\nu-2\beta}tS_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x};t\right\}\right|dt \\
&\qquad = \int_0^1 \left|2^{\lambda-\nu-2\beta}t^{2\nu+1}\int_0^\infty \frac{J_\lambda(xt)}{(xt)^\lambda}x^{\nu+2\beta}h(x)\,dx\right|dt,
\end{aligned}
$$

and this last integral is clearly finite. The proof of (4.4) is similar.

To prove (ii), it will be shown that as continuous linear functionals on $H_{\mu+2\alpha-1/2}$, $S'_{1/2\mu-\alpha,2\alpha}(A)$ agrees with $I_{1/2\mu+\beta-1/2,\mu-\lambda}\{h(t);x\}$, and hence, that as functions

$$S'_{1/2\mu-\alpha,2\alpha}(A) = f_1(x),$$

a.e. for $0 < x < 1$. If $\phi(x) \in H_{\mu+2\alpha-1/2}$ it then suffices to justify the string of equalities:

$$
\begin{aligned}
&\langle S'_{1/2\mu-\alpha,2\alpha}(A), \phi(x)\rangle \\
&= \int_0^\infty tS_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x};t\right\}S_{1/2\mu-\alpha,2\alpha}\{\phi(y);t\}\,dt \\
&= \int_0^\infty 2^{\mu-\lambda}t^{\lambda-\mu}\mathscr{H}_\lambda\{x^{\nu+2\beta-\lambda-1/2}h(x);t\}\mathscr{H}_\mu\{y^{1/2-2\alpha}\phi(y);t\}\,dt \\
&= \int_0^\infty 2^{\mu-\lambda}x^{\nu+2\beta-\lambda-1/2}\mathscr{H}_\lambda\{t^{\lambda-\mu}\mathscr{H}_\mu\{y^{1/2-2\alpha}\phi(y);t\};x\}h(x)\,dx \qquad (4.5)
\end{aligned}
$$

$$(4.6)\qquad = \int_0^\infty h(x)K_{1/2\nu+\beta,\mu-\lambda}\{\phi(y);x\}\,dx$$

$$(4.7)\qquad = \int_0^\infty \phi(y)I_{1/2\nu+\beta-1/2,\mu-\lambda}\{h(x);y\}\,dy.$$

From (4.7) we see that in $H'_{\mu+2\alpha-1/2}$

$$S'_{1/2\mu-\alpha,2\alpha}(A) = I_{1/2\nu+\beta-1/2,\mu-\lambda}\{h(t);x\},$$

but as functions on $0 < x < 1$,

$$\begin{aligned} I_{1/2\nu+\beta-1/2,\mu-\lambda}\{h(t);x\} & \\ &= I_{1/2\nu+\beta-1/2,\mu-\lambda}I_{1/2\mu+\alpha-1/2,\lambda-\mu}\{f_1(t);x\} \\ &= f_1(x), \end{aligned}$$

a.e. by definition.

Line (4.5) is Parseval's relation for Hankel transforms valid for $x^{\nu+2\beta}h(x) \in L^1(0,\infty)$ and $t^{2\lambda-\mu}\mathscr{H}_\mu\{x^{1/2-2\alpha}\phi(x);t\} \in L^1(0,\infty)$. The latter condition is true since

(4.8) (a) $t^{2\lambda-\mu}\mathscr{H}_\mu\{x^{1/2-2\alpha}\phi(x);t\} = t^{2\lambda-\mu+2\alpha+1/2}S_{1/2\mu-\alpha,2\alpha}\{\phi(x);t\}$,

(b) $(4.8) \in H_{2\lambda}$,

(c) $H_{2\delta} \subset L^1(0,\infty)$, whenever $2\lambda > -3/2$.

To prove (4.6) three cases are distinguished, $\lambda = \mu, \lambda - \mu < 0, \lambda - \mu > 0$. When $\lambda = \mu$, (4.6) is obvious from the definitions of the operators. The case $\lambda - \mu < 0$ is treated in the next lemma.

LEMMA 4.3. *If* $\mu - \lambda > 0, \lambda \geqq -1/2, \nu \geqq -1/2$, *then line* (4.6) *is valid.*

Proof. It suffices to demonstrate that

$$\begin{aligned} &2^{\mu-\lambda}x^{\nu+2\beta-\lambda-1/2}\mathscr{H}_\lambda\{t^{\lambda-\mu}\mathscr{H}_\mu\{y^{1/2-2\alpha}\phi(y);t\};x\} \\ (4.9)\qquad &= S_{1/2\nu+\beta,\lambda-\nu-2\beta}S_{1/2\mu-\alpha,2\alpha}\{\phi(y);x\} = K_{1/2\nu+\beta,\mu-\lambda}\{\phi(y);x\}. \end{aligned}$$

This is the content of Lemma 3.2 in Walton [7] for which we need

1. $\mu - \lambda > 0$,
2. $\lambda \geqq -1/2$,
3. $y^{\mu-2\alpha+1}\phi(y) \in L^1(0,\infty)$,
4. $y^{-\nu-2\beta-1}\phi(y) \in L^1(x,\infty)$ for all $x \in (0,\infty)$ and either
5. $\mu - \lambda > 1$ and $y^{1/2-2\alpha}\phi(y) \in L^1(0,\infty)$ or
6. (a) $0 < \mu - \lambda \leqq 1$,
   (b) $y^{1/2-\delta-2\alpha}\phi(y) \in L^1(0,\infty)$,
   (c) $y^{1/2-1/2\delta-2\alpha}\phi(y) \in L^1(0,\infty)$,
   (d) $y^{1/2-2\alpha}\phi(y)(x-y)^{\mu-\lambda-1} \in L^1(0,x)$,
   (e) $y^{1/2-2\alpha}\phi(y)(y-x)^{\mu-\lambda-1} \in L^1(x,\infty)$, where $\delta \in (0,1)$ is such that $\mu - \lambda + \delta > 1$.

To verify requirement 3 recall that

$$y^{\mu-2\alpha+1}\phi(y) \in H_{2\mu+1/2} \subset L^1(0, \infty).$$

Similarly, $y^{1/2-2\alpha}\phi(y) \in H_\mu \subset L^1(0, \infty)$. The proofs of the other conditions on $\phi(x)$ are just as trivial and are omitted.

When $\lambda - \mu > 0$, we have the following.

LEMMA 4.4. *If $\mu - \lambda < 0, \lambda \geqq -1/2, \mu \geqq -1/2$ and $2\mu - \lambda > -3/2$, then line* (4.6) *is valid.*

Proof. As in Lemma 4.3, it suffices to verify (4.9). This is the substance of Lemma 3.3 in Walton [7] the hypotheses of which are

1. $\mu - \lambda < 0$,
2. $\lambda \geqq -1/2, \mu \geqq -1/2$,
3. $2\mu - \lambda > -3/2$,
5. $y^{1/2-2\alpha}\phi(y) \in L^1(0, \infty)$,
6. $t^{1/2+\nu+2\beta-\lambda}S_{1/2\mu-\alpha,2\alpha}\{\phi(y); t\}$ is integrable on $(0, \infty)$,
7. $\phi(y)$ is of bounded variation almost everywhere.

Hypothesis 5 was demonstrated in Lemma 4.3 and hypothesis 7 is obvious since $\phi(y)$ is a testing function. For hypothesis 6 we note that

$$t^{1/2+\nu+2\beta-\lambda}S_{1/2\mu-\alpha,2\alpha}\{\phi(y); t\} \in H_\lambda \subset L^1(0, \infty).$$

It is also necessary to distinguish three cases for equality (4.7). As before, the case $\lambda = \mu$ is trivial. When $\mu - \lambda > 0$, we have the following.

LEMMA 4.5. *If $\mu - \lambda > 0$ and $x^{\nu+2\beta-2\delta}h(x) \in L^1(0, \infty)$ for arbitrary $\delta > 0$, then* (4.7) *is valid.*

Proof. Expanding (4.6) we obtain

$$\int_0^\infty \frac{2h(x)x^{\nu+2\beta}}{\Gamma(\mu-\lambda)}\left[\int_x^\infty (y^2 - x^2)^{\mu-\lambda-1}y^{-\mu-2\alpha+1}\phi(y)\,dy\right]dx$$

$$= \int_0^\infty \frac{2y^{-\mu-2\alpha+1}\phi(y)}{\Gamma(\mu-\lambda)}dy \int_0^y (y^2 - x^2)^{\mu-\lambda-1}x^{\nu+2\beta}h(x)\,dx \tag{4.10}$$

$$= \int_0^\infty y^{\mu-2\alpha+2\delta-2\lambda}\phi(y)I_{\delta-1/2,\mu-\lambda}\{x^{\nu+2\beta-2\delta}h(x); y\}\,dy \tag{4.11}$$

$$= \int_0^\infty \phi(y)I_{1/2\nu+\beta+1/2,\mu-\lambda}\{h(x); y\}\,dy.$$

The inversion of line (4.10) is justified by the absolute convergence of the iterated integrals (4.11).

The case $\mu - \lambda < 0$ is treated in the following lemma.

LEMMA 4.6. *If $\mu - \lambda < 0$, and the Abel equation*

$$h(t)t^{\mu+2\alpha-\delta} = I_{\delta-1/2,\lambda-\mu}\{H(x); t\} \tag{4.12}$$

possess a solution $H(x) \in L^1(0, \infty)$, where $\delta > 0$ is arbitrary, then equality (4.7) *is valid.*

Proof. Let $g(x)/x = I_{1/2\nu+\beta,\mu-\lambda}\{h(t)/t; x\}$, that is, $g(x)/x$ is the solution of the Abel equation,

$$I_{1/2\mu+\alpha,\lambda-\mu}\left\{\frac{g(x)}{x}; t\right\} = \frac{h(t)}{t};$$

and let $\psi(t) = K_{1/2\nu+\beta,\mu-\lambda}\{\phi(y); t\}$ which by Lemma 4.4 is known to be in $H_{\nu+2\beta-1/2}$.

From (4.6) we obtain

$$\begin{aligned} &\int_0^\infty h(t)K_{1/2\nu+\beta,\mu-\lambda}\{\phi(y); t\}\, dt \\ &\quad= \int_0^\infty I_{1/2\mu+\alpha,\lambda-\mu}\left\{\frac{g(x)}{x}; t\right\} t\psi(t)\, dt \\ &\quad= \int_0^\infty g(x)K_{1/2\mu+\alpha,\lambda-\mu}\{\psi(t); x\}\, dx \qquad (4.13) \\ &\quad= \int_0^\infty I_{1/2\nu+\beta-1/2,\mu-\lambda}\{h(t); x\}\phi(x)\, dx. \end{aligned}$$

If the Abel equation (4.12) possesses an integrable solution, then it is easy to see that (4.13) is absolutely convergent thereby justifying the inversion.

The proof that $A(t)$ satisfies (3.2) relies upon Lemma 3.1 and Lemma 3.4 of Walton [7], but the analysis is similar to that given above and is therefore omitted.

5. Regularity. In the preceding section, it was shown that $A(t)$ given by (4.1) satisfies the generalized dual relations (3.1) and (3.2). It will now be proved that $A(t)$ also solves the corresponding conventional problem in which the generalized operators are replaced by the classical integral operators.

The method of proof is to demonstrate that as regular generalized functions

$$S'_{1/2\mu-\alpha,2\alpha}(A) = xS_{1/2\mu-\alpha,2\alpha}\left\{\frac{A(t)}{t}; x\right\}, \qquad (5.1)$$

$$S'_{1/2\nu-\beta,2\beta}(A) = xS_{1/2\mu-\beta,2\beta}\left\{\frac{A(t)}{t}; x\right\}. \qquad (5.2)$$

Then by using the fact that for any μ, H_μ contains the space $\mathscr{D}$ of all complex-valued, infinitely differentiable functions with compact support, we may conclude that (5.1) and (5.2) are valid when equality is in the sense of measurable functions. This together with Theorem 4.1 yields that $A(t)$ satisfies the dual integral equations

$$S_{1/2\mu-\alpha,2\alpha}\left\{\frac{A(t)}{t}; x\right\} = \frac{f_1(x)}{x}, \qquad 0 < x < 1,$$

$$S_{1/2\nu-\beta,2\beta}\left\{\frac{A(t)}{t}; x\right\} = \frac{g_2(x)}{x}, \qquad 1 < x < \infty.$$

By a trivial substitution, it is seen that these last two equations are equivalent to (2.8) and (2.9). As with Theorem 4.2, Theorem 5.1 involves several cases, and the proof is presented in a series of lemmas.

THEOREM 5.1. *If* 1. $\lambda \geqq -1/2$, $\mu > -1/2$, $\nu > -1/2$,
2. $x^{\nu+2\beta}h(x) \in L^1(0, \infty)$ *and either*
3. (a) $\lambda - \mu = 0$,
   (b) $h(x)x^{\nu+2\beta-\lambda-1/2} \in L^1(0, \infty)$ *and*
   (c) $h(x)$ *is of bounded variation almost everywhere, or*
4. (a) $h(x)x^{\nu+2\beta-\varepsilon} \in L^1(0, \infty)$ *for some* $\varepsilon > 0$, *and either*
   (b) (i) $\mu - \lambda > 1$ *and*
       (ii) $x^{-1/2-\lambda+\nu+2\beta}h(x) \in L^1(0, \infty)$, *or*
   (c) (i) $0 < \mu - \lambda \leqq 1$,
       (ii) $x^{\nu+2\beta-\lambda-\delta-1/2}h(x) \in L^1(0, \infty)$,
       (iii) $x^{\nu+2\beta-\lambda-1/2\delta-1/2}h(x) \in L^1(0, \infty)$, *where* $\delta \in (0, 1)$ *is such that* $\mu - \lambda + \delta > 1$, *and*
       (iv) $x^{\nu+2\beta-\lambda-1/2}(t - x)^{\mu-\lambda-1}h(x) \in L^1(0, t)$ *and* $x^{\nu+2\beta-\lambda-1/2}(x - t)^{\mu-\lambda-1}h(x) \in L^1(t, \infty)$ *for almost all* t, *or*
5. (a) $\mu - \lambda < 0$ *and*

   (b) $$\lim_{t\to\infty} t^{\gamma}\left[t^{\lambda-\mu}\int_0^{\infty}(xt)^{1/2}J_{\nu}(xt)x^{\nu+2\beta-\lambda-1/2}h(x)\,dx\right] = 0 \textit{ for some } \gamma \in (0, 1),$$

   and either
6. (a) $\lambda - \nu = 0$,
   (b) $x^{2\beta-1/2}h(x) \in L^1(0, \infty)$ *and*
   (c) $h(x)$ *is of bounded variation almost everywhere, or*
7. (a) $x^{2\beta-\nu}h(x) \in L^1(t, \infty)$, *and either*
   (b) (i) $\lambda - \nu > 1$ *and*
       (ii) $x^{\nu+2\beta-\lambda-1/2}h(x) \in L^1(0, \infty)$, *or*
   (c) (i) $0 < \lambda - \nu \leqq 1$,
       (ii) $x^{\nu+2\beta-\lambda-\delta-1/2}h(x) \in L^1(0, \infty)$,
       (iii) $x^{\nu+2\beta-\lambda-1/2\delta-1/2}h(x) \in L^1(0, \infty)$, *where* $\delta \in (0, 1)$ *is such that* $\lambda - \nu + \delta > 1$, *and*
       (iv) $x^{\nu+2\beta-\lambda-1/2}(x - t)^{\lambda-\nu-1}h(x) \in L^1(t, \infty)$ *and* $x^{\nu+2\beta-\lambda-1/2}(t - x)^{\lambda-\nu-1}h(x) \in L^1(0, t)$ *for almost all* t, *or*
8. (a) $\lambda - \nu < 0$ *and*

   (b) $$\lim_{t\to\infty} t^{\gamma}\left[t^{\nu-\lambda}\int_0^{\infty}(xt)^{1/2}J_{\lambda}(xt)x^{\nu+2\beta-\lambda-1/2}h(x)\,dx\right] = 0,$$

then (5.1) *and* (5.2) *are valid in* $H'_{\mu+2\alpha-1/2}$ *and* $H'_{\nu+2\beta-1/2}$, *respectively, where* $A(t)$ *is given by* (4.1).

Proof. The validity of line (5.1) is demonstrated in the following string of equalities. Let $\phi(y) \in H_{\mu+2\alpha-1/2}$. Then

$$\langle S'_{1/2\mu-\alpha,2\alpha}(A), \phi(y)\rangle$$

$$= \int_0^{\infty} tS_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x}; t\right\}S_{1/2\mu-\alpha,2\alpha}\{\phi(y); t\}\,dt$$

$$
\begin{aligned}
&= \int_0^\infty 2^{\mu-\lambda} t^{\lambda-\mu} \mathscr{H}_\lambda\{x^{\nu+2\beta-\lambda-1/2}h(x); t\} \mathscr{H}_\mu\{y^{1/2-2\alpha}\phi(y); t\}\, dt \\
(5.3)\qquad &= \int_0^\infty 2^{\mu-\lambda} \mathscr{H}_\mu\{t^{\lambda-\mu}\mathscr{H}_\lambda\{x^{\nu+2\beta-\lambda-1/2}h(x); t\}; y\} y^{1/2-2\alpha}\phi(y)\, dy \\
&= \int_0^\infty y S_{1/2\mu-\alpha,2\alpha}\left\{\frac{A(t)}{t}; y\right\}\phi(y)\, dy \\
&= \left\langle y S_{1/2\mu-\alpha,2\alpha}\left\{\frac{A(t)}{t}; y\right\}, \phi(y)\right\rangle.
\end{aligned}
$$

To verify the inversion of line (5.3) we distinguish three cases: $\lambda = \mu$, $\lambda - \mu < 0$ and $\lambda - \mu > 0$. When $\lambda = \mu$, hypothesis 3 implies that

$$\mathscr{H}_\mu\{\mathscr{H}_\mu\{x^{\nu+2\beta-\lambda-1/2}h(x); t\}; y\} = y^{\nu+2\beta-\lambda-1/2}h(y).$$

The inversion of line (5.3) may now be routinely verified.

The case $\lambda - \mu < 0$ is treated in Lemma 5.2.

LEMMA 5.2. *If hypotheses* 1, 2 *and* 4 *of Theorem* 5.1 *hold, then the inversion of line* (5.3) *is valid.*

Proof. The proof is established in the following string of equalities:

$$
\begin{aligned}
&\lim_{N\to\infty} \int_0^N t^{\lambda-\mu}\mathscr{H}_\lambda\{x^{\nu+2\beta-\lambda-1/2}h(x); t\}\mathscr{H}_\mu\{y^{1/2-2\alpha}\phi(y); t\}\, dt \\
&\quad = \lim_{N\to\infty}\int_0^N t^{\lambda-\mu}\left[\int_0^\infty (xt)^{1/2}J_\lambda(xt)x^{\nu+2\beta-\lambda-1/2}h(x)\, dx\right] \\
&\qquad \cdot\left[\int_0^\infty (yt)^{1/2}J_\mu(yt)y^{1/2-2\alpha}\phi(y)\, dy\right] dt \\
(5.4)\qquad &\quad = \lim_{N\to\infty}\int_0^\infty y^{1/2-2\alpha}\phi(y)\, dy \\
&\qquad \cdot\int_0^N t^{\lambda-\mu}(ty)^{1/2}J_\mu(ty)\left[\int_0^\infty (tx)^{1/2}J_\lambda(tx)x^{\nu+2\beta-\lambda-1/2}h(x)\, dx\right] dt \\
(5.5)\qquad &\quad = \int_0^\infty y^{1/2-2\alpha}\phi(y)\, dy \\
&\qquad \cdot\int_0^\infty t^{\lambda-\mu}(ty)^{1/2}J_\mu(ty)\left[\int_0^\infty (tx)^{1/2}J_\lambda(tx)x^{\nu-2\beta-\lambda-1/2}h(x)\, dx\right] dt \\
&\quad = \int_0 y^{1/2-2\alpha}\phi(y)\mathscr{H}_\mu\{t^{\lambda-\mu}\mathscr{H}_\lambda\{x^{\nu+2\beta-\lambda-1/2}h(x); t\}; y\}\, dy.
\end{aligned}
$$

The absolute convergence of (5.4) is easily established, thereby justifying the inversion. In operator notation, (5.5) is the integral

$$\int_0^\infty \phi(y)yS_{1/2\mu-\alpha,2\alpha}S_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x}; y\right\} dy.$$

Hypotheses 1, 2 and 4 insure the applicability of Lemma 3.1 in Walton [7], from which we obtain

$$S_{1/2\mu-\alpha,2\alpha}S_{1/2\nu+\beta,\lambda-\nu-2\beta}\left\{\frac{h(x)}{x};y\right\} = I_{1/2\nu+\beta,\mu-\lambda}\left\{\frac{h(x)}{x};y\right\}.$$

Therefore, for any $\varepsilon > 0$, the expression in line (5.5) is equal to

$$\int_0^\infty y\phi(y)I_{1/2\nu+\beta,\mu-\lambda}\left\{\frac{h(x)}{x};y\right\}dy$$
$$= \int_0^\infty y^{-\nu-2\beta+\varepsilon}\phi(y)I_{1/2\varepsilon-1/2,\mu-\lambda}\{h(x)x^{\nu+2\beta-\varepsilon};y\}\,dy,$$

and this last integral is easily seen to converge absolutely. The existence of the integral in line (5.5) is now evident.

The interchange of the limiting process and the y-integration may now be justified by considering the asymptotic expansions of the Bessel functions. However, the analysis is routine and tedious, and is therefore omitted.

To prove (5.3) for $\mu - \lambda < 0$, we appeal to hypothesis 5. This proof is omitted since it also involves merely considering the asymptotic expansion of $(ty)^{1/2}J_\mu(ty)$.

Equation (5.2) is established in a manner entirely analogous to that used for (5.1), and therefore, the details are omitted. It should be noted, though, that here the relevant hypotheses are 6, 7 and 8, and as usual, three distinct cases must be considered. This completes the regularity result.

At this time two remarks are in order.

Remark 1. In the existence and regularity theorems, restrictions were placed upon the transformed data $h(x)$, which can be considered as being of two types—growth conditions and smoothness conditions. In general, it is a simple matter to relate the growth conditions to the original data f_1 and g_2. For the smoothness conditions, (e.g., requiring $h(x)$ to be of bounded variation) the situation is not quite as clear. However, we omit a thorough treatment of these considerations since it would be lengthy and routine.

Remark 2. The solvability of the dual relation problem was contingent in some cases upon the existence of integrable solutions to certain Abel integral equations. No attempt will be made here to give a thorough discussion of what restrictions on f_1 and g_2 will insure solutions to these equations, since the theory of such problems is extensive. We do, however, remark that imposing smoothness conditions on f_1 and g_2 does guarantee solutions. In the next section an example is considered from which it is clear that for polynomial data the problem is solvable.

6. An example. In Theorem 5.1, one of the alternative hypotheses supposed that for $\mu - \lambda < 0$,

$$\lim_{t\to\infty} t^\gamma\left[t^{\lambda-\mu}\int_0^\infty (tx)^{1/2}J_\lambda(tx)x^{\nu+2\beta-\lambda-1/2}h(x)\,dx\right] = 0, \tag{6.1}$$

where $\gamma > 0$. An example is given here which illustrates that for an important class of data f_1 and g_2, this assumption is easily verified. For simplicity, assume $0 \leqq \lambda - \mu < 1$, and define

$$f_1(x) = c_1 x^n, \qquad 0 < x < 1,$$

$$g_2(x) = c_2 x^m, \qquad 1 < x < \infty,$$

where n, m are real and c_1 and c_2 are constants.

To compute $h(x)$, we must consider

$$I_{1/2\mu+\alpha-1/2,\lambda-\mu}\{f_1(y); x\} \tag{6.2}$$

and

$$K_{1/2\mu-\alpha+1/2,\nu-\lambda}\{g_2(y); x\}. \tag{6.3}$$

It is easy to see that (6.2) $= C_1' x^n$, where, to insure the convergence of (6.2) when $\lambda - \mu > 0$, we require $n > -1 - \mu - 2\alpha$. The constant C_1' has the value C_1 when $\lambda - \mu = 0$ and the value

$$\frac{2}{\Gamma(\lambda - \mu)} C_1 \int_0^1 (1 - y^2)^{\lambda-\mu-1} y^{\mu+2\alpha+n}\, dy \quad \text{for } \lambda - \mu > 0.$$

Moreover,

$$x^{\nu+2\beta-\lambda-1/2} I_{1/2\mu+\alpha-1/2,\lambda-\mu}\{f_1(y); x\} = C_1' x^{n-\nu+2\beta-\lambda-1/2},$$

and hence this last expression is in $L^1(0, 1)$ when $n > -1/2 + \lambda - \nu - 2\beta$.

Similarly it is easy to see that (6.3) $= C_2' x^m$. In this case, $C_2' = C_2$ when $\nu = \lambda$; when $\nu - \lambda > 0$,

$$C_2' = C_2 \frac{2}{\Gamma(\nu - \lambda)} \int_1^\infty (t^2 - 1)^{\nu-\lambda-1} t^{m-\nu+2\beta}\, dt,$$

where, for convergence, $m < 1 + \mu - 2\alpha$; and when $\nu - \lambda < 0$,

$$C_2' = \frac{C_2 \Gamma(\lambda - \nu)}{2[\int_1^\infty (x^2 - 1)^{\lambda-\nu-1} x^{m-\mu+2\alpha}\, dx]},$$

where, for convergence, $m < 1 + \nu - 2\beta$. Further, it is clear that

$$x^{\nu+2\beta-\lambda-1/2} K_{1/2\mu-\alpha+1/2,\nu-\lambda}\{g_2(y); x\} \in L^1(1, \infty),$$

whenever $m < -\frac{1}{2} + \lambda - \nu - 2\beta$.

The Hankel transform of $x^{\nu+2\beta-\lambda-1/2} h(x)$ is calculated by considering the two integrals

$$\int_0^1 (xt)^{1/2} J_\lambda(xt) C_1' x^{n+\nu+2\beta-\lambda-1/2}\, dx, \tag{6.4}$$

$$\int_1^\infty (xt)^{1/2} J_\lambda(xt) C_2' x^{m+\nu+2\beta-\lambda-1/2}\, dx. \tag{6.5}$$

For (6.4), we introduce

$$\int_0^1 (tx)^{1/2} \left[\frac{d}{dx}(x^{-\lambda-1/2} C_1' x^{n+\nu+2\beta-\lambda-1/2})\right] x^{\lambda+1/2} J_{\lambda+1}(tx)\, dx. \tag{6.6}$$

Integrating (6.6) by parts and rearranging terms, it is easily seen that

$$\begin{aligned} &t \int_0^1 (xt)^{1/2} J_\lambda(tx) x^{n+\nu+2\beta-\lambda-1/2} C_1' \, dx \\ &= C_1' t^{1/2} J_{\lambda+1}(t) - C_1'(n + \nu + 2\beta - 2\lambda - 1) \\ &\cdot \int_0^1 (tx)^{1/2} J_{\lambda+1}(tx) x^{n+\nu+2\beta-\lambda-3/2} \, dx. \end{aligned} \tag{6.7}$$

Similarly, we obtain the result

$$\begin{aligned} &t \int_1^\infty (xt)^{1/2} J_\lambda(xt) C_2' x^{m+\nu+2\beta-\lambda-1/2} \, dx \\ &= -C_2' t^{1/2} J_{\lambda+1}(t) - C_2'(m + \nu + 2\beta - 2\lambda - 1) \\ &\cdot \int_1^\infty (tx)^{1/2} J_{\lambda+1}(tx) x^{m+\nu+2\beta-\lambda-3/2} \, dx. \end{aligned} \tag{6.8}$$

Combining (6.8) and (6.7) yields the following.

$$t^{\lambda-\mu} \int_0^\infty (xt)^{1/2} J_\lambda(xt) x^{\nu+2\beta-\lambda-1/2} h(x) \, dx$$

$$= (C_1' - C_2') t^{1/2} J_{\lambda+1}(t) t^{\lambda-\mu-1} \tag{6.9}$$

$$-C_1'(n + \nu + 2\beta - 2\lambda - 1) t^{\lambda-\mu-1} \int_0^1 (tx)^{1/2} J_{\lambda+1}(xt) x^{n+\nu+2\beta-\lambda-3/2} \, dx \tag{6.10}$$

$$-C_2'(m + \nu + 2\beta - 2\lambda - 1) t^{\lambda-\mu-1} \int_1^\infty (tx)^{1/2} J_{\lambda+1}(tx) x^{m+\nu+2\beta-\lambda-3/2} \, dx. \tag{6.11}$$

Clearly (6.9) and (6.11) are $O(t^{\lambda-\mu-1})$, and since $\lambda - \mu < 1$, they display the desired growth property as $t \to \infty$. For (6.10), the additional assumption $n > -1/2 - 2\alpha$ must be made.

For polynomial data f_1 and g_2, verifying the other assumptions of the existence and regularity theorems is trivial. Therefore, the solvability of the classical dual relation problem in this case is contingent upon demonstrating a relation between the parameters α, β, μ and ν and the degrees of the polynomial data. The above example illustrates that this relation is quite simple to obtain.

7. A uniqueness result. For a special case of generalized dual relations (3.1) and (3.2), it is possible to prove the existence of a unique solution. The motivation for the additional restrictions lies with the fact that there is a natural generalization of $I_{\eta,\alpha}$ to an isomorphism between certain of the spaces H'_μ whereas, no such extension seems possible for $K_{\eta,\alpha}$. By merely imposing the restriction $\mu - 2\alpha = \nu - 2\beta$ upon the parameters μ, ν, α and β, we may avoid using $K_{\eta,\alpha}$ in the reduction of the dual relations to a single invertible Fredholm integral equation via the formal method outlined in § 2. Therefore, in the generalized case, only isomorphisms are employed, and the solution is then seen to be unique in one of the spaces H'_μ.

In this case, it is not necessary to require the solution to the generalized problem to be a regular generalized function. Consequently, in this section the generalized problem (later referred to as I) is that of finding a generalized function, $A(t)$, in $H'_{\mu-2\alpha-1/2}$ such that

1. $S'_{1/2\mu-\alpha,2\alpha}(A)$ is a regular generalized function in $H'_{\mu+2\alpha-1/2}$ which equals the measurable function $f_1(x)$ almost everywhere on $(0, 1)$ and
2. $S'_{1/2\nu-\beta,2\beta}(A)$ is a regular generalized function in $H'_{\nu+2\beta-1/2}$ which equals the measurable function $g_2(x)$ almost everywhere on $(1, \infty)$.

If the classical dual relation problem (later referred to as II) is then to find a measurable function $A(t)$ of slow growth at infinity, locally integrable on $(0, \infty)$, and such that

1. $\int_0^1 |A(t)| t^{\mu-2\alpha}\, dt < \infty$,
2. $\int_0^N |t^{-2\alpha} A(t) J_\lambda(xt)|\, dt < \infty$,
3. $\int_0^N |t^{-2\beta} A(t) J_\lambda(xt)|\, dt < \infty$,
4. $\lim\limits_{N\to\infty} 2^{2\alpha} x^{-2\alpha} \int_0^N t^{-2\alpha} A(t) J_\mu(xt)\, dt = f_1(x)/x, 0 < x < 1$,
5. $\lim\limits_{N\to\infty} 2^{2\beta} x^{-2\beta} \int_0^N t^{-2\beta} A(t) J_\nu(xt)\, dt = g_2(x)/x, 1 < x < \infty$,

it is easily seen that a solution of the classical problem is necessarily a solution of the generalized problem. Hence, the solution of the classical problem, which exists when the additional hypotheses of the regularity theorem are imposed, is uniquely determined almost everywhere. This follows from the fact that

1. it is unique in the space $H'_{\mu-2\alpha-1/2}$,
2. $H'_{\mu-2\alpha-1/2} \subset \mathscr{D}'$, the dual space of $\mathscr{D}$,
3. any two measurable functions generating the same distribution in $\mathscr{D}'$ must be equal almost everywhere.

No proofs will be supplied in this section since most of the arguments required are similar to those employed in § 4 and § 5. However, all of the pertinent definitions and theorems will be stated.

DEFINITION 7.1. For $2\eta + \alpha \geqq -\frac{1}{2}$ and $2\eta + 2\alpha + \beta \geqq -\frac{1}{2}$, $I'_{\eta,\alpha,\beta}$ is the mapping from $H'_{2\eta-1/2}$ to $H'_{2\eta+2\alpha+2\beta-1/2}$ defined for $f \in H'_{2\eta-1/2}$ by the equation

$$I'_{\eta,\alpha,\beta}(f) \equiv S'_{\eta+\alpha,\beta} S'_{\eta,\alpha}(f).$$

LEMMA 7.2. *If* 1. $\alpha + \beta \geqq 0$,
2. $2\eta + \alpha \geqq -\frac{1}{2}$,
3. $f(x)x^{2\eta-\gamma} \in L^1(0, 1)$ *for some* $\gamma > 0$,
4. $f(x)$ *is of slow growth at infinity, then as generalized functions in* $H'_{2\eta+2\alpha+2\beta-1/2}$,

$$I'_{\eta,\alpha,\beta}(f) = I_{\eta-1/2,\alpha+\beta}(f).$$

LEMMA 7.3. *If* 1. $\alpha + \beta < 0$,
2. $2\eta + 3\alpha + 2\beta > -\frac{3}{2}$,
3. $2\eta + 2\alpha + \beta \geqq -\frac{1}{2}$,
4. $f(x)x^{2\eta+2\alpha+2\beta-\delta} \in L^1(0, 1)$, *where* $\delta > 0$ *is arbitrary*,
5. $f(x)$ *is of slow growth at infinity*,

6. *the Abel equation*

$$f(x)x^{2\eta+2\alpha+2\beta-\delta} = I_{\delta 1/2-1/2,-\alpha-\beta}\{G(y);x\}$$

possesses a solution $G(y) \in L^1(0, 1)$ *which is of slow growth at infinity, then as generalized functions in* $H'_{2\eta+2\alpha+2\beta-1/2}$,

$$I'_{\eta,\alpha,\beta}(f) = I_{\eta-1/2,\alpha+\beta}(f).$$

The above lemmas state that, under suitable restrictions, the generalized operator $I'_{\eta,\alpha,\beta}$ agrees with the conventional integral transform $I_{\eta-1/2,\alpha+\beta}$ on regular generalized functions. The next theorem is the uniqueness result.

THEOREM 7.4. *If* 1. $\mu, \nu \geqq -\frac{1}{2}$,

2. $\mu - 2\alpha = \nu - 2\beta$, *and either*
3. (a) $\alpha - \beta > 0$,
   (b) $2\nu - \mu > -\frac{3}{2}$,
   (c) $x^{\nu+2\beta-\delta}f_1(x) \in L^1(0, 1)$,
   (d) $x^{\nu+2\beta-\delta}g_2(x) \in L^1(1, \infty)$,
   (e) *the Abel equation*

$$x^{\nu+2\beta-\delta}f_1(x) = I_{\delta-1/2,\mu-\nu}\{F(t);x\}$$

possesses a solution $F(t) \in L^1(0, 1)$ *for arbitrary* $\delta > 0$, *or*

4. (a) $\alpha - \beta \underline{<} 0$,
   (b) $2\mu - \overline{\nu - \frac{3}{2}}$,
   (c) $x^{\mu+2\alpha-\delta}f_1(x) \in L^1(0, 1)$,
   (d) $x^{\mu+2\alpha-\delta}g_2(x) \in L^1(1, \infty)$,
   (e) *the Abel equation*

$$h(x)x^{\mu+2\alpha-\delta} = I_{1/2\delta-1/2,\mu-\nu}\{G(y);x\}$$

possesses a solution $G(y) \in L^1(0, \infty)$,

then the generalized dual relation problem I *possesses a solution*

$$A = (S')^{-1}_{1/2\nu-\beta,2\beta}(h(x))$$

unique in the space $H'_{\mu+2\alpha-1/2}$.

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SINGULARLY PERTURBED NONLINEAR BOUNDARY VALUE PROBLEMS WITH TURNING POINTS*

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Abstract. Differential inequalities for second order boundary value problems are used to study the existence and asymptotic behavior as $\varepsilon \to 0^+$ of solutions of $\varepsilon y'' = f(t, y, y', \varepsilon)$, $y(-1)$, $y(1)$ prescribed, when $f_{y'} = \partial f/\partial y'$ vanishes at $t = 0$. Depending on the behavior of $f_{y'}$ at the turning point ($t = 0$), solutions are shown to possess (i) a transition layer at the turning point, (ii) a boundary layer at $t = -1$ or $t = 1$, or (iii) boundary layers at both endpoints. The results extend several considerations of O'Malley and Dorr for nonresonant linear and quasi-linear problems to more general nonlinear problems. In addition, explicit transition layer and boundary layer estimates are given.

1. Introduction. We consider in this paper nonlinear boundary value problems of the form

$$\begin{aligned} &\varepsilon y'' = f(t, y, y', \varepsilon), \qquad -1 < t < 1, \\ &y(-1) = B_1, \qquad y(1) = B_2, \end{aligned} \tag{1.1}$$

in which f possesses a turning point at $t = 0$, i.e., $\partial f/\partial y'$ vanishes at $t = 0$.[1] The parameter ε is assumed to be small and positive. Our main objective is to give sufficient conditions for the existence of solutions of (1.1) and to study the behavior of these solutions, especially in neighborhoods of the turning point and the endpoints, as $\varepsilon \to 0^+$. The principal assumptions are that appropriate reduced problems

$$\begin{aligned} &0 = f(t, u, u', 0), \qquad -1 < t < 0, \quad 0 < t < 1, \quad -1 < t < 1, \\ &u(-1) = B_1 \quad \text{or} \quad u(1) = B_2 \end{aligned}$$

have smooth solutions and that the function f is of class $C^{(1)}$ and sufficiently well-behaved (in a sense to be described below).

Depending on the behavior of the function $f_{y'} = \partial f/\partial y'$ at the turning point $t = 0$, the solution $y(t, \varepsilon)$ of (1.1) is shown to exhibit essentially two types of behavior. If $f_{y'}$ changes its algebraic sign in passing through zero, $y(t, \varepsilon)$ behaves differently on opposite sides of $t = 0$, and the change takes place within a transition layer. However, if the sign of $f_{y'}$ remains the same, the solution possesses a boundary layer at one of the endpoints. Such behavior has been observed by O'Malley [6] for the case of the linear problem

$$\begin{aligned} &\varepsilon y'' + 2t\alpha(t, \varepsilon)y' - \alpha(t, \varepsilon)\beta(t, \varepsilon)y = 0, \qquad -1 < t < 1, \\ &y(-1), y(1) \text{ prescribed}, \end{aligned} \tag{1.2}$$

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[1] Strictly speaking, a "turning point" is only defined for a linear problem; see, e.g., Wasow [9, Chap. 8] or O'Malley [7, Chap. 8]. However, by linearization, i.e., by considering $\varepsilon y'' = (\partial f/\partial y)y + (\partial f/\partial y')y'$, the definition in [9] or [7] may also be applied to nonlinear problems (1.1).

and by Dorr [3] for quasi-linear equations of the form

$$\varepsilon y'' + t^k F(t, y)y' = 0, \qquad -1 < t < 1,$$

where k is a nonnegative integer. We also consider problems (1.1) in which turning point behavior occurs at one of the endpoints.

Using an existence and comparison theorem for second order boundary value problems, we are able to extend the results of O'Malley and Dorr in the nonresonant case (cf. Ackerberg and O'Malley [1]) to nonlinear problems. In addition, we consistently use differential inequalities to construct explicit boundary layer and transition layer estimates for the solutions of (1.1). To our knowledge, such estimates have not been constructed before, even in the case of linear problems.

Our inspiration for this method of attack is a paper of Briš [2] in which such problems with $f_{y'}$ bounded away from zero are considered.

2. Preliminaries. In this section we collect the definitions and results from the theory of second order boundary value problems which are used in the rest of the paper. Our principal reference is Jackson [5].

Consider the problem

$$x'' = F(t, x, x'), \qquad a < t < b, \tag{2.1}$$

$$x(a) = c, \qquad x(b) = d, \tag{2.2}$$

for F a continuous function on $[a, b] \times \mathbb{R}^2$.

DEFINITION 2.1. A function $\alpha(t)$ is called a *lower solution* of (2.1) on $[a, b]$ if $\alpha \in C^{(2)}[a, b]$ and $\alpha''(t) \geqq F(t, \alpha(t), \alpha'(t))$ on (a, b).

DEFINITION 2.2. A function $\beta(t)$ is called an *upper solution* of (2.1) on $[a, b]$ if $\beta \in C^{(2)}[a, b]$ and $\beta''(t) \leqq F(t, \beta(t), \beta'(t))$ on (a, b).

DEFINITION 2.3. A function $F = F(t, x, x')$ is said to *satisfy a Nagumo condition on* $[a, b]$ *with respect to the pair* $\alpha, \beta \in C[a, b]$ in case $\alpha(t) \leqq \beta(t)$ on $[a, b]$ and there exists a positive continuous function $\varphi(s)$ on $[0, \infty)$ such that $|F(t, x, x')| \leqq \varphi(|x'|)$ for all $a \leqq t \leqq b$, $\alpha(t) \leqq x \leqq \beta(t)$, $|x'| < \infty$ and

$$\int_\lambda^\infty \frac{s\,ds}{\varphi(s)} > \max_{a \leqq t \leqq b} \beta(t) - \min_{a \leqq t \leqq b} \alpha(t),$$

where $\lambda(b - a) = \max\{|\alpha(a) - \beta(b)|, |\alpha(b) - \beta(a)|\}$. We come now to the main existence and comparison theorem.

THEOREM 2.1 (Jackson [5, Thm. 7.3]). *Assume that* $F(t, x, x')$ *satisfies a Nagumo condition with respect to the pair* α, β *which are, respectively, lower and upper solutions of* (2.1) *on* $[a, b]$. *Then if* $\alpha(a) \leqq c \leqq \beta(a)$ *and* $\alpha(b) \leqq d \leqq \beta(b)$, *the boundary value problem* (2.1), (2.2) *has a solution* $x = x(t) \in C^{(2)}[a, b]$ *with* $\alpha(t) \leqq x(t) \leqq \beta(t)$ *on* $[a, b]$.

We remark that this theorem is valid under the weaker assumption that α', β' are absolutely continuous on $[a, b]$; see [5, § 2]. This fact will be needed in the next section where we will consider functions α', β' which are differentiable on $[-1, 1] - \{0\}$.

For convenience of notation, we will simply say that a given function satisfies a Nagumo condition in its domain of definition; it is understood that appropriate α, β will be specified.

Finally, it is clear that Theorem 2.1 remains valid when F depends continuously on the parameter ε.

3. Problems exhibiting a transition layer. Consider the boundary value problem

$$\varepsilon y'' = f(t, y, y', \varepsilon), \qquad -1 < t < 1, \tag{3.1}$$

$$y(-1) = B_1, \qquad y(1) = B_2, \tag{3.2}$$

where B_1 and B_2 are, for simplicity, constants independent of ε. Associated with (3.1), (3.2) are the reduced problems

$$0 = f(t, u, u', 0), \qquad -1 < t < 0, \tag*{$(3.3)_1$}$$

$$u(-1) = B_1, \tag*{$(3.4)_1$}$$

$$0 = f(t, u, u', 0), \qquad 0 < t < 1, \tag*{$(3.3)_2$}$$

$$u(1) = B_2. \tag*{$(3.4)_2$}$$

In this section, we use the method of differential inequalities to deduce the existence and asymptotic behavior of solutions of (3.1), (3.2) when f has a certain type of turning point at $t = 0$. The principal result is the following.

THEOREM 3.1. *Assume*

(a) *there are functions* $u_1 \in C^{(2)}[-1, 0]$ *and* $u_2 \in C^{(2)}[0, 1]$ *satisfying* $(3.3)_1$, $(3.4)_1$ *and* $(3.3)_2$, $(3.4)_2$, *respectively*;

(b) f *is continuous in* (t, y, y', ε) *and of class* $C^{(1)}$ *with respect to* y, y' *in a domain* $\mathscr{R}: -1 \leqq t \leqq 1$, $|y - u_1(t)| \leqq d_1(-1 \leqq t \leqq 0)$, $|y - u_2(t)| \leqq d_2(0 \leqq t \leqq 1)$, $|y'| < \infty$, $0 \leqq \varepsilon \leqq \varepsilon_1$, for $d_1, d_2, \varepsilon_1 > 0$;

(c) $f(t, u_i(t), u_i'(t), \varepsilon) = O(\varepsilon)$, $i = 1, 2$;

(d) *there are functions* h_1, h_2 *with the following properties*: h_1 *is continuous on* $[-1, 0]$, *differentiable on* $[-1, 0)$; $h_1'(0^-)$ *exists*; $h_1(0) = 0$; $h_1' \leqq 0$ *and* $h_1 > 0$ *on* $[-1, 0)$; h_2 *is continuous on* $[0, 1]$, *differentiable on* $(0, 1]$; $h_2'(0^+)$ *exists*; $h_2(0) = 0$; $h_2' \leqq 0$ *and* $h_2 < 0$ *on* $(0, 1]$; *further*, $f_{y'} \geqq h_1(t)$ *in* $\mathscr{R}$ *for* $t \in [-1, 0]$ *and* $f_{y'} \leqq h_2(t)$ *in* $\mathscr{R}$ *for* $t \in [0, 1]$;

(e) $f_y \geqq l > 0$ *in* $\mathscr{R}$, *for some constant* l;

(f) f *satisfies a Nagumo condition in* $\mathscr{R}$.

Then for each ε, $0 < \varepsilon \leqq \varepsilon_1$, *there exists a solution* $y = y(t, \varepsilon)$ *of* (3.1), (3.2). *Moreover, we can distinguish three types of asymptotic behavior.*

(i) *If* $u_1'(0) = u_2'(0)$, *then*

$$|y(t, \varepsilon) - u_1(t)| \leqq |u_2(0) - u_1(0)| \exp\left[-\varepsilon^{-1} \int_t^0 h_1(s)\, ds\right] + c\varepsilon, \qquad -1 \leqq t \leqq 0,$$

$$|y(t, \varepsilon) - u_2(t)| \leqq |u_1(0) - u_2(0)| \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] + c\varepsilon, \qquad 0 \leqq t \leqq 1.$$

(ii) *If* $u_1'(0) \neq u_2'(0)$ *and* $u_1(0) = u_2(0)$, *then*

$$|y(t,\varepsilon) - u_1(t)| \leqq c\sqrt{\varepsilon}, \qquad -1 \leqq t \leqq 0,$$
$$|y(t,\varepsilon) - u_2(t)| \leqq c\sqrt{\varepsilon}, \qquad 0 \leqq t \leqq 1.$$

(iii) *If* $u_1'(0) \neq u_2'(0)$ *and* $u_1(0) \neq u_2(0)$, *then*

$$|y(t,\varepsilon) - u_1(t)| \leqq |u_2(0) - u_1(0)| \exp\left[-\varepsilon^{-1}\int_t^0 h_1(s)\,ds\right] + c\sqrt{\varepsilon}, \qquad -1 \leqq t \leqq 0,$$

$$|y(t,\varepsilon) - u_2(t)| \leqq |u_1(0) - u_2(0)| \exp\left[\varepsilon^{-1}\int_0^t h_2(s)\,ds\right] + c\sqrt{\varepsilon}, \qquad 0 \leqq t \leqq 1.$$

The constant c is a generic constant, independent of ε, *whose magnitude can be determined in each of the three cases.*

Proof. We shall only prove the existence of a solution satisfying the prescribed estimate in the case $u_1'(0) = u_2'(0)$. The proofs of the other two cases follow analogously.

Since the function f is assumed to satisfy a Nagumo condition, the proof reduces to the construction of appropriate lower and upper solutions α, β, respectively. Assume first that $u_1(0) \leqq u_2(0)$; then define, for ε in $(0, \varepsilon_1]$,

$$\alpha(t) = \begin{cases} u_1(t) - \varepsilon\gamma l^{-1}, & -1 \leqq t \leqq 0, \\ u_2(t) - (u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1}\int_0^t h_2(s)\,ds\right] - \varepsilon\gamma l^{-1}, & 0 \leqq t \leqq 1, \end{cases}$$

$$\beta(t) = \begin{cases} u_1(t) - (u_1(0) - u_2(0)) \exp\left[-\varepsilon^{-1}\int_t^0 h_1(s)\,ds\right] + \varepsilon\gamma l^{-1}, & -1 \leqq t \leqq 0, \\ u_2(t) + \varepsilon\gamma l^{-1}, & 0 \leqq t \leqq 1. \end{cases}$$

Similarly, if $u_1(0) > u_2(0)$, define, for ε in $(0, \varepsilon_1]$,

$$\alpha(t) = \begin{cases} u_1(t) - (u_1(0) - u_2(0)) \exp\left[-\varepsilon^{-1}\int_t^0 h_1(s)\,ds\right] - \varepsilon\gamma l^{-1}, & -1 \leqq t \leqq 0, \\ u_2(t) - \varepsilon\gamma l^{-1}, & 0 \leqq t \leqq 1, \end{cases}$$

$$\beta(t) = \begin{cases} u_1(t) + \varepsilon\gamma l^{-1}, & -1 \leqq t \leqq 0, \\ u_2(t) - (u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1}\int_0^t h_2(s)\,ds\right] + \varepsilon\gamma l^{-1}, & 0 \leqq t \leqq 1. \end{cases}$$

In both cases, γ is a positive constant to be determined in the course of the proof. Note that, by assumption (a) and the assumption that $u_1'(0) = u_2'(0)$, α', β' exist on $[-1, 1]$ and are differentiable on $[-1, 1] - \{0\}$.

It is now a straightforward exercise to verify that these functions satisfy the hypotheses of Theorem 2.1. For example, if $u_1(0) \leqq u_2(0)$, we demonstrate that $\varepsilon\alpha''(t) \geqq f(t, \alpha(t), \alpha'(t), \varepsilon)$ for $t \in (-1, 1)$. Restricting attention to $(-1, 0)$, we substitute into equation (3.1) and expand by the mean value theorem to obtain

$$\begin{aligned} \varepsilon\alpha'' - f(t,\alpha,\alpha',\varepsilon) &= \varepsilon u_1'' - f(t, u_1, u_1', \varepsilon) - f_y[t][-\varepsilon\gamma l^{-1}] \\ &\geqq -\varepsilon M_1 - \varepsilon\sigma_1 + \varepsilon\gamma \geqq 0, \end{aligned}$$

if we choose $\gamma \geqq M_1 + \sigma_1$, where $|u_1''| \leqq M_1, |f(t, u_1, u_1', \varepsilon)| \leqq \varepsilon\sigma_1$ and $[t] = (t, u_1 - \theta\varepsilon\gamma l^{-1}, u_1', \varepsilon)$ for some $\theta \in (0, 1)$. Similarly, for $t \in (0, 1)$,

$$\begin{aligned}
\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon) = {} & \varepsilon u_2'' - (h_2'(t) + \varepsilon^{-1} h_2^2(t)) \cdot (u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] \\
& - f(t, u_2, u_2', \varepsilon) \\
& - f_y[t]\left[-(u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] - \varepsilon\gamma l^{-1}\right] \\
& - f_{y'}[t]\left[-\varepsilon^{-1} h_2(t)(u_2(0) - u_1(0)) \cdot \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right]\right] \\
\geqq {} & -\varepsilon M_2 - \varepsilon^{-1} h_2^2(t)(u_2(0) - u_1(0)) \cdot \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] \\
& - \varepsilon\sigma_2 + \varepsilon\gamma \\
& + \varepsilon^{-1} h_2^2(t)(u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] \geqq 0,
\end{aligned}$$

provided $\gamma \geqq M_2 + \sigma_2$, where $|u_2''| \leqq M_2, |f(t, u_2, u_2', \varepsilon)| \leqq \varepsilon\sigma_2$ and

$$\begin{aligned}
[t] = \Bigg(t, u_2 - \theta\left\{(u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] + \varepsilon\gamma l^{-1}\right\}, \\
u_2' - \theta\varepsilon^{-1} h_2(t)(u_2(0) - u_1(0)) \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right], \varepsilon\Bigg).
\end{aligned}$$ [2]

Thus by choosing $\gamma = \max\{M_1 + \sigma_1, M_2 + \sigma_2\}$, we insure that α is a lower solution of equation (3.1); similarly, β is an upper solution. The other case, $u_1(0) > u_2(0)$, follows in an analogous fashion. We conclude by Theorem 2.1 that for each ε, $0 < \varepsilon \leqq \varepsilon_1$, the problem (3.1), (3.2) has a solution $y = y(t, \varepsilon)$ satisfying $\alpha(t) \leqq y(t, \varepsilon) \leqq \beta(t)$, i.e.,

$$\begin{aligned}
& |y(t, \varepsilon) - u_1(t)| \leqq |u_2(0) - u_1(0)| \exp\left[-\varepsilon^{-1} \int_t^0 h_1(s)\, ds\right] + \varepsilon\gamma l^{-1}, \\
& \qquad -1 \leqq t \leqq 0, \\
& |y(t, \varepsilon) - u_2(t)| \leqq |u_1(0) - u_2(0)| \exp\left[\varepsilon^{-1} \int_0^t h_2(s)\, ds\right] + \varepsilon\gamma l^{-1}, \\
& \qquad 0 \leqq t \leqq 1.
\end{aligned}$$

Remark 1. The proof shows that the full force of assumptions (d) and (e) is not required; rather, we need only require

$$\text{(i)} \qquad \begin{aligned}
& f_{y'}(t, u_1(t), u_1'(t), \varepsilon) \geqq h_1(t), && -1 \leqq t \leqq 0, \\
& f_{y'}(0, y, y', \varepsilon) = 0, && (0, y, y', \varepsilon) \in \mathscr{R}, \\
& f_{y'}(t, u_2(t), u_2'(t), \varepsilon) \leqq h_2(t), && 0 \leqq t \leqq 1,
\end{aligned}$$

[2] The proof shows that it is enough to assume that, instead of $h_2' \leqq 0$ on $(0, 1]$, $-h_2' + l \geqq 0$ on $(0, 1]$. Similarly, in the case of h_1', $-h_1' + l \geqq 0$ on $[-1, 0)$ is sufficient.

and

$$\text{(ii)}\qquad \begin{aligned} &f_y(t, u_1(t), u_1'(t), \varepsilon) \geqq l > 0, && -1 \leqq t \leqq 0,\\ &f_y(0, y, y', \varepsilon) \geqq l > 0, && (0, y, y', \varepsilon) \in \mathscr{R},\\ &f_y(t, u_2(t), u_2'(t), \varepsilon) \geqq l > 0, && 0 \leqq t \leqq 1. \end{aligned}$$

Remark 2. Some examples of the types of functions h_1, h_2 which satisfy hypothesis (d) are:

$$\text{(i)}\qquad \begin{aligned} &h_1(t) = -k_1 t^{2n+1}, && -1 \leqq t \leqq 0,\\ &h_2(t) = -k_2 t^{2n+1}, && 0 \leqq t \leqq 1, \quad n = 0, 1, 2, \cdots ; \end{aligned}$$

$$\text{(ii)}\qquad \begin{aligned} &h_1(t) = k_1 t^{pq^{-1}}, && -1 \leqq t \leqq 0, p \geqq q > 0 \text{ integers}, p \text{ even}, q \text{ odd},\\ &h_2(t) = -k_2 t^{r}, && 0 \leqq t \leqq 1, r \text{ real and positive}; \end{aligned}$$

$$\text{(iii)}\qquad \begin{aligned} &h_1(t) = k_1 t^{2n}, && -1 \leqq t \leqq 0,\\ &h_2(t) = -k_2 t^{2n}, && 0 \leqq t \leqq 1, \quad n = 1, 2, \cdots . \end{aligned}$$

Here k_1, k_2 are positive constants.

Remark 3. Our final remark concerns the assumption f_y be positively bounded away from zero. This restriction is certainly necessary in order for the method of proof to succeed; however, examples exist (see O'Malley [6] and Pearson [8]) which justify its inclusion. It is nevertheless possible to prove Theorem 3.1 under the assumption that $f_y \geqq 0$ in $\mathscr{R}$. We must then restrict f in equation (3.1) to be independent of ε and assume that the solutions u_i of $(3.3)_i, (3.4)_i$ $(i = 1, 2)$ satisfy $u_i'' \equiv 0$. The estimates in Theorem 3.1, part (i) are then free of the terms of order $O(\varepsilon)$.

Theorem 3.1 includes several earlier results of O'Malley [6] for linear boundary value problems of the form (3.1), (3.2). This is so because linear functions $f(t, y, y', \varepsilon) = a(t, \varepsilon)y' + b(t, \varepsilon)y + c(t, \varepsilon)$ trivially satisfy a Nagumo condition. In the case of quasi-linear functions, i.e., functions $f(t, y, y', \varepsilon) = a(t, y, \varepsilon)y' + b(t, y, \varepsilon)$, a Nagumo condition is also satisfied. However, Theorem 3.1 is not always directly applicable since assumption (e), the nonnegativity restriction, is too severe for certain quasi-linear functions f. For example, Dorr [3] has considered problems of the form

$$\text{(3.5)}\qquad \begin{aligned} &\varepsilon y'' + t^{2n+1}F(t, y)y' = 0, && -1 < t < 1,\\ &y(-1) = B_1, \qquad y(1) = B_2. \end{aligned}$$

with $n = 0, 1, 2, \cdots$ and $F(t, y) \geqq k > 0$. Rewriting this equation as

$$\varepsilon y'' = -t^{2n+1}F(t, y)y' = f(t, y, y', \varepsilon),$$

we see that $f_y = -t^{2n+1}F_y(t, y)y'$ (assuming F_y exists); consequently, f_y is not, in general, nonnegative. Nevertheless, we may still apply Theorem 2.1, without assuming differentiability of F, by defining for each $\varepsilon > 0$,

$$\alpha(t) = \begin{cases} B_1, & -1 \leqq t \leqq 0, \\ B_2 - (B_2 - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], & 0 \leqq t \leqq 1, \end{cases}$$

$$\beta(t) = \begin{cases} B_1 - (B_1 - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], & -1 \leqq t \leqq 0, \\ B_2, & 0 \leqq t \leqq 1, \end{cases}$$

if $B_1 \leqq B_2$, and

$$\alpha(t) = \begin{cases} B_1 - (B_1 - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], & -1 \leqq t \leqq 0, \\ B_2, & 0 \leqq t \leqq 1, \end{cases}$$

$$\beta(t) = \begin{cases} B_1, & -1 \leqq t \leqq 0, \\ B_2 - (B_2 - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], & 0 \leqq t \leqq 1, \end{cases}$$

if $B_1 > B_2$. It is then trivial to see that

$$\varepsilon\alpha''(t) + t^{2n+1}F(t, \alpha(t))\alpha'(t) \geqq 0,$$

and

$$\varepsilon\beta''(t) + t^{2n+1}F(t, \beta(t))\beta'(t) \leqq 0 \quad \text{for } t \in (-1, 1).$$

Consequently (3.5) has a unique solution $y = y(t, \varepsilon)$ satisfying

$$|y(t,\varepsilon) - B_1| \leqq |B_2 - B_1| \exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], \qquad -1 \leqq t \leqq 0,$$

$$|y(t,\varepsilon) - B_2| \leqq |B_1 - B_2| \exp[-\{\varepsilon(2n+2)\}^{-1}kt^{2n+2}], \qquad 0 \leqq t \leqq 1.$$

(Uniqueness follows from the maximum principle; see, e.g., [4, § 2]). These estimates extend results of Dorr [3] who only showed that

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \begin{cases} B_1, & -1 \leqq t < 0, \\ B_2, & 0 < t \leqq 1. \end{cases}$$

For the most general quasi-linear problem

$$\varepsilon y'' + g(t, y, \varepsilon)y' + h(t, y, \varepsilon) = 0, \qquad -1 < t < 1,$$

$$y(-1), \quad y(1) \text{ prescribed},$$

with g satisfying assumption (d) of Theorem 3.1, it is also possible to obtain results like those above. Assumption (e) is here translated into the assumption that $h_y \geqq l > 0$ in $\mathscr{R}$.

4. Problems exhibiting a boundary layer at one endpoint. We consider again the boundary value problem

$$\varepsilon y'' = f(t, y, y', \varepsilon), \qquad -1 < t < 1, \tag{4.1}$$

$$y(-1) = B_1, \qquad y(1) = B_2, \tag{4.2}$$

in which f has a turning point at $t = 0$. By assuming a different type of behavior of $f_{y'}$ near zero, the solutions of (4.1), (4.2) are shown to possess genuine boundary layers at $t = -1$ or $t = 1$. Specifically, $f_{y'}$ is assumed to behave like an even power of t, i.e., $f_{y'}$ does not change sign at $t = 0$. The nonoccurrence of a sign change then precludes the appearance of a transition layer; instead, the nonuniformity manifests itself at an endpoint. We examine, for the sake of definiteness, first the case in which $f_{y'}$ is negative on $[-1, 1] - \{0\}$. Classical boundary layer theory would suggest that the appropriate reduced problem is

$$0 = f(t, u, u', 0), \qquad -1 < t < 1, \tag{4.3}$$

$$u(1) = B_2. \tag{4.4}$$

The principal result of this section is the following.

THEOREM 4.1. *Assume*

(a) *there is a function* $u \in C^{(2)}[-1, 1]$ *satisfying* (4.3), (4.4);

(b) f *is continuous in* (t, y, y', ε) *and of class* $C^{(1)}$ *with respect to* y, y' *in* $\mathscr{R}$: $-1 \leqq t \leqq 1, |y - u(t)| \leqq d, |y'| < \infty, 0 \leqq \varepsilon \leqq \varepsilon_1, d, \varepsilon_1 > 0$;

(c) $f(t, u(t), u'(t), \varepsilon) = O(\varepsilon), -1 \leqq t \leqq 1$;

(d) *there is an* $h \in C^{(2)}[-1, 2]$ *with the properties*: $h < 0$ *in* $[-1, 2] - \{0\}$, $h(0) = 0$, h *is an even function*, $h'(1 + t) \leqq 0$ *in* $[-1, 2]$; *further suppose* $f_{y'} \leqq h(t)$ *in* $\mathscr{R}$;

(e) $f_y \geqq l > 0$ *in* $\mathscr{R}$, *for some constant* l;

(f) f *satisfies a Nagumo condition in* $\mathscr{R}$.

Then there is an $\varepsilon_0 > 0, \varepsilon_0 \leqq \varepsilon_1$, *such that for each* $\varepsilon, 0 < \varepsilon \leqq \varepsilon_0$, *there exists a solution* $y = y(t, \varepsilon)$ *of* (4.1), (4.2). *In addition,*

$$|y(t, \varepsilon) - u(t)| \leqq |B_1 - u(-1)| \exp\left[\varepsilon^{-1} \int_0^{1+t} h(s)\, ds\right] + c\varepsilon, \qquad -1 \leqq t \leqq 1.$$

Proof. To apply the existence theorem, Theorem 2.1, we must construct suitable lower and upper solutions α, β, respectively. Define, for $t \in [-1, 1]$ and ε in $(0, \varepsilon_1]$,

$$\alpha(t) = \begin{cases} u(t) - (u(-1) - B_1) \exp\left[\varepsilon^{-1} \int_0^{1+t} h(s)\, ds\right] - \varepsilon\gamma l^{-1} & \text{if } u(-1) \geqq B_1, \\ u(t) - \varepsilon\gamma l^{-1} & \text{if } u(-1) \leqq B_1, \end{cases}$$

and

$$\beta(t) = \begin{cases} u(t) + \varepsilon\gamma l^{-1} & \text{if } u(-1) \geqq B_1, \\ u(t) - (u(-1) - B_1) \exp\left[\varepsilon^{-1} \int_0^{1+t} h(s)\, ds\right] + \varepsilon\gamma l^{-1} & \text{if } u(-1) \leqq B_1. \end{cases}$$

Clearly, $\alpha(-1) \leqq B_1 \leqq \beta(-1)$ and $\alpha(1) \leqq B_2 \leqq \beta(1)$. In the case that $u(-1) \geqq B_1$, we verify explicitly that α is a lower solution and β an upper solution of (4.1). The case $u(-1) \leqq B_1$ is treated similarly.

Now

$$\alpha(t) = u(t) - (u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right] - \varepsilon\gamma l^{-1},$$

$$\alpha'(t) = u'(t) - \varepsilon^{-1}h(1+t)(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right],$$

$$\varepsilon\alpha''(t) = \varepsilon u''(t) - (h'(1+t) + \varepsilon^{-1}h^2(1+t))(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right].$$

Therefore, for $t \in (-1, 1)$,

$$\begin{aligned}
&\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon)\\
&\quad \geqq -\varepsilon M - \varepsilon^{-1}h^2(1+t)(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right] + O(\varepsilon)\\
&\qquad - f_y[t]\left[-(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right] - \varepsilon\gamma l^{-1}\right]\\
&\qquad - f_{y'}[t]\left[-\varepsilon^{-1}h(1+t)(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right]\right]\\
&\quad \geqq -\varepsilon M - \varepsilon^{-1}h^2(1+t)(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right]\\
&\qquad + O(\varepsilon) + l(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right]\\
&\qquad + \varepsilon\gamma + \varepsilon^{-1}h(t)h(1+t)(u(-1) - B_1)\exp\left[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds\right].
\end{aligned}$$

For $t \in (-1, -\frac{1}{2}]$, $h(t)h(1+t) - h^2(1+t) \geqq 0$, which follows from the fact that $h'(1+t) \leqq 0$ and h is even in $[-1, 2]$. For $t \in [-\frac{1}{2}, 1)$, the factor $\exp[\varepsilon^{-1}\int_0^{1+t} h(s)\,ds]$ is transcendentally small; in particular, it is of order $O(\varepsilon^2)$. Thus by choosing γ sufficiently large, we can insure that $\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon) \geqq 0$ on $[-\frac{1}{2}, 1)$. Combining these observations, we have the desired inequality. Of course, $|u''| \leqq M$ and $[t]$ is the appropriate intermediate point. (It is, of course, possible to give an explicit estimate for the size of γ; however, for brevity, we choose not to do so.)

It is even simpler to show that $f(t, \beta, \beta', \varepsilon) - \varepsilon\beta'' \geqq 0$ on $(-1, 1)$.

$$f(t, \beta, \beta', \varepsilon) - \varepsilon\beta'' \geqq O(\varepsilon) + f_y[t][\varepsilon\gamma l^{-1}] - \varepsilon M \geqq 0$$

for γ sufficiently large.

Remark 1. The proof shows that the assumption that h be an even function may be weakened to:

$$h(t)h(1+t) - h^2(1+t) \geqq 0$$

for $t \in (-1, \delta]$, $\delta + 1 > 0$, $\delta = O(1)$.

Remark 2. Some examples of functions h satisfying assumption (d) are:
(i) $h(t) = -kt^{2n}$, $n = 1, 2, \cdots$;

(ii) $h(t) = -kt^{pq^{-1}}$, $p \geqq q > 0$ integers, p even, q odd, where k is a positive constant.

Remark 3. Theorem 4.1 has been proved under the assumptions that (i) $f_{y'}$ is negative on $[-1, 1] - \{0\}$ and vanishes at $t = 0$; and (ii) the solution u of the reduced equation satisfies $u(1) = B_2 = y(1)$. If now we assume that u satisfies $u(-1) = B_1 = y(-1)$ and that $f_{y'}$ is positive on $[-1, 1] - \{0\}$ and vanishes at $t = 0$, we might expect to have a boundary layer at $t = 1$. Indeed, this is so; we simply make the change of variable $\tau = -t$ and apply Theorem 4.1 to the transformed problem.

Dorr [3] has considered special quasi-linear problems of the form

$$\begin{gathered} \varepsilon y'' + t^{2n}F(t, y)y' = 0, \qquad -1 < t < 1, \\ y(-1) = B_1, \qquad y(1) = B_2, \end{gathered} \tag{4.5}$$

with $n = 1, 2, \cdots$ and $|F(t, y)| \geqq k > 0$. Unfortunately, Theorem 4.1 is not directly applicable to problems like (4.5) since assumption (e) is, in general, not satisfied. However, we may proceed in a manner analogous to that outlined at the end of the previous section. It is then not difficult to see that for each $\varepsilon > 0$, the problem (4.5) has a unique solution $y = y(t, \varepsilon)$ satisfying, for $t \in [-1, 1]$,

$$\begin{aligned} |y(t, \varepsilon) - B_2| &\leqq |B_1 - B_2| \exp[-\{\varepsilon(2n + 1)\}^{-1}k(1 + t)^{2n+1}] \\ &\quad \text{if } F(t, y) \geqq k > 0, \\ |y(t, \varepsilon) - B_1| &\leqq |B_2 - B_1| \exp[-\{\varepsilon(2n + 1)\}^{-1}k(1 - t)^{2n+1}] \\ &\quad \text{if } F(t, y) \leqq -k < 0. \end{aligned}$$

Dorr showed only that

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \begin{cases} B_2, & -1 < t \leqq 1, \quad \text{if } F(t, y) \geqq k > 0, \\ B_1, & -1 \leqq t < 1, \quad \text{if } F(t, y) \leqq -k < 0. \end{cases}$$

5. Problems exhibiting a boundary layer at each endpoint. In some instances no solution of (4.3) may satisfy either of the boundary conditions (4.2). Depending on the behavior of $f_{y'}$ at $t = 0$, the appearance of boundary layers at $t = -1$ and $t = 1$, arising from the loss of both boundary conditions, can be expected. The motivation for the theorems below arose out of a discussion in O'Malley [6] of the linear problem

$$\begin{gathered} \varepsilon y'' + 2\alpha t y' - \alpha\beta y = 0, \qquad -1 < t < 1, \\ y(-1), y(1) \text{ prescribed}, \end{gathered} \tag{5.1}$$

for $\alpha < 0$ and $\beta \neq 2m$, $m = 0, 1, 2, \cdots$. O'Malley showed that whenever β is less than zero, the solution $y(t, \varepsilon)$ of (5.1) converges to zero, the trivial solution of the reduced problem, uniformly in $[-1 + \delta_1, 1 - \delta_2]$, $\delta_1, \delta_2 > 0$. The nonuniformity arises at the endpoints because the zero solution generally satisfies neither boundary condition in (5.1). If β is equal to zero or a positive even integer, $y(t, \varepsilon)$ may behave very strangely. This is the case of resonance in which $y(t, \varepsilon)$ may converge to a nontrivial function in $(-1, 1)$, as $\varepsilon \to 0^+$. As a consequence, $y(t, \varepsilon)$ generally exhibits nonalgebraic singular behavior in $(-1, 1)$. In Theorems 5.1 and 5.2 below we avoid this unpleasant situation by again assuming that the function f_y is positively bounded away from zero. Further we suppose that $f_{y'}$ behaves essentially like the coefficient of y' in (5.1). The precise statement is the following.

THEOREM 5.1. *Assume*

(a) *there is a function* $u \in C^{(2)}[-1, 1]$ *satisfying* (4.3);

(b) f *is continuous in* (t, y, y', ε) *and of class* $C^{(1)}$ *with respect to* y, y' *in* $\mathscr{R}$: $-1 \leqq t \leqq 1, |y - u(t)| \leqq d, |y'| < \infty, 0 \leqq \varepsilon \leqq \varepsilon_1, d, \varepsilon_1 > 0$;

(c) $f(t, u(t), u'(t), \varepsilon) = O(\varepsilon), -1 \leqq t \leqq 1$;

(d) *there are positive constants* k_1, k_2 *such that in* $\mathscr{R}$,

$$f_{y'} \leqq k_1 t^{2n+1}, \qquad t \in [-1, 0],$$

and

$$f_{y'} \geqq k_2 t^{2n+1}, \qquad t \in [0, 1] \quad \textit{for } n = 0, 1, 2, \cdots;$$

(e) $f_y \geqq l > 0$ *in* $\mathscr{R}$;

(f) $f_{y'} = O(\varepsilon^{-N})$ *in* $\mathscr{R}$ *for some* $N \geqq 0$.

Then there is an $\varepsilon_0 > 0$, $\varepsilon_0 \leqq \varepsilon_1$ *such that for each* ε, $0 < \varepsilon \leqq \varepsilon_0$, *there exists a solution* $y = y(t, \varepsilon)$ *of* (4.1), (4.2). *In addition,*

$$\begin{aligned} |y(t, \varepsilon) - u(t)| \leqq{}& |B_1 - u(-1)| \exp[-\{\varepsilon(2n+2)\}^{-1} k_1 (1+t)^{2n+2}] \\ &+ |B_2 - u(1)| \exp[-\{\varepsilon(2n+2)\}^{-1} k_2 (1-t)^{2n+2}] + c\varepsilon, \\ &\qquad\qquad -1 \leqq t \leqq 1. \end{aligned}$$

Proof. Define, for $t \in [-1, 1]$ and ε in $(0, \varepsilon_1]$,

$$\alpha(t) = \begin{cases} u(t) - (u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1} k_1 (1+t)^{2n+2}] - \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \geqq B_1, \quad u(1) \leqq B_2, \\ u(t) - (u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1} k_1 (1+t)^{2n+2}] \\ \qquad - (u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1} k_2 (1-t)^{2n+2}] - \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \geqq B_1, \quad u(1) \geqq B_2, \\ u(t) - \varepsilon\gamma l^{-1}, \qquad \text{if } u(-1) \leqq B_1, \quad u(1) \leqq B_2, \\ u(t) - (u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1} k_2 (1-t)^{2n+2}] - \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \leqq B_1, \quad u(1) \geqq B_2, \end{cases}$$

and

$$\beta(t) = \begin{cases} u(t) - (u(1) - B_2)\exp[-\{2n+2)\}^{-1} k_2 (1-t)^{2n+2}] + \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \geqq B_1, \quad u(1) \leqq B_2, \\ u(t) + \varepsilon\gamma l^{-1}, \qquad \text{if } u(-1) \geqq B_1, \quad u(1) \geqq B_2, \\ u(t) - (u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1} k_1 (1+t)^{2n+2}] \\ \qquad - (u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1} k_2 (1-t)^{2n+2}] + \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \leqq B_1, \quad u(1) \leqq B_2, \\ u(t) - (u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1} k_1 (1+t)^{2n+2}] + \varepsilon\gamma l^{-1}, \\ \qquad\qquad \text{if } u(-1) \leqq B_1, \quad u(1) \geqq B_2. \end{cases}$$

We verify explicitly that for the case $u(-1) \geqq B_1$, $u(1) \leqq B_2$, α is a lower solution and β an upper solution of (4.1).

$$\alpha(t) = u(t) - (u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}] - \varepsilon\gamma l^{-1},$$

so $\alpha(-1) \leqq B_1$ and $\alpha(1) \leqq B_2$. Also

$$\alpha'(t) = u'(t) + \varepsilon^{-1}k_1(1+t)^{2n+1}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}],$$

$$\varepsilon\alpha''(t) = \varepsilon u''(t) + [(2n+1)k_1(1+t)^{2n} - \varepsilon^{-1}k_1^2(1+t)^{4n+2}](u(-1) - B_1) \cdot \exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}].$$

Substituting and expanding,

$$\begin{aligned}
&\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon)\\
&\quad\geqq -\varepsilon M - \varepsilon^{-1}k_1^2(1+t)^{4n+2}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]\\
&\qquad + O(\varepsilon) - f_y[t][-(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}] - \varepsilon\gamma l^{-1}]\\
&\qquad - f_{y'}[t][\varepsilon^{-1}k_1(1+t)^{2n+1}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}],
\end{aligned}$$

where $|u''| \leqq M$ and $[t]$ is the appropriate intermediate point. On $(-1, 0]$ the estimate $f_{y'} \leqq k_1 t^{2n+1}$ holds; whence, for $t \in (-1, 0]$,

$$\begin{aligned}
&\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon)\\
&\quad\geqq -\varepsilon M - \varepsilon^{-1}k_1^2(1+t)^{4n+2}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]\\
&\qquad + O(\varepsilon) + l(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]\\
&\qquad - \varepsilon^{-1}k_1^2t^{2n+1}(1+t)^{2n+1}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}].
\end{aligned}$$

If $t \in (-1, -\frac{1}{2}]$, $-t^{2n+1}(1+t)^{2n+1} - (1+t)^{4n+2} \geqq 0$; if $t \in [-\frac{1}{2}, 0]$, the exponential factor is transcendentally small, in particular, it is of order $O(\varepsilon^2)$. Therefore, in $(-1, 0]$, $\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon) \geqq 0$ for γ sufficiently large. For $t \in [0, 1)$, $f_{y'} \geqq k_2 t^{2n+1}$, so to produce the desired inequality we invoke the assumption that $f_{y'} = O(\varepsilon^{-N})$, $N \geqq 0$. On $[0, 1)$, the term $\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]$ is transcendentally small, so it is at least of order $O(\varepsilon^{N+2})$. Thus, on $[0, 1)$,

$$\begin{aligned}
&\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon)\\
&\quad\geqq -\varepsilon M - \varepsilon^{-1}k_1^2(1+t)^{4n+2}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]\\
&\qquad + O(\varepsilon) + l(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{-1}k_1(1+t)^{2n+2}]\\
&\qquad + \varepsilon\gamma - (K\varepsilon^{-N})(\varepsilon^{-1}k_1)(1+t)^{2n+1}(u(-1) - B_1)\exp[-\{\varepsilon(2n+2)\}^{2n+2}] \geqq 0
\end{aligned}$$

for γ sufficiently large, where $|u''| \leqq M$ and $|f_{y'}| \leqq K \cdot \varepsilon^{-N}$. Thus α is a lower solution.

The demonstration that β is an upper solution proceeds similarly. From the definition, $\beta(-1) \geqq B_1$ and $\beta(1) \geqq B_2$. And differentiating,

$$\beta'(t) = u'(t) - \varepsilon^{-1}k_2(1-t)^{2n+1}(u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}k_2(1-t)^{2n+2}],$$
$$\varepsilon\beta''(t) = \varepsilon u''(t) + [(2n+1)k_2(1-t)^{2n} - \varepsilon^{-1}k_2^2(1-t)^{4n+2}](u(1) - B_2) \cdot \exp[-\{\varepsilon(2n+2)\}^{-1}k_2(1-t)^{2n+2}].$$

Thus, for $t \in (-1, 1)$,

$$\begin{aligned} f(t, \beta, \beta', \varepsilon) - \varepsilon\beta'' &\geqq O(\varepsilon) + f_y[t][-(u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}k_2(1-t)^{2n+2}] \\ &\quad + f_{y'}[t][-\varepsilon^{-1}k_2(1-t)^{2n+1}(u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}k_2(1-t)^{2n+2}] \\ &\quad - \varepsilon M + \varepsilon^{-1}k_2^2(1-t)^{4n+2}(u(1) - B_2)\exp[-\{\varepsilon(2n+2)\}^{-1}k_2(1-t)^{2n+2}]. \end{aligned}$$

As before, $t^{2n+1}(1-t)^{2n+1} - (1-t)^{4n+2} \geqq 0$ for $t \in [\frac{1}{2}, 1)$, and the exponential factor is of order $O(\varepsilon^2)$ for $t \in [0, \frac{1}{2}]$. Thus if $t \in [0, 1)$, $f(t, \beta, \beta', \varepsilon) - \varepsilon\beta'' \geqq 0$ if γ is sufficiently large, completing the demonstration that β is an upper solution. Finally the assumption on $f_{y'}$ means that for each ε, a Nagumo condition is satisfied. This concludes the proof.

The next theorem shows that the conclusion of Theorem 5.1 remains valid if $f_{y'}$ behaves like an even power of t.

THEOREM 5.2. *Make the same assumptions as in Theorem* 5.1 *with assumption* (d) *changed to*: *there are positive constants* k_1, k_2 *such that in* $\mathscr{R}$,

$$f_{y'} \leqq -k_1 t^{2n}, \qquad t \in [-1, 0],$$

and

$$f_{y'} \geqq k_2 t^{2n}, \qquad t \in [0, 1] \quad \textit{for } n = 1, 2, \cdots.$$

Then the conclusion of Theorem 5.1 *follows with the estimate*

$$\begin{aligned} |y(t, \varepsilon) - u(t)| \leqq |B_1 - u(-1)| &\exp[-\{\varepsilon(2n+1)\}^{-1}k_1(1+t)^{2n+1}] \\ &+ |B_2 - u(1)|\exp[-\{\varepsilon(2n+1)\}^{-1}k_2(1-t)^{2n+1}] + c\varepsilon, \\ &\qquad -1 \leqq t \leqq 1. \end{aligned}$$

The proof of Theorem 5.2 closely resembles that of Theorem 5.1 and is omitted.

While more general results are possible, the two above clearly illustrate the kind of behavior to be expected.

6. Problems in which the turning point coincides with an endpoint. For the sake of definiteness we consider problems in which the turning point occurs at the left-hand endpoint. Specifically, the boundary value problem is

$$\varepsilon y'' = f(t, y, y', \varepsilon), \qquad 0 < t < 1, \tag{6.1}$$

$$y(0) = A, \qquad y(1) = B, \tag{6.2}$$

together with the corresponding reduced problem

$$0 = f(t, u, u', 0), \qquad 0 < t < 1, \tag{6.3}$$

$$u(1) = B. \tag{6.4}$$

The function f is assumed to possess a turning point at $t = 0$. We first prove a general result and then examine some of its consequences in the rest of the section.

THEOREM 6.1. *Assume*

(a) *there is a* $u \in C^{(2)}[0, 1]$ *satisfying* (6.3), (6.4);

(b) f *is continuous in* (t, y, y', ε) *and of class* $C^{(1)}$ *with respect to* y, y' *in* $\mathscr{D}: 0 \leqq t \leqq 1, |y - u(t)| \leqq d, |y'| < \infty, 0 \leqq \varepsilon \leqq \varepsilon_1, d, \varepsilon_1 > 0$;

(c) $$f(t, u(t), u'(t), \varepsilon) = O(\varepsilon), \qquad 0 \leqq t \leqq 1;$$

(d) *there is a function* $h \in C^{(1)}[0, 1]$ *with the properties*: $h(0) = 0$, $h < 0$ *on* $(0, 1]$, $h' \leqq 0$ *on* $[0, 1]$, *and satisfying* $f_{y'} \leqq h$ *in* $\mathscr{D}$;

(e) $f_y \geqq l > 0$ *in* $\mathscr{D}$ *for some constant* l;

(f) f *satisfies a Nagumo condition in* $\mathscr{D}$.

Then for each ε, $0 < \varepsilon \leqq \varepsilon_1$, *there exists a solution* $y = y(t, \varepsilon)$ *of* (6.1), (6.2). *Moreover,*

(i) $$|y(t, \varepsilon) - u(t)| \leqq |A - u(0)| \exp\left[\varepsilon^{-1} \int_0^t h(s)\, ds\right] + c_1\varepsilon, \quad 0 \leqq t \leqq 1,$$

(ii) $$|y'(t, \varepsilon) - u'(t)| \leqq \varepsilon^{-1} c' \exp\left[\varepsilon^{-1} \int_0^t h(s)\, ds\right] + c_2\varepsilon,$$

$$0 < \delta \leqq t \leqq 1,$$

where c_1, c_2, c' *are constants independent of* ε, *and* $\delta = O(\eta(\varepsilon))$ *for some gauge function* $\eta(\varepsilon)$ *which depends on* h.

Proof. The result is established provided we can construct suitable lower solutions α and upper solutions β. Define, for $t \in [0, 1]$ and ε in $(0, \varepsilon_1]$,

$$\alpha(t) = \begin{cases} u(t) - (u(0) - A) \exp\left[\varepsilon^{-1} \int_0^t h(s)\, ds\right] - \varepsilon\gamma l^{-1}, & \text{if } u(0) \geqq A, \\ u(t) - \varepsilon\gamma l^{-1}, & \text{if } u(0) \leqq A, \end{cases}$$

and

$$\beta(t) = \begin{cases} u(t) + \varepsilon\gamma l^{-1}, & \text{if } u(0) \geqq A, \\ u(t) - (u(0) - A) \exp\left[\varepsilon^{-1} \int_0^t h(s)\, ds\right] + \varepsilon\gamma l^{-1}, & \text{if } u(0) \leqq A. \end{cases}$$

We treat explicitly the case $u(0) \leqq A$. Trivially, $\beta(0) \geqq A$, $\beta(1) \geqq B$, and $f(t, \beta, \beta', \varepsilon) - \varepsilon\beta'' \geqq 0$. With

$$\alpha(t) = u(t) - (u(0) - A) \exp\left[\varepsilon^{-1} \int_0^t h(s)\, ds\right] - \varepsilon\gamma l^{-1},$$

$$\alpha(0) \leqq A, \qquad \alpha(1) \leqq B,$$

and

$$\alpha'(t) = u'(t) - \varepsilon^{-1}h(t)(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right],$$

$$\varepsilon\alpha''(t) = \varepsilon u''(t) - (h'(t) + \varepsilon^{-1}h^2(t))(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right].$$

So

$$\begin{aligned}
&\varepsilon\alpha'' - f(t, \alpha, \alpha', \varepsilon)\\
&\quad\geqq -\varepsilon M - \varepsilon^{-1}h^2(t)(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right] + O(\varepsilon)\\
&\qquad - f_y[t]\left[-(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right] - \varepsilon\gamma l^{-1}\right]\\
&\qquad - f_{y'}[t]\left[-\varepsilon^{-1}h(t)(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right]\right]\\
&\quad\geqq -\varepsilon M - \varepsilon^{-1}h^2(t)(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right]\\
&\qquad + O(\varepsilon) + l(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right] + \varepsilon\gamma\\
&\qquad + \varepsilon^{-1}h^2(t)(u(0) - A)\exp\left[\varepsilon^{-1}\int_0^t h(s)\,ds\right] \geqq 0
\end{aligned}$$

for γ sufficiently large, where $|u''| \leqq M$.[3] Of course, we assume $f(t, u(t), u'(t), \varepsilon) = O(\varepsilon)$. The case $u(0) \leqq A$ is treated similarly.

To establish the estimate on $y' - u'$, we set $z = y - u$ and substitute into (6.1), (6.2) to obtain

$$\varepsilon z'' - f_{y'}\{t\} \cdot z' - f_y\{t\} \cdot z = O(\varepsilon), \qquad 0 < t < 1,$$
$$z(0) = A - u(0), \qquad z(1) = 0,$$

where $\{t\} = (t, u + \theta z, u + \theta z', \varepsilon)$ for some $\theta \in (0, 1)$. The estimate on z' now follows as in Dorr, Parter and Shampine [4, § 2].

Remark 1. The proof shows that it is possible to weaken assumptions (d) and (e) to:

(d′)
$$f_{y'}(t, u(t), u'(t), \varepsilon) \leqq h(t), \qquad 0 \leqq t \leqq 1,$$
$$f_{y'}(0, y, y', \varepsilon) \leqq 0, \qquad \text{for } (0, y, y', \varepsilon) \in \mathscr{D};$$

(e′)
$$f_y(t, u(t), u'(t), \varepsilon) \geqq l > 0, \qquad 0 \leqq t \leqq 1,$$
$$f_y(0, y, y', \varepsilon) \geqq l > 0, \qquad \text{for } (0, y, y', \varepsilon) \in \mathscr{D}.$$

[3] The inequalities reveal that in assumption (d), it is enough to assume that $-h' + l \geqq 0$ on $[0, 1]$.

Remark 2. A simple example of a function h satisfying the hypotheses in assumption (d) is $h(t) = -kt^r$, $r \geqq 0$, r real, k a positive constant. The corresponding gauge function $\eta(\varepsilon)$ appearing in estimate (ii) of Theorem 6.1 is then easily seen to be $\eta(\varepsilon) = \varepsilon^{(r+1)^{-1}}$.

Remark 3. The assumption that f_y be bounded positively away from zero is obviously necessary in showing the functions α, β are bounding solutions of (6.1). Moreover, as in the case of the other problems considered, examples exist which justify such a restriction. Indeed, Ackerberg and O'Malley [1] have noted the presence of resonant behavior in problems of the form (1.1) whenever the coefficient of y is negative. The situation here is far from simple. In some cases the solutions of the full problem may possess a transition layer at $t = 0$ and a boundary layer at $t = 1$. The solution of the reduced equation (6.3) then satisfies neither boundary condition and may have an exponentially large amplitude.

Remark 4. We can also treat the case in which the turning point occurs at $t = 1$. The solution u of (6.3) is chosen to satisfy the initial condition $u(0) = A$, the algebraic sign of $f_{y'}$ is changed, and consequently, the solution of (6.1), (6.2) exhibits a boundary layer at $t = 1$. To obtain the precise result, we can make the change of variable $\tau = 1 - t$ in (6.1), (6.2) and apply Theorem 6.1 to the transformed problem.

7. Classification of turning points. In the sections above we have studied various problems involving equations with turning points. The very fact that we have observed substantially different types of behavior of solutions indicates that the turning points themselves are of different types. Taking the results of §§ 3 and 4 in particular, we can formulate two heuristic principles in the nonresonant case.

I. *If the function $f_{y'}$ changes its algebraic sign in passing through the turning point, a transition layer appears at the turning point. The solution $y(t, \varepsilon)$ is approximated uniformly to order $O(\varepsilon)$ on the interval to the left of the transition layer by that reduced solution satisfying the left-hand boundary condition. To the right of the layer $y(t, \varepsilon)$ behaves to order $O(\varepsilon)$ like that reduced solution which satisfies the right-hand boundary condition.*

(The solutions of problems containing such turning points behave essentially like the solutions of classical turning point problems, e.g., Airy's equation.)

II. *If the function $f_{y'}$ does not change its algebraic sign in passing through the turning point, then a boundary layer, but no transition layer, appears at one endpoint. The position of the boundary layer is itself dependent on this sign: if $f_{y'}$ is negative (except at the turning point), the layer occurs at the left-hand endpoint: if positive, at the right.*

(The solutions of such problems display no radical behavior near the turning point. Rather they are approximated uniformly to order $O(\varepsilon)$ by a single function in the entire interval except in a narrow band near one endpoint.)

The observations I, II are meant to be employed only heuristically. The precise classification of the behavior of the solution of any turning point problem must be effected analytically, as we have done above. Their consistent use, however, may assist in the selection of appropriate bounding solutions.

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NEWTON'S METHOD TECHNIQUES FOR SINGULAR PERTURBATIONS*

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Abstract. In this paper, existence and uniqueness of solutions to several classes of singularly perturbed initial value and boundary value problems are proven by use of Newton's method. Asymptotic expansions of solutions to some of these problems are developed and analyzed, and proof of their asymptotic correctness is shown based on Newton's method. As a by-product some numerical algorithms for these solutions are obtained.

1. Introduction. Suppose that we are given an operator P, which maps some open set S of a Banach space E_x into another Banach space E_y. We assume there is a linear operator P' from E_x into E_y such that

$$\|P(y + h) - P(y) - P'(y)h\|/\|h\| \to 0 \quad \text{as } \|h\| \to 0.$$

Henceforth, P' will denote the derivative of the operator P in S. Furthermore, assume P' is continuous in S and consider the equation

$$P(y) = 0. \tag{1.1}$$

Choose an "initial guess" $y_0 \in S$, and generate the sequence $\{y_n\}$ by

$$y_{n+1} = y_n - [P'(y_n)]^{-1}P(y_n), \qquad n = 0, 1, 2, \cdots. \tag{1.2}$$

This method of defining the sequence $\{y_n\}$ is called the basic Newton's method. An infinite Newton's sequence will not necessarily be generated for every "initial guess" y_0 because the operator $P'(y_k)$ may not have an inverse for some integer k. To avoid this possible difficulty, we can generate a sequence $\{\bar{y}_n\}$ by the modified Newton method

$$\bar{y}_{n+1} = \bar{y}_n - [P'(y_0)]^{-1}P(\bar{y}_n). \tag{1.3}$$

The following theorem was formulated by Kantorovic [8]. It gives, not only conditions for the existence of a solution y^* to $P(y) = 0$, but also information concerning the regions of existence and uniqueness of y^* and error bounds for $\bar{y}_n$ and y_n as approximations to y^*.

THEOREM 1.1. *Suppose that the operator P in* (1.1) *is defined in some open set $S(\|y - y_0\| < R)$ and has a continuous second derivative in $S_0(\|y - y_0\| \leqq r < R)$. Moreover, suppose that:*

(a) *The operator $T_0 = [P'(y_0)]^{-1}$ exists and is linear,*
(b) $\|T_0\| \leqq B'$ *in some Banach space norm* $\|\cdot\|$,
(c) $\|P(y_0)\| \leqq V'$,
(d) $\|P''(y)\| \leqq K'$, *for all* $y \in S_0$.

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Then, if $h = K'(B')^2V' \leqq \frac{1}{2}$ *and*

$$r \geqq r_0 = \left(\frac{1 - \sqrt{1 - 2h}}{h}\right)B'V',$$

the equation $P(y) = 0$ *has a solution* y^* *in* S_0 *to which both the basic and modified Newton's methods converge, such that*

$$\|y^* - y_0\| \leqq r_0.$$

Furthermore, if $h < \frac{1}{2}$, *then the solution* y^* *is unique in* S_1 *(equals* $\|y - y_0\| < r_1 = ((1 + \sqrt{1 - 2h})/h)B'V')$, *and if* $h = \frac{1}{2}$, *the solution* y^* *is unique in the closed ball* $\|y - y_0\| \leqq 2B'V'$.

The aim of this paper is to use the above theorem to prove existence and uniqueness for several types of singular perturbation problems. Also, the asymptotic correctness of some expansion schemes are proven by the use of Newton's method.

2. The initial value problem $\varepsilon y' = g(t, y)$, $y(0) = A$. Consider the scalar initial value problem

$$\varepsilon y' = g(t, y), \qquad y(t = 0) = A, \quad ' = \frac{d}{dt}, \tag{2.1}$$

where $\varepsilon \to 0^+$, and let $y_0(t^*, \tau)$ satisfy the initial value problem

$$\frac{dy}{d\tau} = g(t^*, y), \qquad y(\tau = 0) = A, \tag{2.2}$$

where t^* is a fixed parameter and $\tau = t/\varepsilon$.

Instead of $y_0(t^*, \tau)$, consider $y_0(t, \tau)$, where t is now a variable. Suppose that

$$g_y(t, y_0(t, \tau)) \leqq -L$$

for some $L > 0$ and for $0 \leqq t \leqq T$ (T finite). Then we have the following.

LEMMA 2.1. $\varepsilon y_0'(t, \tau) - g(t, y_0(t, \tau)) = O(\varepsilon)$ *uniformly in* $0 \leqq t \leqq T$.

Proof.

$$\varepsilon y_0' = \varepsilon\frac{\partial y_0}{\partial t} + \frac{\partial y_0}{\partial \tau} = \varepsilon\frac{\partial y_0}{\partial t} + g(t, y_0),$$

and since $\partial y_0/\partial t = O(1)$ the result follows.

Note also that $y_0(t, \tau)|_{t=0} = A$. We shall use y_0 below as the first iterate in a Newton's method scheme for obtaining the asymptotic solution.

Example.

$$\varepsilon y' = -10y(y - 1)(2t + 1) = g(t, y), \qquad y(0) = 2. \tag{2.3}$$

Here,

$$y_0(t, \tau) = \{1 - \tfrac{1}{2}\exp(-10\tau(2t + 1))\}^{-1}, \quad g_y(t, y_0(t, \tau)) < 0;$$

the actual solution to (2.3) is

$$y(t, \tau) = \{1 - \tfrac{1}{2}\exp(-10\tau(t + 1))\}^{-1},$$

and

$$\varepsilon y_0' - g(t, y_0) = (-40t\, e^{-10\tau(2t+1)})(e^{-10\tau(2t+1)} - 2)^{-2}$$
$$= O(\varepsilon)$$

for bounded $t = \varepsilon\tau$.

Note. $y_0(t, \tau) \to 1$ as $\tau \to \infty$; that is, away from the boundary layer at $t = 0$, y_0 converges to the root $\phi(t) = 1$ of $g(t, y) = 0$ with $g_y(t, \phi(t)) < 0$.

LEMMA 2.2. *Let $z(t, \varepsilon)$ satisfy the linear initial value problem*

$$\varepsilon z' + f(t)z = h(t), \qquad z(0) = 0,$$

where $f(t)$ and $h(t)$ are continuous functions on bounded intervals $t \geqq 0$ and $f(t) \geqq k > 0$. Then

$$|z| \leqq (1/k) \max_t |h(t)|$$

on bounded intervals $t \geqq 0$.

Proof. Integrating, we have

$$z(t) = (1/\varepsilon) \int_0^t \exp\left\{-\int_s^t (f(r)/\varepsilon)\, dr\right\} h(s)\, ds,$$

hence

$$|z(t)| \leqq (1/\varepsilon) \max_t |h(t)| \int_0^t \exp\left\{-\int_s^t (f(r)/\varepsilon)\, dr\right\} ds$$
$$\leqq (1/\varepsilon) \max_t |h(t)| \int_0^t e^{(-k/\varepsilon)(t-s)}\, ds \leqq (1/k) \max_t |h(t)|.$$

Without loss of generality, we can take $A = 0$ in the initial value problem (2.1), since the function $w(t, \varepsilon) = y(t, \varepsilon) - A$ satisfies $\varepsilon w' = F(t, w)$, $w(0, \varepsilon) = 0$, where $F(t, w) = g(t, w + A)$. Note that $F_w(t, w) = g_y(t, y)$.

We select the initial guess $y_0 = y_0(t, \tau)$ of Lemma 2.1 and use the operator

$$P(y) = \varepsilon \frac{dy}{dt} - g(t, y), \qquad \text{where } g_y(t, y_0) \leqq -L < 0.$$

The operator P acts from the space $C'[0, T_0]$ of continuously differentiable functions on $[0, T_0]$ satisfying the initial condition $y(t = 0) = 0$ into the space $C[0, T_0]$ of continuous functions on $[0, T_0]$. On C', the norm is given by

$$\|y\| = \max_{t\in[0,T_0]} \left|\varepsilon \frac{dy}{dt}\right| + \max_{t\in[0,T_0]} |y|, \tag{2.4}$$

while on $C[0, T_0]$ we use

$$\|f\| = \max_{t\in[0,T_0]} |f|. \tag{2.5}$$

Setting $y = y_0 + \Delta y$ we have

$$P'(y_0)\,\Delta y = \varepsilon \frac{d\,\Delta y}{dt} - g_y(t, y_0)\,\Delta y = f(t) \in C[0, T_0], \qquad \Delta y(0) = 0. \tag{2.6}$$

By Lemma 2.2, $P'(y_0)\,\Delta y = f(t)$ has a unique solution Δy such that

$$\max|\Delta y| \leqq \frac{\max|f|}{L} = \frac{1}{L}\|f\|. \tag{2.7}$$

Thus $[P'(y_0)]^{-1}$ exists. Moreover,

$$\left|\varepsilon\frac{dy}{dt}\right| = |f(t) + g_y(t, y_0)\,\Delta y| \leqq \left(1 + \frac{\max|g_y|}{L}\right)\|f\|,$$

so (2.4) implies

$$\|\Delta y\| \leqq C\|f\|, \qquad \text{where } C = 1 + \frac{1}{L} + \frac{\max|g_y|}{L}, \tag{2.8}$$

That is,

$$\|[P'(y_0)]^{-1}\| \leqq C. \tag{2.9}$$

Now from Lemma 2.1,

$$\|P(y_0)\| \leqq C_1\varepsilon, \qquad \text{where } C_1 \geqq \left|\frac{\partial y_0}{\partial t}\right| \text{ for all } t, \tag{2.10}$$

and finally, for Δy and Δz in $C'[0, T_0]$,

$$P''(y)\,\Delta y\,\Delta z = -g_{yy}(t, y)\,\Delta y\,\Delta z$$

so that

$$\|P''(y)\| \leqq C_2, \qquad \text{where } C_2 = \max|g_{yy}(t, y)|, \tag{2.11}$$

where $y \in S_0$ ($\|y - y_0\| \leqq r < R$).

Thus by Theorem 1.1, since $h = C_2C_1C^2\varepsilon < \frac{1}{2}$ for ε sufficiently small, (2.1) has a solution y^* in $\|y - y_0\| \leqq r$, $r \geqq r_0$ such that

$$\|y^* - y_0\| \leqq r_0 = \frac{2C_1C\varepsilon}{1 + (1 - 2C_2C_1C^2\varepsilon)^{1/2}} = O(\varepsilon).$$

It is unique in the open ball $\|y - y_0\| < r_1$, where

$$r_1 = \frac{1 + (1 - 2C_2C_1C^2\varepsilon)^{1/2}}{C_2C}.$$

Remarks. (a) Newton's method is not only an existence and uniqueness proof for the Cauchy problem (2.1), but also an effective method for obtaining an approximation to this solution.

(b) Unlike the usual expansion procedure our method gives both explicit bounds for the difference between the exact solution and the initial guess and rates of convergence of the basic and modified processes; that is, $\|y^* - y\| \leqq r_0$, where r_0 can be obtained explicitly, and the rate of convergence of the basic process is given by the inequality

$$\|y^* - y_n\| \leqq \frac{(2h)^{2^n}}{2^nK'B'} = O(\varepsilon^{2^n}),$$

where y_n is the nth iterate, and of the modified process by

$$\|y^* - y_n\| \leqq \frac{(1 - \sqrt{1 - 2h})^{n+1}}{K'} = O(\varepsilon^{n+1}).$$

(c) Since $|y^*(t, \varepsilon) - y_0| = O(\varepsilon)$, (2.4) implies that y^* is within $O(\varepsilon)$ of y_0 and $\varepsilon(dy^*)/(dt)$ is within $O(\varepsilon)$ of $\varepsilon(dy_0)/(dt)$ pointwise in $[0, T_0]$.

(d) It should be noted that the above method can be applied to equations to which the asymptotic expansion procedures like that of Vasileva [12] do not apply, for example, $\varepsilon y' = -(2 + e^{-1/t})y$ with $y(0) = 1$. Since $e^{-1/t}$ cannot be represented by a Taylor series about $t = 0$, we cannot use the expansion procedure. However,

$$y_0(t, \tau) = e^{-(2+e^{-1/t})\tau}, \qquad e^{-1/t} = 0 \text{ for } t = 0.$$

(e) One can obtain an approximate solution to within $O(\varepsilon)$ for the system

$$\frac{dx}{dt} = f(t, x, y), \qquad x(0) = x^0,$$

$$\varepsilon \frac{dy}{dt} = g(t, x, y), \qquad y(0) = y^0.$$

Let $y_0(t^*, x^*, \tau)$ solve $dy/d\tau = g(t^*, x^*, y)$, $y(0) = y^0$, $\tau = t/\varepsilon$. We then solve

$$\frac{dx}{dt} = f(t, x, y_0(t, x, \tau)), \qquad x(0) = x^0.$$

Since the exact solution y satisfies $y(t, x, \tau) = y_0(t, x, \tau) + O(\varepsilon)$, we have

$$\frac{dx}{dt} = f(t, x, y) = f(t, x, y_0) + O(\varepsilon), \qquad x(0) = x^0,$$

and x can be determined to within $O(\varepsilon)$.

(f) The above method is closely related to the technique of developing an inner expansion near the boundary layer in the stretched variable $\tau = t/\varepsilon$. Alternately, to a lesser extent, it combines ideas of two variable expansions (cf. Cole [5]).

3. Boundary value problems.

3.1. Preliminary results. Consider the two-point quasi-linear boundary value problem

$$\text{(3.1)} \qquad \begin{gathered} \varepsilon y'' + \bar{f}(x, y)y' + \bar{g}(x, y) = 0, \\ y(0, \varepsilon) = A, \qquad y(1, \varepsilon) = B \end{gathered}$$

for $x \in [0, 1]$, ε small and positive and prescribed constants A and B.

Assume the following conditions hold:

(a) The terminal value problem

$$\text{(3.2)} \qquad \bar{f}(x, u)u' + \bar{g}(x, u) = 0, \qquad u(1) = B$$

has a solution $u_0(x)$ for $0 \leqq x \leqq 1$ such that for some $K > 0$, $\bar{f}(x, u_0(x)) \geqq K$ on $0 \leqq x \leqq 1$.

(b) $\bar{f}(0, c) \geqq K$, where $K > 0$ for all c between A and $u_0(0)$.

(c) The functions $\bar{f}(x, y)$ and $\bar{g}(x, y)$ are each infinitely differentiable in both the domains

$$\{(x, y): x = 0, y \text{ between } A \text{ and } u_0(x)\},$$

and

$$\{(x, y): 0 \leqq x \leqq 1,\ y = u_0(x)\}.$$

Under the above hypotheses O'Malley [10] obtains an asymptotic solution to the boundary value problem (3.1). First, an asymptotic solution $u(x, \varepsilon)$ of the terminal value problem

$$\varepsilon u'' + \bar{f}(x, u)u' + \bar{g}(x, u) = 0, \qquad u(1) = B \tag{3.3}$$

of the form

$$u(x, \varepsilon) = \sum_{k=0}^{\infty} u_k(x)\varepsilon^k \tag{3.4}$$

is obtained by formally substituting (3.4) into (3.3) and equating coefficients of like powers of ε to obtain a system of linear differential equations which are solved successively for the u_k. The solution $y(x, \varepsilon)$ of (3.1) and its derivatives differ from $u(x, \varepsilon)$ and its derivatives, respectively, by convergent power series that are asymptotically zero uniformly on any subinterval $0 < d \leqq x \leqq 1$ (Wasow [14]).

To obtain an asymptotic solution of (3.1) which is uniformly valid in $0 \leqq x \leqq 1$, O'Malley introduces the boundary layer correction

$$w(x, \varepsilon) = y(x, \varepsilon) - u(x, \varepsilon), \tag{3.5}$$

which satisfies a boundary value problem of the form

$$\begin{aligned} &\varepsilon w'' + f(x, w)w' + g(x, w)w = 0, \\ &w(0, \varepsilon) = \sum_{k=0}^{\infty} d_k\varepsilon^k = d(\varepsilon), \qquad w(1, \varepsilon) = 0, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} f(x, w) &= \bar{f}(x, w + u), \\ g(x, w)w &= [\bar{f}(x, w + u) - \bar{f}(x, u)]u' + \bar{g}(x, w + u) - \bar{g}(x, u), \\ w(0, \varepsilon) &= A - u(0, \varepsilon). \end{aligned} \tag{3.7}$$

Let τ be the stretched variable $\tau = x/\varepsilon$. Then the boundary value problem (3.6) becomes

$$\begin{aligned} \frac{dw}{d\tau} &= \varepsilon v, \\ \frac{dv}{d\tau} &= -f(\varepsilon\tau, w)v - g(\varepsilon\tau, w)w, \end{aligned} \tag{3.8}$$

with $w(0) = d(\varepsilon)$, $w(\infty) = v(\infty) = 0$, $\tau \in [0, \infty)$. We set

$$w(\tau) = \sum_{k=0}^{\infty} w_k(\tau)\varepsilon^K, \tag{3.9}$$
$$v(\tau) = \sum_{k=-1}^{\infty} v_k(\tau)\varepsilon^k,$$

so that

$$f(\varepsilon\tau, w) = \sum_{k=0}^{\infty} \varepsilon^k f_k(w_0, w_1, \cdots, w_k; \tau), \tag{3.10}$$
$$g(\varepsilon\tau, w) = \sum_{K=0}^{\infty} \varepsilon^k g_k(w_0, w_1, \cdots, w_k; \tau).$$

Thus by formally equating coefficients in (3.8), we have

$$\frac{dw_0}{d\tau} = v_{-1}, \tag{3.11}$$
$$\frac{dv_{-1}}{d\tau} = -f(0, w_0)v_{-1},$$

and for $k \geqq 1$,

$$\frac{dw_k}{d\tau} = v_{k-1}, \tag{3.12}$$
$$\frac{dv_{k-1}}{d\tau} = -\left(\sum_{m=0}^{k} f_m v_{k-m-1} + \sum_{m=0}^{k-1} g_m w_{k-m-1}\right).$$

Equations (3.11) imply that

$$v_{-1}(\tau) = v_{-1}(0)\exp\left(-\int_0^\tau f(0, w_0(s))\right) ds, \tag{3.13}$$
$$w_0(\tau) = -v_{-1}(0)\int_\tau^\infty \exp\left(-\int_0^z f(0, w_0(s))\, ds\right) dz.$$

The w_k, v_{k-1}'s for $k \geqq 1$, satisfy linear differential equations.

THEOREM 3.1. *Let $y(x, \varepsilon)$ satisfy the boundary value problem*

$$\varepsilon y'' + \bar{f}(x, y)y' + \bar{g}(x, y) = 0,$$
$$y(0, \varepsilon) = A, \qquad y(1, \varepsilon) = B.$$

Let $w_j(x/\varepsilon)$ and $u_j(x)$ be the functions defined above, and for each integer $N \geqq 0$, set

$$y^N = \sum_{j=0}^{N} \varepsilon^j(u_j(x) + \phi(x)w_j(x/\varepsilon)),$$

where $\phi(x)$ is an infinitely differentiable function of x such that

$$\phi(x) = \begin{cases} 1, & 0 \leqq x \leqq \lambda, \\ 0, & 2\lambda \leqq x \leqq 1, \end{cases}$$

where λ is a small positive number. Then for sufficiently small ε, there is a unique solution $y(x, \varepsilon)$ such that $|y(x, \varepsilon) - y^N| \leqq C\varepsilon^{N+1}$ throughout $0 \leqq x \leqq 1$ for some constant C independent of ε.

3.2. Existence, uniqueness, and asymptotic correctness. In this section, we prove Theorem 3.1 by Newton's method. Without loss of generality we can take $A = B = 0$ in the boundary value problem (3.1), since the function $\bar{y}(x, \varepsilon) = y(x, \varepsilon) - h(x)$, where $h(x) = A + (A - B)x$ satisfies $\varepsilon\bar{y}'' + F(x, \bar{y})\bar{y}' + G(x, \bar{y}) = 0$, $\bar{y}(0, \varepsilon) = \bar{y}(1, \varepsilon) = 0$, where $F(x, \bar{y}) = \bar{f}(x, y)$ and $G(x, \bar{y}) = (A - B)\bar{f}(x, y) + \bar{g}(x, y)$.

We thus consider the boundary value problem

$$\begin{aligned} &\varepsilon y'' + \bar{f}(x, y)y' + \bar{g}(x, y) = 0, \qquad 0 \leqq x \leqq 1, \\ &y(0, \varepsilon) = y(1, \varepsilon) = 0. \end{aligned} \tag{3.14}$$

We consider the differential equation (3.14) as a functional equation in the space $C^2[0, 1]$ with norm

$$\|y\| = \max_{x \in [0,1]} |\varepsilon y''| + \max_{x \in [0,1]} |y'| + \max_{x \in [0,1]} |y|. \tag{3.15}$$

Consider the operator P defined by

$$z = P(y) = \varepsilon y'' + \bar{f}(x, y)y' + \bar{g}(x, y), \tag{3.16}$$

where $z \in C[0, 1]$ and

$$\|z\| = \max_{x \in [0,1]} |z(x)|. \tag{3.17}$$

The operator P acts from the space $C^2[0, 1]$ of twice continuously differentiable functions on $[0, 1]$ satisfying the homogeneous boundary conditions $y(0) = y(1) = 0$, into the space $C[0, 1]$ of continuous functions on $[0, 1]$.

As our initial guess, we choose

$$y^N = u^N + \phi(x)w^N, \tag{3.18}$$

where

$$u^N = \sum_{j=0}^{N} \varepsilon^j u_j(x), \qquad w^N = \sum_{j=0}^{N} \varepsilon^j w_j(x/\varepsilon),$$

and $\phi(x)$ is the function defined in Theorem 3.1.

LEMMA 3.1. *$\varepsilon(w^N)'' + f(x, w^N)(w^N)' + g(x, w^N)w^N = O(\varepsilon^N)$ or by* (3.7),

$$\varepsilon(w^N)'' + \bar{f}(x, w^N + u)(w^N)' + (\bar{f}(x, w^N + u) - \bar{f}(x, u))u' + \bar{g}(x, w^N + u) - \bar{g}(x, u) = O(\varepsilon^N)$$

for bounded τ.

Proof. Let

$$v^N = \sum_{j=0}^{N} \varepsilon^j v_{j-1}(x/\varepsilon).$$

From (3.12), we have

$$\frac{dw^N(\tau)}{dx} = \frac{v^N(\tau)}{\varepsilon} \tag{3.19}$$

and

$$\frac{d^2w^N(\tau)}{dx^2} = -\frac{1}{\varepsilon^2}\sum_{k=0}^{N}\varepsilon^k\left(\sum_{m=0}^{k} f_m v_{k-m-1} + \sum_{\substack{m=0\\k>0}}^{k-1} g_m w_{k-m-1}\ .\right) \tag{3.20}$$

On letting

$$f^N(\tau) = \sum_{j=0}^{N} \varepsilon^j f_j(w_0, w_1, \cdots, w_j; \tau),$$
$$g^N(\tau) = \sum_{j=0}^{N} \varepsilon^j g_j(w_0, w_1, \cdots, w_j; \tau),$$

equation (3.20) becomes

$$\frac{d^2w^N(\tau)}{dx^2} = -\frac{1}{\varepsilon^2}(f^N(\tau)v^N(\tau) + \varepsilon g^N(\tau)w^N(\tau) + O(\varepsilon^{N+1})),$$

where

$$g(\varepsilon\tau, w^N) - g^N(\tau) = O(\varepsilon^{N+1}),$$
$$f(\varepsilon\tau, w^N) - f^N(\tau) = O(\varepsilon^{N+1})$$

for $\tau \geqq 0$. Thus substituting w^N into the differential equation (3.6), we have

$$\begin{aligned} &\varepsilon(w^N)'' + f(x, w^N)(w^N)' + g(x, w^N)w^N \\ &= -\frac{1}{\varepsilon}(f^N v^N + \varepsilon g^N w^N + O(\varepsilon^{N+1})) + (f^N + O(\varepsilon^{N+1}))\frac{v^N}{\varepsilon} \\ &\quad + (g^N + O(\varepsilon^{N+1}))w^N \\ &= O(\varepsilon^N) \end{aligned} \tag{3.21}$$

for bounded τ.

Remark. Outside the boundary layer, Lemma 3.1 is also valid since the w_j and $v_j \to 0$ exponentially as $\tau \to \infty$.

LEMMA 3.2.

$$P(y^N) = O(\varepsilon^N). \tag{3.22}$$

Proof. By construction of the $u_j(x)$ and the $w_j(x/\varepsilon)$, we have $u = u^N + O(\varepsilon^{N+1})$, $u' = (u^N)' + O(\varepsilon^{N+1})$, $u'' = (u^N)'' + O(\varepsilon^{N+1})$ and $(w^N)' = O(1/\varepsilon)$. Substituting these relationships into the result of Lemma 3.1, recalling (3.3), (3.4) and the definition of the function $\phi(x)$ and using the fact that $\bar{f}(x, y)$ and $\bar{g}(x, y)$ are sufficiently smooth, we obtain our result.

We now consider $P'(y^N)$. We first note that since $A = B = 0$, we have $u_j(1) = 0$ and $w_j(0) + u_j(0) = 0$ for $j \geqq 0$ by (3.3) and (3.7). Furthermore, $\phi(0) = 1$ and $\phi(1) = 0$. Thus, $y^N(0, \varepsilon) = u^N(0, \varepsilon) + \phi(0)w^N(0, \varepsilon) = 0$ and $y^N(1, \varepsilon) = u^N(1, \varepsilon) + \phi(1) \cdot w^N(1/\varepsilon, \varepsilon) = 0$. Setting $y = y^N + \Delta y$, we have

$$P'(y^N)\,\Delta y = \varepsilon(\Delta y)'' + \bar{f}(x, y^N)(\Delta y)' + (\bar{f}_y(x, y^N)(y^N)' + \bar{g}_y(x, y^N))\,\Delta y = \bar{z}(x),$$

$$\Delta y(0) = \Delta y(1) = 0. \tag{3.23}$$

Let $k(x) = (\varepsilon(u^N)' + \phi v^N(\tau) + \varepsilon\phi' w^N)\bar{f}_y(x, y^N) + \varepsilon\bar{g}_y(x, y^N)$. Then by (3.19), the boundary value problem (3.23) becomes

$$\begin{gathered}\varepsilon(\Delta y)'' + \bar{f}(x, y^N)(\Delta y)' = -(1/\varepsilon)(k(x)(\Delta y) - \varepsilon\bar{z}(x)),\\ \Delta y(0) = \Delta y(1) = 0.\end{gathered} \tag{3.24}$$

LEMMA 3.3. *If $\bar{z}(x)$ is bounded, for all sufficiently small values of ε the linear boundary value problem* (3.24) *has a unique bounded solution Δy such that* $(\Delta y)' = O(1/\varepsilon)$.

Proof. (O'Malley [10], Cochran [3]).

Since

$$\Delta y = [P'(y^N)]^{-1}\bar{z}(x), \tag{3.25}$$

we have from (3.15), (3.24) and Lemma 3.3,

$$\|P'(y^N)^{-1}\| = O(1/\varepsilon). \tag{3.26}$$

We next consider $P''(y)$. From (3.16), we have

$$\begin{aligned}P''(y)\,\Delta y\overline{\Delta y} &= (y'\bar{f}_{yy}(x, y) + \bar{g}_{yy}(x, y))\,\Delta y\,\overline{\Delta y} + \bar{f}_y(x, y)(\Delta y\,\overline{\Delta y'} + \Delta y'\,\overline{\Delta y})\\ &= O(1/\varepsilon)(\max|\Delta y| + \max|\Delta y'|)(\max|\overline{\Delta y}| + \max|\overline{\Delta y'}|),\end{aligned}$$

so

$$\|P''(y)\| = O(1/\varepsilon). \tag{3.27}$$

Using the notation of Theorem 1.1 and (3.22), (3.26) and (3.27), we have

$$h = O(\varepsilon^{N-3}) < \tfrac{1}{2}$$

for $N > 3$ and ε sufficiently small,

$$r_0 = O(\varepsilon^{N-1}),$$

$$r_1 = O(\varepsilon^2).$$

We note that (3.27) is valid in the open ball $\|y - y^N\| < r_1$. Thus by Theorem 1.1, the boundary value problem (3.14) has a solution $y(x, \varepsilon)$ which is unique in the region

$$R = \{y : \|y - y^N\| = O(\varepsilon^2)\}, \tag{3.28}$$

and

$$\|y(x, \varepsilon) - y^N\| = O(\varepsilon^{N-1}) \quad \text{for } N > 3. \tag{3.29}$$

We now wish to show that $|y - y^N| = O(\varepsilon^{N+1})$. Instead of y^N we can take y^{N+1} as our initial guess. We then obtain, as above,

$$\|y(x, \varepsilon) - y^N - \varepsilon^{N+1} y_{N+1}\| = O(\varepsilon^N),$$

but

$$\|y(x, \varepsilon) - y^N - \varepsilon^{N+1} y_{N+1}\| \geqq \|y(x, \varepsilon) - y^N\| - \varepsilon^{N+1}\|y_{N+1}\|,$$

so that

$$\|y(x, \varepsilon) - y^N\| = O(\varepsilon^N). \tag{3.30}$$

Repeating the argument, we obtain

$$\|y(x, \varepsilon) - y^N\| = O(\varepsilon^{N+1}),$$

and by (3.15) we have

$$\max |\varepsilon y'' - \varepsilon(y^N)''| + \max |y' - (y^N)'| + \max |y - y^N| = O(\varepsilon^{N+1})$$

so that

$$\begin{aligned} |\varepsilon y'' - \varepsilon(y^N)''| &= O(\varepsilon^{N+1}), \\ |y' - (y^N)'| &= O(\varepsilon^{N+1}), \\ |y - y^N| &= O(\varepsilon^{N+1}) \end{aligned}$$

for $N \geqq 0$. Thus we have not only shown that $y(x, \varepsilon)$ converges to y^N for all $N \geqq 0$, but also that y' and $\varepsilon y''$ converge to $(y^N)'$ and $\varepsilon(y^N)''$, respectively, for all $N \geqq 0$.

Remarks. (a) In the above discussion we have not only proven Theorem 3.1, but also established a uniqueness result.

(b) For a similar result with slightly different hypotheses, see Coddington and Levinson [4].

4. The Dirichlet problem for $\varepsilon^2 y'' = f(x, y)$.

4.1. Introduction. Briš [2] shows that when $f(x, y)$ and $f_y(x, y)$ are continuous in a certain region D, $0 \leqq x \leqq 1$, $|y| < M$ (where M is a constant depending on the form of $f(x, y)$) and $f_y > 0$ in D, then there exists a solution $y(x, \varepsilon)$ to the boundary value problem

$$\varepsilon^2 y'' = f(x, y), \qquad y(0) = y(1) = 0 \tag{4.1}$$

for ε sufficiently small, and that

$$\lim_{\varepsilon \to 0} y(x, \varepsilon) = \phi(x), \qquad 0 < x < 1, \tag{4.2}$$

where $\phi(x)$ lies in the region D and

$$f(x, \phi(x)) = 0. \tag{4.3}$$

Vasil'eva and Tupciev [13] obtained a uniformly valid asymptotic solution to (4.1). We note that as $\varepsilon \to 0$, nonuniform convergence occurs, in general, at both endpoints (both boundary conditions are lost in the limit).

The aim of this section is to prove existence and to obtain an asymptotic solution to

$$\varepsilon^2 y'' = f(x, y), \qquad y(0) = A, \quad y(1) = B \tag{4.4}$$

under the hypotheses:

(i) The reduced problem $f(x, y) = 0$ has a solution $\phi(x)$ for $0 \leqq x \leqq 1$ such that $f_y(x, \phi(x)) \geqq L^2$ throughout $0 \leqq x \leqq 1$ for some $L > 0$,

(ii)

$$f_y(0, y) \geqq L^2 > 0 \quad \text{for all } y$$

$$f_y(1, \bar{y}) \geqq L^2 > 0 \quad \text{for all } \bar{y}$$

(iii) $f(x, y)$ is infinitely differentiable in the following three domains:

$$\{(x, y): 0 \leqq x \leqq 1,\ y = \phi(x)\},$$

$$\{(x, y): x = 0, \text{ for all } y\},$$

$$\{(x, \bar{y}): x = 1, \text{ for all } \bar{y}\}.$$

Note that hypothesis (i) implies that the reduced problem $f(x, y) = 0$ has no other solution near $y = \phi(x)$, that is, the root $\phi(x)$ is "isolated". If there is another function $\theta(x)$ satisfying the above hypotheses then there exists a corresponding solution $y(x, \varepsilon)$ such that $y(x, \varepsilon) \to \theta(x)$, $0 < x < 1$ as $\varepsilon \to 0$. Moreover, $y(x, \varepsilon)$ is unique in the sense that no other solution exists near $\phi(x)$. Our approach is slightly different from that of Vasil'eva and Tupciev, and our expansion procedure is considerably simpler. As a by-product, we will also obtain an algorithm for the numerical solution of (4.4).

To gain some insight into the problem, let us examine a simple illustrative example which can be explicitly integrated, namely

$$\varepsilon^2 y'' = y, \qquad y(0) = 1, \quad y(1) = 2. \tag{4.5}$$

The exact solution

$$y(x, \varepsilon) = (2 - e^{-1/\varepsilon}) \exp((x - 1)/\varepsilon) + (1 - 2e^{-1/\varepsilon}) \exp(-x/\varepsilon)/(1 - e^{-2/\varepsilon})$$

satisfies

$$y(x, \varepsilon) = 2 \exp((x - 1)/\varepsilon) + \exp(-x/\varepsilon) + O(\exp(-1/\varepsilon)) \to 0$$

as $\varepsilon \to 0$ for $0 < k_1 \leqq x \leqq k_2 < 1$. Thus convergence is nonuniform at both endpoints, and $y(x, \varepsilon) \to \phi(x) = 0$ in the interior.

4.2. The linear problem. Consider the linear problem

$$\varepsilon^2 y'' = h^2(x) y, \qquad y(0) = A, \quad y(1) = B, \tag{4.6}$$

with $h(x) > 0$ and infinitely differentiable.

We study the linear problem (4.6) in order to obtain the Green's function needed for the nonlinear problem. The method is similar to the WKB approximation (see, for example, Erdélyi [7], or Bellman [1]). Note that $\phi(x) = 0$ for this problem.

Generalizing from the constant coefficient case, we assume the solution $y(x, \varepsilon)$ to be of the form

$$(4.7)\quad y(x, \varepsilon) = A(x, \varepsilon)\exp\left(-\frac{1}{\varepsilon}\int_0^x h(s)\,ds\right) + B(x, \varepsilon)\exp\left(-\frac{1}{\varepsilon}\int_x^1 h(s)\,ds\right),$$

where

$$(4.8)\quad \begin{aligned} A(x, \varepsilon) &= \sum_{r=0}^{\infty} a_r(x)\varepsilon^r, \\ B(x, \varepsilon) &= \sum_{r=0}^{\infty} b_r(x)\varepsilon^r. \end{aligned}$$

Substituting (4.7) into (4.6), we get

$$\exp\left(-\frac{1}{\varepsilon}\int_0^x h(s)\,ds\right)(\varepsilon^2 A'' - 2\varepsilon hA' - A\varepsilon h') + \exp\left(-\frac{1}{\varepsilon}\int_x^1 h(s)\,ds\right)(\varepsilon^2 B'' + 2\varepsilon hB' + \varepsilon Bh') = 0.$$

Setting the brackets separately to zero and equating coefficients of like powers of ε, we obtain

$$(4.9)\quad \begin{aligned} 2ha_r' + h'a_r &= a_{r-1}'', & r &\geqq 0, & a_{-1}(x) &= 0, \\ 2hb_r' + h'b_r &= -b_{r-1}'' & r &\geqq 0, & b_{-1}(x) &= 0. \end{aligned}$$

Since $y(0) \sim A(0, \varepsilon)$ and $y(1) \sim B(1, \varepsilon)$,

$$(4.10)\quad \begin{aligned} a_0(0) = y(0) = A, &\qquad b_0(1) = y(1) = B, \\ a_r(0) = b_r(1) = 0 &\quad \text{for } r \geqq 1. \end{aligned}$$

Thus the a_r's and b_r's are uniquely determined by (4.9) and (4.10), and the complete expansion (4.7) can be found.

To establish the asymptotic validity of (4.7), we need the following.

THEOREM 4.1. *Let $y(x, \varepsilon)$ satisfy the boundary value problem*

$$(4.11)\quad \varepsilon^2 y'' - h^2(x)y = 0, \quad y(0) = A,\ y(1) = B, \text{ with } h(x) > 0,$$

and let $a_r(x)$ and $b_r(x)$ be the functions defined above. For each integer $N \geqq 0$, let

$$(4.12)\quad \begin{aligned} y(x, \varepsilon) = {} & \left(\sum_{r=0}^{N} a_r(x)\varepsilon^r\right)\exp\left(-\frac{1}{\varepsilon}\int_0^x h(s)\,ds\right) \\ & + \left(\sum_{r=0}^{N} b_r(x)\varepsilon^r\right)\exp\left(-\frac{1}{\varepsilon}\int_x^1 h(s)\,ds\right) \\ & + \varepsilon^{N+1}R_N(x, \varepsilon). \end{aligned}$$

Then $R_N(x, \varepsilon) = O(1)$ for all $x \in [0, 1]$.

Proof. Substituting (4.12) into (4.11) and recalling (4.9), we get

$$(4.13)\quad \begin{aligned} \varepsilon^2 R_N'' - h^2 R_N &= -\varepsilon(a_N''C + b_N''D), \\ R_N(0) = J \sim 0, &\qquad R_N(1) = K \sim 0, \end{aligned}$$

where $C = \exp(-(1/\varepsilon)\int_0^x h(s)\,ds)$ and $D = \exp(-(1/\varepsilon)\int_x^1 h(s)\,ds)$. Boundedness of R_N now follows from the maximum principle (Dorr, Parter, and Shampine [6]). Existence follows by solving the corresponding integral equation by iterative methods (cf. O'Malley [10]).

4.3. Construction of an asymptotic solution. We now seek a solution to the nonlinear problem (4.4) of the form

$$y(x,\varepsilon) = u(x,\varepsilon) + \psi(x)v(\tau,\varepsilon) + \varphi(x)w(\sigma,\varepsilon), \qquad \psi,\varphi \in C^\infty, \tag{4.14}$$

where

$$\varphi(x) = \begin{cases} 0, & 0 \leqq x \leqq \gamma, \\ 1, & 2\gamma \leqq x \leqq 1, \end{cases}$$

$$\psi(x) = \begin{cases} 1, & 0 \leqq x \leqq 1 - 2\delta, \\ 0, & 1 - \delta \leqq x \leqq 1, \end{cases}$$

for small positive numbers γ and δ, $\tau = x/\varepsilon$, $\sigma = (1 - x)/\varepsilon$ and v (and w) $\to 0$ as as τ (and σ) $\to \infty$. Thus in the open interval (0, 1), $y(x,\varepsilon)$ will be asymptotically equal to the outer solution $u(x,\varepsilon)$ which must satisfy

$$\varepsilon^2 u'' = f(x,u). \tag{4.15}$$

4.3.1. The outer solution. We seek a formal solution to (4.15) in the form

$$u(x,\varepsilon) = \sum_{i=0}^{\infty} u_i(x)\varepsilon^i. \tag{4.16}$$

Substituting (4.16) in (4.15) and expanding $f(x,u)$ about (x,u_0) we have

$$\begin{aligned}\varepsilon^2(u_0'' + \varepsilon u_1'' + \varepsilon^2 u_2'' + \cdots) &= f(x,u_0) + f_u(x,u_0)(u_1\varepsilon + u_2\varepsilon^2 + \cdots) \\ &\quad + \tfrac{1}{2} f_{uu}(x,u_0)(u_1\varepsilon + u_2\varepsilon^2 + \cdots)^2 + \cdots.\end{aligned}$$

Thus, u_0 solves the reduced problem $f(x,u_0) = 0$, and u_1 satisfies $u_1 f_u(x,u_0) = 0$. Since $f_u > 0$, we have $u_1 = 0$. Likewise, $u_2 f_u(x,u_0) = u_0''$, which implies that $u_2 = u_0''/f_u(x,u_0)$. In general, $u_{2k+1}(x) = 0$, $k = 0, 1, \cdots$, that is, $u(x,\varepsilon)$ is a power series in ε^2 (as could be expected). The u_{2k}'s can be successively determined in a straightforward manner.

We can thus determine the outer expansion (4.16) to any order.

4.3.2. The boundary layer correction at $x = 0$. Since near $x = 0$ w is negligible and $\psi = 1$, we can locally consider the solution to (4.4) to be of the form

$$y(x,\varepsilon) = u(x,\varepsilon) + v(\tau,\varepsilon). \tag{4.17}$$

Substituting (4.17) into (4.4), we obtain

$$\begin{aligned}&\varepsilon^2 u'' + v_{\tau\tau} = f(x, u + v) = f(x,u) + \frac{1}{v}(f(x, u + v) - f(x,u))v, \\ &v(0,\varepsilon) = A - u(0,\varepsilon), \qquad v(x_0,\varepsilon) = 0 \quad \text{for any fixed } x_0 > 0.\end{aligned} \tag{4.18}$$

Thus

$$(4.19)\qquad \begin{gathered} v_{\tau\tau} = G(\varepsilon\tau, v)v, \\ v(0, \varepsilon) = A - u(0, \varepsilon), \qquad v(\infty, \varepsilon) = 0, \end{gathered}$$

where $G(x, v) = (1/v)(f(x, u + v) - f(x, u))$. We seek a solution of the form

$$(4.20)\qquad v(\tau, \varepsilon) = \sum_{j=0}^{\infty} v_j(\tau)\varepsilon^j,$$

which is negligible at $\tau = \infty$. Using Taylor series, we have

$$(4.21)\qquad G(\varepsilon\tau, v) = \sum_{k=0}^{\infty} G_k(v_0, v_1, \cdots, v_k; \tau)\varepsilon^k,$$

where, for example,

$$G_0(v_0, \tau) = G(0, v_0) = \frac{1}{v_0(\tau)}(f(0, u_0(0) + v_0(\tau)) - f(0, u_0(0))) \geqq L^2 > 0.$$

Substituting (4.20), (4.21) into (4.19) and equating coefficients of like powers of ε, we obtain

$$(4.22)\qquad \begin{gathered} v_{0\tau\tau} = G_0 v_0, \quad v_0(0) = d_0, \quad v_0(\infty) = 0, \\ v_{j\tau\tau} = G_0 v_j + \sum_{m=1}^{j} G_m v_{j-m}, \quad v_j(0) = d_j, \quad v_j(\infty) = 0 \end{gathered}$$

for $j \geqq 1$, where $d_0 = A - u_0(0)$ and $d_j = -u_j(0)$.

LEMMA 4.1. *Let $\bar{w}(x, \varepsilon)$ solve the boundary value problem*

$$\varepsilon^2 \bar{w}'' = g(x, \bar{w})\bar{w},$$

$g(x, \bar{w}) \geqq L^2 > 0$ *throughout* $[0, 1]$,

$$\bar{w}(0) = K > 0,$$

$\bar{w}(x_0) = 0$, x_0 *is any fixed positive value. Then* $\bar{w}(x, \varepsilon) = O(e^{-Lx/\varepsilon})$, *that is,* $\bar{w}(x, \varepsilon)$ *decays exponentially to zero as* $\varepsilon \to 0$ *away from* $x = 0$.

Proof. Without loss of generality, we take $x_0 = 1$ and consider the linear boundary value problem

$$\varepsilon^2 z'' = L^2 z, \qquad z(0) = K, \quad z(1) = 0.$$

Then,

$$(4.23)\qquad z(x, \varepsilon) = K\, e^{-Lx/\varepsilon}(1 - e^{L(x-1)/\varepsilon})/(1 - e^{-2L/\varepsilon}) = O(e^{-Lx/\varepsilon}),$$

and $w = \bar{w} - z$ satisfies

$$\varepsilon^2 w'' - g(x, \bar{w})w = (g(x, \bar{w}) - L^2)z, \qquad w(0) = w(1) = 0.$$

Since $K > 0$, (4.23) implies that $z \geqq 0$. Then by the maximum principle, $w \leqq 0$ for $x \in [0, 1]$. Moreover, $\bar{w} \geqq 0$ for $x \in [0, 1]$. Thus $z(x, \varepsilon) \geqq \bar{w}(x, \varepsilon) \geqq 0$ on $[0, 1]$ and the lemma follows.

COROLLARY 4.1. *Let $\bar{w}(x, \varepsilon)$ solve the boundary value problem*

$$\varepsilon^2 \bar{w}'' = g(x, \bar{w})\bar{w},$$

$g(x, \bar{w}) \geqq L^2 > 0$ *throughout* $[0, 1]$,

$$\bar{w}(0) = 0,$$

$\bar{w}(x_0) = K > 0$ *for any fixed positive* x_0. *Then* $\bar{w}(x, \varepsilon) = O(e^{-L/\varepsilon(x - x_0)})$, *that is*, $\bar{w}(x, \varepsilon)$ *decays exponentially to zero as* $\varepsilon \to 0$ *for* $x < x_0$.

Proof. Make the change of variable $t = x_0 - x$ and apply Lemma 4.1.

Remark. The above results are also valid if $K < 0$.

Returning to the boundary value problems (4.22), we see that Lemma 4.1 implies that $v_0(\tau)$ decays exponentially away from $x = 0$ as $\tau \to \infty$. Moreover, the v_j for all j decay exponentially. For suppose the v_j decay exponentially for $j < n$. Then by (4.22), $v_n'' = G_0 v_n + h(\tau)$, where $h(\tau) = O(e^{-c\tau})$ as $\tau \to \infty$ for some $c > 0$. We now consider $z''(\tau) = L^2 z(\tau) + h(\tau)$, $G_0 \geqq L^2 > 0$, and use a comparison analysis as in Lemma 4.1. We can then show that $v_n(\tau) = O(e^{-c\tau})$, since $z(\tau) = O(e^{-c\tau})$ as $\tau \to \infty$.

Thus to determine $v(\tau, \varepsilon) = \sum_{j=0}^{\infty} v_j(\tau)\varepsilon^j$, we must solve a nonlinear problem (in general) on the infinite interval $\tau \geqq 0$ to obtain $v_0(\tau)$. The $v_j(\tau)$ for $j > 0$ are then determined successively from the linear problems given by (4.22) for $j > 0$.

4.3.3. The boundary layer correction at $x = 1$. Since $v(\tau, \varepsilon)$ decays exponentially away from $x = 0$ and $\varphi = 1$ near $x = 1$, we see that $y(x, \varepsilon) = u(x, \varepsilon) + w(\sigma, \varepsilon)$ near $x = 1$. Proceeding as we did to obtain $v(\tau, \varepsilon)$, we require that $w(\sigma, \varepsilon)$ satisfies

$$\begin{aligned} &w_{\sigma\sigma} = H(x, w)w, \quad \text{where } H(x, w) \geqq L^2 > 0 \text{ throughout } [0, 1], \\ &w(0, \varepsilon) = B - u(1, \varepsilon) = \sum_{i=0}^{\infty} c_i \varepsilon^i, \qquad w(\infty, \varepsilon) = 0. \end{aligned} \tag{4.24}$$

Expanding $H(1 - \sigma\varepsilon, w)$ in a Taylor series about $(1, w_0)$, we have

$$w_{0\sigma\sigma} = H_0 w_0, \quad w_0(0) = c_0, \quad w_0(\infty) = 0,$$

$$w_{j\sigma\sigma} = H_0 w_0 + \sum_{m=1}^{j} H_m w_{j-m}, \quad w_j(0) = c_j, \quad w_j(\infty) = 0$$

in the obvious notation. Here, the $w_j(\sigma)$ all decay exponentially as $\sigma \to \infty$.

It should be noted that, without loss of generality, we can take $A = B = 0$ in the boundary value problem (4.4). This is so because the function $\bar{y}(x, \varepsilon) = y(x, \varepsilon) - h(x)$, where $h(x) = A + (A - B)x$, satisfies the boundary value problem $\varepsilon^2 \bar{y}'' = F(x, \bar{y})$, with $\bar{y}(0, \varepsilon) = \bar{y}(1, \varepsilon) = 0$, where $F(x, \bar{y}) = f(x, y)$ and $F_{\bar{y}}(x, \bar{y}) = f_y(x, y)$. We thus consider the boundary value problem with homogeneous boundary conditions

$$\varepsilon^2 y'' = f(x, y), \qquad y(0) = y(1) = 0.$$

4.4. Existence, uniqueness, and asymptotic validity. To prove the asymptotic correctness of our formal expansion, we need only show the following.

THEOREM 4.2. *Consider the nonlinear two-point boundary value problem*

$$\varepsilon^2 y'' = f(x, y), \qquad 0 < x < 1, \tag{4.25}$$
$$y(0, \varepsilon) = 0, \qquad y(1, \varepsilon) = 0,$$

with $f_y > 0$ *along* $(x, u_0(x))$ *for all* $x \in [0, 1]$ *and along* $(0, y)$ *and* $(1, \bar{y})$ *for all* y *and* $\bar{y}$, *and where* $f(x, u_0(x)) = 0$. *Furthermore, assume* $f(x, y)$ *is infinitely differentiable in the regions indicated. Let* $v_j(\tau)$, $w_j(\sigma)$, $u_j(x)$, $\varphi(x)$ *and* $\psi(x)$ *be the functions defined above* ($\tau = x/\varepsilon$, $\sigma = (1 - x)/\varepsilon$). *For each integer* $N \geqq 0$, *set*

$$y^N = \sum_{j=0}^{N} \varepsilon^j (u_j(x) + \varphi(x) w_j(\sigma) + \psi(x) v_j(\tau)). \tag{4.26}$$

Then for any $N \geqq 0$, *there is a solution* $y(x, \varepsilon)$ *to the boundary value problem* (4.25) *and constant* C *independent of* ε *such that for* ε *sufficiently small*

$$|y(x, \varepsilon) - y^N| \leqq C\varepsilon^{N+1} \quad \textit{for } 0 \leqq x \leqq 1.$$

Proof. Consider the operator $P(y)$ defined by

$$P(y) = \varepsilon^2 \frac{d^2 y}{dx^2} - f(x, y).$$

$P(y)$ acts from the space $C^2[0, 1]$ of twice continuously differentiable functions on $[0, 1]$ satisfying the homogeneous boundary conditions into the space $C[0, 1]$ of continuous functions on $[0, 1]$.

On $C^2[0, 1]$ we take

$$\|y\| = \max_{x \in [0,1]} \left| \varepsilon^2 \frac{d^2 y}{dx^2} \right| + \max_{x \in [0,1]} |y|. \tag{4.27}$$

On $C[0, 1]$ we use

$$\|g(x)\| = \max_{x \in [0,1]} |g(x)|. \tag{4.28}$$

We will use Newton's method to prove the theorem. To this end, consider the linearized boundary value problem

$$P'(y^N)\,\Delta y = \varepsilon^2 \frac{d^2 \Delta y}{dx^2} - f(x, y^N)\,\Delta y = h(x), \tag{4.29}$$

where $\Delta y(0) = 0$, $\Delta y(1) = 0$, $h(x) \in C[0, 1]$. We assume that P is defined in some open set $S(\|y - y^N\| < R)$ and a continuous second derivative in $S_0(\|y - y^N\| \leqq r < R)$, and seek an approximation to $\|(P'(y^N))^{-1}\|$. By Theorem 4.1, we know that the homogeneous boundary value problem corresponding to (4.29) has only the asymptotically zero solution. By the same theorem, recalling (4.9), the approximate general solution to the homogeneous equation has the form

$$\begin{aligned} \Delta y(x) = {} & C_1 \left(\frac{\exp\left(-\frac{1}{\varepsilon} \int_0^x \sqrt{f_y(s, y^N)}\, ds \right)}{(f_y(x, y^N))^{1/4}} + O(\varepsilon) \right) \\ & + C_2 \left(\frac{\exp\left(-\frac{1}{\varepsilon} \int_x^1 \sqrt{f_y(s, y^N)}\, ds \right)}{(f_y(x, y^N))^{1/4}} + O(\varepsilon) \right). \end{aligned} \tag{4.30}$$

The Green's function for the operator

$$\varepsilon^2(d^2\,\Delta y)/(dx^2) - f_y(x, y^N)\,\Delta y, \tag{4.31}$$

under the conditions $\Delta y(0) = \Delta y(1) = 0$, then has the form

$$G(x,t) = \begin{cases} (1/D(x,\varepsilon))F(t)H(x) + O(\varepsilon) & \text{for } t \leqq x \leqq 1, \\ (1/D(x,\varepsilon))F(x)H(t) + O(\varepsilon) & \text{for } 0 \leqq x \leqq t, \end{cases} \tag{4.32}$$

where

$$D(x,\varepsilon) = 2\varepsilon\left(1 - \exp\left(-(2/\varepsilon)\int_0^1 g(s)\,ds\right)\right)(f_y(x, y^N(x))^{1/4}(f_y(t, y^N(t))^{1/4}),$$

$$F(r) = \exp\left((1/\varepsilon)\int_0^r g(s)\,ds\right) - \exp\left((-1/\varepsilon)\int_0^r g(s)\,ds\right),$$

$$H(r) = \exp\left((-1/\varepsilon)\int_0^r g(s)\,ds\right)\left(1 - \exp\left((-2/\varepsilon)\int_r^1 g(s)\,ds\right)\right),$$

$$g(s) = (f_y(s, y^N(s)))^{1/2}.$$

Thus a solution to the nonhomogeneous problem (4.29) is

$$y(x) = \int_0^1 G(x,t)h(t)\,dt, \tag{4.33}$$

where $G(x,t)$ is approximately zero for x away from t and $G(x,t) = O(1/\varepsilon)$ for x sufficiently close to t.

It should be noted that $\max_{x\in[0,1]}|\Delta y(x)| \leqq C\max_{x\in[0,1]}|h(x)|$, where C is a finite constant, even though $\max|G(x,t)| = O(1/\varepsilon)$. This is because when the integration of $G(x,t)$ is performed, the factor ε in the denominator cancels with the ε coming from the exponent. From the differential equation, then, we have $\max_{x\in[0,1]}|\varepsilon^2(d^2\,\Delta y/dx^2)| \leqq K\max_{x\in[0,1]}|h(x)|$ for some finite constant K. Thus $\|\Delta y\| \leqq K_1\|h\|$, and

$$\|(P'(y^N))^{-1}\| \leqq K_1 = O(1). \tag{4.34}$$

We next obtain estimates for $P(y^N) = \varepsilon^2(d^2y^N/dx^2) - f(x, y^N)$. Since away from $x = 1$, the w_j decay exponentially, we have

$$y^N = \sum_{j=0}^N u_j(x)\varepsilon^j + \sum_{j=0}^N \psi(x)v_j(\tau)\varepsilon^j = u^N + \psi(x)v^N(\tau).$$

By construction of the u_j and v_j, we see that $P(y^N) = O(\varepsilon^{N+1})$ for x away from 1. Similarly, this holds away from $x = 0$. We thus have

$$\|P(y^N)\| \leqq K_2\varepsilon^{N+1} \quad \text{for all } x\in[0,1], \tag{4.35}$$

where K_2 is a constant independent of ε. Furthermore,

$$P''(y)\,\Delta y\,\overline{\Delta y} = -f_{yy}(x,y)\,\Delta y\,\overline{\Delta y},$$

which implies that

$$\|P''(y)\| \leqq K_3, \qquad \text{where } K_3 = \max_R |f_{yy}(x, y)|,$$

$$R = \{(x, y) | 0 \leqq x \leqq 1, \|y - y^N\| \leqq r\}. \tag{4.36}$$

Referring to Theorem 1.1 on Newton's method, we see that (4.34), (4.35), and (4.36) imply that $h = K_1^2 K_3 K_2 \varepsilon^{N+1} < \frac{1}{2}$ for all integers $N \geqq 0$ and ε sufficiently small. Hence, the boundary value problem (4.25) has a solution $y(x, \varepsilon)$, to which the basic and modified Newton methods converge, such that

$$\|y(x, \varepsilon) - y^N\| \leqq r_0 = C\varepsilon^{N+1},$$

where C is a constant independent of ε, and by (4.27), the theorem follows.

Remarks. (a) If we assume that (4.36) holds in the open ball

$$\|y - y^N\| < r_1, \quad \text{where } r_1 = (1 + \sqrt{1 - 2h})/(K_1 K_3),$$

then the solution $y(x, \varepsilon)$ is unique in $\{(x, y) : 0 \leqq x \leqq 1, \|y - y^N\| < r_1\}$, that is, it is the unique solution near $\phi(x)$ $(= u_0(x))$. If $f(x, y) = 0$ has more than one solution, then in a sufficiently small neighborhood of each one, a unique solution to (4.25) exists.

(b) By taking $y^0 = u_0 + \psi(x)v_0 + \varphi(x)w_0$ as our initial approximation, the proof shows that Newton's method will converge to the solution, thus giving an algorithm for numerical solution.

(c) If $f(x, y)$ is less differentiable (we had assumed $f(x, y)$ to be infinitely differentiable in the relevant domains), the expression (4.36) may be obtained through only a finite number of terms. In any case, as long as $f(x, y)$, $f_y(x, y)$, and $f_{yy}(x, y)$ are continuous, we have that $y(x, \varepsilon) \to u_0(x)$ as $\varepsilon \to 0$ for $x \in (0, 1)$, where $f(x, u_0) = 0$.

(d) Lyubcenko [9] obtained a Green's function $G(x, t)$ for the operator (4.31), and claimed that

$$\max_{0 \leqq x,t \leqq 1} |G(x, t)| = O(\varepsilon).$$

That this is incorrect can be seen by considering the linear boundary value problem

$$\varepsilon^2 z'' - z = 1, \qquad z(0) = z(1) = 0,$$

where $z = \int_0^1 G(x, t)\, dt$. But $z \to -1$ as $\varepsilon \to 0$, for $x \in (0, 1)$ implies that $G(x, t)$ cannot be $O(\varepsilon)$ everywhere.

(e) No additional difficulties arise if the f, A, and B of (4.4) are represented by asymptotic power series expansions valid as $\varepsilon \to 0$.

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COMPLETELY CONVEX AND POSITIVE HARMONIC FUNCTIONS*

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Abstract. A completely convex function is a positive real-valued function on a real interval whose even derivatives alternate in sign. The author shows that every completely convex function is the restriction to the real line of a positive harmonic function in a vertical strip which is completely convex in x for each y. An integral representation for certain of these extensions is presented, and its relation to the integral representation of a completely convex function found by R. P. Boas is discussed. Finally, conformal transformations are used to derive analogous differentiability conditions providing similar extensions for functions defined on other curves in the plane.

1. Introduction. A completely convex function is an infinitely differentiable function $f(x)$ on a real interval (a, b) which satisfies $(-1)^k f^{(2k)}(x) \geqq 0$ for $k \geqq 0$ in that interval. Any completely convex function on the interval $(0, 1)$ is the restriction of an entire function of exponential type π [4, pp. 177–179]. A minimal completely convex function on $(0, 1)$ is a completely convex function such that there is no positive constant c such that $f(x) - c \sin \pi x$ is completely convex. Every completely convex function on $(0, 1)$ is of the form $f(x) + c \sin \pi x$, where $f(x)$ is a minimal completely convex function and $c \geqq 0$.

The study of infinitely differentiable functions with derivatives having certain fixed signs in an interval began around 1914. At that time, S. Bernstein proved that if $f^{(k)}(x) \geqq 0$ for $k \geqq 0$ and for all x on $[a, b]$, then f is the restriction of a function which is analytic in a disk centered at a and of radius $b - a$, [2, p. 1086]. If $f(x)$ was defined on (a, b) and had all its derivatives nonnegative, Bernstein called it an "absolutely monotonic" function. Similarly, a positive function $f(x)$ on an interval whose successive derivatives alternate in sign was called "completely monotonic". By analogy, the name "completely convex function" came to be applied to functions as defined above. A summary of work concerning these relations and their generalizations is given in [2].

R. P. Boas [1] used Lidstone series to show that every minimal completely convex function on $[0, 1]$ has the integral representation

$$
\begin{aligned}
f(x) = \tfrac{1}{2} \sin \pi x \bigg\{ \int_0^\infty & [f(1 + it) + f(1 - it)][\cosh \pi t + \cos \pi x]^{-1}\, dt \\
& + \int_0^\infty [f(it) + f(-it)][\cosh \pi t - \cos \pi x]^{-1}\, dt \bigg\}.
\end{aligned} \tag{1}
$$

Further, he showed that a function is completely convex on $[0, 1]$ if and only if it has the integral representation

$$
\begin{aligned}
f(x) = c \sin \pi x + \sin \pi x \bigg\{ \int_0^\infty & \varphi(t)[\cosh \pi t + \cos \pi x]^{-1}\, dt \\
& + \int_0^\infty \psi(t)[\cosh \pi t - \cos \pi x]^{-1}\, dt \bigg\},
\end{aligned} \tag{2}
$$

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where $c \geqq 0$, and φ and ψ are even entire functions of exponential type π with nonnegative Maclaurin coefficients.

In this paper, we show that a completely convex function on (a, b) is the restriction of a positive harmonic function in the strip $a < \operatorname{Re} z < b$ to the interval (a, b). We obtain conditions on the positive harmonic function for its restriction to the real line to be completely convex. We relate the integral representations of positive harmonic functions in the strip to those of completely convex functions. Finally, we obtain analogous differentiability conditions which make functions defined on other curves in the plane have similar extensions.

Since there is no loss in generality, we consider functions which are completely convex on the interval $(0, 1)$. I shall refer to the region in the z-plane defined by $0 < \operatorname{Re} z < 1$ as the strip, and its closure in the plane as the closed strip. The letter D will be used for the standard differentiation operator, and (∂_x) and (∂_y) for the partial differential operators $\partial/\partial x$ and $\partial/\partial y$, respectively.

2. The extension theory.

LEMMA 1. *Let $f(z) = u(z) + iv(z)$ be entire. Then*

$$\text{(a)} \qquad (\partial_y)^{2n}u(x, y) = (-1)^n(\partial_x)^{2n}u(x, y)$$

and

$$\text{(b)} \qquad (\partial_y)^{2n-1}u(x, y) = (-1)^n(\partial_x)^{2n-1}v(x, y) \quad \textit{for } n \geqq 1.$$

This is easily deduced from the Cauchy–Riemann equations by induction on n.

THEOREM 1. *Let $f(x)$ be an infinitely differentiable real-valued function on $(0, 1)$. Then $f(x)$ is completely convex on $(0, 1) \Leftrightarrow f(x) = u(x, 0)$ where $u(x, y)$ is a positive harmonic function in the strip, and*

$$(-1)^k(\partial_x)^{2k}u(x, y) \geqq 0$$

there, $k \geqq 0$.

Proof. Suppose first that $f(x)$ is completely convex on $(0, 1)$. Then, $f(x)$ is the restriction of the entire function $f(z)$ to $0 < x < 1$. Thus $f(z) = u(z) + iv(z)$, and $u(x, 0) = f(x)$ where $u(x, y)$ is harmonic in the plane. All the partial derivatives of $u(x, y)$ are continuous in the closed strip, and $u(x, y)$ is the real part of an entire function whose Taylor series converges everywhere, so we may apply Taylor's theorem:

$$u(x_0 + h, y_0 + k) = \sum_{n=0}^{\infty} \frac{1}{n!}(h\partial_x + k\partial_y)^n u(x_0, y_0).$$

Let $0 < x_0 < 1$, $y_0 = 0$, $h = 0$ and $k = t$. Then

$$u(x_0, t) = \sum_{n=0}^{\infty} \frac{1}{n!}(t\partial_y)^n u(x_0, 0).$$

By Lemma 1,

$$u(x_0, t) = \sum_{n=0}^{\infty} (-1)t^n\left\{\frac{1}{(2n)!}(\partial_x)^{2n}u(x_0, 0) - \frac{1}{(2n+1)!}(\partial_x)^{2n+1}v(x_0, 0)\right\}.$$

Since $f(z)$ is real on the real axis, $v(x_0, 0) = 0$ for $0 < x_0 < 1$. Moreover, $(\partial_x)^{2n+1}v \cdot (x_0, 0) = 0$ there for $n \geqq 0$. Thus

$$u(x_0, t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}(\partial_x)^{2n}u(x_0, 0)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} D^{2n}f(x_0).$$

Let $(-1)^n D^n f(x_0) = a_n$. Then $a_{2n} \geqq 0$ for every n, since $f(x)$ is completely convex. Thus

$$u(x_0, t) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} a_{2n} t^{2n},$$

where $a_{2n} \geqq 0$ for every n. Therefore, $u(x_0, t) \geqq 0$ for all t, and $0 < x_0 < 1$; i.e., $u(x, y)$ is a positive harmonic function in the strip.

Now, the second partial derivative $u_{xx} = (\partial_x)^2 u$ is the real part of the entire function $D^2 f(z)$. We may again, then, apply Taylor's theorem in two variables:

$$u_{xx}(x_0, t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\partial_y)^n u_{xx}(x_0, 0).$$

Since the partials of u are continuous, $(\partial_y)^{2n-1}u_{xx} = (\partial_x)^2(\partial_y)^{2n-1}u$ for $n \geqq 1$, and by Lemma 1, $(\partial_y)^{2n-1}u(x_0, 0) = (-1)^n(\partial_x)^{2n}v(x_0, 0) = 0$ for $n \geqq 1$, since the harmonic conjugate of u vanishes on the real axis. Thus

$$u_{xx}(x_0, t) = \sum_{n=0}^{\infty} (\partial_y)^{2n}u_{xx}(x_0, 0)\frac{t^{2n}}{(2n)!}.$$

But

$$(\partial_y)^{2n}u_{xx} = (\partial_x)^2(\partial_y)^{2n}u = (\partial_x)^2((-1)^n(\partial_x)^{2n}u) = (-1)^n(\partial_x)^{2n+2}u,$$

so that

$$u_{xx}(x_0, t) = \sum_{n=0}^{\infty} (-1)^n(\partial_x)^{2n+2}u(x_0, 0)\frac{t^{2n}}{(2n)!}$$
$$= \sum_{n=0}^{\infty} (-1)^n D^{2n+2}f(x_0)\frac{t^{2n}}{(2n)!}.$$

Since $f(x)$ is completely convex, $(-1)^{n+1}D^{2n+2}f(x_0) \geqq 0$ for every n, so

$$(-1)^n D^{2n+2}f(x_0) = -a_{2n+2} \leqq 0 \quad \text{for every } n.$$

Thus

$$u_{xx}(x_0, t) = -\sum_{n=0}^{\infty} \frac{a_{2n+2}}{(2n)!}t^{2n} \leqq 0$$

for all t, $0 < x_0 < 1$; that is, $-u_{xx}(x, y) \geqq 0$ in the strip.

One may complete the proof by iteration of this procedure for the even partial derivatives of u with respect to x. Since the converse is clear, the theorem is proved.

3. The integral representation.

DEFINITION. A harmonic function $u(x, y)$ in the strip which satisfies $(-1)^k(\partial_x)^{2k} \cdot u(x, y) \geqq 0$ there for $k \geqq 0$, will be called a *completely convex harmonic function.*

D. V. Widder [3] showed that any positive harmonic function in the strip has the representation

$$\begin{aligned} u(x, y) = (A\,e^{\pi y} + B\,e^{-\pi y}) \sin \pi x + \int_{-\infty}^{+\infty} P(x, t - y)\, d\alpha(t) \\ + \int_{-\infty}^{+\infty} P(1 - x, t - y)\, d\beta(t), \end{aligned} \tag{3}$$

where $A, B \geqq 0$, $\alpha(t)$ and $\beta(t)$ are nondecreasing, the integrals converge in the open strip and $P(x, y) = \sin(\pi x)[2(\cosh \pi y - \cos \pi x)]^{-1}$.

Since the completely convex harmonic functions in the strip are positive there, they must have this form. Note also that a positive harmonic function which vanishes on the boundary of the strip is a completely convex harmonic function, since such a function has the representation (3) with $\alpha(t)$ and $\beta(t)$ constant [3].

A particular completely convex function may have many completely convex harmonic extensions. For example, $\sin(\pi x)$ has the extensions $u(x, y) = (A\,e^{\pi y} + B\,e^{-\pi y}) \sin \pi x$, where $A + B = 1$, and A and B are nonnegative. However, one can show that a completely convex function $f(x)$ on $(0, 1)$ has a unique completely convex harmonic extension $u(x, y)$ such that $(\partial_y)u(x, y) = 0$ if $y = 0$. We will exhibit the precise integral representation for this function.

Sufficient conditions [3] have been established for a harmonic function in the strip to be represented by its Poisson integral. Further, an application of Green's formula to a suitably chosen contour in the strip, and careful evaluation of the appropriate integrals as that contour approaches the boundary of the strip shows that the partial derivative of the function with respect to x also has this representation, i.e., the following lemma holds.

LEMMA 2. *Let $u(x, y)$ be a harmonic function in the closed strip which is represented by its Poisson integral there, and such that $u(0, t)$ and $u(1, t)$ are in $L^1(e^{-\pi|t|}\,dt)$. Then*

$$\begin{aligned} (\partial_x)u(x, y) = \int_{-\infty}^{+\infty} P(x, t - y)(\partial_x)u(0, t)\, dt \\ + \int_{-\infty}^{+\infty} P(1 - x, t - y)(\partial_x)u(1, t)\, dt. \end{aligned} \tag{4}$$

Further, if $u(0, t) = \psi(t)$ and $u(1, t) = \varphi(t)$ are such that $D^n\psi(t)$ and $D^n\varphi(t)$ are in $L^1(e^{-\pi|t|}\,dt)$ for every n, then every partial derivative of $u(x, y)$ with respect to x is then represented by its Poisson integral.

PROPOSITION 3.1. *Let $f(x)$ be a minimal completely convex function on $[0, 1]$. If $u(z)$ is the real part of the entire extension $f(z)$ to $f(x)$, then $u(z)$ is represented by its Poisson integral for the strip.*

Proof. We first show that $u(0, t) \in L^1(e^{-\pi|t|}\,dt)$. We have

$$\left| \int_{-\infty}^{+\infty} u(0, t)\, e^{-\pi|t|}\, dt \right| = \left| \int_{-\infty}^{+\infty} \sum_{k=0}^{\infty} (-1)^k(\partial_t)^{2k}u(0, 0)\frac{t^{2k}}{(2k)!}\, e^{-\pi|t|}\, dt \right|$$

as in the proof of Theorem 1. This is bounded by

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} |(\partial_t)^{2k} u(0,0)| \int_{-\infty}^{+\infty} t^{2k} e^{-\pi|t|} \, dt.$$

By integration by parts,

$$\int_{-\infty}^{+\infty} t^{2k} e^{-\pi|t|} \, dt = 2\pi^{-(2k+1)}(2k)!,$$

so

$$\begin{aligned}\int_{-\infty}^{+\infty} |u(0,t)| \, e^{-\pi|t|} \, dt &\leqq 2\pi^{-1} \sum_{k=0}^{\infty} \pi^{-2k} |(\partial_y)^{2k} u(0,0)| \\ &= 2\pi^{-1} \sum_{k=0}^{\infty} \pi^{-2k} |D^{2k} f(0,0)|,\end{aligned}$$

and the last sum converges by [1, (5)]. A similar argument shows that $u(1,t) \in L^1 \cdot (e^{-\pi|t|} \, dt)$.

Theorem 1 of [3] now shows that

$$F(x,y) = \int_{-\infty}^{+\infty} P(x, t-y) u(0,t) \, dt + \int_{-\infty}^{+\infty} P(1-x, t-y) u(1,t) \, dt$$

is a harmonic function in the strip which equals $u(x,y)$ on the boundary of the strip. Corollaries 2.1 and 2.2 of [3] can then be used to show that $F(x,y)$ is $u(x,y)$, which proves the proposition.

Finally, we have the following theorem.

THEOREM 2. *Let $u(x,y)$ be a harmonic function in the closed strip such that $(\partial_y)u(x,0) = 0$, $0 < x < 1$. Then $u(x,y)$ is a completely convex harmonic function iff*

$$\begin{aligned}u(x,y) = C(e^{\pi y} + e^{-\pi y}) \sin \pi x &+ \int_{-\infty}^{+\infty} P(x, t-y) \psi(t) \, dt \\ &+ \int_{-\infty}^{+\infty} P(1-x, t-y) \varphi(t) \, dt,\end{aligned} \tag{5}$$

where $C \geqq 0$, and $\psi(t)$, $\varphi(t)$ have the following properties:

(i) *They are even entire functions of exponential type π,*

(ii) *are real on the real line,*

(iii) *have nonnegative Maclaurin coefficients and*

(iv) *are in $L^1(e^{-\pi|t|} \, dt)$.*

Proof. If $u(x,0)$ is a minimal completely convex function, the condition $(\partial_y)u(x,0) = 0$ and Proposition 3.1 show that $u(x,y)$ has form (5) with $C = 0$. If $u(x,0)$ is not a minimal completely convex function, then $U(x,0) = u(x,0) - C_1 \sin \pi x$ is, for some $C_1 \geqq 0$. Let $C = \frac{1}{2}C_1$, and define

$$U(x,y) = u(x,y) - C(e^{\pi y} + e^{-\pi y}) \sin \pi x.$$

This function is harmonic in the closed strip, equals $u(x, y)$ on the boundary and satisfies $(\partial_y)U(x, 0) = 0$. It follows that $u(x, y)$ has form (5) with $C > 0$.

Conversely, a function of the form (5) is harmonic and positive in the closed strip. Lemma 2 implies that it is a completely convex harmonic function. Since $(\partial_y)P(x, t - y)$ and $(\partial_y)P(1 - x, t - y)$ are odd functions for $y = 0$, the result follows.

Note. The integral representation (5) is clearly the extension to the representation (2) of a completely convex function on [0, 1] found by R. P. Boas.

Finally, the kernel $P(x, y)$ itself shows that the completely convex harmonic functions are a strict subclass of the positive harmonic functions in the strip, since $(\partial_x)^2 P(x, y) > 0$ for $y = 0, 0 < x < 1$.

4. Analogous differentiability conditions on other regions in the plane.

4.1. Analytic functions. Suppose $\varphi(z)$ is a conformal map from the strip $0 < \operatorname{Re} z < 1$ onto some other domain. If $f(z)$ is a function on the strip, define

$$h(w) = h(\varphi(z)) = f(z),$$

or conversely,

$$f(\varphi^{-1}(w)) = h(w).$$

Then,

$$\frac{df}{dz} = \frac{dh}{dw}\frac{dw}{dz} = \frac{dh}{dw}\psi(w),$$

where $\psi(w) = 1/(dz/dw)$ is a function of w. Further,

$$\frac{d^2f}{dz^2} = \frac{d}{dw}\left(\frac{dh}{dw}\psi(w)\right)\frac{dw}{dz} = \psi^2(w)\frac{d^2h}{dw^2} + \psi(w)\psi'(w)\frac{dh}{dw}.$$

But

$$\left[\psi(w)\frac{d}{dw}\right]^2 h(w) = \psi(w)\left[\frac{d}{dw}\left\{\psi(w)\frac{dh}{dw}\right\}\right] = \psi(w)\psi'(w)\frac{dh}{dw} + \psi^2(w)\frac{d^2h}{dw^2},$$

so that

$$\frac{d^2f}{dz^2} = \left[\psi(w)\frac{d}{dw}\right]^2 h(w).$$

To show that this formula holds for higher order even derivatives, let

$$h_1(w) = \left[\psi(w)\frac{d}{dw}\right]^2 h(w), \qquad g(z) = \frac{d^2f}{dz^2} = h_1(w).$$

Differentiating $g(z)$ twice with respect to z gives

$$\frac{d^4f}{dz^4} = \left[\psi(w)\frac{d}{dw}\right]^2 h_1(w) = \left[\psi(w)\frac{d}{dw}\right]^4 h(w),$$

and repetition of this process shows that this formula holds for any even derivative.

If $f(x)$ is completely convex with the entire extension $f(z)$, the condition $(-1)^k f^{(2k)}(x) \geqq 0$ for $0 < x < 1$ has the analogue $(-1)^k[\psi(w)(d/dw)]^{2k}h(w) \geqq 0$ for w on the w-image of $(0, 1)$ under $\varphi(z)$.

The function $h(w)$ satisfying this condition thus extends to a function which is analytic except for those points which $\varphi^{-1}(w)$ maps to infinity.

For the special cases of the unit disc and the upper half-plane, computation yields the following results.

PROPOSITION 4.1. *If* $(-1)^k[(1 + x^2)(d/dx)]^{2k}h(x) \geqq 0$ *for* $k \geqq 0$ *and* $-1 < x < +1$, *then* $h(x)$ *has an extension to the plane which is analytic except for the points* $\pm i$.

PROPOSITION 4.2. *If* $[w(d/dw)]^{2k}h(w) \geqq 0$ *for* $k \geqq 0$ *and* w *on the upper half of the unit circle, then* $h(w)$ *has an extension to the plane which is analytic except for the origin.*

Example. $h(w) = i\pi^{-1}\ln(-iw)$, since $h(e^{i\theta}) = 1 - \theta\pi^{-1}$ for $0 < \theta < \pi$ is positive, and $[w(d/dw)]^{2k}h(w) = 0$ for $k \geqq 1$.

4.2. Harmonic functions. Suppose that $u(w)$ is a real-valued function defined on some curve in the w-plane, which is contained in the domain Ω in the w-plane. Also suppose that $\varphi(z) = w$ maps the strip conformally onto Ω in such a way that the parameterization of the w-plane in either polar or Cartesian coordinates (v_1, v_2) makes these coordinates into functions of x and y, where $z = x + iy$. If $\varphi(z)$ maps $[0, 1]$ to the curve, then on the curve of definition $u(v_1, v_2) = f(x), 0 < x < 1$.

Thus

$$\frac{df}{dx} = \frac{\partial u}{\partial v_1}\frac{\partial v_1}{\partial x} + \frac{\partial u}{\partial v_2}\frac{\partial v_2}{\partial x}.$$

If $\partial v_1/\partial x = 0$ for $y = 0$, then

$$\frac{df}{dx} = \frac{\partial u}{\partial v_2}\frac{\partial v_2}{\partial x}.$$

If it is also the case that

$$\frac{\partial^2 v_2}{\partial x^2} = 0 \quad \text{for } y = 0,$$

then

$$\frac{d^2 f}{dx^2} = \frac{\partial^2 u}{\partial v_2^2}\left(\frac{\partial v_2}{\partial x}\right)^2,$$

or in general,

$$\frac{d^{2k} f}{dx^{2k}} = \frac{\partial^{2k} u}{\partial v_2^{2k}}\left(\frac{\partial v_2}{\partial x}\right)^{2k} \quad \text{for } k \geqq 1.$$

For the special cases of the unit disc, the upper half-plane, and a horizontal strip, computation yields the following results.

PROPOSITION 4.3. *Let* $u(x)$ *be real-valued for* $-1 < x < +1$, *and suppose that*

$$(-1)^k\left[(1 + x^2)\frac{d}{dx}\right]^{2k} u(x) \geqq 0,$$

$$-1 < x < +1, \quad k \geqq 0.$$

Then there is a positive harmonic function $u(x, y)$ in the unit disc such that $u(x, 0) = u(x)$.

PROPOSITION 4.4. *Let $u(w)$ be a real-valued function defined on $w = e^{i\theta}, 0 < \theta < \pi$. Further suppose that*

$$(-1)^k \frac{\partial^{2k} u(1, \theta)}{\partial \theta^{2k}} \geqq 0, \qquad k \geqq 0, \quad 0 < \theta < \pi.$$

Then there exists a positive harmonic function $u(re^{i\theta})$ in the upper half-plane which is equal to the original function on the upper half of the unit circle and which has the property

$$(-1)^k \frac{\partial^{2k} u(r, \theta)}{\partial \theta^{2k}} \geqq 0, \qquad k \geqq 0, \quad 0 < \theta < \pi, \quad r > 0.$$

Example. Let $u(1, \theta) = \sin \theta$, $0 < \theta < \pi$. Then $u(r, \theta) = [(r^2 + 1)/2r] \sin \theta = \operatorname{Re} \sin(-i \ln w)$ for $w = r e^{i\theta}$ is the required extension. Finally, we have the following proposition.

PROPOSITION 4.5. *Let $u(w)$ be a real-valued function on $w = iy$, $-1 < y < +1$, and suppose that*

$$(-1)^k \frac{\partial^{2k} u(0, y)}{\partial y^{2k}} \geqq 0 \quad \textit{for } k \geqq 0 \textit{ there.}$$

Then $u(w)$ extends to an entire function. Further, there is a harmonic function $U(x, y)$ in the horizontal strip $-1 < y < +1$, $-\infty < x < +\infty$, such that $U(0, y) = u(0, y)$, and $U(x, y)$ has the above differentiability property at every point in this strip.

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UNIFORM BOUNDEDNESS IN A CLASS OF VOLTERRA EQUATIONS*

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Abstract. Conditions are found under which solutions of the Volterra integral equation $x'(t) + \int_0^t a_\lambda(t-s)x(s)\,ds = k$ are bounded on $\{0 \leqq t < \infty\}$, uniformly in λ, when each a_λ is nonnegative, nonincreasing and convex in t.

The results generalize earlier work of the author which did not admit certain piecewise linear kernels. The main proof uses a method of Shea and Wainger involving transforms of H^1 functions. Applications to equations in Hilbert space are indicated.

1. Introduction. Let $\mathscr{A} = \{a_\lambda | \lambda \in \Lambda\}$ be a set of functions, each satisfying

$$a \in C(0,\infty) \cap L^1(0,1), \tag{1.1}$$

a is nonnegative, nonincreasing, and convex on $(0,\infty)$, *and* $a(t) \not\equiv a(\infty)$.

Let $x_\lambda(t; x_0, k)$ denote the solution of

$$x'(t) + \int_0^t a_\lambda(t-s)x(s)\,ds = k, \qquad x(0) = x_0, \tag{1.2}$$

$t \geqq 0$, $' = d/dt$, and set $u_\lambda(t) = x_\lambda(t;1,0)$, $w_\lambda(t) = x_\lambda(t;0,1)$. Then $x_\lambda(t;x_0,k) = x_0u_\lambda(t) + kw_\lambda(t)$ and $u_\lambda = w'_\lambda$. In this paper we find conditions under which

$$|w_\lambda(t)| \leqq K < \infty, \qquad 0 \leqq t < \infty, \quad \lambda \in \Lambda. \tag{1.3}$$

The proof of Theorem 2 of [7] shows that if (1.1) holds ($a \in \mathscr{A}$), then

$$|u_\lambda(t)| \leqq \sqrt{2}, \qquad 0 \leqq t < \infty, \quad \lambda \in \Lambda. \tag{1.4}$$

We show in § 2 that (1.3) and (1.4) yield a representation theorem and results on asymptotic behavior for the equation

$$\mathbf{x}(t) + \int_0^t \mathbf{A}(t-s)\mathbf{x}(s)\,ds = t\boldsymbol{\eta} + \boldsymbol{\xi} \tag{1.5}$$

in a separable Hilbert space $\mathscr{H}$. Here,

$$\mathbf{A}(t) = \int_{-\infty}^{\infty} A_\lambda(t)\,d\mathbf{E}_\lambda,$$

where $A_\lambda(t) = \int_0^t a_\lambda(s)\,ds$ and $\{\mathbf{E}_\lambda\}$ is the spectral family [14] corresponding to a fixed self-adjoint linear operator $\mathbf{L}$ defined on a dense subspace $\mathscr{D}$ of $\mathscr{H}$. $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are prescribed elements of $\mathscr{D}$.

THEOREM 1. *Let* a_λ *satisfy* (1.1). *Suppose there is a number* $T > 0$ *such that*

$$A_\lambda(T) \geqq M^{-1} > 0. \tag{1.6}$$

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Then there is a $C < \infty$, *depending only on* T, *such that*

$$|w_\lambda(t)| \leqq C(M + 1), \qquad 0 \leqq t < \infty. \tag{1.7}$$

Thus (1.3) *holds if* M *and* T *can be chosen the same for all* λ *in* Λ.

We shall prove Theorem 1 in § 3.

We shall write

$$\hat{a}(\zeta) = \frac{ia(\infty)}{\zeta} + \int_0^\infty e^{i\zeta t}[a(t) - a(\infty)]\,dt$$

for the Fourier transform of a function $a(t)$ satisfying (1.1).

Let Λ_0 be the subset of Λ for which

$$\hat{a}(\tau) \neq i\tau, \qquad \tau \text{ real}, \quad \tau \neq 0, \tag{1.8}$$

with $a = a_\lambda$, and let $\Lambda^* = \Lambda \diagdown \Lambda_0$.

Then [5] $\lambda \in \Lambda^*$ if and only if $a_\lambda(0+) < \infty$ and a_λ is piecewise linear with changes of slope only at integer multiples of

$$t_0(\lambda) = 2\pi[a_\lambda(0+)]^{-1/2}. \tag{1.9}$$

Theorem 1 generalizes Theorem 3 of [8], where we established (1.3) under hypotheses implying that $\Lambda = \Lambda_0$. When $a_\lambda(t) = \lambda a(t)$, $\lambda \geqq \lambda_0 > 0$, (1.3) is a consequence of [9, Theorem 1]. For the more general case considered here, we use methods and certain important estimates introduced by Shea and Wainger [15], who showed (as part of a much broader result) that $u_\lambda \in L^1(0, \infty)$, if $\lambda \in \Lambda_0$.

We can generate examples for Theorem 1, not covered by the results in [8], by perturbing a kernel of the type considered in [9]. Thus let $b(t)$ be a piecewise linear kernel satisfying (1.1) with $b(\infty) = 0$ and with changes of slope at and only at the integer multiples of $t_0 = 2\pi/\sqrt{b}\,(0)$. Let

$$a_\lambda(t) = \lambda b(t) + [1 - \cos(2\pi\sqrt{\lambda})](t + 1)^{-1}, \qquad \lambda \in \Lambda = [1, \infty).$$

Then the hypotheses of Theorem 1 hold, and $\Lambda^* = \{k^2; k = 1, 2, 3, \cdots\}$. This kernel also satisfies the hypotheses of Theorem 2 below ($\mathbf{L} \geqq \mathbf{I}$). [8] will suggest further examples.

Levin and Nohel [11] used the properties of transforms of kernels which satisfy conditions implying (1.1) to study asymptotic behavior of solutions of equations like (1.2). Subsequently, many authors ([4], [10], [12], [13], [15], for example) have studied linear and nonlinear integrodifferential equations with kernels satisfying (1.1).

2. Equations in Hilbert space. In this section, let Λ be the spectrum of the operator $\mathbf{L}$ of § 1, and assume that

$$\lim_{\mu \to \lambda, \mu \in \Lambda} \int_0^t \left| a_\lambda(s) - a_\mu(s) \right| ds = 0, \qquad t > 0. \tag{2.1}$$

The following technical result will be proved at the end of this section.

LEMMA 2.1. $u_\lambda(t)$ *and* $w_\lambda(t)$ *are continuous in* λ *for each fixed* t, *and* Λ^* *is closed.*

If (1.3) and (1.4) hold, it follows that the operators $\mathbf{U}(t) = \int_{-\infty}^{\infty} u_\lambda(t)\,d\mathbf{E}_\lambda$ and $\mathbf{W}(t) = \int_{-\infty}^{\infty} w_\lambda(t)\,d\mathbf{E}_\lambda$ are bounded in norm on H by $\sqrt{2}$ and K respectively. As shown in the proof of Theorem 1 of [8], $\mathbf{U}(t)$ and $\mathbf{W}(t)$ are strongly continuous and map $\mathscr{D}$ into $\mathscr{D}$; moreover, if

$$A_\lambda(t) \leqq (1 + |\lambda|)\alpha(t), \qquad \lambda \in \Lambda, \tag{2.2}$$

where $\alpha(t)$ is bounded on compact subsets of $[0, \infty)$, then the unique solution of (1.5) is given by

$$\mathbf{x}(t) = \mathbf{U}(t)\boldsymbol{\xi} + \mathbf{W}(t)\boldsymbol{\eta}, \qquad \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathscr{D}.$$

(We think of this as a weak solution of (1.5) if $\boldsymbol{\xi}$ or $\boldsymbol{\eta}$ is in $\mathscr{H} \setminus \mathscr{D}$.)

For $\lambda \in \Lambda^*$, we know from [5] that

$$u_\lambda(t) - u_\lambda^*(t) \to 0, \qquad w_\lambda(t) - w_\lambda^*(t) \to 0, \qquad t \to \infty,$$

where

$$u_\lambda^*(t) = 2\gamma_\lambda^{-1} \cos \omega_\lambda t,$$

$$w_\lambda^*(t) = 2[\gamma_\lambda \omega_\lambda]^{-1} \sin \omega_\lambda t + A_\lambda^{-1}(\infty).$$

Here $\omega_\lambda = [a_\lambda(0+)]^{1/2}$ is the unique positive τ where $\hat{a}(\tau) = i\tau$, and

$$\gamma = 3 - \left[\frac{a_\lambda(\infty)}{a_\lambda(0+)}\right].$$

(We interpret $A_\lambda^{-1}(\infty)$ as zero when $A_\lambda(\infty) = \infty$.) If $\lambda \in \Lambda_0$, $u_\lambda(t) \to 0$ and $w_\lambda(t) \to A_\lambda^{-1}(\infty)$ as $t \to \infty$.

Now fix $\boldsymbol{\xi}, \boldsymbol{\eta} \in \mathscr{H}$, and let

$$F(\rho) = \int_{-\infty}^{\rho} [d(\mathbf{E}_\lambda\boldsymbol{\xi}, \boldsymbol{\xi}) + d(\mathbf{E}_\lambda\boldsymbol{\eta}, \boldsymbol{\eta})].$$

Let $\varepsilon > 0$ and choose $N > 0$ so that $F(-N) + F(\infty) - F(N) < \varepsilon^2$. Let Γ be a relatively open subset of $[-N, N]$ such that $[-N, N] \cap \Lambda^* \subset \Gamma$ and

$$\int_{\Gamma \setminus \Lambda^*} dF(\lambda) < \varepsilon^2.$$

Set

$$\boldsymbol{\Omega}(t) = \int_{\Lambda^*} [u_\lambda^*(t)\, d\mathbf{E}_\lambda\boldsymbol{\xi} + w_\lambda^*(t)\, d\mathbf{E}_\lambda\boldsymbol{\eta}] + \int_{\Lambda_0} \mathbf{A}_\lambda^{-1}(\infty)\, d\mathbf{E}_\lambda\boldsymbol{\eta}.$$

Then straightforward estimates show that

$$\begin{aligned} \|\mathbf{x}(t) - \boldsymbol{\Omega}(t)\| \leqq Q\varepsilon &+ \left\| \int_{[-N,N] \setminus \Gamma} [u_\lambda(t)\, d\mathbf{E}_\lambda\boldsymbol{\xi} + (w_\lambda(t) - A_\lambda^{-1}(\infty))\, d\mathbf{E}_\lambda\boldsymbol{\eta}] \right\| \\ &+ \left\| \int_{\Gamma \cap \Lambda^*} ([u_\lambda(t) - u_\lambda^*(t)]\, d\mathbf{E}_\lambda\boldsymbol{\xi} + [w_\lambda(t) - w_\lambda^*(t)]\, d\mathbf{E}_\lambda\boldsymbol{\eta}) \right\|, \end{aligned}$$

norms in $\mathscr{H}$ $Q = 3\sqrt{2} + 5K$. Here both integrands tend to zero ($t \to \infty$) for each λ. If we assume further that

(2.4) $\mathscr{H}$ *$u_\lambda(t), w_\lambda(t)$ are continuous in λ on Λ_0, uniformly in t, $0 \leqq t < \infty$,*

and

(2.5) *$u_\lambda(t) - u_\lambda^*(t), w_\lambda(t) - w_\lambda^*(t)$ are continuous in λ on Λ^*, uniformly in t, $0 \leqq t < \infty$,* then the integrands in (2.3) converge to zero ($t \to \infty$), uniformly on the compact sets $[-N, N] \setminus \Gamma$ and $\Gamma \cap \Lambda^*$. Then $\|\mathbf{x}(t) - \mathbf{\Omega}(t)\| \leqq 2Q\varepsilon$ for large t. Thus we have the following conditional result.

THEOREM 2. *Assume* (1.1), $a \in \mathscr{A}$, (2.1), *and* (2.2). *If* (1.3), (1.4), (2.4), *and* (2.5) *hold, then* $\|\mathbf{x}(t) - \mathbf{\Omega}(t)\| \to 0$ *as* $t \to \infty$.

If, for example, Λ^* consists of a sequence of points $\lambda_n \to \infty$ (as when $a_\lambda(t) = \lambda a(t)$, $\lambda \geqq \mu > 0$, $\mathbf{A}(t) \equiv A(t)\mathbf{L}$ with $\mathbf{L} \geqq \mu$), then (2.5) certainly holds. In [8], we showed that (2.1) implies (2.4), provided $A_\lambda^{-1}(\infty)$ is continuous on Λ_0, $a_\lambda(\infty) = 0$, and $a_\lambda \in C^1$, $\lambda \in \Lambda_0$. (The last restriction can be dropped; in the proof one need only change the last conclusion of Lemma 4.1 of [8] to $a^*(\zeta, \lambda) + i\zeta \neq 0$, $\zeta \in Z$.) We hope to study (2.4) and (2.5) further in future work.

In [8], we give examples of kernels $\mathbf{A}(t)$ to which our method applies, and we discuss briefly the methods of C. M. Dafermos [1], A. Friedman and M. Shinbrot [3], and R. C. MacCamy and J. S. W. Wong [13] for related equations. The main features of our approach are

(i) the restriction to kernels represented in terms of $\mathbf{L}$,
(ii) the inclusion of piecewise linear a_λ,
(iii) the inclusion of cases where $a_\lambda(\infty) = 0$, but $A_\lambda(\infty) = \infty$.

Proof of Lemma 2.1. Integrating (1.2), we obtain

$$x(t) + \int_0^t A_\lambda(t - s)x(s)\, ds = kt + x_0.$$

Equation (2.1) implies that $A_\lambda(t)$ is continuous in λ, uniformly on compact subsets of $\{0 \leqq t < \infty\}$. Our first assertion then follows by standard arguments for Volterra integral equations.

For the second assertion, suppose $\lambda_n \to \lambda$, $\lambda_n \in \Lambda^*$ as $n \to \infty$. Then $\lambda \in \Lambda$. We let $a_n = a_\mu$, with $\mu = \lambda_n$. Choose $t_1 > 0$ such that $a_\lambda(2t_1) > 0$. Then by (2.1), $\int_0^{t_1} a_n(t)\, dt$ is bounded and for large n,

$$\int_0^{t_1} a_n(t)\, dt \geqq \tfrac{1}{2} t_1 a_n(0+).$$

Then $a_n(0+) \leqq \beta < \infty$, $n = 1, 2, \cdots$, and from (1.9) we see that $t_0(\lambda_n) \geqq T_1 > 0$, $n = 1, 2, 3, \cdots$. It follows easily that $a_n(0+) \to a_\lambda(0+) \leqq \beta$, so

$$t_0(\lambda_n) \to 2\pi[a_\lambda(0+)]^{-\frac{1}{2}} \equiv t_0(\lambda), \qquad n \to \infty. \tag{2.6}$$

Now suppose $a_\lambda(t)$ is not linear at $t = \tau$; that is, for every $\delta > 0$,

$$2a_\lambda(\tau) < a_\lambda(\tau + \delta) + a_\lambda(\tau - \delta).$$

Then, (2.1) implies that there are sequences $\tau_k \to \tau$ and $n(k) \to \infty$ such that $a_{n(k)}$ changes slope at τ_k. Since τ_k is an integer multiple of $t_0(\lambda_{n(k)})$, (2.6) shows that τ is an integer multiple of $t_0(\lambda)$. Thus $\lambda \in \Lambda^*$, and our proof is complete.

3. Proof of Theorem 1. Let $a(t)$ be any function satisfying (1.1) and (1.8) with a $(0+) - a'(0+) < \infty$; $a(t) = e^{-t}$ will do. Let $w(t)$ be the solution $x(t; 0, 1)$ of (1.2) with kernel $a(t)$, and set $v = w_\lambda - w$. With $a(t)$ fixed, C denotes, generically, a positive constant depending only on T.

We shall assume that

$$a_\lambda(0+) - a'_\lambda(0+) < \infty. \tag{3.1}$$

If not, approximate a_λ with a sequence of kernels b_n for which (1.1) holds and $b_n(0+) - b'_n(0+) < \infty$, but $A_\lambda(T) = \int_0^T b_n(s)\,ds$, $b_n(t) = a_\lambda(t)$, $t \geqq T/n$, and $\int_0^T |b_n(t) - a_\lambda(t)|\,dt \to 0$, $n \to \infty$. Then the corresponding $w_n(t)$ tend to $w_\lambda(t)$ pointwise as $n \to \infty$ and $|w_n(t)| \leqq C(M + 1)$, $n = 1, 2, \cdots$. Thus we can assume (3.1) without loss of generality.

Let $\tilde{a}(z) = \int_0^\infty e^{-zt}a(t)\,dt$, $\operatorname{Re} z > 0$, denote the Laplace transform of $a(t)$; similarly we have $\tilde{a}_\lambda(z)$, $\tilde{v}(z)$.

Using (1.2), we see that

$$\tilde{v}(z) = \frac{1}{\tilde{a}(z) + z}\left(1 - \frac{\tilde{a}(z)}{\tilde{a}_\lambda(z)}\right)\frac{\tilde{a}_\lambda(z)}{z[\tilde{a}_\lambda(z) + z]}, \qquad \operatorname{Re} z > 0.$$

We shall show below that there exists a function $\varphi_\lambda(t)$ such that

$$\int_0^\infty |\varphi_\lambda(t)|\,dt \leqq C(M + 1) \tag{3.2}$$

and

$$\tilde{\varphi}_\lambda(z) = \frac{\tilde{a}(z)}{\tilde{a}_\lambda(z)}\frac{1}{z + \tilde{a}(z)}, \qquad \operatorname{Re} z > 0. \tag{3.3}$$

Set

$$y(t) = \int_0^t [u(s) - \varphi_\lambda(s)][1 - u_\lambda(t - s)]\,ds.$$

Then $\tilde{y}(z) \equiv \tilde{v}(z)$, so $y(t) \equiv v(t)$; but by (1.4) and (3.2),

$$|y(t)| \leqq (1 + \sqrt{2})\left[C(M + 1) + \int_0^\infty |u(t)|\,dt\right],$$

so $|v(t)| \leqq C(M + 1)$. Since $|w(t)| \leqq C$, our result follows for $w_\lambda = v + w$. To complete the proof, we need only find φ satisfying (3.2) and (3.3).

Our method here follows the procedure of Shea and Wainger [15, (1.23)], as outlined in the proof of the following lemma.

LEMMA 3.1. *Suppose $f(\zeta)$ is continuous and bounded in $\{\operatorname{Im} \zeta \geqq 0\}$ and analytic in $\{\operatorname{Im} \zeta > 0\}$. Assume that $f(\tau) = O(\tau^{-1})$, $\tau \to \pm\infty$ and that $f(\tau)$ is absolutely continuous in $\{-\infty < \tau < \infty\}$ with*

$$\int_{-\infty}^\infty \left|\frac{df}{d\tau}\right| d\tau \leqq m.$$

Then there exists $r(t)$ such that $\int_0^\infty |r(t)|\,dt \leqq m/2$ and $\tilde{r}(-i\zeta) = f(\zeta)$, $\operatorname{Im} \zeta > 0$.

Proof. Using the Poisson integral representation for f in $\{\operatorname{Im} \zeta > 0\}$ and integrating by parts [15], one obtains the estimate

$$\int_{-\infty}^\infty |f'(\tau + i\sigma)|\,d\tau \leqq \int_{-\infty}^\infty |f'(\tau)|\,d\tau, \qquad \sigma \geqq 0,$$

so that $f' \in H^1$ (=Hardy space on $\{\operatorname{Im} z > 0\}$) and $f'(\tau) = \lim_{\sigma \to 0+} f'(\tau + i\sigma)$ a.e. Now let

$$
\begin{aligned}
r(t) &= \frac{-i}{2\pi t} \int_{-\infty}^{\infty} e^{-i\tau t} f'(\tau)\, d\tau \\
&= \lim_{N \to \infty} \frac{1}{2\pi} \int_{-N}^{N} e^{-i\tau t} f(\tau)\, d\tau .
\end{aligned} \tag{3.4}
$$

Then $r(t) = 0$, $t < 0$ [2, p. 197], and by the half-plane version of a theorem of Hardy and Littlewood [2, p. 198],

$$\int_0^\infty |r(t)|\, dt \leqq m/2 .$$

By (3.4), $f(\tau)$ and $\tilde{r}(-i\tau)$ are equal in $L^2(-\infty, \infty)$; since both are continuous, they are identical. Then $F(\zeta) = f(\zeta) - \tilde{r}(-i\zeta)$ is analytic in $\{\operatorname{Im} \zeta > 0\}$ and continuous in $\{\operatorname{Im} \zeta \geqq 0\}$ with $F(\tau) = 0$ for real τ, so $F(\zeta) \equiv 0$, and Lemma 3.1 is proved.

We set

$$f(\zeta) = \frac{\hat{a}(\zeta)}{\hat{a}_\lambda(\zeta)} \frac{1}{\hat{a}(\zeta) - i\zeta}, \qquad \operatorname{Im} \zeta \geqq 0, \quad \zeta \neq 0. \tag{3.5}$$

Then

$$f(\zeta) = \frac{1}{\hat{a}_\lambda(\zeta)} \left[1 + \frac{i\zeta}{\hat{a}(\zeta) - i\zeta} \right], \tag{3.6}$$

and $f(iz)$ is the right-hand side of (3.3). Also note that $f(-\bar{\zeta}) = \bar{f}(\zeta)$. We show that $f(\zeta)$ satisfies the hypotheses of Lemma 3.1.

As shown in [5], $\hat{a}$ and $\hat{a}_\lambda$ are analytic in $\{\operatorname{Im} \zeta > 0\}$ and can be extended as continuous functions to $S = \{\operatorname{Im} \zeta \geqq 0, \zeta \neq 0\}$; moreover

$$\hat{a}(\zeta), \hat{a}_\lambda(\zeta) = o(1), \qquad \zeta \to \infty, \operatorname{Im} \zeta \geqq 0, \tag{3.7}$$

$$\operatorname{Re} \hat{a}(\zeta) > 0, \qquad \operatorname{Re} \hat{a}_\lambda(\zeta) > 0, \qquad \operatorname{Im} \zeta > 0, \tag{3.8}$$

$$\hat{a}(\zeta) \to A(\infty), \qquad \hat{a}_\lambda(\zeta) \to A_\lambda(\infty), \qquad \zeta \to 0, \quad \zeta \in S. \tag{3.9}$$

(Here $A(t) = \int_0^t a(s)\, ds$.)

The following estimates are essentially those of [15, Lemma 1]. (Compare [5], [6].)

LEMMA 3.2. *Let* (1.1) *hold with* $a = c$. *Then if* $\sigma \geqq 0, \tau > 0$ *we have*

$$|\hat{c}(\tau + i\sigma)| \leqq 4 \int_0^{1/\tau} e^{-\sigma t} c(t)\, dt, \tag{3.10}$$

$$2\sqrt{2} |\hat{c}(\tau + i\sigma)| \geqq \int_0^{1/\tau} e^{-\sigma t} c(t)\, dt . \tag{3.11}$$

If, moreover, $c(0+) - c'(0+) < \infty$, *then*

$$|c'(\tau)| \leqq 40 \int_0^{1/\tau} t c(t)\, dt, \qquad \tau \geqq 0, \tag{3.12}$$

Proof. We may assume that $\sigma = 0$, since $e^{-\sigma t} c(t)$ satisfies (1.1) if $c(t)$ does. Estimates (3.10) and (3.12) then follow directly from [15, Lemma 1] (with a little computation since $c(\infty) = 0$ is assumed in [15]). For (3.11) we modify the proof in [15] as follows.

$$\begin{aligned}
\sqrt{2}\,|\hat{c}(\tau)| &\geqq \operatorname{Re}\hat{c}(\tau) + \operatorname{Im}\hat{c}(\tau) \\
&= \frac{c(\infty)}{\tau} + \int_0^\infty [\cos\tau t + \sin\tau t][c(t) - c(\infty)]\,dt \\
&= \frac{c(\infty)}{\tau} + \int_0^{\pi/2\tau} \cos\tau t[c(t) - c(\infty)]\,dt + \int_0^\infty \sin\tau t\left[c(t) - c\left(t + \frac{\pi}{2\tau}\right)\right]dt \\
&\geqq \frac{c(\infty)}{\tau} + \frac{1}{2}\int_0^{\pi/3\tau} [c(t) - c(\infty)]\,dt \\
&\geqq \frac{1}{2}\int_0^{1/\tau} c(t)\,dt.
\end{aligned}$$

This proves Lemma 3.2.

In view of (1.8), (3.8), (3.9), and (3.11) for $c = a_\lambda$, (3.6) shows that $f(\zeta)$ is analytic in $\{\operatorname{Im}\zeta > 0\}$, continuous in $\{\operatorname{Im}\zeta \geqq 0\}$.

Consider $\hat{a}/\hat{a}_\lambda$. If $\tau \geqq T^{-1}$, Lemma 3.2 implies that

$$\begin{aligned}
\left|\frac{\hat{a}(\sigma + i\tau)}{\hat{a}_\lambda(\sigma + i\tau)}\right| &\leqq C\frac{\int_0^{1/\tau} a(t)\,e^{-\sigma t}\,dt}{\int_0^{1/\tau} a_\lambda(t)\,e^{-\sigma t}\,dt} \\
&= C\frac{A(\tau^{-1})\,e^{-\sigma/\tau} + \sigma\int_0^{1/\tau} A(t)\,e^{-\sigma t}\,dt}{A_\lambda(\tau^{-1})\,e^{-\sigma/\tau} + \sigma\int_0^{1/\tau} A_\lambda(t)\,e^{-\sigma t}\,dt} \\
&\leqq CM\left[\frac{a(0+)/\tau}{1/T\tau} + \frac{\int_0^{1/\tau} a(0+)t\,e^{-\sigma t}\,dt}{\int_0^{1/\tau} (t/T)\,e^{-\sigma t}\,dt}\right] \\
&\leqq CM.
\end{aligned} \tag{3.13}$$

Here we have used (1.6) and the fact that $A_\lambda(t) \geqq tA_\lambda(T)/T$, $0 \leqq t \leqq T$ (a consequence of (1.1)).

Thus from (3.5), (3.7) and the symmetry of $f(\sigma + i\tau)$ in τ, we conclude that $f(\zeta)$ is bounded in $\{\operatorname{Im}\zeta \geqq 0, |\operatorname{Re}\zeta| \geqq T^{-1}\}$.

For $0 < \tau \leqq T^{-1}$ we have, similarly,

$$\left|\frac{\hat{a}(\sigma + i\tau)}{\hat{a}_\lambda(\sigma + i\tau)}\right| \leqq C\frac{\int_0^{1/\tau} a(0+)\,e^{-\sigma t}\,dt}{\sigma\int_0^T (t/T)A_\lambda(T)\,e^{-\sigma t}\,dt} \leqq CM, \qquad \sigma \geqq 1.$$

Since $\hat{a}(\zeta) \to 0$ as $\zeta \to \infty$ and $f(\tau + i\sigma)$ is conjugate-symmetric in τ, (3.5) now shows that $f(\zeta)$ is bounded in $\{\operatorname{Im}\zeta \geqq 0\}$ and $O(\zeta^{-1})$, $\zeta \to \infty$.

To apply Lemma 3.1 we need only estimate $\int_{-\infty}^{\infty} |f'(\tau)|\,d\tau$.

When $0 < |\tau| \leqq T^{-1}$ we differentiate (3.6) to show that

$$\begin{aligned}
f'(\tau) &= \frac{-\hat{a}_\lambda'(\tau)}{\hat{a}_\lambda^2(\tau)}\left[1 + \frac{i\tau}{\hat{a}(\tau) - i\tau}\right] \\
&\quad + \frac{1}{\hat{a}_\lambda(\tau)}\left[\frac{i}{\hat{a}(\tau) - i\tau} - i\tau\frac{\hat{a}'(\tau) - i}{(\hat{a}(\tau) - i\tau)^2}\right].
\end{aligned} \tag{3.14}$$

Using Lemma 3.2 we see that

$$C\left|\frac{\hat{a}_\lambda'(\tau)}{\hat{a}_\lambda^2(\tau)}\right| \leqq \frac{\int_0^{1/\tau} t a_\lambda(t)\,dt}{[\int_0^{1/\tau} a_\lambda(t)\,dt]^2} \equiv \psi(\tau^{-1}), \qquad \tau > 0. \tag{3.15}$$

As is shown in [15], the integral of ψ can be estimated as follows. We note that

$$\psi'(y) = 2a_\lambda(y)\left(\int_0^y a_\lambda(t)\,dt\right)^{-3}\int_0^y (\tfrac{1}{2}y - t)a_\lambda(t)\,dt.$$

Since $a_\lambda(t)$ is nonnegative and decreasing, we conclude that $\psi'(y) \geqq 0$, $y > 0$. Thus for $0 < \delta < T^{-1}$,

$$\begin{aligned}\int_\delta^{1/T} \psi(\tau^{-1})\,d\tau &= \int_T^{1/\delta} y^{-2}\psi(y)\,dy\\ &= -\delta\psi(\delta^{-1}) + \psi(T)/T + \int_T^{1/\delta} y^{-1}\psi'(y)\,dy\\ &\leqq \psi(T)/T + \int_T^{1/\delta} a_\lambda(y)\left(\int_0^y a_\lambda(t)\,dt\right)^{-2} dy\\ &\leqq 2A_\lambda^{-1}(T) \leqq CM.\end{aligned}$$

Similarly, $\int_0^{1/T} |\hat{a}'(\tau)\hat{a}^{-2}(\tau)|\,d\tau \leqq C$. Then since (3.9) holds and $|f'(-\tau)| = |f'(\tau)|$, (3.14) shows that

$$\int_{-1/T}^{1/T} |f'(\tau)|\,d\tau \leqq C(M+1). \tag{3.16}$$

When $\tau > T^{-1}$, we differentiate (3.5) to show that

$$\begin{aligned}f'(\tau) &= -\frac{\hat{a}_\lambda'(\tau)}{\hat{a}_\lambda(\tau)}\frac{\hat{a}(\tau)}{\hat{a}_\lambda(\tau)}\frac{1}{\hat{a}(\tau) - i\tau}\\ &\quad + \frac{\hat{a}'(\tau)}{\hat{a}_\lambda(\tau)}\frac{1}{\hat{a}(\tau) - i\tau} - \frac{\hat{a}(\tau)}{\hat{a}_\lambda(\tau)}\frac{\hat{a}'(\tau)}{[\hat{a}(\tau) - i\tau]^2} \qquad (3.17)\\ &= I_1(\tau) + I_2(\tau) + I_3(\tau).\end{aligned}$$

By Lemma 3.2,

$$\left|\frac{\hat{a}_\lambda'(\tau)}{\hat{a}_\lambda(\tau)}\right| \leqq \frac{C\int_0^{1/\tau} t a_\lambda(t)\,dt}{\int_0^{1/\tau} a_\lambda(t)\,dt} \leqq C/\tau, \qquad \tau \geqq T^{-1}.$$

Using this together with (3.7) and (3.13), we get

$$|I_1(\tau)| \leqq C(M+1)/\tau^2, \qquad \tau \geqq T^{-1}.$$

Writing $\hat{a}'/\hat{a}_\lambda = (\hat{a}'/\hat{a})(\hat{a}/\hat{a}_\lambda)$, we use similar estimates to see that

$$|I_2(\tau)| \leqq C(M+1)/\tau^2, \qquad \tau \geqq T^{-1}.$$

Finally, we use (3.7), (3.13), and (3.12) to show that

$$|I_3(\tau)| \leqq CM\tau^{-2}, \qquad \tau \geqq T^{-1}.$$

Then by (3.16) and (3.17) and symmetry,

$$\int_{-\infty}^{\infty} |f'(\tau)|\, d\tau \leqq C(M + 1).$$

Thus the hypotheses of Lemma 3.1 hold with $m = C(M + 1)$, and $r = \varphi_\lambda$ satisfies (3.2) and (3.3). This completes our proof.

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SOME EXTENSIONS OF HARDY'S INEQUALITY*

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Abstract. A number of integral inequalities involving the Hardy maximal function in certain L^p-spaces, $0 < p < \infty$, are obtained and their discrete analogues are given. Some of the results are applied to give an estimate of the Laplace transform of exponentially integrable functions.

1. Introduction. There is an extensive literature on the properties of the well-known maximal operators of Hardy and Littlewood and their applications in various branches of mathematics. (See, e.g., [1], [2], [3], [4], [5], [8].) In this note we give variants and extensions of maximal functions and show them continuous between weighted Lebesgue spaces. The results are applied to give new estimates for the Laplace transform, while the discrete analogue establishes extensions of Hardy's inequality [4, Thm. 346], Carleman's inequality [2] and a variant of an inequality of Konyuškov [7].

We distinguish between two sets of integral inequalities. The first gives conditions under which variants of integral means are continuous between certain Lebesgue spaces, while the second set is based on a result of C. Fefferman and E. M. Stein [3, Lemma 1] involving the Hardy–Littlewood maximal function. We use their argument to establish some weak type weighted estimates for the maximal functions which are used to prove continuity of these operators in weighted L^p-spaces.

In order to apply the results to the Laplace transform, as well as establishing their discrete analogues, we confine ourselves to scalar-valued functions defined on the real line instead of $\mathbb{R}^n$.

In the next section we prove a number of integral inequalities and apply them to obtain some new estimates involving the Laplace transform. In § 3 we give the discrete analogue of these results and various estimates involving means.

Throughout, A or A_p (with possibly other subscripts) denote constants depending only on the parameters under consideration and may be different at different appearances. The conjugate index of p is $p' = p/(p-1)$, $p > 1$, and the characteristic function of a set E is denoted by X_E.

2. Integral inequalities.

THEOREM 1. *Let p, s, λ be real numbers satisfying $p + s > \lambda$, $p > 0$. If*

$$\int_0^\infty t^{\lambda - s}|f(t)|\,dt < \infty,$$

then

$$\int_0^\infty x^\lambda \exp\left[px^{-p}\int_0^x t^{p-1}\log|x^{-s}f(t)|\,dt\right]dx \leqq e^{1/p}\cdot A\int_0^\infty t^{\lambda-s}|f(t)|\,dt, \tag{1}$$

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where $A = p/(p + s - \lambda)$.

Proof. Since

$$e^{1/p} = \exp\left[-p \int_0^1 y^{p-1} \log y \, dy\right],$$

a change of variable shows that (1) has the form

$$\int_0^\infty x^\lambda \exp\left[p \int_0^1 y^{p-1} \log |x^{-s} f(xy)| \, dy\right] dx$$
$$\leqq A \exp\left[-p \int_0^1 y^{p-1} \log y \, dy\right] \int_0^\infty t^{\lambda - s} |f(t)| \, dt,$$

which is equivalent to

$$\int_0^\infty x^\lambda \exp\left[p \int_0^1 y^{p-1} \log |x^{-s} y f(xy)| \, dy\right] dx \leqq A \int_0^\infty t^{\lambda - s} |f(t)| \, dt. \tag{2}$$

But by Jensen's inequality [5, p. 202] the left side of (2) is dominated by

$$p \int_0^\infty x^\lambda \left[\int_0^1 y^p x^{-s} |f(xy)| \, dy\right] dx = p \int_0^1 y^p \left[\int_0^\infty x^{\lambda - s} |f(xy)| \, dx\right] dy$$
$$= p \int_0^1 y^{p+s-\lambda-1} \left[\int_0^\infty t^{\lambda - s} |f(t)| \, dt\right] dy.$$

The last term is obtained by an interchange of order of integration which is justified by Fubini's theorem.

A result similar to Theorem 1 is the following.

THEOREM 2. *Let* $2p - 1 > \lambda - sp$, $p > 0$, *and*

$$\int_0^\infty t^{\lambda - sp} |f(t)|^p \, dt < \infty.$$

Then

$$\int_0^\infty x^\lambda \exp\left[p^2 x^{-p} \int_0^x t^{p-1} \log |x^{-s} f(t)| \, dt\right] dx \leqq eA \int_0^\infty t^{\lambda - sp} |f(t)|^p \, dt, \tag{3}$$

where $A = p/(2p + sp - \lambda - 1)$.

Proof. On writing

$$e = \exp\left[-p^2 \int_0^1 y^{p-1} \log y \, dy\right],$$

a change of variable shows that (3) is equivalent to

$$\int_0^\infty x^\lambda \exp\left[p^2 \int_0^1 y^{p-1} \log |x^{-s} y f(xy)| \, dy\right] dx \leqq A \int_0^\infty t^{\lambda - sp} |f(t)|^p \, dt.$$

By Jensen's inequality, the left side is dominated by

$$p\int_0^\infty x^\lambda\left[\int_0^1 y^{p-1}|x^{-s}yf(xy)|^p\,dy\right]dx$$
$$= p\int_0^1 y^{2p-1}\left[\int_0^\infty x^{\lambda-sp}|f(xy)|^p\,dx\right]dy$$
$$= p\int_0^1 y^{2p+sp-\lambda-2}\left[\int_0^\infty t^{\lambda-sp}|f(t)|^p\,dt\right]dy,$$

where the interchange of integration is justified by Fubini's theorem. Hence the result.

We shall now obtain some extensions of inequalities of G. H. Hardy and J. E. Littlewood involving the maximal function. The results follow along the lines of the proofs given in [5] and [3, Lemma 1].

Let f be a nonnegative locally integrable function defined on $\mathbb{R} = (-\infty, \infty)$. We define f_1, f_2 and f^*, the maximal functions of f, by

$$f_1(x) = \sup_{x<\xi} \frac{1}{\xi - x}\int_x^\xi f(t)\,dt,$$
$$f_2(x) = \sup_{\xi<x} \frac{1}{x-\xi}\int_\xi^x f(t)\,dt$$

and

$$f^*(x) = \max\{f_1(x), f_2(x)\}.$$

If $y > 0$, E_y, E_y^i, $i = 1, 2$, and E_y^* denote the sets

$$\{x : f(x) > y\}, \quad \{x : f_i(x) > y\} \quad \text{and} \quad \{x : f^*(x) > y\},$$

respectively. If ψ is a nonnegative locally integrable function on $\mathbb{R}$, then the distribution function of a function f with respect to ψ is defined by

$$D_f^\psi(y) = \int_{E_y} \psi(x)\,dx, \quad y > 0.$$

One defines similarly $D_{f_i}^\psi(y)$, $i = 1, 2$, and $D_{f^*}^\psi(y)$.

Our first lemma is essentially contained in the proof of Lemma 1 of [3]. It yields a result of Kaneko [6, Thm. 1] when $\psi(x) = (1 + |x|^\alpha)^{-1}$, $0 < \alpha \leqq 1$. We need this result to prove Theorem 3 and the weak type estimate given in Lemma 2.

LEMMA 1. *If f and ψ are nonnegative functions defined on $\mathbb{R}$, then for $y > 0$,*

$$D_{f_i}^\psi(y) \leqq \frac{1}{y}\int_{E_y^i} f(x)\psi_i(x)\,dx, \qquad i = 1, 2, \tag{4}$$

and

$$D_{f^*}^\psi(y) \leqq \frac{2}{y}\int_{E_y^*} f(x)\psi^*(x)\,dx, \tag{5}$$

where ψ_i^*, $i = 1, 2$, *and* ψ^* *are the maximal functions of* ψ.

Proof. Since $E_y^* = E_y^1 \cup E_y^2$, it follows that $D_{f^*}^{\psi}(y) \leqq D_{f_1}^{\psi}(y) + D_{f_2}^{\psi}(y)$, so that (5) is a consequence of (4).

To prove (4) let $i = 2$. The result for $i = 1$ is similar and, therefore, omitted.

By the Riesz–Calderon–Zygmund lemma ([9, pp. 17–18]), $E_y^2 = \bigcup_k I_k$, where $I_k = (\alpha_k, \beta_k)$ are pairwise disjoint intervals satisfying

$$y \leqq \frac{1}{\beta_k - \alpha_k} \int_{\alpha_k}^{\beta_k} f(x)\, dx.$$

Now for any such interval I_k we have

$$\begin{aligned}\int_{\alpha_k}^{\beta_k} f(x)\psi_2(x)\, dx &\geqq \int_{\alpha_k}^{\beta_k} \psi(x)\left[\frac{1}{\beta_k - \alpha_k}\int_{\alpha_k}^{\beta_k} f(t)\, dt\right] dx \\ &\geqq y \int_{\alpha_k}^{\beta_k} \psi(t)\, dt.\end{aligned}$$

Therefore

$$\begin{aligned} yD_{f_2}^{\psi}(y) &= y\int_{E_y^2} \psi(t)\, dt = y\int_{\cup_k I_k} \psi(t)\, dt = y\sum_k \int_{\alpha_k}^{\beta_k} \psi(t)\, dt \\ &\leqq \sum_k \int_{\alpha_k}^{\beta_k} f(x)\psi_2(x)\, dx = \int_{\cup_k I_k} f(x)\psi_2(x)\, dx \\ &= \int_{E_y^2} f(x)\psi_2(x)\, dx, \end{aligned}$$

which is the result.

THEOREM 3. *If* f *and* ψ *are nonnegative real-valued functions and* $p > 1$, *then*

$$\int_{-\infty}^{\infty} f^*(x)^p \psi(x)\, dx \leqq A_p \int_{-\infty}^{\infty} f(x)^p \psi^*(x)\, dx, \tag{6}$$

provided the right side exists.

Proof. Since the map $*: L^\infty(\mathbb{R}, \psi^*) \to L^\infty(\mathbb{R}, \psi)$ is bounded and by (5) $*$ is of weak type (1, 1), the Marcinkiewicz interpolation theorem ([10, p. 183 ff.]) produces the result.

COROLLARY 1. *If* f *and* ψ *are nonnegative real-valued functions defined on* $(0, \infty)$ *and* ψ *is nonincreasing, then for* $p > 1$,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, dt\right)^p \psi(x)\, dx \leqq A_p \int_0^\infty f(x)^p \left(\frac{1}{x}\int_0^x \psi(t)\, dt\right) dx,$$

provided the right side is finite.

Proof. By (6) with f and ψ restricted to the right half-line,

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\, dt\right)^p \psi(x)\, dx \leqq \int_0^\infty f^*(x)^p \psi(x)\, dx \leqq A_p \int_0^\infty f(x)^p \psi^*(x)\, dx.$$

But since ψ is nonincreasing,

$$\frac{1}{x}\int_0^x \psi(t)\,dt = \psi^*(x),$$

and the result follows.

The next result reduces to Hardy's inequality for integrals when $\psi(x) \equiv 1$.

COROLLARY 2. *If f and ψ satisfy the conditions of Corollary* 1, *then for* $r > 0$, $p \geqq 1$,

$$\int_0^\infty x^{-r-1}\psi(x)\left(\int_0^x f(t)\,dt\right)^p dx \leqq A_{p,r}\int_0^\infty t^{-r-1}[tf(t)]^p\left(\frac{1}{t}\int_0^t \psi(x)\,dx\right) dt$$

whenever the right side exists.

Proof. For $p = r + 1$ this is Corollary 1.

If $p < r + 1$, Hölder's inequality and an interchange of the order of integration yields

$$\begin{aligned}\int_0^\infty x^{-r-1}\psi(x)\left(\int_0^x f(t)\,dt\right)^p dx &\leqq \int_0^\infty x^{-r-1+p}\psi(x)\left(\frac{1}{x}\int_0^x f(t)^p\,dt\right) dx\\ &= \int_0^\infty f(t)^p\left(\int_t^\infty x^{-r-2+p}\psi(x)\,dx\right) dt\\ &\leqq \frac{1}{r+1-p}\int_0^\infty t^{-r-1}[tf(t)]^p\psi(t)\,dt,\end{aligned}$$

and since ψ is nonincreasing, the result follows.

If $1 < r + 1 < p$, let $f(t) = t^{(r/p)-1}g(t)$. Then again by Hölder's inequality,

$$\begin{aligned}\int_0^\infty x^{-r-1}\psi(x)\left(\int_0^x f(t)\,dt\right)^p dx &= \int_0^\infty x^{-r-1}\psi(x)\left(\int_0^x g(t)t^{(r/p)-1}\,dt\right)^p dx\\ &\leqq \left(\frac{p}{r}\right)^{p-1}\int_0^\infty x^{r((1/p')-1)-1}\psi(x)\left(\int_0^x g(t)^p t^{(r/p)-1}\,dt\right) dx\\ &= \left(\frac{p}{r}\right)^{p-1}\int_0^\infty t^{(r/p)-1}g(t)^p\left(\int_t^\infty x^{r((1/p')-1)-1}\psi(x)\,dx\right) dt\\ &\leqq \left(\frac{p}{r}\right)^{p-1}\int_0^\infty t^{-1}g(t)^p\psi(t)\,dt = \left(\frac{p}{r}\right)^p\int_0^\infty t^{-r-1}[tf(t)]^p\psi(t)\,dt,\end{aligned}$$

and again the result follows since ψ is nonincreasing.

To investigate the situation when $p = 1$ or $0 < p < 1$, we need the following.

LEMMA 2 [5]. *If* $0 < k < 1$, *then for* $f > 0$,

$$D_{f^*}^{\psi}(y) \leqq \frac{2}{(1-k)y}\int_{E_{yk}} f(x)\psi^*(x)\,dx.$$

Proof. Define $g(x) = f(x)$ if $f(x) > yk$ and zero otherwise. Then

$$\begin{aligned} f_2(x) &= \sup_{\xi < x} \left[\frac{1}{x-\xi} \int_\xi^x f(t)\chi_{E_{yk}}(t)\,dt + \frac{1}{x-\xi} \int_\xi^x f(t)(1 - \chi_{E_{yk}}(t))\,dt \right] \\ &= \sup_{\xi < x} \left[\frac{1}{x-\xi} \int_\xi^x g(t)\,dt + \frac{1}{x-\xi} \int_{[x,\xi]E_{yk}} f(t)\,dt \right] \\ &\leqq g_2(x) + yk \end{aligned}$$

Let $N_s^2 \equiv \{x : g_2(x) > s\}$, $s > 0$. Then

$$E_y^2 = \{x : f_2(x) > y\} \subset \{x : g_2(x) + yk > y\} = N_{y(1-k)}^2.$$

Applying (4) with f replaced by g, we obtain

$$\begin{aligned} yD_{f_2}^{\psi}(y) &\equiv y \int_{E_y^2} \psi(x)\,dx \leqq y \int_{N_{y(1-k)}^2} \psi(x)\,dx \\ &\leqq \frac{1}{1-k} \int_{N_{y(1-k)}^2} g(x)\psi_2(x)\,dx = \frac{1}{1-k} \int_{N_{y(1-k)}^2 \cap E_{yk}} f(x)\psi_2(x)\,dx \\ &\leqq \frac{1}{1-k} \int_{E_{yk}} f(x)\psi_2(x)\,dx. \end{aligned}$$

Similarly,

$$yD_{f_1}^{\psi}(y) \leqq \frac{1}{1-k} \int_{E_{yk}} f(x)\psi_1(x)\,dx.$$

Therefore,

$$\begin{aligned} yD_{f^*}^{\psi}(y) &\equiv y \int_{E_y^*} \psi(x)\,dx = y \int_{E_y^1 \cup E_y^2} \psi(x)\,dx \\ &\leqq y \int_{E_y^1} \psi(x)\,dx + y \int_{E_y^2} \psi(x)\,dx \leqq \frac{2}{1-k} \int_{E_{yk}} f(x)\psi^*(x)\,dx, \end{aligned}$$

which proves the lemma.

THEOREM 4 [5, 21.80]. *If $E \subset \mathbb{R} \equiv (-\infty, \infty)$ and $0 < k < 1$, then with $\tilde{\psi} = \psi\chi_E$,*

$$\int_{\mathbb{R}} f^*(x)\tilde{\psi}(x)\,dx \leqq \frac{1}{k} \int_{\mathbb{R}} \tilde{\psi}(x)\,dx + \frac{2}{1-k} \int_{\mathbb{R}} [f(x)\log^+ f(x)]\tilde{\psi}^*(x)\,dx,$$

where $\log^+ x = \log x$ *if* $x > 1$ *and zero otherwise.*

Proof. By Lemma 2,

$$\begin{aligned} \int_E f^*(x)\psi(x)\,dx &= \int_0^\infty D_{f^*}^{\psi}(y)\,dy = \left\{ \int_0^{1/k} + \int_{1/k}^\infty \right\} D_{f^*}^{\psi}(y)\,dy \\ &\leqq \int_0^{1/k} \left(\int_{E_y^*} \psi(x)\,dx \right) dy + \frac{2}{1-k} \int_{1/k}^\infty \frac{1}{y} \left\{ \int_{E_{yk}} f(x)\psi^*(x)\,dx \right\} dy \\ &\leqq \frac{1}{k} \int_E \psi(x)\,dx + \frac{2}{1-k} \int_{\mathbb{R}} f(x)\psi^*(x) \left\{ \int_{1/k}^\infty \chi_{E_{yk}}(x) \frac{dy}{y} \right\} dx. \end{aligned}$$

But since

$$\int_{1/k}^{\infty} \chi_{E_{yk}}(x)\frac{dy}{y} = \int_{1/k}^{f(x)/k} \frac{dy}{y} \leqq \log^+ f(x),$$

the result follows.

As a special case of this theorem we have the following.

COROLLARY 3. *If* $1 < s < \infty, f(x) \geqq 0$ *and* $f(x)\log^+ f(x)$ *is integrable, then*

$$\int_0^\infty e^{-sx} f^*(x)\,dx \leqq \frac{1}{s-1} + A_s \int_0^\infty f(x)\log^+ f(x)\,dx.$$

Proof. Let $\psi(x) = e^{-x/(1-k)}$, $0 < k < 1$, in Theorem 4 with $E = (0, \infty) = \mathbb{R}^+$. Then for $s = 1/(1-k)$ we obtain

$$\int_0^\infty e^{-sx} f^*(x)\,dx \leqq \frac{1}{s-1} + 2\int_0^\infty f(x)\log^+ f(x)\left[\frac{1-e^{-sx}}{x}\right] dx.$$

Since the bracketed term on the right is bounded, the result holds.

Next we give an estimate for the maximal function when $0 < p < 1$.

THEOREM 5 [5]. *If* f *and* ψ *are nonnegative functions and* $E \subset \mathbb{R}$, *then for* $0 < p < 1$,

$$\int_E f^*(x)^p \psi(x)\,dx \leqq A_p\left(\int_E \psi(x)\,dx\right)^{1-p}\left(\int_{\mathbb{R}} f(x)\psi^*(x)\,dx\right)^p,$$

provided the right side is finite.

Proof. Let $\alpha > 0$ and $0 < k < 1$. Then by definition of $D_{f^*}^{\psi}(y)$ and Lemma 2,

$$\begin{aligned}
\int_E f^*(x)^p \psi(x)\,dx &= p\int_0^\infty y^{p-1} D_{f^*}^{\psi}(y)\,dy \\
&= p\left\{\int_0^{\alpha/k} + \int_{\alpha/k}^\infty\right\} y^{p-1} D_{f^*}^{\psi}(y)\,dy \\
&\leqq p\int_0^{\alpha/k} y^{p-1}\left\{\int_{E_y^*} \psi(x)\,dx\right\} dy + p\int_{\alpha/k}^\infty y^{p-1} D_{f^*}^{\psi}(y)\,dy \\
&\leqq \left(\frac{\alpha}{k}\right)^p \int_E \psi(x)\,dx + \frac{2p}{1-k}\int_{\alpha/k}^\infty y^{p-2}\int_{E_{yk}} f(x)\psi^*(x)\,dx\,dy \\
&= \left(\frac{\alpha}{k}\right)^p \int_E \psi(x)\,dx + \frac{2p}{1-k}\int_{\mathbb{R}} f(x)\psi^*(x)\,dx \int_{\alpha/k}^\infty y^{p-2}\chi_{E_{yk}}(x)\,dy.
\end{aligned}$$

But

$$\begin{aligned}
\int_{\alpha/k}^\infty y^{p-2}\chi_{E_{yk}}(x)\,dy &= \int_{\alpha/k}^{f(x)/k} y^{p-2}\,dy \\
&= \begin{cases} \dfrac{k^{1-p}}{1-p}[\alpha^{p-1} - f(x)^{p-1}] & \text{if } f(x) > \alpha, \\ 0 & \text{if } 0 < f(x) < \alpha, \end{cases}
\end{aligned}$$

so that

$$\int_E f^*(x)^p \psi(x)\,dx \leqq \left(\frac{\alpha}{k}\right)^p \int_E \psi(x)\,dx + \frac{2p}{(1-k)(1-p)}\left(\frac{\alpha}{k}\right)^{p-1}\int_{\mathbb{R}} f(x)\psi^*(x)\,dx.$$

Minimizing the right side with respect to α, we obtain the theorem.

We single out the special case with $\psi(x) = e^{-x}$ as the following.

COROLLARY 4. *If f is nonnegative and integrable, then for $0 < p < 1$,*

$$\int_0^\infty e^{-x} f^*(x)^p \, dx \leqq A_p \left\{ \int_0^\infty f(x) \, dx \right\}^p .$$

We conclude this section with some applications.

Let f be a locally integrable function on $(0, \infty)$. We say that F is the Laplace transform of f if

$$F(x) = \int_0^\infty e^{-xt} f(t) \, dt, \qquad x > 0,$$

and the integral converges.

THEOREM 6. *Let ψ be a nonnegative, decreasing function and ψ^* its maximal function. If for $p > 1$,*

$$\int_0^\infty |f(t)|^p \psi^*(t) \, dt < \infty,$$

then the Laplace transform F of f exists and

$$\left\{ \int_0^\infty \psi(x) |x^{-1} F(x^{-1})|^p \, dx \right\}^{1/p} \leqq A_p \left\{ \int_0^\infty |f(t)|^p \psi^*(t) \, dt \right\}^{1/p} .$$

Proof. Since

$$|x^{-1} F(x^{-1})| \leqq \frac{1}{x} \int_0^x e^{-t/x} |f(t)| \, dt + \frac{1}{x} \int_x^\infty e^{-t/x} |f(t)| \, dt,$$

Minkowski's inequality yields

$$\begin{aligned} \left\{ \int_0^\infty \psi(x) |x^{-1} F(x^{-1})|^p \, dx \right\}^{1/p} &\leqq \left\{ \int_0^\infty \psi(x) \left[\frac{1}{x} \int_0^x |f(t)| \, dt \right.\right. \\ &\qquad \left.\left. + \frac{1}{x} \int_0^\infty e^{-t/x} |f(t)| \, dt \right]^p dx \right\}^{1/p} \\ &\leqq \left\{ \int_0^\infty \psi(x) \left[\frac{1}{x} \int_0^x |f(t) \, dt \right]^p dx \right\}^{1/p} \\ &\quad + \left\{ \int_0^\infty \psi(x) \left[\frac{1}{x} \int_x^\infty e^{-t/x} |f(t)| \, dt \right] dx \right\}^p \\ &\equiv I_1 + I_2 . \end{aligned}$$

By Corollary 1,

$$I_1 \leqq A_p \left\{ \int_0^\infty |f(t)|^p \psi^*(t) \, dt \right\}^{1/p} .$$

To estimate I_2 we use Hölder's inequality and the fact that $x^{-1} e^{-t/x} \leqq t^{-1} e^{-1}$. Thus

$$\begin{aligned} I_2^p &\leqq e^{1-p} \int_0^\infty \psi(x) \left[\frac{1}{x} \int_x^\infty e^{-t/x} |f(t)|^p \, dt \right] \left[\frac{1}{x} \int_x^\infty e^{-t/x} \, dt \right]^{p-1} dx \\ &= e^{2(1-p)} \int_0^\infty \frac{\psi(x)}{x} \left\{ \int_x^\infty e^{-t/x} |f(t)|^p \, dt \right\} dx \\ &= e^{2(1-p)} \int_0^\infty |f(t)|^p \left\{ \int_0^t e^{-t/x} x^{-1} \psi(x) \, dx \right\} dt \\ &\leqq e^{1-2p} \int_0^\infty |f(t)|^p \psi^*(t) \, dt. \end{aligned}$$

Note that for $\psi(x) \equiv 1$ this is a result of Titchmarsh [11, p. 397; prob. 16].

For $p = 1$ we obtain the following.

COROLLARY 5. *Let ψ be nonnegative decreasing and ψ^* its maximal function, such that*

$$\int_0^\infty \psi^*(x) \, dx = 1.$$

If

$$\int_0^\infty |f(t)| \log^+ |f(t)| \psi^*(t) \, dt = \infty,$$

then F, the Laplace transform of f, exists and

$$\int_0^\infty \psi(x) |x^{-1} F(x^{-1})| \, dx < \infty.$$

Proof. By Theorem 4,

$$\begin{aligned} \int_0^\infty \psi(x) |x^{-1} F(x^{-1})| \, dx &\leqq \int_0^\infty \psi(x) \left[\frac{1}{x} \int_0^x |f(t)| \, dt + \frac{1}{x} \int_x^\infty e^{-t/x} |f(t)| \, dt \right] dx \\ &\leqq \frac{1}{k} \int_0^\infty \psi(x) \, dx + \frac{2}{1-k} \int_0^\infty |f(t)| \log^+ |f(t)| \psi^*(t) \, dt \\ &\quad + \int_0^\infty |f(t)| \left(\int_0^t \frac{e^{-t/x} \psi(x)}{x} dx \right) dt \\ &\leqq \frac{1}{k} \int_0^\infty \psi^*(x) \, dx + \frac{2}{1-k} \int_0^\infty |f(t)| \log^+ |f(t)| \psi^*(t) \, dt \\ &\quad + e^{-1} \int_0^\infty |f(t)| \psi^*(t) \, dt. \end{aligned}$$

Let $\Phi(x) = x \log^+ x, x > 0$. Then $\Phi(x)$ is convex, and by Jensen's inequality,

$$\Phi\left[\int_0^\infty |f(t)| \psi^*(t) \, dt \right] \leqq \int_0^\infty |f(t)| \log^+ |f(t)| \psi^*(t) \, dt,$$

so that

$$\int_0^\infty |f(t)|\psi^*(t)\,dt$$

exists. Therefore F exists and hence the corollary holds.

We now show that the result of Titchmarsh may be extended in the sense that the exponent p is replaced by the exponential function. In fact we show that if F is the Laplace transform of f, then

$$\int_0^\infty \exp\,[s^{-1}F(s^{-1})]\,ds \leqq A\int_0^\infty \exp\,[f(x)]\,dx,$$

provided the right side exists. In order to prove this inequality we need the following theorem.

THEOREM 7. *If $f \in L^1(0, \infty)$ and*

$$F(s) = \int_0^\infty e^{-sx}\log|f(x)|\,dx, \qquad s > 0,$$

then F exists and

$$\int_0^\infty s^{-2}\exp\,[sF(s)]\,ds \leqq A\int_0^\infty |f(x)|\,dx,$$

where

$$A = \exp\left[-\int_0^\infty e^{-t}\log t\,dt\right].$$

The above inequality is now obtained by replacing $\log|f(x)|$ by $f(x)$ in Theorem 7.

Proof. A change of variable yields

$$\begin{aligned}\int_0^\infty s^{-2}\exp\,[sF(x)]\,ds &= \int_0^\infty \exp\,[s^{-1}F(s^{-1})]\,ds\\ &= \int_0^\infty \exp\left[\frac{1}{s}\int_0^\infty e^{-x/s}\log|f(x)|\,dx\right]ds\\ &= \int_0^\infty \exp\left[\int_0^\infty e^{-t}\log|f(ts)|\,dt\right]ds \qquad (x = st).\end{aligned}$$

This last term is less or equal to $A\|f\|_1$ if and only if

$$\int_0^\infty \exp\left[\int_0^\infty e^{-t}\log\,[t|f(ts)|]\,dt\right]ds \leqq \|f\|_1.$$

But by Jensen's inequality the left side is dominated by

$$\begin{aligned}\int_0^\infty\int_0^\infty e^{-t}t|f(ts)|\,dt\,ds &= \int_0^\infty e^{-t}t\int_0^\infty |f(ts)|\,ds\,dt\\ &= \int_0^\infty e^{-t}\,dt\int_0^\infty |f(x)|\,dx = \int_0^\infty |f(x)|\,dx,\end{aligned}$$

which we set out to show.

3. Discrete analogues. For the discrete analogue of Theorem 1 it is convenient to introduce the following notation. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers and $p > 0$. Then we write

$$\pi(a_n, p) = [a_1 a_2^{2^{p-1}} \cdots a_n^{n^{p-1}}]^{p/n^p}, \qquad n = 1, 2, \cdots .$$

Note that in the following analogue the parameters are somewhat more restricted than in Theorem 1.

THEOREM 8. *Suppose* $\{a_n\}_{n=1}^{\infty}$ *is a nonnegative sequence and* $s > 0$, $p \geqq 1$, $0 \leqq \lambda < s + p$. *If*

$$\sum_{n=1}^{\infty} n^{\lambda - s} a_n = M < \infty,$$

then

$$\sum_{n=1}^{\infty} n^{\lambda - sp} \pi(a_n, p) \leqq e^{1/p} A \sum_{n=1}^{\infty} n^{\lambda - s} a_n, \tag{7}$$

where $A = p[1 + 1/(p + s - \lambda)]$.

Proof. Observe that we may assume without loss of generality that $n^{-s} a_k \leqq 1$, $k = 1, 2, \cdots, n$. If $0 < M \leqq 1$ this is obvious. If $M > 1$ this is obvious. If $M > 1$, divide both sides of (7) by M to obtain

$$\frac{1}{M} \sum_{n=1}^{\infty} n^{\lambda - sp} \pi\left(\frac{a_n}{M}, p\right) M^{\nu} \leqq e^{1/p} A \sum_{n=1}^{\infty} n^{\lambda - s} \frac{a_n}{M},$$

where $\nu = (p \sum_{k=1}^{n} k^{p-1})/n^p$ and where the sum on the right is now equal to 1. From the easily established inequalities

$$1 \leqq \frac{p}{n^p} \sum_{k=1}^{n} k^{p-1} \leqq p, \tag{8}$$

it follows that

$$\frac{1}{M} \sum_{n=1}^{\infty} n^{\lambda - sp} \pi\left(\frac{a_n}{M}, p\right) \leqq A\, e^{1/p} \sum_{n=1}^{\infty} n^{\lambda - s} \frac{a_n}{M}.$$

Replacing a_n/M by a_n, we obtain $n^{\lambda - s} a_n \leqq 1$ and hence

$$n^{-s} a_n \leqq n^{-\lambda} \leqq 1,$$

which implies $n^{-s} a_k \leqq 1$, $k = 1, 2, \cdots, n$.

To prove (7) observe that by (8),

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{\lambda - sp} \pi(a_n, p) &\leqq \sum_{n=1}^{\infty} n^{\mu} \pi(a_n, p) \\
&= \sum_{n=1}^{\infty} n^{\lambda} \exp\left\{\frac{p}{n^p} \log [(n^{-s} a_1)(n^{-s} a_2)^{2^{p-1}} \cdots (n^{-s} a_n)^{n^{p-1}}]\right\} \\
&= \sum_{n=1}^{\infty} n^{\lambda} \exp\left\{\frac{p}{n^p} \sum_{k=1}^{n} k^{p-1} \log (n^{-s} a_k)\right\} \\
&= \sum_{n=1}^{\infty} n^{\lambda} \exp\left\{\frac{p}{n^p} \sum_{k=1}^{n} k^{p-1} \int_{k-1}^{k} \log (n^{-s} f(t))\, dt\right\},
\end{aligned}$$

where $\mu = \lambda - (sp \sum_{k=1}^{n} k^{p-1})/n^p$ and where

$$f(t) = \begin{cases} a_k, & k-1 < t \leqq k, \quad k = 1, 2, \cdots n, \\ 0, & \text{otherwise}. \end{cases}$$

But since $n^{-s}f(t) \leqq 1$, the last sum is dominated by

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\lambda} \exp\left\{\frac{p}{n^p} \sum_{k=1}^{n} \int_{k-1}^{k} t^{p-1} \log(n^{-s}f(t))\, dt\right\} \\ &= \sum_{n=1}^{\infty} n^{\lambda} \exp\left\{\frac{p}{n^p} \int_{0}^{n} t^{p-1} \log(n^{-s}f(t))\, dt\right\} \\ &= \sum_{n=1}^{\infty} n^{\lambda} \exp\left\{p \int_{0}^{1} y^{p-1} \log(n^{-s}f(ny))\, dy\right\} \qquad \left(y = \frac{t}{n}\right). \end{aligned}$$

Clearly this is less than or equal to

$$A\, e^{1/p} \sum_{n=1}^{\infty} n^{\lambda-s} a_n$$

if and only if

$$\sum_{n=1}^{\infty} n^{\lambda} \exp\left\{p \int_{0}^{1} y^{p-1} \log[n^{-s} y f(ny)]\, dy\right\} \leqq A \sum_{n=1}^{\infty} n^{\lambda-s} a_n. \tag{9}$$

By Jensen's inequality the left side of (9) is dominated by

$$\begin{aligned} &p \sum_{n=1}^{\infty} n^{\lambda} \int_{0}^{1} y^p n^{-s} f(ny)\, dy \\ &= p \sum_{n=1}^{\infty} n^{\lambda-s-p-1} \int_{0}^{n} t^p f(t)\, dt \qquad (ny = t) \\ &= p \sum_{n=1}^{\infty} n^{\lambda-s-p-1} \sum_{k=1}^{n} a_k \int_{k-1}^{k} t^p\, dt \\ &\leqq p \sum_{n=1}^{\infty} n^{\lambda-s-p-1} \sum_{k=1}^{n} k^p a_k \\ &= p \sum_{k=1}^{\infty} k^p a_k \sum_{n=k}^{\infty} n^{\lambda-s-p-1} \leqq A \sum_{k=1}^{\infty} k^{\lambda-s} a_k, \end{aligned}$$

which proves the theorem.

As a special case, we obtain the following generalization of Carleman's inequality.

COROLLARY 6. *Under the hypotheses of Theorem* 8 *with* $p = 1$,

$$\sum_{n=1}^{\infty} n^{\lambda-s}[a_1 a_2 \cdots a_n]^{1/n} \leqq A \cdot e \sum_{n=1}^{\infty} n^{\lambda-s} a_n.$$

For $s = \lambda$ this is of course Carleman's inequality [2, VI, § 11, prob. 37], although with $A = 2$, this constant is not best possible.

Next we prove the discrete version of Corollary 1.

THEOREM 9. *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two nonnegative sequences. If* $\{b_n\}$ *is* *nonincreasing and* $1 < p < \infty$, *then*

$$\sum_{n=1}^{\infty} b_n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leqq A_p \sum_{n=1}^{\infty} a_k^p\left(\frac{1}{k}\sum_{n=1}^{k} b_n\right) \tag{10}$$

whenever the right side is finite.

Proof. Let

$$f(t) = \begin{cases} a_k, & k-1 \leqq t < k, \quad k = 1,2,\cdots,n, \quad n = 1,2,\cdots, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$\psi(x) = \begin{cases} b_n, & n-1 \leqq x < n, \quad n = 1,2,\cdots, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\begin{aligned}
\sum_{n=1}^{\infty} b_n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p &= \sum_{n=1}^{\infty} b_n n^{-p}\left[\sum_{k=1}^{n-1} a_k + a_n\right]^p \\
&\leqq \sum_{n=1}^{\infty} b_n \int_{n-1}^{n} x^{-p}\left[\sum_{k=1}^{n-1} a_k + a_n(x-n+1)\right]^p dx \qquad (11)\\
&= \sum_{n=1}^{\infty} b_n \int_{n-1}^{n} x^{-p}\left[\sum_{k=1}^{n-1}\int_{k-1}^{k} f(t)\,dt + \int_{n-1}^{x} f(t)\,dt\right]^p dx \\
&= \sum_{n=1}^{\infty}\int_{n-1}^{n} \psi(x)\left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx \\
&= \int_0^{\infty} \psi(x)\left[\frac{1}{x}\int_0^x f(t)\,dt\right]^p dx \leqq A\int_0^{\infty} f(t)^p\left(\frac{1}{t}\int_0^t \psi(x)\,dx\right)^p dt. \qquad (12)
\end{aligned}$$

The inequality (12) follows from Corollary 1, and we obtain the inequality (11) as follows. Let

$$A_{n-1} = \sum_{k=1}^{n-1} a_k.$$

Then by Hölder's inequality,

$$\begin{aligned}
&\sum_{n=1}^{\infty} b_n \int_{n-1}^{n} x^{-p}\left[\sum_{k=1}^{n-1} a_k + a_n(x-n+1)\right]^p dx \\
&\quad = b_1\int_0^1 x^{-p}(a_1x)^p\,dx + \sum_{n=2}^{\infty} b_n\int_{n-1}^{n}\left[\frac{A_{n-1}}{x} + a_n - \frac{(n-1)a_n}{x}\right]^p dx \qquad (13)\\
&\quad \geqq b_1a_1^p + \sum_{n=2}^{\infty} b_n\left[\int_{n-1}^{n}\left(\frac{A_{n-1}}{x} + a_n - \frac{(n-1)a_n}{x}\right)dx\right]^p.
\end{aligned}$$

But since for $t > 0$, $1 - t \leqq \log(1/t)$,

$$\int_{n-1}^{n} \left[\frac{A_{n-1}}{x} + a_n - \frac{(n-1)a_n}{x}\right] dx = A_{n-1}\log\frac{n}{n-1} + a_n - a_n(n-1)\log\frac{n}{n-1}$$

$$= \log\left(\frac{n}{n-1}\right)[A_{n-1} - a_n(n-1)] + a_n$$

$$\geqq \left(1 - \frac{n-1}{n}\right)[A_{n-1} - a_n(n-1)] + a_n$$

$$= \frac{1}{n}[A_{n-1} + a_n]$$

$$= \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Therefore (13) dominates

$$b_1 a_1^p + \sum_{n=2}^{\infty} b_n\left[\frac{1}{n}\sum_{k=1}^{n} a_k\right]^p = \sum_{n=1}^{\infty} b_n\left[\frac{1}{n}\sum_{k=1}^{n} a_k\right]^p,$$

showing that (11) holds.

To complete the proof, we note that the right side of (12) has the form

$$A\sum_{k=1}^{\infty}\int_{k-1}^{k} f(t)^p\left(\frac{1}{t}\int_0^t \psi(x)\,dx\right)dt$$

$$\leqq A\left\{a_1^p\int_0^1\left(\frac{1}{t}\int_0^t b_1\,dx\right)dt + \sum_{k=2}^{\infty} a_k^p\left(\frac{1}{k-1}\sum_{n=1}^{k} b_n\right)\right\}$$

$$= A\left\{a_1^p b_1 + 2\sum_{k=2}^{\infty} a_k^p\left(\frac{1}{k}\sum_{n=1}^{k} b_n\right)\right\}$$

$$\leqq 2A\sum_{k=1}^{\infty} a_k^p\left(\frac{1}{k}\sum_{n=1}^{k} b_n\right).$$

COROLLARY 7. *If $c > 1$, $p \geqq 1$, then under the hypotheses of Theorem 9,*

$$\sum_{n=1}^{\infty} b_n n^{-c}\left(\sum_{k=1}^{n} a_k\right)^p \leqq A_{p,c}\sum_{k=1}^{\infty} k^{-c}(ka_k)^p\left(\frac{1}{k}\sum_{n=1}^{k} b_k\right).$$

Proof. For $c = p$, this is Theorem 9. The cases $1 \leqq p < c$ and $1 < c < p$ are proved as in Corollary 2 and are therefore omitted here.

This corollary generalizes [4, Thm. 346] and reduces to Hardy's theorem when $b_n = 1$ for all n.

For $0 < p < 1$ we have an inequality which follows from Theorem 5.

THEOREM 10. *Let $\{a_n\}$, $\{b_n\}$ be nonnegative, nonincreasing sequences such that*

$$\sum_{n=1}^{\infty} b_n = M < \infty.$$

If $0 < p < 1$, *then*

$$\sum_{n=1}^{\infty} b_n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leqq A_p M^{1-p}\left\{\sum_{k=1}^{\infty} a_k\left(\frac{1}{k}\sum_{n=1}^{k} b_n\right)\right\}^p$$

provided that the right side exists.

Proof. Let f and ψ be defined as in the proof of Theorem 9. Then

$$\begin{aligned}\int_0^{\infty} \psi(x)\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx &= \sum_{n=1}^{\infty} b_n \int_{n-1}^{n} \left[\frac{1}{x}\left(\sum_{k=1}^{n-1} a_k + a_n(x-n+1)\right)\right]^p dx \\ &\geqq \sum_{n=1}^{\infty} b_n\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p,\end{aligned} \tag{14}$$

since the integrand on the right side of (14) is decreasing as a function of x. Now by Theorem 5 the left side of (14) is dominated by

$$\begin{aligned}&A_p M^{1-p}\left\{\sum_{k=1}^{\infty} a_k \int_{k-1}^{k} \frac{dt}{t}\int_0^t \psi(x)\,dx\right\}^p \\ &\leqq A_p M^{1-p}\left\{a_1\int_0^1 \frac{dt}{t}\int_0^t b_1\,dx + \sum_{k=2}^{\infty} a_k\left(\frac{1}{k-1}\sum_{n=1}^{k} b_n\right)\right\}^p \\ &\leqq A_p M^{1-p}\left\{a_1 b_1 + \sum_{k=2}^{\infty} a_k\left(\frac{1}{k}\sum_{n=1}^{k} b_n\right)\right\}^p \\ &\leqq 2A_p M^{1-p}\left\{\sum_{k=1}^{\infty} a_k\left(\frac{1}{k}\sum_{n=1}^{\infty} b_n\right)\right\}^p,\end{aligned}$$

which proves the result.

Observe that with $b_n = n^{-c+p}$, $c > p + 1$, $0 < p < 1$, this theorem yields

$$\sum_{n=1}^{\infty} n^{-c}\left(\sum_{k=1}^{n} a_k\right)^p \leqq A_p\left(\sum_{k=1}^{\infty} k^{-c+p} a_k\right)^p.$$

Comparing this inequality with Theorem 346(β) in [4], we obtain

$$\sum_{n=1}^{\infty} n^{-c+p} a_n^p \leqq A\left\{\sum_{n=1}^{\infty} n^{-c+p} a_n\right\}^p.$$

If we assume further that $a_k \leqq 1$ for all k, then we obtain

$$\sum_{n=1}^{\infty} n^{-c}\left(\sum_{k=1}^{n} a_k\right) \leqq A_p\left\{\sum_{k=1}^{\infty} k^{-c}(k a_k)^p\right\}^p,$$

which under somewhat more relaxed conditions is an inequality of the type discussed by Konyuškov [7].

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AN EXTENSION OF PARSEVAL'S EQUATION*

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Abstract. Let $\{\alpha_k(x)\}_1^\infty$ be any real complete orthonormal system on $[a, b]$. Then Parseval's equation asserts that

$$\sum_{k=1}^{\infty} \int_a^b \int_a^b f(x)g(y)\alpha_k(x)\alpha_k(y)\,dy\,dx = \int_a^b f(x)g(x)\,dx$$

holds for any square integrable functions f and g on $[a, b]$. This paper gives conditions on $K(x, y)$ under which

$$\sum_{k=1}^{\infty} \int_a^b \int_a^b K(x, y)\alpha_k(x)\alpha_k(y)\,dy\,dx = \int_a^b K(x, x)\,dx$$

holds. Also shown is that for a large class of complete orthogonal systems $\{v_k(x)\}_1^\infty$ on $[a, b]$, the identity

$$\sum_{k=1}^{\infty} \int_a^b \int_a^b K(x, y)v_k(x) \int_a^x v_k(x')\,dx'\,dx\,dy = \frac{1}{2}\int_a^b \int_a^b K(x, y)\,dx\,dy$$

holds.

1. Introduction. Let $L^2[a, b]$ denote the class of all real-valued square integrable functions on $[a, b]$. If $[\alpha_k(x)\}_1^\infty$ is a real complete orthonormal system (C.O.N.S.) on $[a, b]$, then Parseval's equation gives

$$\sum_{k=1}^{\infty} \int_a^b f(x)\alpha_k(x)\,dx \int_a^b g(x)\alpha_k(x)\,dx = \int_a^b f(x)g(x)\,dx$$

for any f and g in $L^2[a, b]$.

On rewriting the left-hand side of the above equation in the form $\sum_{k=1}^{\infty}\int_a^b\int_a^b f(x)g(y)\alpha_k(x)\alpha_k(y)\,dy\,dx$, it is possible to interpret Parseval's equation as follows. If $K(x, y)$ is a square integrable function on $[a, b] \times [a, b]$ and if its variables are separable, then

$$\sum_{k=1}^{\infty} \int_a^b \int_a^b K(x, y)\alpha_k(x)\alpha_k(y)\,dy\,dx = \int_a^b K(x, x)\,dx.$$

A natural question that arises is: when the variables of the function $K(x, y)$ are not separable, will such an equation still hold under certain assumptions on $K(x, y)$?

The main purpose of this paper is to answer this question in the affirmative. In fact the equation holds for a large and important class of functions. The result and others are used to introduce a certain type of stochastic integral as an application.

2. Main results and their proofs. A real-valued function $S(x, y)$ defined on $[a, b] \times [a, b]$ is called symmetric if $S(y, x) = S(x, y)$, and is semipositive definite if $\int_a^b\int_a^b S(x, y)f(y)f(x)\,dy\,dx \geqq 0$ for every f in $L^2[a, b]$. By Mercer's theorem (see [7]) a continuous symmetric semipositive definite function $S(x, y)$ has a uniformly

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convergent series expansion

$$S(x, y) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x) \overline{\phi_j(y)},$$

where λ_j and ϕ_j are the characteristic values and characteristic elements, respectively, and $\lambda_j \geqq 0$.

LEMMA. *If $\{\alpha_k(x)\}_1^\infty$ is a real C.O.N.S. on $[a, b]$ and if $K(x, y) = f(x)f(y)S(x, y)$, where $f \in L^2[a, b]$ and $S(x, y)$ is a continuous symmetric semipositive definite function on $[a, b] \times [a, b]$, then*

$$\sum_{k=1}^{\infty} \int_a^b \int_a^b K(x, y)\alpha_k(x)\alpha_k(y)\, dx\, dy = \int_a^b K(x, x)\, dx.$$

Proof. By the uniform convergence of Mercer's expansion $S(x, y) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)\overline{\phi_j(y)}$, it follows that

$$\begin{aligned} &\sum_{k=1}^{\infty} \int_a^b \int_a^b f(x)f(y)\left[\sum_{j=1}^{\infty} \lambda_j \phi_j(x)\overline{\phi_j(y)}\right]\alpha_k(x)\alpha_k(y)\, dy\, dx \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j \int_a^b \int_a^b f(x)f(y)\phi_j(x)\overline{\phi_j(y)}\alpha_k(x)\alpha_k(y)\, dy\, dx \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j \left|\int_a^b f(x)\phi_j(x)\alpha_k(x)\, dx\right|^2. \end{aligned}$$

Since $\lambda_j \geqq 0$, the series is positive, and hence the order of the summation can be reversed to obtain

$$\sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\infty} \left|\int_a^b f(x)\phi_j(x)\alpha_k(x)\, dx\right|^2,$$

that is,

$$\sum_{j=1}^{\infty} \lambda_j \int_a^b |f(x)\phi_j(x)|^2\, dx.$$

Again by the uniform convergence of the series $\sum_{j=1}^{\infty} \lambda_j |\phi_j(x)|^2 = S(x, x)$, this becomes

$$\int_a^b f^2(x)S(x, x)\, dx = \int_a^b K(x, x)\, dx,$$

thus completing the proof.

Remarks 1. The function $S(x, y)$ in the lemma does not have to be semipositive definite. It is sufficient that the eigenvalues λ_j have identical sign except for finitely many of them.

2. If $\{\alpha_k(x)\}_1^\infty$ is a complex C.O.N.S. on $[a, b]$, then the lemma should be replaced by: *If $K(x, y) = f(x)\overline{f(y)}H(x, y)$, where $\int_a^b |f(x)|^2\, dx < \infty$ and $H(x, y)$ is continuous and Hermitian symmetric, that is, $\overline{H(y, x)} = H(x, y)$, and is semipositive*

definite[1] *then*

$$\sum_{k=1}^{\infty}\int_a^b\int_a^b K(x,y)\alpha_k(x)\overline{\alpha_k(y)}\,dy\,dx=\int_a^b K(x,x)\,dx.$$

THEOREM 1. *If* $\{\alpha_k(x)\}_{k=1}^{\infty}$ *is a real C.O.N.S. on* $[a,b]$ *and if* $K(x,y)=f(x)\cdot g(y)S(x,y)$ *with* $f,g\in L^2[a,b]$ *and* $S(x,y)$ *is a continuous symmetric semipositive definite function on* $[a,b]\times[a,b]$, *then*

$$\sum_{k=1}^{\infty}\int_a^b\int_a^b K(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx=\int_a^b K(x,x)\,dx. \tag{1}$$

Proof. Since

$$\int_a^b\int_a^b f(x)g(y)S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx=\int_a^b\int_a^b f(y)g(x)S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx$$

and

$$[f(x)+g(x)][f(y)+g(y)]=f(x)f(y)+g(x)g(y)+f(x)g(y)+f(y)g(x),$$

it follows that

$$\begin{aligned}
&2\sum_{k=1}^{\infty}\int_a^b\int_a^b f(x)g(y)S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx\\
&\quad=\sum_{k=1}^{\infty}\int_a^b\int_a^b [f(x)+g(x)][f(y)+g(y)]S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx\\
&\qquad-\sum_{k=1}^{\infty}\int_a^b\int_a^b f(x)f(y)S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx\\
&\qquad-\sum_{k=1}^{\infty}\int_a^b\int_a^b g(x)g(y)S(x,y)\alpha_k(x)\alpha_k(y)\,dy\,dx.
\end{aligned}$$

Hence by the use of the lemma, the right-hand side becomes

$$\begin{aligned}
&\int_a^b [f(x)+g(x)]^2S(x,x)\,dx\\
&\qquad-\int_a^b f^2(x)S(x,x)\,dx-\int_a^b g^2(x)S(x,x)\,dx\\
&\quad=2\int_a^b f(x)g(x)S(x,x)\,dx\\
&\quad=2\int_a^b K(x,x)\,dx.
\end{aligned}$$

Thus the theorem follows.

[1] A complex function $H(x,y)$ is semipositive definite on $[a,b]^2$ if and only if

$$\int_a^b\int_a^b H(x,y)f(x)\overline{f(y)}\,dx\,dy\geqq 0$$

for every f with $\int_a^b|f|^2<\infty$.

For the case when $S(x, y)$ is a complex function, the theorem still holds if the real part and the imaginary part satisfy the conditions on $S(x, y)$ stated in the theorem.

COROLLARY 1.1. *If $\{\alpha_k(x)\}_{k=1}^{\infty}$ is a real C.O.N.S. on $[a, b]$ and if $K(x, y) = \sum_{j=1}^{n} K_j(x, y)$, where each $K_j(x, y)$ satisfies the assumptions on $K(x, y)$ in the theorem, then* (1) *holds for this $K(x, y)$.*

COROLLARY 1.2. *Let $\{\alpha_k(x)\}_1^{\infty}$ be a real C.O.N.S. on $[a, b]$ and let $K(x, y) \in L^2([a, b]^2)$ have an L^2-expansion*

$$K(x, y) \sim \sum_{j=1}^{\infty} a_j \beta_j(x) \beta_j(y)$$

with the $a_j \geqq 0$ and $\{(\beta_j(x)\}_1^{\infty}$ a real C.O.N.S. on $[a, b]$.

If $b_{ij} = \int_a^b \int_a^b K(x, y)\alpha_i(x)\alpha_j(y)\, dx\, dy$, then

$$\sum_{i=1}^{\infty} b_{ii} = \sum_{i=1}^{\infty} a_i.$$

Proof.

$$\begin{aligned}\sum_{i=1}^{\infty} b_{ii} &= \sum_{i=1}^{\infty} \int_a^b \int_a^b K(x, y)\alpha_i(x)\alpha_i(y)\, dx\, dy \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_j \int_a^b \int_a^b \beta_j(x)\beta_j(y)\alpha_i(x)\alpha_i(y)\, dx\, dy.\end{aligned}$$

Since the series is positive, the order of the summation may be reversed to yield

$$\begin{aligned}\sum_{i=1}^{\infty} b_{ii} &= \sum_{j=1}^{\infty} a_j \sum_{i=1}^{\infty} \int_a^b \beta_j(x)\alpha_i(x)\, dx \int_a^b \beta_j(y)\alpha_i(y)\, dy \\ &= \sum_{j=1}^{\infty} a_j \int_a^b \beta_j^2(x)\, dx = \sum_{j=1}^{\infty} a_j.\end{aligned}$$

In cases when the kernel $K(x, y)$ has either a jump discontinuity on the diagonal $y = x$ or is nonsymmetric, we have the following.

THEOREM 2. *If $S(x, y) = K_1(x, y) + K_1(y, x)$ is a continuous symmetric semi-positive definite function on $[a, b] \times [a, b]$ and if $K(x, y) = f(x)f(y)K_1(x, y)$, $f \in L^2[a, b]$, then* (1) *holds for this $K(x, y)$.*

This theorem follows immediately by observing that

$$\int_a^b \int_a^b f(x)f(y)S(x, y)\alpha_k(x)\alpha_k(y)\, dx\, dy = 2\int_a^b \int_a^b f(x)f(y)K_1(x, y)\alpha_k(x)\alpha_k(y)\, dx\, dy.$$

COROLLARY 2.1. *If $\{\alpha_k(x)\}_1^{\infty}$ is a real C.O.N.S. on $[a, b]$, then*

$$\sum_{k=1}^{\infty} \int_a^b \alpha_k(x) \int_a^x \alpha_k(y)\, dy\, dx = \frac{b - a}{2}.$$

The corollary follows from Theorem 2 by setting $f(x) \equiv 1$ and $K_1(x, y)$

$= \xi(x, y)$, where

$$\xi(x, y) = \begin{cases} 1, & x > y, \\ \frac{1}{2}, & x = y, \\ 0, & x < y. \end{cases} \tag{2}$$

COROLLARY 2.2. *If $\{\alpha_k(x)\}_1^\infty$ is a real C.O.N.S. on $[a, b]$, then*

$$\operatorname*{l.i.m.}_{n \to \infty} \sum_{k=1}^{n} \alpha_k(x) \int_a^y \alpha_k(u)\, du = \xi(y, x)$$

on $[a, b]^2$ in the L^2-sense, where $\xi(x, y)$ is defined by (2).

The following result is rather interesting and useful.

THEOREM 3. *Let $\{v_k(x)\}_1^\infty$ be the complete orthonormal characteristic functions of the Sturm–Liouville system*

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv}{dx}\right] + [\lambda g(x) - l(x)] = 0, \qquad \lambda > 0, \\ &v'(a) - c_1 v(a) = 0, \qquad v'(b) + c_2 v(b) = 0, \end{aligned} \tag{3}$$

where on $[a, b]$ the functions $k(x)$, $g(x)$, and $l(x)$ are continuous, $k(x)$ and $g(x)$ do not vanish, $k(x)$ is continuously differentiable, and $g(x) \cdot k(x)$ has a continuous second derivative. Also let each $v_k(x)$ correspond to the characteristic number λ_k arranged in increasing order. Then

$$\operatorname*{l.i.m.}_{n \to \infty} \sum_{k=1}^{n} v_k(x) \int_a^x v_k(y)\, dy = \tfrac{1}{2} \tag{4}$$

on $[a, b]$ in the L^2-sense.

Proof. A direct computation yields that if $\{\alpha_k(x)\}_1^\infty$ is the set of C.O.N. cosine functions on $[a, b]$, given by $\alpha_1(x) = (b - a)^{-1/2}$, $\alpha_k(x) = [2/(b - a)]^{1/2} \cos[(k - 1) \cdot \pi(x - a)/(b - a)]$, $k \geqq 2$, then

$$\int_a^b \left[\frac{1}{2} - \sum_{k=1}^{n} \alpha_k(x) \int_a^x \alpha_k(y)\, dy\right]^2 dx \to 0 \quad \text{as } n \to \infty.$$

The same result can be established for the C.O.N. sine functions, the C.O.N. trigonometric functions, and the C.O.N. Legendre polynomials. To prove the general case, we use the asymptotic expressions for the C.O.N.S. It is known that the Sturm–Liouville system (3) in the theorem can be normalized by appropriate transformations (see [3, pp. 270–271]) into the following system on $[0, \pi]$:

$$\begin{aligned} &\frac{d^2u}{dx^2} + [\rho^2 - q(x)]u = 0, \\ &u'(0) - c_1 u(0) = 0, \qquad u'(\pi) + c_2 u(\pi) = 0, \end{aligned} \tag{5}$$

where $\rho^2 = K^2\lambda$ for some constant K. Let $\{u_k(x)\}_1^\infty$ be the C.O.N. characteristic functions of this system. For each fixed $x \in (0, \pi)$ define $I_x(s)$ by

$$I_x(s) = \begin{cases} 1, & 0 \leqq s \leqq x, \\ 0, & x < s \leqq \pi. \end{cases}$$

Then the orthogonal development of $I_x(s)$ in s, given by

$$\sum_{k=1}^{\infty} u_k(s) \int_0^{\pi} I_x(y) u_k(y)\, dy = \sum_{k=1}^{\infty} u_k(s) \int_0^x u_k(y)\, dy,$$

converges to the average value of $I_x(s)$ at the jump (see [2, p. 772]), and hence

$$\sum_{k=1}^{\infty} u_k(x) \int_0^x u_k(y)\, dy = \tfrac{1}{2} \quad \text{for every } x \in (0, \pi).$$

Owing to the fact that if limit and l.i.m. both exist, then they must be the same, it remains only to show the existence of

$$\operatorname*{l.i.m.}_{n\to\infty} \sum_{k=1}^{n} u_k(x) \int_0^x u_k(y)\, dy.$$

To verify this we use the asymptotic expression given in [3, p. 272]; namely, the characteristic functions of (5) corresponding to the characteristic numbers ρ_k are given by

$$\tilde{u}_k(x) = \{1 + O(k^{-2})\} \cos kx + \{K^{-1} w(x) + O(k^{-2})\} \sin kx, \tag{6}$$

where

$$w(x) = c_1 + \tfrac{1}{2} \int_0^x q(y)\, dy - cx,$$

c is a constant, and $O(k^{-2})$ denotes a function whose absolute value is bounded by a constant times K^{-2} on $[0, \pi]$, the constant being independent of k and x. To normalize $\tilde{u}_k(x)$, we rewrite (6) in the form

$$\tilde{u}_k(x) = \cos kx + k^{-1} w(x) \sin kx + O(k^{-2}). \tag{7}$$

Then with $\|\tilde{u}_k\|^2 = \int_0^\pi v_k^2(x)\, dx$, we have

$$u_k(x) \equiv \tilde{u}_k(x)/\|\tilde{u}_k(x)\| = (2/\pi)^{1/2} \tilde{u}_k(x)[1 + O(k^{-2})]^{-1/2}.$$

Thus there exists a number K such that

$$u_k(x) = (2/\pi)^{1/2} \tilde{u}_k(x)[1 + O(k^{-2})], \qquad k \geqq K. \tag{8}$$

From (7) and (8) it follows that

$$u_k(x) = (2/\pi)^{1/2}(\cos kx + k^{-1} w(x) \sin kx) + O(k^{-2}), \qquad k \geqq K.$$

Therefore

$$u_k(x) \int_0^x u_k(y)\, dy = \frac{\sin 2kx}{k\pi} + O(k^{-2}), \qquad k \geqq K.$$

Thus for $n \geqq m \geqq K > 1$, we have

$$\int_0^\pi \left[\sum_{k=m}^{n} u_k(x) \int_0^x u_k(y)\, dy \right]^2 dx$$
$$= \int_0^\pi \left[\sum_{k=m}^{n} \frac{\sin 2kx}{k\pi} \right]^2 dx + 2 \int_0^\pi \left[\sum_{k=m}^{n} \frac{\sin 2kx}{k\pi} \right] \left[\sum_{k=m}^{n} O(k^{-2}) \right] dx$$
$$+ \int_0^\mu \left[\sum_{k=m}^{n} O(k^{-2}) \right]^2 dx.$$

Now $(k\pi)^{-1}\sin 2kx = (2/\pi)^{1/2}\cos kx \int_0^x (2/\pi)^{1/2}\cos ky\,dy$, and from the comment made for the C.O.N. cosine functions in the beginning of the proof, we see that

$$\lim_{m,n\to\infty}\int_0^\pi\left[\sum_{k=m}^n (2/\pi)^{1/2}\cos kx\int_0^x (2/\pi)^{1/2}\cos ky\,dy\right]^2 dy = 0,$$

and hence

$$\lim_{m,n\to\infty}\int_0^\pi\left[\sum_{k=m}^n u_k(x)\int_0^4 u_k(y)\,dy\right]^2 dx = 0,$$

which completes the proof.

COROLLARY 3.1. *If $\{v_k(x)\}_{k=1}^\infty$ is a C.O.N.S. on $[a,b]$ satisfying the condition*

$$\underset{n\to\infty}{\text{l.i.m.}}\sum_{k=1}^n v_k(x)\int_a^x v_k(y)\,dy = \tfrac{1}{2} \quad \text{on } [a,b],$$

then the C.O.N.S. $\{\alpha_k(x,y)\}_{k=1}^\infty \equiv \{v_i(x)v_j(y)\}_{i,j=1}^\infty$ on $[a,b]^2$ satisfies

$$\underset{n\to\infty}{\text{l.i.m.}}\sum_{k=1}^n \alpha_k(x,y)\int_a^y\int_a^x \alpha_k(u,v)\,\mathrm{du}\,dv = \tfrac{1}{4},$$

$$\underset{n\to\infty}{\text{l.i.m.}}\sum_{k=1}^n \alpha_k(x,w)\int_a^y\int_a^x \alpha_k(u,v)\,du\,dv = \tfrac{1}{2}\xi(y,w),$$

and

$$\underset{n\to\infty}{\text{l.i.m.}}\sum_{k=1} \alpha_k(x',y')\int_a^y\int_a^x \alpha_k(u,v)\,du\,dv = \xi(x,x')\xi(y,y'),$$

where $\xi(x,y)$ is defined by (2).

This corollary follows immediately from Corollary 2.2.

3. Some applications to stochastic integrals. Let $\{X(\bar t):\bar t\in R_+^N\}$ denote the N-parameter Wiener process on $R_+^N \equiv [0,\infty)^N$ satisfying the conditions:

(i) this process is a separable real Gaussian stochastic process,

(ii) $X(\bar t) = 0$ almost surely for every $\bar t\in R_+^N - (0,\infty)^N$,

(iii) the expected value $E(x(\bar t)) = 0$ for every $\bar t\in R_+^N$,

(iv) the covariance $E(X(\bar s)X(\bar t)) = \prod_{1\leqq i\leqq N}\min(s_i,t_i)$ for every pair of points $\bar s = (s_1,\cdots,s_N)$ and $\bar t = (t_1,\cdots,t_N)$ in R_+^N.

In particular, the one-parameter Wiener process is the well-known Brownian motion process (see [1]). In [5] the author has introduced a stochastic integral on the N-parameter Wiener process which is entirely different from Ito's stochastic integral in [4] even for $N = 1$. Zimmerman [8] extends the Ito-type stochastic integral of a one-parameter process into that of a two-parameter process. According to Zimmerman (see [8, pp. 1239–43]), the Ito-type stochastic integral

$$\int_0^T\int_0^S F(s,t;\omega)\,dX(s,t)$$

for the two-parameter Wiener process $\{X(s,t):(s,t)\in[0,\infty)^2\}$ exists if $F(s,t;\omega)$ is square integrable on $[0,S]\times[0,T]\times\Omega$ and $F(s,t;\omega)$ is measurable with respect to $\mathscr{F}_{s,t} \equiv \sigma\{X(u,v;\omega):(u,v) \leqq (s,t)\}$ for all $(s,t) \in [0,S] \times [0,T]$, where $\sigma\{\cdot\}$ denotes the smallest σ-field generated by $\{\cdot\}$, and $(u,v) \leqq (s,t)$ means $u \leqq s$ and $v \leqq t$.

Consider the stochastic integral

$$\int_0^1 \int_0^1 h(s,t)\left[\int_0^1 X(s,u)\,du\right] dX(s,t), \tag{9}$$

where $h(s,t)[\int_0^T X(s,u;\omega)\,du]$ with $h(s,t) \in L^2([0,1]^2)$ stands for $F(s,t;\omega)$. However this $F(s,t;\omega)$ is not measurable with respect to $\mathscr{F}_{s,t}$ unless $t = 1$, and hence the Ito-type stochastic integral does not apply for it.

Let $\{v_k(x)\}_{k=1}^{\infty}$ be a C.O.N.S. on $[0,1]$ with each $v_k(x)$ of bounded variation, which satisfies the condition (4). Also let $\{\alpha_k(x,y)\}_{k=1}^{\infty} = \{v_i(x)v_j(y)\}_{i,j=1}^{\infty}$. For $F(s,t;\omega) \in L^2(I^2 \times \Omega)$, $I = [0,1]$, let $F_n(s,t)$ denote the nth partial sum of the Fourier expansion of $F(s,t;\omega)$ with respect to $\{\alpha_k(x,y)\}$. Define the Paley–Wiener–Zygmund-type (PWZ-type) stochastic integral by

$$\int_{I^2} F(s,t;\omega)\,dX(s,t) = \operatorname*{l.i.m.}_{n\to\infty} \int_{I^2} F_n(s,t)\,dX(s,t), \tag{10}$$

if the limit in the mean exists in the L^2-sense over the Wiener process $\{X(s,t):(s,t) \in [0,\infty)^2\}$. We shall show that the stochastic integral (9) exists in the PWZ-sense with the expected value

$$\tfrac{1}{2}\int_{I^2} h(s,t)(1-t)\,ds\,dt,$$

and the variance

$$\tfrac{1}{3}\int_{I^2} h^2(s,t)s\,ds\,dt.$$

To see this, we observe that for $n > m$,

$$\begin{aligned}
I_{m,n} &\equiv E\left[\int_{I^2} F_n(s,t)\,dX(s,t) - \int_{I^2} F_m(s,t)\,dX(s,t)\right]^2 \\
&= E\left[\sum_{k=m+1}^{n} \int_{I^2} h(s,t)\left(\int_I X(s,u)\,du\right)\alpha_k(s,t)\,ds\,dt \int_{I^2} \alpha_k(s,t)\,dX(s,t)\right]^2 \\
&= E\left[\sum_{j,k=m+1}^{n} \int_{I^6} h(s,t)h(s',t')\alpha_j(s,t)\,\alpha_k(s',t')\right. \\
&\qquad \left.\cdot\left(X(s,u)X(s',u')\int_{I^2}\alpha_j\,dX \int_{I^2}\alpha_k\,dX\right)\right] d\mu_6,
\end{aligned} \tag{11}$$

where μ_6 denotes the Lebesgue measure on I^6. Now recall the well-known formula[2]

$$\begin{aligned}
&E\left[\int_{I^2} h_1\,dX \int_{I^2} h_2\,dX \int_{I^2} h_3\,dX \int_{I^2} h_4\,dX\right] \\
&\quad = \int_{I^2} h_1h_2 \cdot \int_{I^2} h_3h_4 + \int_{I^2} h_1h_3 \cdot \int_{I^2} h_2h_4 + \int_{I^2} h_1h_4 \cdot \int_{I^2} h_2h_3,
\end{aligned} \tag{12}$$

[2] The formula (12) can be easily established by looking at the approximating Riemann–Stieltjes sums for the $\int_{I^2} h_i\,dX$, $i = 1, 2, 3, 4$.

where $h_i(s,t)$, $i = 1,2,3,4$, are functions of bounded variation on I^2. Since $X(s,u) = \int_{I^2} \xi(s,s')\xi(u,u')\,dX(s',u')$, we may use (12) and Fubini's theorem to obtain from (11)

$$
\begin{aligned}
I_{m,n} = \sum_{j,k=m+1}^{n} \int_{I^6} & h(s,t)h(s',t')\alpha_j(s,t)\alpha_k(s',t') \\
& \cdot \Big[\min(s,s') \cdot \min(u,u')\,\delta_{jk} \\
& \quad + \int_0^u \int_0^s \alpha_j(x,y)\,dx\,dy \cdot \int_0^{u'} \int_0^{s'} \alpha_k(x,y)\,dx\,dy \\
& \quad + \int_0^u \int_0^s \alpha_k(x,y)\,dx\,dy \cdot \int_0^{u'} \int_0^{s'} \alpha_j(x,y)\,dx\,dy \Big]\,d\mu_6 \\
= \sum_{k=m+1}^{n} \int_{I^6} & h(s,t)h(s',t)\min(s,s')\min(u,u')\alpha_k(s,t)\alpha_k(s',t')\,d\mu_6 \\
+ \sum_{j,k=m+1}^{n} \int_{I^6} & h(s,t)h(s',t')\alpha_j(s,t) \int_0^u \int_0^s \alpha_j(s,y)\,dx\,dy \\
& \cdot \alpha_k(s',t') \int_0^{u'} \int_0^{s'} \alpha_k(x,y)\,dx\,dy\,d\mu_6 \\
+ \sum_{j,k=m+1}^{n} \int_{I^6} & h(s,t)h(s',t')\alpha_j(s,t) \int_0^{u'} \int_0^{s'} \alpha_j(x,y)\,dx\,dy \\
& \cdot \alpha_k(s',t') \int_0^u \int_0^s \alpha_k(x,y)\,dx\,dy\,d\mu_6 .
\end{aligned}
$$

Upon using Theorem 1 for the first sum, and Corollary 3.1 for the second and third sums with $m = 0$, as $n \to \infty$ we get

$$
\begin{aligned}
I_{0,\infty} &= \int_{I^2} h^2(s,t)s\,ds\,dt \cdot \int_{I^2} \min(u,u')\,du\,du' \\
&\quad + \tfrac{1}{4} \int_{I^6} h(s,t)h(s',t')\xi(u,t)\xi(u',t')\,d\mu_6 \\
&\quad + \int_{I^6} h(s,t)h(s',t)\xi(u',t)\xi(s',s)\xi(s,s')\xi(u,t')\,d\mu_6 \\
&= \tfrac{1}{3} \int_{I^2} h^2(s,t)s\,ds\,dt + \tfrac{1}{4}\left[\int_{I^2} h(s,t)(1-t)\,ds\,dt\right]^2 .
\end{aligned}
$$

This shows that $\lim_{m,n\to\infty} I_{m,n} = 0$, and hence that the stochastic integral exists in the PWZ-sense. Furthermore, the expectation

$$
\begin{aligned}
E&\left\{\int_{I^2} h(s,t)\left[\int_I X(s,u)\,du\right] dX(s,t)\right\} \\
&= \sum_{k=1}^{\infty} E\left\{\int_{I^2} h(s,t)\left[\int_I X(s,u)\,du\right]\alpha_k(s,t)\,ds\,dt \cdot \int_{I^2} \alpha_k(s,t)\,dX(s,t)\right\} \\
&= \sum_{k=1}^{\infty} \int_{I^3} h(s,t)\alpha_k(s,t) \int_0^u \int_0^s \alpha_k(s',t')\,ds'\,dt'\,d\mu_3 \\
&= \tfrac{1}{2}\int_{I^3} h(s,t)\xi(u,t)\,du\,ds\,dt = \tfrac{1}{2}\int_{I^2} h(s,t)(1-t)\,ds\,dt,
\end{aligned}
$$

and the variance is $3^{-1}\int_{I^2} h^2(s,t)s\,ds\,dt$. It is interesting to note that

$$
\operatorname{var} \int_{I^2} h(s,t)\left[\int_I X(s,u)\,du\right] dX(s,t) = \int_{I^2} h^2(s,t) E\left[\int_I X(s,u)\,du\right]^2 ds\,dt,
$$

which is always the case for the Ito-type stochastic integrals. The existence of various PWZ-type stochastic integrals plays a central role in [6] to generalize Cameron–Martin's linear transformation theorem into higher dimensions.

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CONVERGENCE OF NONCOMMUTATIVE CONTINUED FRACTIONS*

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Abstract. A sufficient condition is given for the convergence of continued fractions whose elements are noncommutative linear operators on a Banach space. The theorem generalizes a corresponding result for conventional continued fractions.

1. Introduction. A formal theory of noncommutative continued fractions, whose elements obey a noncommutative law of multiplication, has been presented by Wynn [1]. Subsequently, a number of convergence theorems have been established by Wynn [2] and Fair [3], [4]. Here we present a simple sufficient condition for convergence of noncommutative continued fractions, which is a generalization of a well-known theorem [5] for corresponding conventional (scalar) commutative continued fractions.

Continued fractions are commonly employed in the solution of three-term recurrence relations arising in the treatment of higher transcendental functions, such as Mathieu functions. In mathematical physics, recurrence relations with five or more terms are generally treated as an infinite system of linear equations. Such recurrence relations can be cast into three-term vector recurrence relations, the solutions of which may be expressed in terms of continued fractions of, in general, noncommutative matrix elements.

In this paper, we present a proof of a simple sufficiency condition for convergence of a class of such noncommutative continued fractions. This theorem facilitates the applicability of continued fractions to solutions of recurrence relations with a greater than three (but finite) number of terms and thus significantly enhances the scope of the conventional procedure.

2. Statement of the problem; definitions and notations. We consider the convergence of continued fractions of the form

$$K_1 = (B_1 + (B_2 + (\cdots)^{-1})^{-1})^{-1}, \tag{1}$$

where B_n, $n = 1, 2, \cdots$, are bounded, linear, noncommutative operators on a Banach space $[X]$ over the complex field. We formally define a set of continued fractions

$$K_n = (B_n + (B_{n+1} + (\cdots)^{-1})^{-1})^{-1}, \qquad n = 1, 2, \cdots, \tag{2}$$

which satisfy the recurrence relation

$$K_n = (B_n + K_{n+1})^{-1} \tag{3}$$

This recurrence relation will be the starting point for the proof of the theorem to be presented here.

The nth convergent of (1) is defined [1], [4] as

$$\begin{aligned} S_n &= (B_1 + (B_2 + \cdots (B_{n-1} + B_n^{-1})^{-1} \cdots)^{-1})^{-1} \\ &= Q_n^{-1} P_n, \end{aligned} \tag{4}$$

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where P_n and Q_n are determined from the recurrence relations

$$(5)\qquad \begin{aligned} P_n &= B_nP_{n-1} + P_{n-2} \quad \text{for } n \geqq 2, \\ Q_n &= B_nQ_{n+1} + Q_{n-2} \quad \text{for } n \geqq 2, \end{aligned}$$

with the initial conditions

$$P_0 = 0, \quad P_1 = 1, \quad Q_0 = 1, \quad Q_1 = B_1.$$

The continued fraction K_1 converges if the limit of S_n, as $n \to \infty$, exists.

3. Convergence theory.

LEMMA. *If B and C are operators in $[X]$ such that the norms $\|B^{-1}\| \leqq \frac{1}{2}\rho$, $\rho < 1$, and $\|C\| < 1$, then the norm of the operator*

$$(6)\qquad K = (B + C)^{-1}$$

is less than unity, specifically $\|K\| < \rho$.

Proof. The assumption that $\|B^{-1}\| \leqq \frac{1}{2}\rho$, $\rho < 1$, implies that B^{-1} exists. Therefore, (6) may be rewritten as

$$(7)\qquad K = (1 + B^{-1}C)^{-1}B^{-1}.$$

Recognizing that $\|B^{-1}C\| \leqq \|B^{-1}\|\,\|C\| < \frac{1}{2}\rho$, one deduces by means of power series expansions of $(1 + B^{-1}C)^{-1}$, along with the triangular inequality, that

$$\|(1 + B^{-1}C)^{-1}\| < 2.$$

Invoking the Schwarz inequality, one then obtains from (7)

$$\|K\| \leqq \|(1 + B^{-1}C)^{-1}\|\,\|B^{-1}\| \leqq \rho < 1.$$

We now prove the following.

THEOREM. *If $\|B_n^{-1}\| < \frac{1}{2}\rho$, $\rho < 1$, for every $n \geqq 1$, then the continued fraction* (1) *converges.*

Proof. We introduce in the space $[X]$ the operators $K_n^{(m)}$ via the recurrence relation[1]

$$(8)\qquad K_n^{(m)} = (B_n + K_{n+1}^{(m-1)})^{-1}, \qquad n \geqq 1, \quad m = 1, 2, \cdots,$$

with

$$K_n^{(0)} = 0, \qquad n \geqq 1.$$

It is easily shown from (8) that

$$(9)\qquad K_n^{(m)} = (B_n + (B_{n+1} + \cdots + (B_{n+m-2} + B_{n+m-1}^{-1})^{-1} + \cdots)^{-1})^{-1},$$

which is the mth convergent of the continued fraction K_n in (2). By the repeated use of the lemma in each iteration in (8) for every $m \geqq 1$ and every $n \geqq 1$, it follows that $\|K_n^{(m)}\| \leqq \rho$. Therefore $\{K_n^{(m)} | m = 1, 2, \cdots\}$ is a bounded sequence of operators in $[X]$, for every $n \geqq 1$.

---

[1] The choice of $K_n^{(0)}$ is made for convenience, in order that the $K_n^{(m)}$ be the mth approximant to K_n. The result holds for any $\|K_n^{(0)}\| < 1$.

To prove that $\{K_n^{(m)} | m = 1, 2, \cdots\}$ is a convergent sequence for $n \geqq 1$, we first construct, from (8), the differences

$$K_n^{(m+1)} - K_n^{(m)} = -K_n^{(m+1)}[K_{n+1}^{(m)} - K_{n+1}^{(m-1)}]K_{n+1}^{(m)},$$

from which we then obtain

$$\|K_n^{(m+1)} - K_n^{(m)}\| \leqq \rho^2 \|K_{n+1}^{(m)} - K_{n+1}^{(m-1)}\|, \quad m \geqq 1, n \geqq 1 \text{ integers}. \tag{10}$$

Iterating on (10), one has

$$\|K_n^{(m+1)} - K_n^{(m)}\| \leqq \rho^{2m} \|K_{n+m}^{(1)} - K_{n+m}^{(0)}\| \leqq \tfrac{1}{2}\rho^{2m+1}. \tag{11}$$

Here use has been made of the property

$$\|K_{n+m}^{(1)} - K_{n+m}^{(0)}\| = \|K_{n+m}^{(1)}\| = \|B_{n+m}^{-1}\| \leqq \tfrac{1}{2}\rho. \tag{12}$$

Consider now

$$\|K_n^{(m+p)} - K_n^{(m)}\| \quad \text{for } m > 1, n > 1, p \geqq 1,$$

$$\|K_n^{(m+p)} - K_n^{(m)}\| \leqq \|K_n^{(m+p)} - K_n^{(m+p-1)}\| + \|K_n^{(m+p-1)} - K_n^{(m+p-2)}\| + \cdots + \|K_n^{(m+1)} - K_n^{(m)}\|.$$

Repeated application of (11) yields

$$\|K_n^{(m+p)} - K_n^{(m)}\| \leqq \frac{1}{2}[\rho^{2m+2p-1} + \rho^{2m+2p-3} + \cdots + \rho^{2m+1}] = \frac{1}{2}\frac{\rho^{2m+1}(1 - \rho^{2p})}{1 - \rho^2}$$

$$\leqq \frac{1}{2}\frac{\rho^{2m+1}}{1 - \rho^2}. \tag{13}$$

Consequently, by choosing the value of m to be sufficiently large, $\|K_n^{m+p} - K_n^{(m)}\|$ can be made arbitrarily small for all $p \geqq 1$. This establishes the convergence in the norm of the sequence $\{K_n^{(m)} | m = 1, 2, \cdots\}$ for every $n \geqq 1$ and, consequently, the convergence of all continued fractions K_n in (2) for $n \geqq 1$, under the stated conditions.

4. Example: A comparison with the commutative case. To see how the convergence condition established above relates to the well-known result for scalar continued fractions, let us consider the simple case of an $N \times N$ matrix continued fraction, given by

$$K = (B + (B + (\cdots)^{-1})^{-1})^{-1}, \tag{14}$$

the elements of which are obviously commutative. In this case, the matrix K can be diagonalized and expressed in terms of scalar continued fractions of eigenvalues of B, as follows. Let λ_i, $i = 1, \cdots, N$, be the set of eigenvalues of B. Then the eigenvalues of K are given by

$$\kappa_i = (\lambda_i + (\lambda_i + (\cdots)^{-1})^{-1})^{-1}, \qquad i = 1, \cdots, N.$$

To ensure that these scalar continued fractions converge for all i, it is sufficient to require

$$|\lambda_{\min}| = \min_i |\lambda_i| > 2. \tag{15}$$

Now, applying the convergence condition for noncommutative continued fractions, which is obviously valid also in the case of (14), one finds that the continued fraction (14) converges if

$$\|B^{-1}\| \leqq \tfrac{1}{2}\rho, \qquad \rho < 1.$$

Defining the norm of a matrix by its maximum eigenvalue, we have

$$\|B^{-1}\| = \max_i \frac{1}{|\lambda_i|} = \frac{1}{|\lambda_{\min}|} < \frac{1}{2},$$

which is identical with (15).

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NONLINEAR VOLTERRA EQUATIONS IN A HILBERT SPACE*

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Abstract. The existence and asymptotic behavior of the solutions of the equation

$$u(t) + \int_0^t a(t-s)g(u(s))\,ds = f(t), \qquad 0 < t < \infty,$$

in a Hilbert space H is investigated. Here $a(t)$ is a prescribed real function on $[0, \infty[$ and g belongs to a certain class of nonlinear maximal monotone operators in $H \times H$.

1. Introduction. This paper deals with the integral equation

$$u(t) + \int_0^t a(t-s)g(u(s))\,ds = f(t), \qquad 0 \leqq t < \infty, \tag{1.1}$$

in a Hilbert space H, where $a(t)$ is a real function on $[0, \infty[$ and g is a nonlinear (multivalued) operator from H into itself.

Equation (1.1) occurs in the study of mechanical systems with memory and in several problems of physical interest. The special case, $H = R^1$, of this equation has been extensively studied and we refer the reader to [5] for significant results and references on this subject. More recently, (1.1) was studied in [6], [8] under general monotonicity hypotheses on $a(t)$. These results were partially generalized to Hilbert spaces by MacCamy [11] and Londen [9] by the use of Laplace transforms. In these papers the operator g is assumed to satisfy certain boundedness conditions which considerably restrict the application field where usually g is a nonlinear elliptic partial differential operator. At the expense of assuming the positivity of the kernel $a(t)$, we are able in Theorems 1 and 2 to obtain results on the existence and asymptotic behavior of the solutions under weaker hypotheses on g. Specifically, g is the subdifferential of some convex lower semicontinuous function from H to $]-\infty, +\infty]$. The method we have used here is closely related to that used by Brézis (see, for example, [2], [3]) in the study of evolution equations associated with *subgradient* mappings. In Theorems 3 and 4, g is *monotone*, *demicontinuous* and coercive and satisfies some growth condition from a Hilbert space $V \subset H$ into its dual V'. Assuming that $a(t)$ is a positive kernel, that is, $\operatorname{Re} \tilde{a}(i\omega) \geqq 0$ on R^1, we prove the existence in $L^2(0, \infty; V)$ for the solutions of (1.1). In § 6, some examples are presented. Comparing these results with the results of [9] and [11], we observe that the assumptions we imposed on $a(t)$ are quite different from those used in [9] and [11]. In fact in essence our main condition on a is that the operator $u \xrightarrow{L} \int_0^t a(t-s)u(s)\,ds$ is positive on $L^2(0, \infty; H)$, while in [9] and [11] this condition (actually much stronger) is imposed on $u \to (d/dt)Lu$.

2. The main result. Throughout this paper H will denote a real Hilbert space with the norm denoted $|\cdot|$ and the inner product $(\cdot, \cdot)$.

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We begin by recalling some definitions and elementary results concerning maximal monotone operators in Hilbert spaces. For other results on this subject we refer the reader to [2].

Let g be a nonlinear (multivalued) operator from H to itself. We shall use the following notations:

$$D(g) = \{u \in H; g(u) \neq \emptyset\}, \qquad R(g) = U\{g(u); u \in D(g)\},$$
$$g^{-1}(u) = \{v \in H; u \in g(v)\}.$$

The operator g is said to be *monotone in* $H \times H$ if

$$(y_1 - y_2, x_1 - x_2) \geqq 0 \quad \text{for all } x_i \in D(g), y_i \in g(x_i), \quad i = 1, 2.$$

A monotone operator g which admits no proper monotone extension is called *maximal monotone*. According to a well-known result due to Minty, a monotone operator g is maximal monotone if and only if $R(I + \lambda g) = H$ for all (or equivalently for some) $\lambda > 0$. In this case, $(I + \lambda g)^{-1}$ is a contraction defined on all of H.

Let g be maximal monotone; for every $\lambda > 0$ the operator g_λ (*Yosida approximation* of g) defined by

$$g_\lambda = \lambda^{-1}(I - (I + \lambda g)^{-1}) \tag{2.1}$$

is monotone and Lipschitzian on H. Moreover, for all $\lambda > 0$ and $u \in H$, $g_\lambda(u) \in g((I + \lambda g)^{-1}u)$ and

$$\lim_{\lambda \to 0} g_\lambda(u) = g^0(u), \qquad u \in D(g), \tag{2.2}$$

where $g^0(u)$ is the element of minimal norm in $g(u)$.

An important class of maximal monotone operators in $H \times H$ is the subdifferentials of lower semicontinuous convex functions defined on H. More precisely, let φ be a convex, lower semicontinuous function from H to $]-\infty, +\infty]$, nonidentically $+\infty$. Let

$$D(\varphi) = \{u \in H; \varphi(u) < +\infty\} \tag{2.3}$$

and

$$\partial\varphi(u) = \{y \in H; \varphi(u) - \varphi(v) \leqq (y, u - v), \forall v \in H\}. \tag{2.4}$$

The multivalued operator $u \to \partial\varphi(u)$ is called the *subdifferential* of φ. If φ is Gâteaux differentiable at u, then $\partial\varphi(u)$ is reduced to a single point and coincides with the Gâteaux differential at u.

We shall denote by $L^2_{\text{loc}}(0, \infty; H)$ the space of all H-valued measurable functions u: $]0, \infty[\to H$ such that $\int_0^T |u(t)|^2 \, dt < +\infty$ for all $T > 0$. Denote also

$$H^1_{\text{loc}}(0, \infty) = \{u \in L^2_{\text{loc}}(0, \infty; H), u' \in L^2_{\text{loc}}(0, \infty; H)\}, \tag{2.5}$$

where $' = d/dt$ is taken in the sense of H-valued distributions on $]0, +\infty[$. We recall that every $u \in H^1_{\text{loc}}(0, \infty)$ coincides a.e. on $]0, \infty[$ with a locally absolutely continuous function, a.e. differentiable on $]0, \infty[$ and with its first derivative (which coincides with u') in $L^2_{\text{loc}}(0, \infty; H)$ (see Appendix in [2] for detailed information on this subject).

We now state the basic assumptions:

(i) $a(t)$ is continuous on $[0, \infty[$ and locally absolutely continuous on $]0, \infty[$. Moreover,

$$(-1)^k a^{(k)}(t) \geqq 0 \quad \text{for } k = 0, 1, \quad \text{a.e. } t > 0 \tag{2.6}$$

and

$$\operatorname{Re} \tilde{a}(\lambda) > 0 \quad \text{in } \operatorname{Re} \lambda > 0, \tag{2.7}$$

where

$$\tilde{a}(\lambda) = \int_0^\infty \exp(-\lambda t) a(t)\, dt.$$

(ii) $g = \partial\varphi$ where $\varphi : H \to]-\infty, +\infty]$ is convex, lower semicontinuous and nonidentically $+\infty$.

A function $u :]0, \infty[\to H$ is called a solution of (1.1) if $u \in L^2_{\mathrm{loc}}(0, \infty; H)$ and there exists a function $w :]0, \infty[\to H$ such that

$$w \in L^2_{\mathrm{loc}}(0, \infty; H), \qquad w(t) \in g(u(t)) \quad \text{a.e. } t > 0, \tag{2.8}$$

$$u(t) + \int_0^t a(t - s) w(s)\, ds = f(t) \quad \text{a.e. } t > 0. \tag{2.9}$$

For simplicity we shall write $g(u)$ instead of w.

THEOREM 1. *Suppose that* (i) *and* (ii) *hold. Let* $f \in H^1_{\mathrm{loc}}(0, \infty)$ *be such that*

$$f(0) \in D(\varphi). \tag{2.10}$$

Then (1.1) *has a solution* $u \in H^1_{\mathrm{loc}}(0, \infty)$ *satisfying*

$$t \to \varphi(u(t)) \quad \textit{is absolutely continuous on every } [0, T]. \tag{2.11}$$

In addition, if g is strictly monotone in $H \times H$, then the solution u is unique.

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied. In addition, suppose that* $a(\infty) > 0$, $f' \in L^2(0, \infty; H)$ *and*

$$\varphi(u) \to +\infty \quad \textit{as } |u| \to +\infty. \tag{2.12}$$

Then any solution $u(t)$ *of* (1.1) *satisfies*

$$|u(t)| \quad \textit{bounded on } [0, \infty[, \tag{2.13}$$

$$u', g(u) \in L^2(0, \infty; H) \tag{2.14}$$

and

$$\lim_{t \to +\infty} \varphi(u(t)) = \varphi_\infty = \min\{\varphi(v); v \in H\}. \tag{2.15}$$

If in addition we suppose that $\sqrt{t}\, a' \in L^1(0, \infty)$, $\varphi(f) \in L^1(0, \infty)$ *and* $\sqrt{t} f' \in L^2(0, \infty; H)$, *then*

$$\sqrt{t}\, u', \sqrt{t}\, g(u) \in L^2(0, \infty; H) \tag{2.16}$$

and

$$t\varphi(u(t)) \quad \textit{is bounded on } [0, \infty[. \tag{2.17}$$

Let V be a real Hilbert space, densely and continuously imbedded in H. We identify H with its own dual and denote V' the dual of V. Then the following relation holds:

$$V \subset H \subset V'.$$

Let (v', v) be the pairing between an element $v' \in V'$ and $v \in V$; if $v, v' \in H$, this is the ordinary inner product in H. We denote by $\|\cdot\|$ and $\|\cdot\|_*$ the norms of V and V' respectively.

A nonlinear operator g from V to V' is said to be monotone if

$$(v_1 - v_2, u_1 - u_2) \geqq 0 \quad \text{for all } v_i \in g(u_i), \quad i = 1, 2,$$

and maximal monotone if it is monotone and admits no proper monotone extensions in $V \times V'$.

The operator (single-valued) $g : V \to V'$ is said to be *demicontinuous* if $D(g) = V$ and it is continuous from the strong topology of V into the weak topology of V'. We recall that a demicontinuous monotone operator from V to V' is maximal monotone. Moreover, if g is coercive, then $R(g) = V'$ (see, for example, [4]).

Now we list the hypotheses which will be used in the next theorems:

(i) $a \in L^1(0, \infty)$ and the following inequality holds:

$$\operatorname{Re} \tilde{a}(i\omega) \geqq 0 \quad \text{for all } \omega \in R^1. \tag{2.18}$$

Here $\tilde{a}(i\omega) = \int_0^\infty \exp(-i\omega t) a(t)\, dt$.

(ii) The operator g is monotone, single-valued and demicontinuous from V to V'. There exists some positive constant C_1 such that

$$\|g(u)\|_* \leqq C_1 \|u\| \quad \text{for all } u \in V. \tag{2.19}$$

(iii) There exists $C_2 > 0$ such that

$$(g(u), u) \geqq C_2 \|u\|^2 \quad \text{for all } u \in V. \tag{2.20}$$

(iv) $a \in L^1(0, \infty) \cap L^2(0, \infty)$ and there exists $q > 0$ such that

$$|\tilde{a}(i\omega)|^2 \geqq q \operatorname{Re} \tilde{a}(i\omega) \quad \text{for all } \omega \in R^1. \tag{2.21}$$

(v) There exist α real and $C_2 > 0$ such that

$$(g(u), u) + \alpha |u|^2 \geqq C_2 \|u\|^2 \quad \text{for all } u \in V. \tag{2.22}$$

THEOREM 3. *Suppose that* (i), (ii) *and* (iii) *hold. Then for every* $f \in L^2(0, \infty; V)$, *the equation* (1.1) *has a solution* $u \in L^2(0, \infty; V)$.

The next result is a variant of Theorem 3 where instead of (iii) the weaker condition (v) is assumed.

THEOREM 4. *Assume that* (ii), (iv) *and* (v) *hold with* $\alpha q < 1$. *Then, for every* $f \in L^2(0, \infty; V)$ *the equation* (1.1) *has a solution* $u(t)$ *satisfying*

$$u \in L^2(0, \infty; V). \tag{2.23}$$

3. Proof of Theorem 1. Let $\varphi_\lambda : H \to]-\infty, +\infty]$ be the convex function defined by

$$\varphi_\lambda(u) = \inf\left\{\frac{|u-v|^2}{2\lambda} + \varphi(v); v \in H\right\}, \quad \lambda > 0, \quad u \in H. \tag{3.1}$$

It is well known (see, for example, [2]) that φ_λ is Fréchet differentiable on H and $\partial\varphi_\lambda = g_\lambda$ for all $\lambda > 0$. As g_λ is Lipschitzian on H, for every $\lambda > 0$ the approximating equation

$$u_\lambda(t) + \int_0^t a(t-s) g_\lambda(u_\lambda(s))\, ds = f(t), \qquad 0 \leqq t < \infty, \tag{3.2}$$

has a unique solution $u_\lambda \in H^1_{\mathrm{loc}}(0, \infty; H)$. The equation (3.2) can be written as

$$u'_\lambda(t) + a(0) g_\lambda(u_\lambda(t)) + \int_0^t a'(t-s) g_\lambda(u_\lambda(s))\, ds = f'(t), \quad \text{a.e. } t > 0. \tag{3.3}$$

Noting that

$$\frac{d}{dt}\varphi_\lambda(u_\lambda(t)) = (g_\lambda(u_\lambda(t)), u'_\lambda(t)), \quad \text{a.e. } t > 0,$$

by (3.3) it follows that

$$\frac{d}{dt}\varphi_\lambda(u_\lambda(t)) + a(0)|g_\lambda(u_\lambda(t))|^2 \leqq |f'(t)|\,|g_\lambda(u_\lambda(t))|,$$
$$+ |g_\lambda(u_\lambda(t))| \int_0^t |a'(t-s)|\,|g_\lambda(u_\lambda(s))|\, ds.$$

By integrating this inequality on $]0, T[$ and using (3.2), we get

$$\varphi_\lambda(u_\lambda(t)) + a(0)\int_0^t |g_\lambda(u_\lambda(s))|^2\, ds \leqq \varphi_\lambda(f(0)) + \int_0^t |f'(s)|\,|g_\lambda(u_\lambda(s))|\, ds$$
$$+ \int_0^t |g_\lambda(u_\lambda(s))|\, ds \int_0^s |a'(s-\tau)|\,|g_\lambda(u_\lambda(\tau))|\, d\tau.$$

By Schwartz's inequality we obtain

$$\int_0^t |g_\lambda(u_\lambda(s))|\, ds \int_0^s |a'(s-\tau)|\,|g_\lambda(u_\lambda(\tau))|\, d\tau$$
$$\leqq \int_0^t |g_\lambda(u_\lambda(s))|^2\, ds \int_0^t |a'(\tau)|\, d\tau$$

because the L^2-norm on $]0, t[$ of $\int_0^s |a'(s-\tau)|\,|g_\lambda(u_\lambda(\tau))|\, d\tau$ is majorized by $\int_0^t |a'(\tau)|\, d\tau \,(\int_0^t |g_\lambda(u_\lambda(\tau))|^2\, d\tau)^{1/2}$. Since $a' \leqq 0$ we finally obtain

$$\varphi_\lambda(u_\lambda(t)) + a(t)\int_0^t |g_\lambda(u_\lambda(s))|^2\, ds \leqq \varphi_\lambda(f(0)) + \int_0^t |f'(s)|\,|g_\lambda(u_\lambda(s))|\, ds.$$

Let $0 < T < \infty$ be such that $a(T) > 0$. Then the last estimate yields

$$\varphi_\lambda(u_\lambda(t)) + \tfrac{1}{2}a(t)\int_0^t |g_\lambda(u_\lambda(s))|^2\,ds \leqq \varphi(f(0)) \tag{3.4}$$
$$+ \frac{1}{2a(t)}\int_0^t |f'(s)|^2\,ds, \qquad 0 \leqq t \leqq T,$$

because $\varphi_\lambda(u) \leqq \varphi(u)$ for all $u \in H$.

Let u_0 be arbitrary in $D(g)$. Then the inequality

$$\varphi_\lambda(u_\lambda) \geqq \varphi_\lambda(u_0) - (g_\lambda(u_0), u_0 - u_\lambda)$$

gives, with some constants α and β,

$$\varphi_\lambda(u_\lambda(t)) \geqq \alpha|u_\lambda(t)| + \beta \quad \text{for all } \lambda > 0 \tag{3.5}$$

because $|g_\lambda(u_0)| \leqq |g^0(u_0)|$ and $\varphi_\lambda(u_0)$ is bounded.

Applying Schwartz's inequality in (3.2), we find that

$$|u_\lambda(t)| \leqq |f(t)| + \varepsilon\int_0^t |g_\lambda(u_\lambda(s))|^2\,ds + C_\varepsilon,$$

where ε is positive and can be chosen arbitrary small. Combining this inequality with (3.4) and (3.5), we find that $\{g_\lambda(u_\lambda)\}$ is bounded in $L^2(0, T; H)$ and $\{u_\lambda\}$ is bounded in $L^\infty(0, T; H)$.

Therefore, we may assume that as $\lambda \to 0$,

$$\begin{aligned} &u_\lambda \to u \quad \text{weakly in } L^2(0, T; H),\\ &g_\lambda(u_\lambda) \to g \quad \text{weakly in } L^2(0, T; H). \end{aligned} \tag{3.6}$$

Let L denote the integral operator

$$(Lv)(t) = \int_0^t a(t-s)v(s)\,ds, \qquad 0 < t < T. \tag{3.7}$$

We recall that assumption (2.7) implies that L is positive on $L^2(0, T; H)$ (see [10], [12] for details). Then by (3.2) we get

$$\int_0^T (g_\lambda(u_\lambda(t)) - g_\mu(u_\mu(t)), u_\lambda(t) - u_\mu(t))\,dt \leqq 0 \qquad \text{for all } \lambda, \mu > 0. \tag{3.8}$$

Since g is maximal monotone, (3.6) and (3.8) imply (see, for example, [2]) that $u(t) \in D(g)$ a.e. $t \in\,]0, T[$ and

$$g(t) \in g(u(t)) \quad \text{a.e. } t \in\,]0, T[,$$

which shows that $u(t)$ is a solution of (1.1) on the interval $[0, T]$. Obviously, u' and $g(u)$ belong to $L^2(0, T; H)$, so that the function $t \to \varphi(u(t))$ is absolutely continuous on $[0, T]$ (see [2, Lemma 3.3]) and

$$\frac{d}{dt}\varphi(u(t)) = (u'(t), g(u(t))) \quad \text{a.e. } t \in\,]0, T[. \tag{3.9}$$

To conclude the existence it remains to show that $u(t)$ can be continued on $]T, +\infty[$.

We consider the equation

$$v(t) + \int_0^t a(t-s)g(v(s))\,ds \ni f(t+T) - \int_0^T a(T+t-s)g(u(s))\,ds, \tag{3.10}$$
$$0 \leqq t \leqq T.$$

Let $f_1(t)$ denote the right side of (3.10). Clearly $f_1' \in L^2(0, T; H)$ and $f_1(0) = u(T) \in D(\varphi)$. Thus, according to the first part of the proof, there exists a solution v of (3.10) such that $v' \in L^2(0, T; H)$. It now follows readily that $\tilde{u}:[0, 2T] \to H$ defined by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } 0 \leqq t \leqq T, \\ v(t-T) & \text{if } T < t \leqq 2T, \end{cases}$$

satisfies the equation (1.1) on $[0, 2T]$.

Thus, by repeatedly using this argument, one obtains a solution $u(t)$ of (1.1) defined on the whole half-axis and satisfying the claimed conditions.

If g is strictly monotone, that is, $(g(u) - g(v), u - v) = 0$ iff $u = v$, the uniqueness of the solution u is an immediate consequence of the positivity of L. Thus the proof of Theorem 1 is complete.

4. Proof of Theorem 2. Let u be any solution of (1.1). Clearly, $u \in H^1_{\mathrm{loc}}(0, \infty)$ and

$$u'(t) + a(0)g(u(t)) + \int_0^t a'(t-s)g(u(s))\,ds = f'(t), \quad \text{a.e. } t > 0. \tag{4.1}$$

As we have noted before, this implies that

$$\frac{d}{dt}\varphi(u(t)) = (g(u(t)), u'(t)), \quad \text{a.e. } t > 0. \tag{4.2}$$

Multiplying (4.1) by $g(u(t))$ and integrating gives

$$\varphi(u(t)) + \frac{a(t)}{2}\int_0^t |g(u(s))|^2\,ds \leqq \varphi(f(0)) + \frac{1}{2a(t)}\int_0^t |f'(s)|^2\,ds, \qquad t \geqq 0. \tag{4.3}$$

By hypothesis (2.12) we have

$$|u(t)| \quad \text{bounded on } [0, \infty[. \tag{4.4}$$

As φ is bounded from below by an affine function, by (4.3) we obtain

$$g(u) \in L^2(0, \infty; H) \tag{4.5}$$

because $a(t) \geqq a(\infty) > 0$.

Next, by (4.1) and (4.2) we have

$$u' \in L^2(0, \infty; H), \qquad \frac{d}{dt}\varphi(u) \in L^1(0, \infty). \tag{4.6}$$

Thus $\varphi_\infty = \lim_{t\to+\infty} \varphi(u(t))$ exists. But by the definition of $\partial\varphi$, we have

$$\varphi(u(t)) \leqq \varphi(v) + (g(u(t)), u(t) - v) \quad \text{for all } v \in H.$$

Hence by (4.4) and (4.5),

$$\varphi_\infty = \lim_{t\to+\infty} \varphi(u(t)) \leqq \varphi(v) \quad \text{for all } v \in H,$$

as claimed.

Suppose now that $\varphi(f) \in L^1(0, \infty)$ and

$$\sqrt{t}\, a' \in L^1(0, \infty), \qquad \sqrt{t}\, f' \in L^2(0, \infty; H). \tag{4.7}$$

Multiplying (4.1) by $tg(u(t))$, and integrating over $]0, t[$ yields

$$\begin{aligned} &t\varphi(u(t)) + a(0) \int_0^t s|g(u(s))|^2\, ds \\ &\leqq -\int_0^t s\, ds \int_0^s (g(u(\tau)), a'(s-\tau)g(u(s)))\, d\tau + \int_0^t \varphi(u(s))\, ds \\ &\quad + \left(\int_0^t s|f'(s)|^2\, ds\right)^{1/2} \left(\int_0^t s|g(u(s))|^2\, ds\right)^{1/2}, \qquad t \geqq 0. \end{aligned} \tag{4.8}$$

Now we multiply (1.1) by $g(u(t))$ and integrate over $]0, t[$. Since L is positive, we obtain

$$\int_0^t (g(u(s)), u(s) - f(s))\, ds \leqq 0 \quad \text{for all } t \geqq 0.$$

In other words,

$$\int_0^t \varphi(u(s))\, ds \leqq \int_0^t \varphi(f(s))\, ds \quad \text{for all } t \geqq 0. \tag{4.9}$$

We now put

$$I(t) = -\int_0^t s\, ds \int_0^s (g(u(\tau)), a'(s-\tau)g(u(s)))\, d\tau$$

and

$$I_g^2(t) = \int_0^t s|g(u(s))|^2\, ds.$$

Interchange of the order of integration, which is easily justified by the hypotheses and Fubini's theorem, yields

$$|I(t)| \leqq -\int_0^t a'(\tau)\, d\tau \left(\int_\tau^t s|g(u(s-\tau))|^2\, ds\right)^{1/2} I_g(t), \qquad t \geqq 0.$$

After some calculation, we get

$$\begin{aligned} &|I(t)| \leqq (a(0) - a(t)) I_g^2(t) \\ &\quad + \left(\int_0^t |g(u(s))|^2\, ds\right)^{1/2} I_g(t) \int_0^t \sqrt{s}|a'(s)|\, ds, \qquad t \geqq 0. \end{aligned} \tag{4.10}$$

Combining (4.5), (4.7), (4.8), (4.9) and (4.10), we obtain

$$t\varphi(u(t)) + \int_0^t s|g(u(s))|^2\,ds \leqq C, \qquad t \geqq 0, \tag{4.11}$$

which completes the proof.

Remark 1. Theorem 1 remains valid if instead of (2.6) we assume that

$$a(0) > 0, \quad a'(t) \geqq 0 \quad \text{a.e. } t > 0. \tag{4.12}$$

This follows by observing that in the proof of Theorem 1 the inequality (3.4) holds in this case with $2a(0) - a(t)$ instead of $a(t)$. Similarly, assuming that $a(\infty) < 2a(0)$, we see that Theorem 2 follows with (4.12) instead of (2.6).

5. Proofs of Theorem 3 and 4. We introduce the following notations:

$$\mathscr{V} = L^2(0, \infty; V), \quad \mathscr{H} = L^2(0, \infty; H), \quad \mathscr{V}' = L^2(0, \infty; V').$$

We shall also denote by $\langle \cdot , \cdot \rangle$ the natural pairing between $\mathscr{V}$ and $\mathscr{V}'$, that is,

$$\langle u, v\rangle = \int_0^\infty (u(t), v(t))\,dt, \qquad u \in \mathscr{V}, \quad v \in \mathscr{V}'.$$

Proof of Theorem 3. Let us denote by $\tilde{g}: \mathscr{V} \to \mathscr{V}'$ the operator defined by

$$(\tilde{g}(u))(t) = g(u(t)), \qquad u \in \mathscr{V}, \quad \text{a.e. } t > 0.$$

By (ii) the operator $\tilde{g}$ is monotone and demicontinuous from $\mathscr{V}$ to $\mathscr{V}'$. As we have already noted, this implies that $\tilde{g}$ is maximal monotone in $\mathscr{V} \times \mathscr{V}'$. Moreover, by (iii) we have

$$\langle \tilde{g}(u), u\rangle \geqq C_2 \|u\|^2_{\mathscr{V}}, \qquad u \in \mathscr{V}. \tag{5.1}$$

Since $a \in L^1(0, \infty)$, the operator $(Lu)(t) = \int_0^t a(t - s)u(s)\,ds$ is continuous on $\mathscr{V}$, $\mathscr{H}$ and $\mathscr{V}'$. Moreover, as $\operatorname{Re} \tilde{a}(i\omega) \geqq 0$ for all $\omega \in R^1$, it follows that L is positive on these spaces (see, for example, [5]). In particular, this implies that $(\lambda I + L)^{-1}$ is continuous on $\mathscr{V}$ and $\mathscr{H}$ for all $\lambda > 0$.

We consider the equation

$$u_\lambda + \lambda \tilde{g}(u_\lambda) + L\tilde{g}(u_\lambda) = f. \tag{5.2}$$

Equivalently,

$$\tilde{g}(u_\lambda) + (\lambda I + L)^{-1} u_\lambda = (\lambda I + L)^{-1} f. \tag{5.3}$$

The operator $(\lambda I + L)^{-1}$ is obviously positive and continuous from $\mathscr{V}$ to $\mathscr{V}'$. Hence it is maximal monotone. Since g is maximal monotone and by (5.1) coercive from $\mathscr{V}$ to $\mathscr{V}'$, it follows that (5.3) has at least a solution $u \in \mathscr{V}$, (see, for example [4]). From (4.2) it now follows readily that this solution is unique.

Multiplying (5.2) with $\tilde{g}(u_\lambda)$ and using (5.1), we derive

$$C_1 \|u_\lambda\|^2_{\mathscr{V}} + \lambda \|\tilde{g}(u_\lambda)\|^2_{\mathscr{H}} \leqq \langle f, \tilde{g}(u_\lambda)\rangle.$$

Next by (2.20),

$$\|u_\lambda\|^2_{\mathscr{V}} + \lambda \|\tilde{g}(u_\lambda)\|^2_{\mathscr{H}} \leqq C \|f\|^2_{\mathscr{V}} \tag{5.4}$$

for some positive constant C.

As L is positive from $\mathscr{V}$ to itself, we have

$$\|(\lambda I + L)^{-1}u\|_{\mathscr{V}} \leqq \lambda^{-1}\|u\|_{\mathscr{V}} \quad \text{for all } \lambda > 0, \quad u \in \mathscr{V}.$$

Then by (5.3) we get

$$\lambda\|\tilde{g}(u_\lambda)\|_{\mathscr{V}} \leqq \|u_\lambda\|_{\mathscr{V}} + \|f\|_{\mathscr{V}} \quad \text{for all } \lambda > 0. \tag{5.5}$$

We now set

$$v_\lambda = u_\lambda + \lambda\, \tilde{g}(u_\lambda).$$

By (5.4) and (5.5) it follows that

$$(u_\lambda - v_\lambda) \to 0 \quad \text{in } \mathscr{H} \quad \text{as } \lambda \to 0$$

and

$$\{v_\lambda\} \quad \text{bounded in } \mathscr{V}.$$

Therefore a subsequence (denoted again $\{u_\lambda\}$) can be extracted from $\{u_\lambda\}$ such that as $\lambda \to 0$,

$$\begin{aligned} u_\lambda &\to u \quad \text{weakly in } \mathscr{V}, \\ v_\lambda &\to u \quad \text{weakly in } \mathscr{V}, \\ \tilde{g}(u_\lambda) &\to w \quad \text{weakly in } \mathscr{V}'. \end{aligned} \tag{5.6}$$

To conclude the proof it suffices to show that $w = \tilde{g}(u)$. Again using the positivity of L in $\mathscr{H}$, by (5.2) it follows that

$$\langle \tilde{g}_\lambda(v_\lambda) - \tilde{g}_\mu(v_\mu), v_\lambda - v_\mu \rangle \leqq 0 \quad \text{for all } \lambda, \mu > 0, \tag{5.7}$$

where $\tilde{g}_\lambda = \tilde{g}_{\mathscr{H}}(I + \lambda\tilde{g}_{\mathscr{H}})^{-1}$ and $\tilde{g}_{\mathscr{H}} : D(\tilde{g}) \subset \mathscr{H} \to \mathscr{H}$ is defined by

$$\tilde{g}_{\mathscr{H}}(u) = \tilde{g}(u) \quad \text{for } u \in D(\tilde{g}) = \{u \in \mathscr{H}\,;\, \tilde{g}(u) \in \mathscr{H}\}. \tag{5.8}$$

Extracting a further subsequence if necessary, we may assume that

$$\lim_{\lambda \to 0} \langle \tilde{g}_\lambda(v_\lambda), v_\lambda \rangle = l. \tag{5.9}$$

Then the inequality (5.7) yields

$$l \leqq \langle w, u \rangle. \tag{5.10}$$

Let v be arbitrary in $D(\tilde{g})$. We have

$$\langle \tilde{g}_\lambda(v_\lambda) - \tilde{g}_\lambda(v), v_\lambda - v \rangle \geqq 0$$

because $\tilde{g}_\lambda$ is monotone in $\mathscr{H}$ (see § 2). As $\tilde{g}_\lambda(v) \to \tilde{g}_{\mathscr{H}}(v)$ strongly in $\mathscr{H}$ and $v_\lambda \to u$ weakly in $\mathscr{H}$, by (5.9) and (5.10) we deduce that

$$\langle w - \tilde{g}(v), u - v \rangle \geqq 0 \quad \text{for all } v \in D(\tilde{g}). \tag{5.11}$$

We observe that under our assumptions, $D(\tilde{g})$ is a dense subset of $\mathscr{V}$. Indeed, for any $v \in \mathscr{V}$, the solution $v_\varepsilon \in D(\tilde{g})$ of the equation

$$v_\varepsilon + \varepsilon\tilde{g}(v_\varepsilon) = v$$

satisfies

$$C_1\|v_\varepsilon\|_{\mathscr{V}}^2 + \varepsilon\|g(v_\varepsilon)\|_{\mathscr{H}}^2 \leqq C_2\|v\|_{\mathscr{V}}\|v_\varepsilon\|_{\mathscr{V}}.$$

Hence $\{v_\varepsilon\}$ is bounded in $\mathscr{V}$. As for $\varepsilon \to 0$, $v_\varepsilon = (I + \varepsilon\tilde{g}_{\mathscr{H}})^{-1}v$ converges to v in $\mathscr{H}$ (see § 2) we conclude that $v_\varepsilon \to v$ weakly in $\mathscr{V}$.

Now, let v be an arbitrary element of $\mathscr{V}$ and let $\{v_n\} \subset D(g)$ be such that $v_n \to v$ strongly in $\mathscr{V}$. As $\tilde{g}$ is demicontinuous from V to $\mathscr{V}'$, this implies that $\tilde{g}(v_n) \to \tilde{g}(v)$ weakly in $\mathscr{V}'$. Thus the inequality (5.11) extends to all $v \in \mathscr{V}$. As $\tilde{g}$ is maximal monotone in $\mathscr{V} \times \mathscr{V}'$, we deduce that $w = \tilde{g}(u)$.

Since $\lambda\tilde{g}(u_\lambda) \to 0$ in $\mathscr{H}$ as $\lambda \to 0$, by (5.2) and (5.6) we deduce that

$$u + L\tilde{g}(u) = f,$$

which concludes the proof.

Remark 2. For applications, it would be desirable to extend Theorem 3 in the case when V is a reflexive Banach space. It is clear that the preceding proof is applicable to this general case if in addition we assume that

$$\lambda\,\|(\lambda I + L)^{-1}\|_{L(\mathscr{V},\mathscr{V})} \text{ is bounded as } \lambda \to 0, \tag{5.12}$$

where $L(\mathscr{V}, \mathscr{V})$ is the space of all linear continuous operators from $\mathscr{V}$ into itself. In particular, this holds if L is accretive on $\mathscr{V}$. We do not know whether or not (5.12) is implied by (i).

Proof of Theorem 4. This proof reproduces, with minor changes, that of Theorem 3. We observe that condition (2.20) was used only for proving the estimate (5.4).

But if (iv) holds, then we have [1]

$$\int_0^\infty |Lu(t)|^2\,dt \leqq q\int_0^\infty (Lu(t), u(t))\,dt \quad \text{for all } u \in \mathscr{H}. \tag{5.13}$$

Since $\alpha q < 1$, by (2.22), (5.2) and (5.12) we obtain

$$\|u_\lambda\|_{\mathscr{V}}^2 + \|u_\lambda + \tilde{g}(u_\lambda) - f\|_{\mathscr{H}}^2 + \lambda\|g(u_\lambda)\|_{\mathscr{H}}^2 \leqq C\|f\|_{\mathscr{V}}^2 \tag{5.14}$$

with some positive C. In particular, (5.14) implies (5.4). Thus arguing as in the proof of Theorem 3, we deduce that a sequence denoted again $\{u_\lambda\}$ can be extracted from $\{u_\lambda\}$ such that

$$u_\lambda \to u \quad \text{weakly in } \mathscr{V},$$

where u is a solution of (1.1).

6. Examples. Throughout this section, Ω will denote a bounded open subset of R^n with sufficiently smooth boundary Γ. $H^k(\Omega)$ and $H_0^k(\Omega)$, $k = 1, 2$, are the usual Sobolev spaces on Ω.

Example 1. Let β be a maximal monotone graph in $R^1 \times R^1$. Then there exists a lower semicontinuous convex function $j: R^1 \to]-\infty, +\infty]$ such that $\beta = \partial j$. We recall that j is uniquely determined up to an additive constant.

We take $H = L^2(\Omega)$ and $\varphi: H \to]-\infty, +\infty]$, defined by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\text{grad}\, u|^2\,dx + \int_\Omega j(u)\,dx & \text{if } u \in H_0^1(\Omega), j(u) \in L^1(\Omega), \\ +\infty & \text{otherwise}. \end{cases} \tag{6.1}$$

We recall (see [3]) that $\partial\varphi(u)(x) = -\Delta u(x) + \beta(u(x))$ a.e. $x \in \Omega$ and

$$|\Delta u|_H \leqq |(\partial\varphi)^0(u)|_H \quad \text{for all } u \in D(\partial\varphi). \tag{6.2}$$

Taking $g = \partial\varphi$, (1.1) becomes

$$\begin{gathered} u(t,x) - \int_0^t a(t-s)\Delta u(s,x)\,ds + \int_0^t a(t-s)\beta(u(s,x))\,ds \ni f(t,x), \qquad x \in \Omega, \quad t \geqq 0, \\ u(t,x) = 0, \qquad x \in \Gamma, \qquad t \geqq 0. \end{gathered} \tag{6.3}$$

Suppose that $a(t)$ satisfies the assumptions of Theorem 2. Let $f \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ be such that

$$\frac{\partial f}{\partial t} \in L^2(0, \infty; L^2(\Omega)), \qquad f(0, \cdot) \in H_0^1(\Omega)$$

and

$$j(f(0, \cdot)) \in L^1(\Omega).$$

Then (6.3) has a solution $u(t, x)$ satisfying

$$u \in L^2(0, \infty; H_0^1(\Omega) \cap H^2(\Omega)), \qquad \frac{\partial u}{\partial t} \in L^2(0, \infty; L^2(\Omega)), \tag{6.4}$$

$$\|u(t, \cdot)\|_{H_0^1(\Omega)} \quad \text{bounded on } [0, \infty[. \tag{6.5}$$

In particular, (6.5) implies that $\{u(t); t \geqq 0\}$ is a compact subset of $L^2(\Omega)$. Let u_∞ be an arbitrary limit point of $u(t)$ as $t \to \infty$. By (2.15), $\varphi(u_\infty) = \inf\{\varphi(u); u \in L^2(\Omega)\}$, that is, $u_\infty \in (\partial\varphi)^{-1}(0)$. This shows that $u_\infty \in H_0^1(\Omega) \cap H^2(\Omega)$ is the solution (clearly unique) of the nonlinear boundary problem

$$\begin{aligned} \Delta u - \beta(u) &\ni 0 \quad \text{on } \Omega, \\ u &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{6.6}$$

Hence

$$\lim_{t \to \infty} u(t,x) = u_\infty(x) \quad \text{in } L^2(\Omega). \tag{6.7}$$

Example 2. Let β be as in the preceding example and let $\varphi: L^2(\Omega) \to]-\infty, +\infty]$ be defined by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\text{grad } u|^2\,dx + \int_\Gamma j(u)\,d\sigma & \text{if } u \in H^1(\Omega), j(u) \in L^1(\Gamma), \\ +\infty & \text{otherwise}. \end{cases}$$

We note [3] that

$$\partial\varphi = -\Delta \text{ on } D(\partial\varphi) = \left\{ u \in H^2(\Omega); \frac{\partial u}{\partial n} \in -\beta(u) \quad \text{a.e. on } \Gamma \right\}.$$

Here $\partial/\partial n$ denotes the outward normal derivative.

Taking $H = L^2(\Omega)$ and $g = \partial\varphi$, equation (1.1) can be written as

$$(6.8)\qquad \begin{aligned} &u(t,x) - \int_0^t a(t-s)\Delta u(s,x)\,ds = f(t,x) && \text{on }]0,\infty[\, \times \Omega, \\ &\frac{\partial u}{\partial n} \in -\beta(u) && \text{on }]0,\infty[\, \times \Gamma, \end{aligned}$$

and we can apply Theorems 1 and 2.

In particular, if $j: R^1 \to]-\infty, +\infty]$ is defined by $j(r) = 0$ if $r \geqq 0$ and $j(r) = +\infty$ if $r < 0$, then (5.8) becomes

$$\begin{aligned} &u(t,x) - \int_0^t a(t-s)\Delta u(s,x)\,ds = f(t,x) && \text{on }]0,\infty[\, \times \Omega, \\ &u \geqq 0, \quad \frac{\partial u}{\partial n} \geqq 0 && \text{on }]0,\infty[\, \times \Gamma, \\ &u\frac{\partial u}{\partial n} = 0 && \text{on }]0,\infty[\, \times \Gamma. \end{aligned}$$

We note that problems of this type occur in the theory of heat conduction in materials with memory.

Example 3. Let V be a closed subspace of $H^m(\Omega)$ such that

$$H_0^m(\Omega) \subset V \subset H^m(\Omega)$$

and let $g: V \to V'$ be defined by

$$(6.9)\qquad (g(u), v) = \sum_{|\alpha| \leqq m} \int_\Omega A_\alpha(x, D^\gamma u(x)) D^\alpha v(x)\,dx, \qquad v \in V, \quad |\gamma| \leqq m,$$

where $A_\alpha(x, \xi)$ are continuous in ξ and measurable in x on Ω. We impose on A_α the growth conditions

$$(6.10)\qquad |A_\alpha(x,\xi)| \leqq C \sum_{|\alpha| \leqq m} |\xi_\alpha|, \quad \text{a.e. } x \in \Omega; \quad \xi \in R^m,$$

and the monotonicity condition

$$(6.11)\qquad \sum_{|\alpha| \leqq m} (A_\alpha(x,\xi) - A_\alpha(x,\eta)(\xi_\alpha - \eta_\alpha)) \geqq 0, \quad \text{a.e. } x \in \Omega.$$

Finally we assume

$$(6.12)\qquad \sum_{|\alpha| \leqq m} A_\alpha(x,\xi)\xi_\alpha \geqq C_2 \sum_{|\alpha| \leqq m} |\xi_\alpha|^2, \qquad C_2 > 0.$$

Then the operator g satisfies conditions (ii) and (iii) (see, for example, [4], [7]). Now, assuming that the kernel $a(t)$ satisfies (i), by Theorem 3 it follows that for any $f \in L^2(0, \infty; V)$ the equation

$$(6.13)\qquad u(t,x) + \sum_{|\alpha| \leqq m} (-1)^\alpha \int_0^t a(t-s) D^\alpha A_\alpha(x, D^\gamma u(s,x))\,ds = f(t,x)$$

has a solution $u \in L^2(0, \infty; V)$. In particular, if $V = H_0^m(\Omega)$, then $u(t,x)$ satisfies the Dirichlet boundary conditions

$$D^\alpha u = 0 \qquad \text{on }]0, \infty[\times \Gamma, \quad |\alpha| \leqq m - 1. \tag{6.14}$$

Now, if the condition (iv) holds instead of (i), by Theorem 4, it follows that the above existence result remains valid only assuming that

$$\sum_{|\alpha| \leqq m} A_\alpha(x, \xi)\xi_\alpha \geqq C_2 \sum_{|\alpha| = m} |\xi_\alpha|^2, \quad \text{a.e. } x \in \Omega, \quad \xi \in R^m.$$

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SEPARATION THEOREMS FOR SELF-ADJOINT LINEAR DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER*

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Abstract. This paper is devoted principally to the development of separation theorems for two solutions of a self-adjoint linear differential equation of the fourth order whose coefficients are continuous.

The main portion of this paper is concerned with the behavior of two solutions in the neighborhood of zeros of a certain determinant which identifies the focal points of an end-plane. We first look at two conjugate solutions and prove a separation theorem for them.

The problem is studied further since not all pairs of solutions of a differential equation are conjugate. Given two nonconjugate solutions, there is a solution linearly independent of and conjugate to both given solutions. Using theorems which relate the number of focal points to conjugate families in an interval and of problems with different end-conditions, we obtain a separation theorem for non-conjugate solutions.

1. Introductory remarks. There has been a revival in the study of separation, comparison, and oscillation theorems of self-adjoint linear differential equations of the fourth order of the form

$$L(y) = [(r(x)y'')' + q(x)y']' + p(x)y = 0,$$

where $r(x)$ is positive and all three coefficients are continuous on an interval I.

Leighton and Nehari [10] were among the first to rekindle the study, but they required that $r(x)$, $q(x)$, $p(x)$ be functions of C^2, C^1, C^0 respectively and for the most part $q(x) \equiv 0$. Barrett [1], [2] required that the coefficients just be continuous and did include the middle term $q(x)$, but most of his results require that $q \geqq 0$ and $p \geqq 0$. Sherman [14] establishes the existence of first conjugate and focal points for solutions of the general nth order linear differential equation. Hinton [7] proves several oscillation theorems with regard to functions which satisfy clamped end boundary conditions. Ladas [9] proves some comparison and oscillation theorems for ordinary self-adjoint differential equations of order $2n$. In particular, he studies the oscillatory behavior of fourth order differential equations containing a middle term which is wider than that studied by Leighton and Nehari.

More recently papers by Bradley [4] and Keener [8] have studied properties of solutions of fourth order differential equations with restrictions on r and p and $q \equiv 0$.

Most of the abovementioned articles are concerned with disconjugacy on I (that is, no nontrivial solution of $L(y) = 0$ has four zeros on I). Rarely do the above papers examine the difference between the number of zeros of two linearly independent nontrivial solutions of $L(y) = 0$. Leighton and Nehari and Keener do prove some separation theorems for the case $q \equiv 0$ and p of one sign.

The purpose of this paper is to investigate the differential equation $L(y) = 0$ with no restrictions on $r(x)$, $q(x)$, $p(x)$ other than continuity and to derive several

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separation theorems which state the difference between the number of zeros of two solutions in an interval.

This paper consists of three parts. First there is a brief outline of work leading to an index theorem. This theorem makes it possible to count the number of negative characteristic roots rather than the number of focal points when the former is more convenient to compute. Second, the major separation theorem is derived which states the difference between the number of zeros of two conjugate solutions in terms of the number of focal points. Finally, by comparing two nonconjugate solutions u and v to a solution w which is conjugate to both u and v, an upper bound of the difference between the number of zeros of two nonconjugate solutions in an interval is obtained.

Part I. We are going to study the fourth order self-adjoint linear differential equation

$$[(r(x)y'')' + q(x)y']' + p(x)y = 0, \tag{1.1}$$

where $r(x)$, $q(x)$, $p(x)$ are functions of C^0 and $r(x) > 0$ on the interval $[a, b]$. Corresponding to a function y, set $D_0 y = y$, $D_1 y = y'$, $D_2 y = ry''$, $D_3 y = (ry'')' + qy'$.

A solution of (1.1) is a function y of class C^2 such that $D_2 y$ and $D_3 y$ are of C^1 and the differential equation is satisfied. This result is easily obtained by writing (1.1) in phase-vector form and applying standard uniqueness and existence theorems.

Let Y and Z be the matrices

$$Y = [y(a), y(b), y'(a), y'(b)]^T,$$
$$Z = [-D_3 y(a), D_3 y(b), D_2 y(a), -D_2 y(b)]^T,$$

where T indicates transpose.

We now state the following

THEOREM 1.1. *Any set of self-adjoint boundary conditions can be given in the form*

$$Y = 0 \quad \text{or} \quad \begin{matrix} Y - Cu = 0, \\ C^T Z - Bu = 0, \end{matrix}$$

where u is a column of t parameters with $0 < t \leqq 4$, C is a matrix of 4 rows and t columns having rank t, and B is a symmetric matrix of t rows and columns. Conversely, any set of conditions of this form is self-adjoint.

2. Classification of boundary conditions. The self-adjoint boundary conditions will be represented in the form

$$y^s = c_h{}^s u_h, \qquad y'^s = d_h{}^s u_h \qquad (s = 1, 2 \text{ not summed}, \quad h = 1, \cdots, t), \tag{2.1}$$

$$[-D_3^s y c_h^s + D_2^s y d_h^s]_1^2 + b_{hk} u_k = 0, \tag{2.2}$$

where $y^1 = y(a)$, $y^2 = y(b)$, $D_2^1 y = r(a)y''(a)$, etc. and $C = [c_h^s, d_h^s]^T$ is of rank t, and $B = [b_{hk}]$ is symmetric, and $]_1^2$ is used to indicate the difference between the values at $x = a$ and $x = b$. One terms (2.1) the *end-plane* π_t in the space of the four variables y^s, y'^s $(s = 1, 2)$ regarding the variables (u) as parameters. The symmetric quadratic form $b_{hk}u_h u_k$ will be called the *end-form*. Conditions (2.2) are called *transversality* conditions. The π_0-plane corresponding to $t = 0$ is included as a special case called the *null* end-plane. This case has no transversality condition.

This paper is concerned with boundary value problems of the following type. In all cases the differential equation will be

$$[(ry'')' + qy']' + py - \lambda y = 0. \tag{2.3}$$

Problem A. Boundary conditions $y(a) = y'(a) = 0$, $y(b) = y'(b) = 0$.

Problem B. Boundary conditions

$$\begin{aligned} y(a) &= c_h^1 u_h, & y(b) &= 0, \\ y'(a) &= d_h^1 u_h, & y'(b) &= 0, \\ D_3^1 c_h^1 &- D_2^1 d_h^1 + b_{hk}u_k = 0. \end{aligned} \qquad (h = 1, t; t \leqq 2)$$

Problems A′ and B′. Same as Problems A and B respectively with condition $y(b) = y'(b) = 0$ omitted.

Problems A′ and B′ will be called *focal boundary problems* and the end conditions at $x = a$ will be termed the *end-plane* π_t' in the space of two variables y^1, y'^1. If $t = 0$, the end conditions $y(a) = y'(a) = 0$ apply.

3. Conjugate and focal points. The index theorem. In order to prove the index theorem, one uses the "broken" extremal technique. The steps used in deducing this theorem are a modification by Miller [11, pp. 11–35] of work done by Gottlieb [5]. More recently, some of these results can be shown to be special cases of self-adjoint differential systems as developed by Reid [13, Chap. 7]. Essentially it consists of minimizing the functional

$$I(y, \lambda) = b_{hk}u_h u_k + \int_a^b 2\omega(y, y', y'', \lambda)\, dx,$$

where $2\omega(y, y', y'', \lambda) = r(x)y''^2 - q(x)y'^2 + p(x)y^2 - \lambda y^2$.

Following Reid [13], let $y^0(x) = [y, y', -D_3 y, D_2 y]^T$ and J denote the 4×4 symplectic matrix

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Suppose y and z are two solutions of a self-adjoint linear differential equation of the fourth order; it is well known that

$$z^{0T} J y^0 = zD_3 y - z'D_2 y + y'D_2 z - yD_3 z = \text{const.} \tag{3.1}$$

If the constant is zero, then the solutions are said to be *conjugate*; otherwise they are *nonconjugate*.

The following two theorems are useful.

THEOREM 3.1. *Any two solutions of a focal boundary problem are mutually conjugate.*

THEOREM 3.2. *Given two mutually conjugate solutions of* (2.3) *there exists a set of end and transversality conditions satisfying Problem* A$'$ *or Problem* B$'$ *which is satisfied by the two solutions.*

DEFINITION 3.1. If, for a given value of λ, there exists a nonnull extremal which has a zero of order at least two at $x = x_1$ and $x = x_2$ $(a \leqq x_1 < x_2 \leqq b)$, then the two points x_1 and x_2 are said to be *conjugate* with respect to $I(y, \lambda)$.

DEFINITION 3.2. If, for a given value of λ, an extremal which satisfies the conditions of Problem B$'$ has a zero of order at least two at $x = x_1$ in $(a, b]$ without vanishing identically, then the point $x = x_1$ is said to be a *focal point* of the end-plane π_t' at $x = a$ with respect to B'.

LEMMA 3.1. *Suppose u and w are two linearly independent solutions of Problem* B$'$. *Then the focal points of the end-plane* π_t' *are the zeros of the determinant*

$$\alpha_{01} = \begin{vmatrix} u & w \\ u' & w' \end{vmatrix}.$$

Proof. If $x = x_1$ is a focal point of π_t', then there exists a solution y satisfying the endpoint and transversality conditions of Problem B$'$ at $x = a$ and having a double zero at $x = x_1$. Now every solution y satisfying these endpoint and transversality conditions is a linear combination of solutions u and w. Then at $x = x_1$,

$$y = au + bw = 0, \qquad y' = au' + bw' = 0,$$

where a, b are constants not both zero. It follows that

$$\alpha_{01}(x_1) = \begin{vmatrix} u(x_1) & w(x_1) \\ u'(x_1) & w'(x_1) \end{vmatrix} = 0.$$

The lemma is proved.

A similar lemma holds for Problem A$'$ with focal point replaced by conjugate point. An effect of Lemma 3.1 and Theorem 3.2 is that focal points can be associated directly with any pair u, w of conjugate solutions.

DEFINITION 3.3. The *index* of a conjugate point or focal point at $x = c$ is defined as the nullity of the matrix

$$A = \begin{bmatrix} u & w \\ u' & w' \end{bmatrix}$$

at $x = c$, where u and w are two linearly independent conjugate solutions. Conjugate and focal points will be counted a number of times equal to their index.

Combining these ideas with the index of a functional and a count of the number of characteristic roots less than λ^*, we are led to the index theorem.

THEOREM 3.3. *The number of characteristic roots less than* λ^* *in the Problem* A *is equal to the number of conjugate points of* $x = a$ *on* (a, b). *The number of characteristic roots less than* λ^* *in the problem* B *is equal to the number of focal points of the end-plane* π_t' *on* (a, b).

Part II.

4. Orders of zeros of two functions. In this section the behavior of the zeros of two solutions of a self-adjoint linear differential equation of the fourth order will be examined in relation to the associated focal points. Part of this investigation pertains to two functions in more generality, and the problem will be attacked in this broader form.

Consider the class of functions C^{*3} having the following properties on an interval (a, b):

(i) each function u is of C^2,

(ii) $D_2 u$ is of C^1,

(iii) $D_3 u$ is of C^1.

Now let two functions u and w belonging to C^{*3} have the following properties:

(a) zeros on (a, b) of the following type:

$$D_0 u(c) = 0,$$

$$D_0 u(c) = D_1 u(c) = 0,$$

$$D_0 u(c) = D_1 u(c) = D_2 u(c) = 0,$$

but not

$$D_i u(c) = 0, \qquad i = 0, 1, 2, 3;$$

(b) no zeros of u and w in common such that

$$D_i u = D_i w = 0, \qquad i = 0, 1, 2;$$

(c) no linear combination u and w having a zero such that

$$D_i(c_1 u + c_2 w) = 0, \qquad i = 0, 1, 2, 3.$$

Generalizing on the notation used by Barrett [3], let

$$A = \begin{bmatrix} u & w \\ u' & w' \end{bmatrix}.$$

The following determinants are related to A:

$$\alpha_{ij} = \begin{vmatrix} D_i u & D_i w \\ D_j u & D_j w \end{vmatrix}, \quad i \neq j, i < j, \quad i = 0, 1, 2, \quad j = 1,2,3.$$

More specifically, they are $\alpha_{01}, \alpha_{02}, \alpha_{12}, \alpha_{03}, \alpha_{13}, \alpha_{23}$.

In Theorems 4.1, 5.2, 6.1 the order of the zeros of u is taken to be greater than or equal to the order of zeros of w. Obviously, the roles can be reversed.

THEOREM 4.1. *The only possible combinations of zeros of u and w at zeros of α_{01} are shown in Table* 1.

Proof. It is known that if α_{01} is not zero in an open interval, the zeros of the functions u and w separate each other on such an interval. Therefore, consider the point $x = c$, where $\alpha_{01}(c) = 0$. In what follows, $k = \text{const}$.

Case 1. If $u(c) \neq 0$, $w(c) \neq 0$, then at $x = c$ either $w = ku$, $w' = ku'$ or $w = ku$, $w' = ku'$, $rw'' = kru''$. These conclusions are shown in Table 1, Case B, column 2(a) and Case D, column 2(a) respectively.

Case 2. If $u(c) = 0$, $w(c) \neq 0$, then at $x = c$ either $u = u' = 0$ or $u = u' = ru'' = 0$. See Table 1, Case B column 2(b) and Case D, column 2(b).

Case 3. If $u(c) = w(c) = 0$, $u'(c) \neq 0$, $w'(c) \neq 0$, then at $x = c$ either $w' = ku'$, $rw'' \neq kru''$ or $w' = ku'$, $rw'' = kru''$. See Table 1, Case C, column 2(a) and Case E, column 2(a).

TABLE 1

1							2(a)		2(b)		3(a)		3(b)		4
		Zero at $x = c$					No. of zeros at $x = c$				Under				Relation between
Case	Nullity	α_{01}	α_{02}	α_{12}	α_{03}	α_{13}	u	w	u	w	$x < c$	$x > c$	$x < c$	$x > c$	u and w
A	0						0	0	1	0			u	w	C/NC
B	1	*					0	0	2	0	u	w	u	w	C/NC
C	1	*	*		*		1	1	2	1	u	u	u	w	NC
D	1	*	*	*			0	0	3	0	u	u	u	w	NC
E	1	*	*	*	*		1	1	3	1	u	w	u	w	C
F	2	*	*	*	*	*	2	2	3	2	u	u	u	w	C

Column 4: C = conjugate, NC = nonconjugate.

Case 4. If $u(c) = u'(c) = w(c) = 0$, $w'(c) \neq 0$, then at $x = c$ either $u = u' = 0$ or $u = u' = ru'' = 0$. See Table 1, Case C, column 2(b) and Case E, column 2(b).

Case 5. If $u(c) = u'(c) = w(c) = w'(c) = 0$, then at $x = c$ either $rw'' \neq 0$, $ru'' \neq 0$ or $u = u' = ru'' = 0$. See Table 1, Case F, column 2(a) and Case F, column 2(b). These are the only possible combinations of u and w at zeros of α_{01}.

If the nullity of A at $x = c$ is zero, then A is nonsingular at $x = c$ and $\alpha_{01}(c) \neq 0$. The converse is also true.

If the nullity of $A(c)$ is one, then $A(c)$ is singular and $\alpha_{01}(c) = 0$. Since $A(c) \neq 0$, Cases 1, 2, 3, 4 apply. Conversely, if Cases 1, 2, 3, 4 apply, the nullity of $A(c)$ will be one.

If the nullity of $A(c)$ is two, then $A(c) = 0$, and Case 5 applies. Conversely, if Case 5 applies, then $A(c)$ has a nullity of two.

5. Foreknowledge of zeros of two functions. In the count of the numbers of zeros of two solutions, it is necessary to predict which of the two solutions will vanish next if either vanishes or which has the higher order zero if both vanish. The theory developed in this section will apply to two functions u and w with properties (a), (b), (c) of § 4.

DEFINITION 5.1. Let u and w be two functions and suppose u, w, and α_{01} do not vanish at $x = a$. Then u is *under* w if and only if $\alpha_{01}(a)$ and $u(a)w(a)$ have the same sign.

This somewhat artificial definition is made clearer by the following theorem.

THEOREM 5.1. *Suppose u is under w at $x = a$ and the first zero of w following $x = a$ is at $x = b$. Then if $\alpha_{01} \neq 0$ in $(a, b]$, u vanishes in $(a, b]$.*

Proof. Without loss of generality it can be assumed $u(a) > 0$, $w(a) > 0$ and therefore $\alpha_{01}(a) > 0$. Suppose u does not vanish in $(a, b]$. Then at $x = b$,

$$\alpha_{01}(b) = u(b)w'(b) < 0$$

since $w(b) = 0$, $u(b) > 0$, and $w'(b) < 0$. The fact contradicts the hypothesis that α_{01} does not change sign in $(a, b]$.

THEOREM 5.2. *The various cases of one function being under another just before and just after a zero of α_{01} are shown in columns* 3(a) *and* 3(b) *in Table* 1.

Proof. It can be assumed that the zero of α_{01} occurs at $x = c$ and for $x < c$ but sufficiently close to c, $u(x)$, $w(x)$, $\alpha_{01}(x)$ are all positive, that is, u is under w just before c in all cases.

The proof consists of considering the five cases ($\alpha_{01}(c) = 0$) indicated in Table 1 individually, and then repeatedly applying the law of the mean and the following identities: $\alpha'_{01} = \alpha_{02}/r, \alpha'_{02} = \alpha_{12} + \alpha_{03} - q\alpha_{01}, \alpha'_{12} = \alpha_{13}, \alpha''_{12} = \alpha_{23}/r + p\alpha_{01}$, and $\alpha_{12} = \alpha_{03}$ if u and w are conjugate.

For example, consider Case C: if $\alpha_{01}(c) = \alpha_{02}(c) = \alpha_{03}(c) = 0$ and $\alpha_{12}(c) > 0$, then $\alpha'_{02}(x) = \alpha_{12}(x) + \alpha_{03}(x) - q(x)\alpha_{01}(x) > 0$ if x is near enough to c. If $x < X < c$, then using the law of the mean, $\alpha_{02}(x) = \alpha'_{02}(X)(x - c)$, which implies $\alpha_{02}(x) < 0$, x near c. Since $\alpha'_{01}(x) = \alpha_{02}(x)/r(x) < 0$ for x near c, it follows from the law of the mean that $\alpha_{01}(x) = \alpha'_{01}(X)(x - c)$ implies $\alpha_{01}(x) > 0$ for x near c. For $c < x$ but near c, one can show similarly that $\alpha_{01}(x) > 0$. In the general case, u and w both have simple zeros at $x = c$, hence for $c < x$, u and w are both negative and uw and α_{01} have the same sign for $c < x$, that is, u is under w. In the special case, u has a double zero and w a single zero at $x = c$, hence for $c < x$, u is positive and w is negative, and uw and α_{01} have opposite signs, that is, w is under u.

6. Conjugate and nonconjugate solutions and zeros of α_{01}. In this section the functions u and w used in §§ 4 and 5 are restricted to solutions of a self-adjoint linear differential equation of the fourth order. The purpose is to show the relation between the property of two solutions being conjugate or not at points where α_{01} is zero.

Using the identity (3.1) it is possible in many instances to tell at a zero of α_{01} whether two solutions are conjugate or not.

THEOREM 6.1. *The property of two solutions being conjugate or not at zeros of α_{01} is shown on Table* 1, *column* 4.

Proof. As in Theorem 5.2, one considers the five cases separately. In each particular case there is a relationship between the two solutions u and w at a zero of α_{01}. From these relations one can usually determine whether u and w are conjugate or not. For example (assuming $x = c$ is a zero of α_{01}), consider Cases B and E:

Case B. In both the general case and special case there is not enough information to tell whether the constant in (3.1) is zero or not.

Case E. In the general case, since $\alpha_{01}(c) = \alpha_{02}(c) = \alpha_{12}(c) = \alpha_{03}(c) = 0$, $u(c) = w(c) = 0$, there exists a number k such that $D_1u(a) = kD_1w(a)$, $D_2u(a) = kD_2w(a)$. Substituting these conditions in (3.1) makes the constant zero. The solutions are conjugate.

In the special case, since u has a triple zero and w has a simple zero at $x = c$, it follows easily from (3.1) that the solutions are conjugate.

7. A separation theorem for points on a line. A major theorem is concerned with the difference between the number of zeros of two conjugate solutions. Although the theorem is to apply to conjugate solutions of a differential equation, it can be proved more easily if one considers points and intervals on the x-axis having properties similar to the zeros and one solution being under another of conjugate solutions.

DEFINITION 7.1. Consider a set of points on the x-axis. A point is called a *single* or *simple point* if there is only one point in the set having a given value of x. A point is called a *double point* if there are exactly two points in the set having a given value of x. The definition is similar for a triple point.

Consider three sets of points on the interval (a, b) of the x-axis with the following properties:

u is a finite set of points (possibly double or triple),

w is a finite set of points (possibly double or triple),

A is a finite set of points (possibly double).

The points in set u or w correspond to the zeros of the conjugate solution u or w respectively in (a, b). A simple point corresponds to a simple zero, a double point to a double zero, etc.

The points in set A correspond to the nullity of matrix A. A simple point corresponds to a matrix with a nullity of one while a double point corresponds to a matrix with a nullity of two.

At any point c in the interval (a, b), only certain coincidences of points from sets u, w, A will be allowed. This information is gleaned from Table 1, and the coincidences are listed in columns 2, 3, and 4 of Table 2 (e.g., in Case E, Table 1, column 2(a), one sees at $x = c$ that there is a simple zero of u and a simple zero of w and the nullity of A is one. This situation is listed in line 4, Table 2, where it can be seen that if there is a point of u and a point of w at c, there is also a point of A.)

Let $S = u \cup w \cup A$. The intervals between consecutive points are labeled (u) or (w) in the following way. Let $x = c$ be a point in S. If the solution u is under the solution w just to the left of $x = c$, the interval to the left of c in set S will be marked (u). Similarly, if the solution w is under solution u just to the left of $x = c$, the interval to the left of c in set S will be marked (w). The interval to the right of c in set S is treated in a similar fashion. No interval can be labeled (u) and (w) simultaneously, as seen from Definition 5.1. The permissible labels are listed in columns 5 and 6 of Table 2. These entries are easily found from Table 1 (e.g., in Case E, Table 1, using the zeros of u and w found in column 2(a), one sees in column 3(a) for $x < c$, that u is under w while for $x > c$, w is under u). Thus one labels the interval to the left of c (u) and the interval to the right of c (w). These labels are found in line 4b.

Let $N(u)$ represent the number of points in set u, each point counted according to its multiplicity. A similar definition applies to sets w and A.

LEMMA 7.1. *There can be no consecutive points of S which are simple points of u.*

Proof. Suppose there are two consecutive points of S both of which are simple points of u, namely, x_1 and x_2, $x_1 < x_2$. This case appears in line 2 of Table 2. The label appearing in the interval to the right of $x = x_1$ is (u) whereas the label appearing in the interval to the left of $x = x_2$ is (w). This is a contradiction, for the interval (x_1, x_2) cannot be labeled both (u) and (w).

LEMMA 7.2. *Suppose $N(A) = n$. Then $|N(w) - N(u)|$ is bounded above.*

Proof. Mark the points of u and A. Then there are at most $N(u) + n + 1$ intervals. Since there cannot be two consecutive points of S which are simple points of w, there are at most $N(u) + n + 1$ points of w inside the intervals. It follows from Table 2 that all other possible points of w occur at points of A. At such points, the number of points of w can exceed the number of points of u by at most

TABLE 2

Line	Number of points at $x = c$: u	w	A	Left label	Right label
1	0	1	0	w	u
2	1	0	0	u	w
3a	0	0	1	w	u
3b	0	0	1	u	w
4a	1	1	1	w	u
4b	1	1	1	u	w
5	0	2	1	w	u
6	2	0	1	u	w
7	1	3	1	w	u
8	3	1	1	u	w
9a	2	2	2	u	u
9b	2	2	2	w	w
10	2	3	2	w	u
11	3	2	2	u	w

Left label = Label appearing in interval just to left of $x = c$.
Right label = Label appearing in interval just to right of $x = c$.

2. Since there are n such points, the number of points of w can exceed the number of points of u at points of A in (a, b) by at most $2n$. Combining these two results, one can write the following inequality:

$$N(w) \leqq N(u) + n + 1 + 2n,$$

which yields

$$N(w) - N(u) \leqq 3n + 1.$$

If u and w are interchanged, then

$$N(u) - N(w) \leqq 3n + 1.$$

Combining the two inequalities, one has

$$|N(w) - N(u)| \leqq 3n + 1,$$

and the lemma is proved.

Let $Q(n) = \max |N(w) - N(u)|$. Then for all u, w,

$$|N(w) - N(u)| \leqq Q(n). \tag{7.1}$$

Furthermore, there exists a u, w, A with $N(A) = n$ such that either

$$N(u) = N(w) - Q(n) \quad \text{or} \quad N(w) = N(u) - Q(n). \tag{7.2}$$

It will be assumed in what follows that $N(w) \geqq N(u)$. Let u^*, w^*, A^* be sets of points with $N(A^*) = n$ which achieve equality (7.2).

LEMMA 7.3. *If c is a point of A^*, there exist sets u^*, w^*, A^* with $N(A^*) = n$ and $\bar{u}$, $\bar{w}$, $\bar{A}$ with $N(\bar{A}) = n$ and a neighborhood B of c such that outside of B the sets $\bar{u}$, $\bar{w}$, $\bar{A}$ are identical to the sets u^*, w^*, A^* respectively and inside B, the number of points of sets $\bar{u}$, $\bar{w}$, $\bar{A}$ equals the number of points of u^*, w^*, A^* respectively, but in B there are no coincident points or multiple points of $\bar{u}$, $\bar{w}$, $\bar{A}$. Furthermore, the labeling in the intervals between consecutive points of S^* and $\bar{S}$ is consistent with the rules set down in Table 2.*

Proof. The lemma is proved by considering all cases. The cases correspond to the conditions found in similarly numbered lines in Table 2. Use is made of the following code:

$$(\cdot)\ \text{label on interval},$$

$$\underline{\underset{\cdot}{u}}\ \ \text{points from set } u.$$

Case 1 and *Case* 2 do not fall under this lemma.

Case 3. The lemma is obviously true.

Case 4(a).

$$\begin{array}{ccc} & u^* & \\ & w^* & \\ \hline (u^*) & \underset{\cdot}{A^*} & (w^*) \end{array} \text{ is replaced by } \underline{(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{A}}_{\cdot}(\bar{w})}.$$

Case 4(b) is the same, interchanging u and w.

Case 5.

$$\begin{array}{ccc} & u^* & \\ & u^* & \\ \hline (u^*) & \underset{\cdot}{A^*} & (w^*) \end{array} \text{ is replaced by } \underline{(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{A}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})}.$$

Case 6. Similar to Case 5, u and w interchanged.

Case 7.

$$\begin{array}{ccc} & u^* & \\ & u^* & \\ & u^* & \\ & w^* & \\ \hline (u^*) & \underset{\cdot}{A^*} & (w^*) \end{array} \text{ is replaced by } \underline{(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{A}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})}.$$

Case 8. Similar to Case 7, u and w interchanged.

Case 9(a).

$$\begin{array}{ccc} & u^* & \\ & u^* & \\ & w^* & \\ & w^* & \\ & A^* & \\ \hline (u^*) & \underset{\cdot}{A^*} & (u^*) \end{array} \text{ is replaced by } \underline{(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{A}}_{\cdot}(\bar{w})^{\bar{A}}_{\cdot}(\bar{u})}.$$

Case 9(b). Similar to Case 9(a), u and w interchanged.

Case 10.

$$\begin{array}{ccc} & u^* & \\ & u^* & \\ & u^* & \\ & w^* & \\ & w^* & \\ & A^* & \\ \end{array}$$

$\underline{(u^*)\quad \underset{\cdot}{A^*}\quad (w^*)}$ is replaced by $\underline{(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{w}}_{\cdot}(\bar{u})^{\bar{u}}_{\cdot}(\bar{w})^{\bar{A}}_{\cdot}(\bar{u})^{\bar{A}}_{\cdot}(\bar{w})}$.

Case 11. Similar to Case 10 with u and w interchanged.

The proof of the lemma is complete.

THEOREM 7.1. *If in an open interval* (a, b), u, w, A *are sets of points having the properties described in Table* 2, *then the difference in the number of points of* u *and* w *is at most* $N(A) + 1$.

Proof. Let $N(A) = n$. It follows from Lemma 7.3 that u^*, w^*, A^* with $N(A^*) = n$ can be replaced by $\bar{u}$, $\bar{w}$, $\bar{A}$ with $N(\bar{A}) = n$ in which none of the points $\bar{u}$, $\bar{w}$, $\bar{A}$ are coincident or multiple, and

$$N(\bar{u}) = N(\bar{w}) - Q(n). \tag{7.3}$$

Mark the points of $\bar{w}$. It follows from Lemma 7.1 that there is a point of $\bar{u}$ or a point of $\bar{A}$ in each interval between points of $\bar{w}$. It follows, therefore, that

$$N(\bar{u}) + n \geqq N(\bar{w}) - 1 \quad \text{or} \quad N(\bar{w}) - N(\bar{u}) \leqq n + 1. \tag{7.4}$$

Using 7.3, it follows from (7.4) that

$$Q(n) \leqq n + 1. \tag{7.5}$$

Applying the inequality (7.5) to the result (7.1), one has

$$|N(w) - N(u)| \leqq Q(n) \leqq n + 1 = N(A) + 1. \tag{7.6}$$

Theorem 7.1 can be improved if one happens to know a number of points of A which are double points.

THEOREM 7.2. *If in an open interval* (a, b) u, w, A *are sets having the properties described in Table* 2, *and* r *is the cardinal number of double points of* A, *then the difference in the number of points* u *and* w *in* (a, b) *is at most* $N(A) + 1 - 2r$.

Proof.

Case 1. Suppose c is a double point of each of the sets u, w, A. Let u', w', A' be the sets u, w, A with point c excluded. Furthermore, this can be done without changing the labels on the intervals to the left and right of c. From (7.6) it follows that

$$|N(w') - N(u')| \leqq N(A') + 1,$$

$$|N(w') + 2 - N(u') - 2| \leqq N(A') + 1.$$

But since $N(w') + 2 = N(w)$ and $N(u') + 2 = N(u)$,

$$|N(w) - N(u)| \leqq N(A') + 1. \tag{7.7}$$

Now $N(A') = N(A) - 2$, hence (7.7) becomes

$$|N(w) - N(u)| \leqq N(A) - 1.$$

Case 2. Suppose c is a double point of each of the sets w, A and a triple point of set u. Let u', w', A' be the sets u, w, A with two points of c of each set excluded. There is, of course, a simple point c left in set u'. Again this can be done without changing the labels on the intervals to the left and right of c. Applying the same reasoning as in the previous case, it follows that

$$|N(w) - N(u)| \leqq N(A) - 1.$$

Case 3. A similar argument obtains for w having a triple point at c while u has only a double point.

It can be observed in each case that for each double point of A, the number $N(A) + 1$ appearing on the right-hand side of the inequality (7.6) can be reduced by 2.

8. The separation theorem for conjugate solutions. Suppose one has two conjugate solutions of a differential equation. It was shown in § 7 that the properties of the zeros and focal points of two conjugate solutions correspond to the properties of sets u, w, A in Table 2. In addition, the property of being "under" as applied to solutions agrees with the notion of labeled intervals. Thus the theorem which was proved for point sets is also true for two conjugate solutions. Rewriting Theorem 7.2 in terms of conjugate solutions and focal points, one has the following.

THEOREM 8.1. *If in an open interval (a, b), Γ is the sum of the indices of the focal points of two conjugate solutions, and if r is the cardinal number of focal points of index two, then the difference between the number of zeros in (a, b) of the two solutions is at most $\Gamma + 1 - 2r$.*

This is the best theorem one can get, even for two conjugate solutions of a fourth order differential equation as shown by the following example.

Consider the self-adjoint differential equation

$$y^{\text{iv}} + y'' = 0.$$

A fundamental set of solutions for this equation is

$$y_1 = 1, \quad y_2 = x, \quad y_3 = \sin x, \quad y_4 = \cos x.$$

The solutions y_1 and y_3 are conjugate as seen by substituting in (3.1). The focal points of y_1 and y_3 are the zeros of the determinant

$$\alpha_{01}(x) = \begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} = \cos x = 0,$$

hence the focal points occur at $x = \pm(2n - 1)\pi/2$ and are of index 1. Suppose there are n focal points. In the interval $[0, n\pi]$, y_3 has $n + 1$ zeros while y_1 has none. Thus

$$|N(y_3) - N(y_1)| = n + 1$$

and equality is actually achieved in Theorem 8.1.

Part III.

9. Relations between nonconjugate solutions. In studying the separation properties of two nonconjugate solutions, it is convenient to compare the two nonconjugate solutions to a common conjugate solution.

Suppose u_1 and u_2 are two conjugate solutions of the self-adjoint differential equation of the fourth order

$$L(y) = [(ry'')' + qy']' + py = 0. \tag{9.1}$$

According to Hadamard [6, p. 347] one can always find two other solutions v_1 and v_2 conjugate to each other such that the solutions u_1, u_2, v_1, v_2 form a fundamental set.

LEMMA 9.1. *Given two nonconjugate solutions y_1 and y_2 of the self-adjoint linear differential equation of the fourth order, there exist two other solutions y_3 and y_4 both of which are conjugate to and linearly independent of y_1 and y_2.*

Proof. Let y_1 and y_2 be two solutions of (9.1) which are not conjugate. If w is a 4×1 vector and x_0 is any selected value, the pair of vector equations

$$y_a^{0T}(x_0)Jw = 0 \quad (a = 1, 2)$$

has a two-parameter family of solutions with basis (w_3, w_4). Now if y_k is the solution of $L(y) = 0$ satisfying $y_k^0(x_0) = w_k$ $(k = 3, 4)$ then each of the triples (y_1, y_2, y_k) is a linearly independent set of solutions of $L(y) = 0$ and the solutions of each of the pairs (y_1, y_k) and (y_2, y_k) are mutually conjugate.

10. Most general conjugate family. It is convenient to make use of the following family of solutions in deriving comparison theorems for focal points. These results are a particular case of a conjoined family of solutions of dimension 2 as found in Reid [13, p. 306].

The number of linearly independent solutions of a self-adjoint linear differential equation of the fourth order which are mutually conjugate is at most 2 (see Reid [13, Thm. 2.1, p. 306]). A system of two linearly independent mutually conjugate solutions will be called a *conjugate base.* The set of all solutions linearly dependent on the solutions of a conjugate base will be called a *conjugate family.* The matrix

$$A = \begin{bmatrix} u & w \\ u' & w' \end{bmatrix}$$

will be called the matrix of that base. The nullity of two different bases of the same conjugate family is the same.

11. Comparison theorems for focal points. The following two theorems are used in finding a separation theorem for two nonconjugate solutions. They were originally adapted from Morse [12] but can now be considered as special cases of Reid [13, Thm. 7.9, particularly problem 2, p. 367, and Thm. 12.3].

THEOREM 11.1. *If two conjugate families F and G have σ linear independent solutions in common, then the number of focal points of F on any interval γ (open or closed) differs from the corresponding number for G by at most $2 - \sigma$.*

Consider two boundary value problems B_1 and B_2 with the same differential form $2\omega(y, y', y'', \lambda)$ in common. The end-plane in B_1 and B_2 will be denoted π_{t_1} and π_{t_2} respectively, where t_1 and t_2 are the dimensions of these end-planes. Let π_{t_1} and π_{t_2} be respectively represented by means of parameters (u) and (v). Let the corresponding end-form be

$$b_{hk}u_h u_k b^*_{pq} v_p v_q, \qquad (h, k = 1, \cdots, t_1), \quad (p, q = 1, \cdots, t_2).$$

If π_{t_2} is a section of π_{t_1}, and if

$$b^*_{pq} v_p v_q = b_{hk} u_h u_k$$

when (v) and (u) determine the same point of π_{t_2}, then B_2 will be called a *subproblem* of B_1. In particular, if the end-plane in B_2 is the null end-plane, B_2 is a subproblem of B_1.

THEOREM 11.2. *Let there be given a problem B_1 and a subproblem B_2 with end-planes π_{t_1} and π_{t_2} respectively. If λ_1 and λ_2 are respectively the number of characteristic roots of B_1 and B_2 less than a given constant λ, then*

$$\lambda_1 - (t_1 - t_2) \leqq \lambda_2 \leqq \lambda_1.$$

Theorem 11.2 and Theorem 3.3 can be applied to Problem B, § 2 and to a subproblem as follows.

COROLLARY 11.1. *Suppose π'_{t_1} is an end-plane at $x = a$ alone and π'_{t_2} is a section of π'_{t_1} and suppose that to each is adjoined the conditions $y(b) = y'(b) = 0$ to yield a Problem* B *and a subproblem. Then the number of focal points of π'_{t_1} on (a, b) exceeds the number of focal points of π'_{t_2} on (a, b) by at most $t_1 - t_2$.*

12. The separation theorem for nonconjugate solutions. We now have the necessary theorems to find the difference between the number of zeros of two nonconjugate solutions u and v. Our approach will be to find a solution w conjugate to the two given nonconjugate solutions. One then compares the number of focal points in the family (u, w) with the number of focal points in (v, w). With this information one can derive the difference in the number of zeros between u and v.

THEOREM 12.1. *Let u and v be two nonconjugate solutions of the self-adjoint linear differential equation of the fourth order. Then there exists a solution w linearly independent of and conjugate to both u and v such that if Γ is the sum of the indices of the conjugate points of $x = a$ in the open interval (a, b) and r is the cardinal number of focal points of index two on (a, b) in both families together, then the difference between the number of zeros of the two solutions in the interval (a, b) or $[a, b]$ is at most $2\Gamma + 5 - 2r$.*

THEOREM 12.2. *If u, v, w, Γ, and r are as described in Theorem* 12.1, *and if u and v are nonconjugate solutions listed under one of the cases in Table* 3, *then the maximum difference between the number of zeros of u and v on $[a, b]$ is listed under the corresponding case in Table* 4. *Table* 5 *lists similar properties on the interval (a, b).*

Proof. Consider the solutions u and v. Let $x = a$ be the first zero of either solution, and let u be the name given to the solution having the higher order zero at $x = a$. Seven situations can arise as shown in the first two columns of Table 3. To show that w can be picked as in column 3, Table 3, the following lemma is inserted.

TABLE 3
Number of Zeros at $x = a$

Case	u	v	w	$x < a$
(a)	1	0	1	u under w
(b)	1	0	1	w under u
(c)	1	1	3	w under u
(d)	2	0	2	u under w
(e)	2	0	2	w under u
(f)	2	1	3	w under u
(g)	3	0	2	u under w

LEMMA 12.1. *A nontrivial solution w can be found such that the pair (u, w) and (v, w) are conjugate pairs and w has the order zero at $x = a$ indicated in the third column of Table* 3.

Proof. The seven cases are considered separately, e.g., Case (a). Using the condition that two solutions be conjugate, it follows that at $x = a$,

$$u^{0T}Jw^0 = -(D_2u)w' + u'D_2w = 0,$$

$$v^{0T}Jw^0 = -(D_2v)w' + v'D_2w - vD_3w = 0.$$

Since there are two equations in three unknowns, this system always has nontrivial solutions for $w'(a)$, $D_2w(a)$, $D_3w(a)$. The lemma is proved.

In order to prove the main theorem, we shall consider each case separately. Furthermore, it will be more convenient as in § 7 to consider points and intervals on the x-axis having properties similar to the zeros and one solution being under another of the solutions.

According to Lemma 7.3 we can "pull apart" multiple zeros of u, v, w at $x = a$ leaving the u-point or w-point at $x = a$, and pulling the rest of the points to the right of $x = a$. Clearly w is under v in all cases, but the property of being under for u and w varies from case to case. See column 4, Table 3.

Now let $x = b$ be a point such that $b > a$ and b is not in the sets u, v, w, A and b is not a conjugate point of a. (A consists of focal points in either family.)

We shall justify the theorem for Case (a). Let Γ be the number of conjugate points of a on the open interval (a, b). Then according to Corollary 11.1, the conjugate pair of solutions (u, w) has (Γ or $\Gamma + 1$) focal points on (a, b) or ($\Gamma + 1$ or $\Gamma + 2$) on $[a, b)$.

By Theorem 11.1 the number of focal points of the conjugate family (u, w) differs from the number of focal points of the conjugate family (v, w) by at most one. But since at $x = a$ the family (u, w) has at least one focal point more than the family (v, w) except in Case (c), it follows that the number of focal points in family (v, w) is less than or equal to the number of focal points in family (u, w). This is also true for Case (c) by calling the solution u, that solution of the nonconjugate pair which makes the number of focal points of the family $(u, w) \geqq$ the number of focal points of the family (v, w). Thus the number of focal points of (v, w) on $[a, b)$ is [(Γ or $\Gamma + 1$)

or ($\Gamma + 1$ or $\Gamma + 2$)]. Mark the points of w. Then on $[a, b)$,

$$N(w) - 1 \leqq N(u) - 1 + \begin{pmatrix} \Gamma + 1 \\ \Gamma + 2 \end{pmatrix} - 2r_u, \tag{12.1}$$

where r_u is the cardinal number of focal points of the (u, w) family on (a, b).

The term -1 in the right-hand side of (12.1) is due to the fact that there is one point u to the left of the first w point. Now mark the points of u. Then on $[a, b)$,

$$N(u) - 1 \leqq N(w) + \begin{pmatrix} \Gamma + 1 \\ \Gamma + 2 \end{pmatrix} - 2 - 2r_u. \tag{12.2}$$

The term -2 on the right-hand side of (12.2) is due to the fact that there are three w- or A-points together with no u-point. Thus the number of w- or A-points between u-points can be reduced by 2. Combining results (12.1) and (12.2), we have on $[a, b)$,

$$-\begin{pmatrix} \Gamma + 1 \\ \Gamma + 2 \end{pmatrix} + 2r_u \leqq N(u) - N(w) \leqq \begin{pmatrix} \Gamma \\ \Gamma + 1 \end{pmatrix} - 2r_u. \tag{12.3}$$

Now consider the (v, w)-family. First mark the w-points. Then on $[a, b)$,

$$N(w) - 1 \leqq N(v) + \begin{pmatrix} \Gamma & \text{or} & \Gamma + 1 \\ \Gamma + 1 & \text{or} & \Gamma + 2 \end{pmatrix} - 2r_v. \tag{12.4}$$

Next mark the v-points, and we have on $[a, b)$,

$$N(v) - 1 \leqq (N(w) - 1) + \begin{pmatrix} \Gamma & \text{or} & \Gamma + 1 \\ \Gamma + 1 & \text{or} & \Gamma + 2 \end{pmatrix} - 2r_v. \tag{12.5}$$

The term -1 on the right-hand side is due to the fact there is one w-point to the left of the first v-point. Combining results (12.4) and (12.5), we have on $[a, b)$,

$$-\begin{pmatrix} \Gamma & \text{or} & \Gamma + 1 \\ \Gamma + 1 & \text{or} & \Gamma + 2 \end{pmatrix} + 2r_v \leqq N(w) - N(v) \leqq \begin{pmatrix} \Gamma + 1 & \text{or} & \Gamma + 2 \\ \Gamma + 2 & \text{or} & \Gamma + 3 \end{pmatrix} - 2r_v. \tag{12.6}$$

From adding (12.3) and (12.6) it follows that

$$-\begin{pmatrix} 2\Gamma + 1 & \text{or} & 2\Gamma + 2 \\ 2\Gamma + 3 & \text{or} & 2\Gamma + 4 \end{pmatrix} + 2r \leqq N(u) - N(v) \leqq \begin{pmatrix} 2\Gamma + 1 & \text{or} & 2\Gamma + 2 \\ 2\Gamma + 3 & \text{or} & 2\Gamma + 4 \end{pmatrix} - 2r.$$

Since $x = b$ is neither a point of u, v, w, A nor a conjugate point of $x = a$, the difference between solutions (conjugate or nonconjugate) on $[a, b]$ is the same as on $[a, b)$.

To calculate the difference between the number of zeros on the open interval (a, b), note that at the point $x = a$, there is a u-point but no v-point. Hence in the open interval (a, b), the number of u-points is one less. Letting $N^*(u)$, $N^*(v)$ be the number of u-points and v-points respectively in (a, b), it follows that

$$-\begin{pmatrix} 2\Gamma + 2 & \text{or} & 2\Gamma + 3 \\ 2\Gamma + 4 & \text{or} & 2\Gamma + 5 \end{pmatrix} + 2r \leqq N^*(u) - N^*(v) \leqq \begin{pmatrix} 2\Gamma & \text{or} & 2\Gamma + 1 \\ 2\Gamma + 2 & \text{or} & 2\Gamma + 3 \end{pmatrix} - 2r.$$

The count of zeros in the other six cases is made similarly.

Suppose one has two nonconjugate solutions of a self-adjoint linear differential equation of the fourth order. It can be observed that the properties of the zeros and focal points of the conjugate families (u, w) and (v, w) correspond to the properties of the sets u, v, w, A. In addition, the property of being "under" as applied to solutions agree with the notion of labeled intervals. Thus the results which were proved for point sets are also true for two nonconjugate solutions. Thus Theorem 12.2 is proved.

The tabulation of these results is listed in Table 4.

Analyzing these results one sees that the maximum difference between two nonconjugate solutions on (a, b) or $[a, b]$ is $2\Gamma + 5 - 2r$, the conclusion of Theorem 12.1. This is the best theorem one can get for two nonconjugate solutions of a fourth order differential equation by considering the equation

$$y^{\mathrm{iv}} + 10y'' + 9y = 0.$$

The question as to the necessity of the factor 2 in the term 2Γ is also answered by the above example. If the factor 2 is dropped, it can easily be shown that the theorem would be false. Therefore, the factor 2 of 2Γ is necessary.

TABLE 4

Maximum difference between number of zeros of two conjugate solutions on $[a, b]$

Case

$$\left.\begin{array}{ll}
(a) & -(2\Gamma + 4)\\
(b) & -(2\Gamma + 3)\\
(c) & -(2\Gamma + 3)\\
(d) & -2\Gamma\\
(e) & -(2\Gamma + 1)\\
(f) & -(2\Gamma + 1)\\
(g) & -2\Gamma
\end{array}\right\} + 2r \leqq N(u) - N(v) \leqq \left.\begin{array}{r}
2\Gamma + 4\\
2\Gamma + 5\\
2\Gamma + 3\\
2\Gamma + 4\\
2\Gamma + 3\\
2\Gamma + 3\\
2\Gamma + 4
\end{array}\right\} - 2r$$

TABLE 5

Maximum difference between number of zeros of two nonconjugate solutions on (a, b)

Case

$$\left.\begin{array}{ll}
(a) & -(2\Gamma + 5)\\
(b) & -(2\Gamma + 4)\\
(c) & -(2\Gamma + 3)\\
(d) & -(2\Gamma + 2)\\
(e) & -(2\Gamma + 3)\\
(f) & -(2\Gamma + 2)\\
(g) & -(2\Gamma + 3)
\end{array}\right\} + 2r \leqq N^*(u) - N^*(v) \leqq \left.\begin{array}{r}
2\Gamma + 3\\
2\Gamma + 4\\
2\Gamma + 3\\
2\Gamma + 2\\
2\Gamma + 1\\
2\Gamma + 2\\
2\Gamma + 1
\end{array}\right\} - 2r$$

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CONCERNING $\int_0^\infty e^{-at}J_\mu(bt)J_\nu(ct)t^{\mu-\nu}\,dt$*

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Abstract. A method is presented to reduce the integral in the title, for the case where $\mu > -\frac{1}{2}$, $\nu > -\frac{1}{2}$, 2μ and 2ν integers and the $\mathscr{R}(a \pm ib \pm ic) > 0$, to an integral of Jacobian elliptic functions. By this means results can be expressed in terms of the complete elliptic integrals K and E with modulus $k = 2\sqrt{bc}/\sqrt{a^2 + (b + c)^2}$. The summary at the end of the paper contains a table for the cases which have been worked out.

1. Introduction. The integral in the title will be referred to as $I_{\mu,\nu}$. This integral and a more general form, in which the power of t is unrestricted, occur in several places in the literature [1], [2], [3], [4]. It is the purpose of this paper to show that $I_{\mu,\nu}$ can be reduced to an integral of Jacobian elliptic functions and to obtain some results by this method.

2. Reduction of $I_{\mu,\nu}$. Since the Bessel function of the first kind is given by

$$J_K(vt) = \frac{(vt/2)^K}{\sqrt{\pi}\Gamma(K + \frac{1}{2})}\int_0^\pi e^{\pm ivt\cos\theta}\sin^{2K}\theta\,d\theta,$$

let the integral with $+$ sign be substituted in $I_{\mu,\nu}$ for $J_\mu(at)$ and that with a $-$ sign for $J_\nu(bt)$. If $\mu > -\frac{1}{2}$, then the order of integration can be changed, and we obtain

$$I_{\mu,\nu} = \frac{(a/2)^\mu(b/2)^\nu}{\pi\Gamma(\mu + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}\int_0^\pi \sin^{2\nu}\varphi \cdot\left\{\int_0^\pi \sin^{2\mu}\theta\left(\int_0^\infty \exp\left(-(p + ib\cos\varphi - ia\cos\theta)t\right)t^{2\mu}\,dt\right)d\theta\right\}d\varphi$$

provided that the real part $R(p \pm ia \pm ib) > 0$, a restriction needed to make the inner integral converge.

Let $A = p + ib\cos\varphi$; then

$$\int_0^\infty e^{-(A - ia\cos\theta)t}t^{2\mu}\,dt = \frac{\Gamma(2\mu + 1)}{(A - ia\cos\theta)^{2\mu+1}}$$

so

$$I_{\mu,\nu} = \frac{(a/2)^\mu(b/a)^\nu\Gamma(2\mu + 1)}{\pi\Gamma(\mu + \frac{1}{2})\Gamma(\nu + \frac{1}{2})}\int_0^\pi \sin^{2\nu}\varphi\left(\int_0^\pi \frac{\sin^{2\mu}\theta\,d\theta}{(A - ia\cos\theta)^{2\mu+1}}\right)d\varphi.$$

Denote the inner integral by J and let $\theta = 2\psi$. Then

$$J = \int_0^{\pi/2} \frac{2^{2\mu+1}}{(A - ia)^{2\mu+1}}\frac{\sin^{2\mu}\psi\cos^{2\mu}\psi\,d\psi}{(1 - k\sin^2\psi)^{2\mu+1}},$$

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where $k = -2ia/(A - ia)$. If $\sin^2 \psi = x$, then

$$J = \frac{2^{2\mu}}{(A - ia)^{2\mu+1}} \int_0^1 x^{\mu-1/2}(1 - x)^{\mu-1/2}(1 - kx)^{-2\mu-1}\,dx$$

$$= \frac{2^{2\mu}}{(A - ia)^{2\mu+1}}\,{}_2F_1(2\mu + 1, \mu + \tfrac{1}{2}; 2\mu + 1; k)\frac{[\Gamma(\mu + \frac{1}{2})]^2}{\Gamma(2\mu + 1)}$$

on using [7, p. 243] with $a = 2\mu + 1, b = \mu + \frac{1}{2}, c = 2\mu + 1$. Hence

$$J = \frac{2^{2\mu}[\Gamma(\mu + \frac{1}{2})]^2}{(A - ia)^{2\mu+1}\Gamma(2\mu + 1)}(1 - k)^{-\mu-1/2}.$$

But $1 - k = (A + ia)/(A - ia)$ so

$$J = \frac{2^{2\mu}[\Gamma(\mu + \frac{1}{2})]^2}{\Gamma(2\mu + 1)} \cdot \frac{1}{(A^2 + a^2)^{\mu+1/2}}.$$

Now

$$2^{2\mu}\Gamma(\mu + \tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2\mu + 1)}{\Gamma(\mu + 1)},$$

so that

$$(2.1)\quad I_{\mu,\nu} = \frac{(a/2)^\mu (b/2)^\nu \Gamma(2\mu + 1)}{\sqrt{\pi}\,\Gamma(\nu + \frac{1}{2})\Gamma(\mu + 1)} \int_0^\pi \frac{\sin^{2\nu}\varphi\,d\varphi}{(p^2 + 2ipb\cos\varphi - b^2\cos^2\varphi + a^2)^{\mu+1/2}}.$$

Both Gegenbauer (Wiener Sitzungsberichte LXXXVIII (2), pp. 975–1003, formula on p. 995) and Watson [6, p. 390] omit the factor $\Gamma(\mu + 1)$ in the denominator which appears in the above work.

If G denotes the integral in (2.1) and $x = \cos\varphi$, then

$$G = \int_{-1}^1 \frac{(1 - x^2)^\nu}{(p^2 + a^2 + 2ipbx - b^2x^2)^\mu} \cdot \frac{dx}{\sqrt{(1 - x^2)(p^2 + a^2 + 2ipbx - b^2x^2)}}.$$

This integral can be reduced to an elliptic integral of standard type (see [7, 22.7–22.735]. Let $f_1 = ((p^2 + a^2)/b^2) + 2i(p/b)x - x^2$ and $f_2 = 1 - x^2$. Also let

$$P = \sqrt{p^2 + (a + b)^2}, \quad Q = \sqrt{p^2 + (a - b)^2}, \quad \alpha = \frac{2pb}{b^2 - p^2 - a^2 + PQ},$$

$$\beta = \frac{2pb}{b^2 - p^2 - a^2 - PQ}, \quad k = \frac{P - Q}{P + Q}$$

and

$$u = (p^2 + a^2 - b^2 + PQ)^{1/2}(p^2 + a^2 - b^2 - PQ)^{-1/2} \cdot \frac{x - i\beta}{x - i\alpha}.$$

This is the transformation which reduces the integral to the type form. The coefficient of the fraction will be denoted by L. Now

$$\pm\left(\frac{p^2 + a^2 - b^2 + PQ}{p^2 + a^2 - b^2 - PQ}\right)^{1/2} = L = \pm\sqrt{\frac{\alpha}{\beta}}$$

but

$$\alpha\beta = \frac{4p^2b^2}{(p^2+a^2-b^2)^2 - p^2Q^2} = \frac{4p^2b^2}{(p^2+a^2-b^2)^2 - (p^2+a^2+b^2)^2 + 4a^2b^2} = -1,$$

hence $L = \pm\sqrt{-\alpha^2} = e^{\pm i\pi/2}\alpha$. If $z = x + iy$, then the integral is along the real axis in the z-plane. The transformation

$$w = e^{-i\pi/2}\alpha \cdot \frac{z + i/\alpha}{z - i\alpha} \tag{2.2}$$

changes the line $y = 0$ from $x = -1$ to $x = 1$ into an arc C of the circle $u^2 + (v+\alpha)^2 - (\alpha + 1/\alpha)(v+\alpha) = 0$ from $w = -1$ to $w = 1$ lying above the real axis ($w = i/\alpha$ is on the arc). So $L = e^{-i\pi/2}\alpha$. From (2.2) it follows that

$$(z - e^{i\pi/2}\alpha)(w - e^{-i\pi/2}\alpha) = 1 + \alpha^2 = 1 - L^2 = -\frac{PQL}{ipb} = e^{i\pi/2}\frac{PQL}{pb}$$

or $(z - i\alpha)^2 = (e^{i\pi}P^2Q^2L^2/p^2b^2(w-L)^2)$, which is needed to obtain

$$f_1(z) = \frac{e^{i\pi}PQ(P+Q)^2(p^2+a^2-b^2+PQ)}{8p^2b^2}\cdot\frac{1-k^2w^2}{(w-L)^2},$$

$$f_2(z) = \frac{e^{i\pi}PQ(p^2+a^2-b^2+PQ)}{2p^2b^2}\cdot\frac{1-w^2}{(w-L)^2},$$

$$\frac{dz}{\sqrt{f_1f_2}} = \frac{2b}{P+Q}\,\frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}},$$

$$G = \int_{-1}^{1}\frac{f_2^\nu}{b^{2\mu+1}f_1^\mu}\cdot\frac{dx}{\sqrt{f_1f_2}} = \int_C \frac{f_2^\mu}{b^{2\mu+1}f_1^\mu}\cdot\frac{2b}{(P+Q)}\cdot\frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}$$

$$G = \int_C \frac{2^{3\mu-\nu+1}(p^2b^2)^{\mu-\nu}\,e^{-i\pi(\mu-\nu)}}{(P+Q)^{2\mu+1}[PQ(p^2+a^2-b^2+PQ)]^{\mu-\nu}} \cdot\frac{(1-w^2)^\nu(w-L)^{2\mu-2\nu}\,dw}{(1-k^2w^2)^\mu\sqrt{(1-w^2)(1-k^2w^2)}}.$$

Since $\nu > -\frac{1}{2}$, an integral $\int A(w)\,dw/(1-w^2)^{-\nu+1/2}$, in which $A(w)$ is analytic at $w = \pm 1$ and the path is a circular arc of decreasing radius about $w = \pm 1$, must vanish. Moreover, the only possible singularities of the integrand of G are at ± 1, $\pm 1/k$, L, hence there are no singularities between the u-axis and C. Therefore by Cauchy's theorem,

$$G = \frac{2^{3\mu-\nu+1}(p^2b^2)^{\mu-\nu}\,e^{-i\pi(\mu-\nu)}}{(P+Q)^{2\mu+1}[PQ(p^2+a^2-b^2+PQ)]^{\mu-\nu}} \cdot\int_{-1}^{1}\frac{(1-u^2)^\nu(u-L)^{2\mu-2\nu}\,du}{(1-k^2u^2)^\mu\sqrt{(1-u^2)(1-k^2u^2)}}.$$

Finally let $u = sn(s, k)$, so that

$$G = \frac{2^{3\mu-\nu+1}(p^2b^2)^{\mu-\nu}\, e^{-i\pi(\mu-\nu)}}{(P+Q)^{2\mu+1}[PQ(p^2+a^2-b^2+PQ)]^{\mu-\nu}} \int_{-K}^{K} \frac{cn^{2\nu}s(sns - L)^{2\mu-2\nu}\, ds}{dn^{2\mu}s}.$$

Thus the result obtained is given by

$$\begin{aligned} I_{\mu,\nu} &= \frac{2^{2\mu-2\nu+1}\Gamma(2\mu+1)p^{2\mu-2\nu}a^{\mu}b^{2\mu-\nu}\, e^{-i\pi(\mu-\nu)}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})\Gamma(\mu+1)(P+Q)^{2\mu+1}[PQ(p^2+a^2-b^2+PQ)]^{\mu-\nu}} \\ &\quad \cdot \int_{-K}^{K} \frac{cn^{2\nu}S(sns-L)^{2\mu-2\nu}\, ds}{dn^{2\mu}s}, \end{aligned} \tag{2.3}$$

where $k = (P - Q)/(P + Q)$ is the modulus of the elliptic functions and K is the real quarter period. Also

$$L = -i\alpha = \frac{2ipb}{p^2 + a^2 - b^2 - PQ}.$$

The more simple cases of $I_{0,0}$, $I_{1,0}$, and $I_{1,1}$, which occur in the first four references, were worked out by this method and found to agree with our results.

3. Special cases of (2.3). If μ and ν are integers, then the integral in (2.3) can be reduced to a sum of integrals of powers of a Jacobian elliptic function and these can be evaluated by the reduction formula given in Neville [5, 14.203].

Since $k = (P - Q)/(P + Q)$ involves p in a complicated way, it is convenient to make a Landen transformation changing k to $\bar{k} = \sqrt{P^2 - Q^2}/P = 2(ab)^{1/2} \cdot [p^2 + (a+b)^2]^{-1/2}$. If K, E are the complete integrals with modulus $\bar{k}$, then

$$K = \frac{P+Q}{2P}\bar{K}, \qquad E = \frac{P\bar{E} + Q\bar{K}}{P+Q}.\text{[1]}$$

After some tedious algebraic reduction the answers in the tabld were obtained. If $\mu = m + \frac{1}{2}$ or $\nu = n + \frac{1}{2}$ or both, where m, n are integers, then the integrand of

$$\int_{-1}^{1} \frac{(1-u^2)^{\nu}(u-L)^{2\mu-2\nu}}{(1-k^2u^2)^{\mu}} \cdot \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}$$

will contain only $\sqrt{1-u^2}$ or $\sqrt{1-k^2u^2}$ or be a rational function, and the elliptic functions are no longer needed.

The results for $I_{0,1/2}$, $I_{1/2,0}$, $I_{1/2,1}$, $I_{1,1/2}$ which are in Table 1 were obtained by this method.

4. Acknowledgment. The author wishes to express his sincere thanks to the referee, who furnished valuable references and who corrected several errors in the original work.

[1] These do not agree with Neville [5, 17.626] but the 4 in the first fraction there should be a 2. The difficulty seems to arise from the identification of

$$\int_0^{u_{n+1}} dn^2(K_{n+1} - u_{n+1})\, du_{n+1} \quad \text{with } Dn(K_{n+1} - u_{n+1}).$$

TABLE 1

Values of $I_{\mu,\nu} = \int_0^\infty e^{-pt}J_\mu(at)J_\nu(bt)t^{\mu-\nu}\,dt$

$$P = \sqrt{p^2+(a+b)^2}, \qquad Q = \sqrt{p^2+(a-b)^2}, \qquad A^2 = p^2+a^2+b^2,$$

$$\bar k = \frac{\sqrt{P^2-Q^2}}{P} = \frac{2\sqrt{ab}}{\sqrt{p^2+(a+b)^2}},$$ $\bar K, \bar E$ are the complete elliptic integrals with modulus $\bar k$.

$\mu = 0$	$\nu = 0$	$I_{0,0} = \dfrac{2\bar K}{\pi P}$
	$\nu = \frac{1}{2}$	$I_{0,1/2} = \sqrt{2}\,(\pi b)^{-1/2}\arcsin[2b/(P+Q)]$
$\mu = \frac{1}{2}$	$\nu = 0$	$I_{1/2,0} = \sqrt{2a/\pi}\,[PQ(P+Q)]^{-1}\sqrt{(P+Q)^2-4b^2}$
	$\nu = 1$	$I_{1/2,1} = \sqrt{2a/\pi}\,b^{-1}(P+Q)^{-1}[P+Q-\sqrt{(P+Q)^2-4b^2}]$
$\mu = 1$	$\nu = 0$	$I_{1,0} = (\pi a P Q^2)^{-1}[Q^2\bar K - (p^2-a^2+b^2)\bar E]$
	$\nu = \frac{1}{2}$	$I_{1,1/2} = \{\sqrt{\pi}\,PQ(P+Q)^2\}^{-1}\cdot 2a\sqrt{2b[(P+Q)^2-4b^2]}$
	$\nu = 1$	$I_{1,1} = [\pi abP]^{-1}\cdot\{A^2\bar K - P^2\bar E\}$
$\mu = 2$	$\nu = 0$	$I_{2,0} = (2/\pi)(aP^2Q^2)^{-2}\{(p^2a^2+P^2Q^2)PQ^2\bar K - [P^2Q^2(A^2-3a^2)+2p^2a^2(P^2+Q^2)]P\bar E\}$
	$\nu = 1$	$I_{2,1} = (\pi a^2bP)^{-1}(2A^2-a^2)\bar K - (\pi a^2bPQ^2)^{-1}[2A^4-a^2(A^2+6b^2)]\bar E$
	$\nu = 2$	$I_{2,2} = 2(3\pi a^2b^2P)^{-1}[(A^4-a^2b^2)\bar K - A^2P^2\bar E]$
$\mu = 3$	$\nu = 3$	$I_{3,3} = (15\pi a^3b^3P)^{-1}\{A^2(8A^4-17a^2b^2)\bar K - (8A^4-9a^2b^2)P^2\bar E\}$

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ASYMPTOTIC EVALUATION OF FRACTIONAL INTEGRAL OPERATORS WITH APPLICATIONS*

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Abstract. A technique is developed which yields an asymptotic expansion in the two limits $\lambda \to 0^+$ and $\lambda \to \infty$ for the fractional integral operator of order μ with respect to the function λ^p given by

$$I^{\mu}_{\lambda^p} f(\lambda) = \frac{1}{\Gamma(\mu)} \int_0^{\lambda} (\lambda^p - \xi^p)^{\mu-1} p\xi^{p-1} f(\xi)\, d\xi,$$

under general conditions that f be algebraically dominated near 0 and ∞. Representing $I^{\mu}_{\lambda^p} f(\lambda)$ as a convolution of Mellin transforms, the domain of the transform is extended by analytic continuation. By moving the contour of integration to the right or the left an asymptotic expansion for $I^{\mu}_{\lambda^p} f(\lambda)$ can be systematically generated for $\lambda \to \infty$ or $\lambda \to 0^+$. The technique is illustrated by the asymptotic expansion of fractional integral operators derived from the Euler–Poisson–Darboux equation and generalized axially symmetric potential theory.

1. Asymptotic expansions of fractional integrals. In a series of recent papers [1], [2] and [3], Handelsman and Lew have developed a theory which yields asymptotic expansions of integrals of the form

$$I(\lambda) = \int_0^{\infty} g(t) f(\lambda t)\, dt \tag{1.1}$$

in the limits $\lambda \to \infty$ and $\lambda \to 0^+$. In the present paper we apply that theory to investigate the asymptotic behavior, as $\lambda \to \infty$, of the fractional integrals

$$I^{\mu}_{\lambda^p} f(\lambda) = \frac{1}{\Gamma(\mu)} \int_0^{\lambda} (\lambda^p - \xi^p)^{\mu-1} p\xi^{p-1} f(\xi)\, d\xi. \tag{1.2}$$

Here, $f(\xi)$ is a given locally integrable function, $\operatorname{Re}(\mu) > 0$ and $p > 0$.

If, in (1.2) we set $\xi = \lambda t$, then we obtain

$$I^{\mu}_{\lambda^p} f(\lambda) = \frac{p\lambda^{p\mu}}{\Gamma(\mu)} \int_0^1 (1 - t^p)^{\mu-1} t^{p-1} f(\lambda t)\, dt. \tag{1.3}$$

This integral is of the form (1.1) with

$$g = \begin{cases} \dfrac{p\lambda^{p\mu}}{\Gamma(\mu)}(1 - t^p)^{\mu-1} t^{p-1}, & 0 \leqq t \leqq 1, \\ 0, & 1 < t < \infty. \end{cases}$$

Hence, we can apply the method of [3].

We first note that under suitable conditions on f, (see Titchmarsh [4]), the Parseval relation for Mellin transforms,

$$I^{\mu}_{\lambda^p} f(\lambda) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z} M[f; z] M[g; 1 - z]\, dz,$$

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yields from (1.3),

$$(1.4) \qquad I^{\mu}_{\lambda p} f(\lambda) = \frac{\lambda^{p\mu}}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda^{-z} M[f; z] \frac{\Gamma(1 - z/p)}{\Gamma(\mu + 1 - (z/p))} dz.$$

Here

$$(1.5) \qquad M[g; z] = \lambda^{p\mu} \frac{\Gamma(1 + (z - 1)/p)}{\Gamma(\mu + 1 + ((z - 1)/p))}$$

represents the Mellin transform of $g(t)$ evaluated at z, and the contour of integration is a Bromwich contour in the common domain of analyticity of $M[f; z]$ (the Mellin transform of $f(t)$) and $M[g; 1 - z]$.[1]

We now suppose that, as $t \to \infty$,

$$(1.6) \qquad f \sim e^{-\alpha t} \sum_{m=0}^{\infty} d_m t^{-r_m}$$

with $\alpha \geqq 0$ and $\operatorname{Re}(r_m) \uparrow \infty$ as $m \to \infty$. If in (1.6) we have $\alpha > 0$, then $M[f; z]$ is analytic in $\operatorname{Re}(z) > x_0$ for some x_0. We also have from (1.5) that the analytic continuation of $M[g; 1 - z]$ into the right half-plane $\operatorname{Re}(z) \geqq p(n + 1)$ has simple poles at the points $z = p(n + 1)$, $n = 0, 1, 2, \cdots$. Moreover, $M[g; 1 - z]$ has Laurent expansions about these points with singular parts

$$(1.7) \qquad \frac{(-1)^{n+1} \lambda^{p\mu}}{n! \Gamma(\mu - n)(z - p(n + 1))}.$$

Upon displacing the contour in (1.4) arbitrarily far to the right and computing the appropriate residues, we formally obtain the asymptotic expansion

$$(1.8) \qquad I^{\mu}_{\lambda p} f(\lambda) \sim \sum_{n=0}^{\infty} \frac{M[f; p(n + 1)]}{n! \Gamma(\mu - n)} (-1)^n \lambda^{-p(n-\mu+1)}.$$

If in (1.6), $\alpha = 0$, then, as is shown in Handelsman and Lew [3], $M[f; z]$ can be analytically continued into the right half-plane $\operatorname{Re}(z) \geqq r_0$ as a meromorphic function with simple poles at the points $z = r_m$ and Laurent expansions about these points having singular parts

$$d_m/(z - r_m).$$

We first suppose that $r_m \neq p(n + 1)$ for any nonnegative integers m, n. Then the poles of $M[g; 1 - z]$ and $M[f; z]$ are disjoint and the residue calculation yields

$$(1.9) \qquad \begin{aligned} I^{\mu}_{\lambda p} f(\lambda) \sim & \sum_{n=0}^{\infty} \frac{M[f; p(n + 1)]}{n! \Gamma(\mu - n)} (-1)^n \lambda^{p(\mu-n-1)} \\ & - \sum_{m=0}^{\infty} d_m \frac{\Gamma(1 - (r_m/p)) \lambda^{p\mu - r_m}}{\Gamma(\mu + 1 - (r_m/p))}. \end{aligned}$$

Finally, if $\alpha = 0$ in (1.7) and $r_m = p(n + 1)$ for at least one pair of nonnegative integers n, m, then poles of $M[g; 1 - z]$ and $M[f; z]$ coincide producing logarithmic

[1] For simplicity, we shall assume that $M[f; z]$ exists in the ordinary sense.

terms in the expansion. Indeed, suppose for definiteness we assume that $r_0 = p(l + 1)$ for some nonnegative integer l. The residue calculation then yields

$$I^{\mu}_{\lambda p} f(\lambda) = \sum_{n=0}^{l-1} \frac{(-1)^n M[f; p(n+1)]}{n!\Gamma(\mu - n)} \lambda^{p(\mu-n-1)} + \frac{(-1)^l d_0 \lambda^{p(\mu-l-1)} \ln \lambda}{l!\Gamma(\mu - l)} + O(\lambda^{p(\mu-l-1)}). \tag{1.10}$$

The case $\lambda \to 0^+$ is much simpler to treat. This is because the desired expansion is obtained by displacing the contour in (1.4) to the left and the Mellin transform

$$\frac{\Gamma(1 - (z/p))}{\Gamma(\mu + 1 - (z/p))}$$

is analytic in the left half-plane $\mathrm{Re}(z) < p$. Thus to calculate the residue series, we need only analytically continue $M[f; z]$ into the left half-plane $\mathrm{Re}(z) < p$ as a meromorphic function.

We again apply the theory of Handelsman and Lew [3] and find that if, as $t \to 0^+$,

$$f \sim \sum_{m=0}^{\infty} b_m t^{a_m}, \qquad \mathrm{Re}(a_m) \uparrow \infty, \tag{1.11}$$

then $M[f; z]$ has simple poles at the points $z = -a_m$ with corresponding Laurent expansions about these poles having singular parts

$$b_m/(z + a_m). \tag{1.12}$$

It then follows from (1.4) and (1.12) and the assumption that we are justified in displacing the Bromwich contour in (1.4) arbitrarily far to the left, that hence

$$I^{\mu}_{\lambda p}(f) \sim \sum_{m=0}^{\infty} \lambda^{a_m + p\mu} b_m \frac{\Gamma(1 + (a_m/p))}{\Gamma(1 + \mu + (a_m/p))}, \qquad \lambda \to 0^+. \tag{1.13}$$

To rigorously establish the validity of the above asymptotic expansions, we need only justify the displacement of the contour of integration in (1.4) arbitrarily far to the right to generate expansions such as (1.8), (1.9), (1.10) or arbitrarily far to the left to generate expressions such as (1.13).

From (1.5) we find that

$$M[g; 1 - x - iy] = O(|y|^{-\mu}),. \qquad |y| \to \infty.$$

Thus, to justify the displacement, f must be such that

$$M[f, x + iy] = O(|y|^{\mu - 1 - \varepsilon})$$

for some $\varepsilon > 0$ and for either $x > c$ or $x < c$ depending on whether expansions for $\lambda \to \infty$ or $\lambda \to 0^+$ are to be generated. To insure this behavior we suppose that (1.6) and (1.11) hold. One finds that if for $x > -\mathrm{Re}(a_0)$, $(t\, d/dt)^n(t^x f)$ is in $L_1(0, \infty)$ and vanishes at 0 and ∞ for $n = 1, 2, \cdots, p$, then

$$M[f; x + iy] = O(|y|^{-p}), \qquad |y| \to \infty,$$

for $x > -\mathrm{Re}(a_0)$. If in addition the asymptotic expansion of $(d^m/dt^m)f(t)$ for $m = 0, \cdots, p$, as $t \to 0^+$ is obtained by successively differentiating (1.11) term by term, then

$$M[f; z] = O(|y|^{-p}), \qquad |y| \to \infty, \tag{1.14}$$

for *all* x. Here by $M[f; z]$ we understand the analytic continuation of the Mellin transform into the entire z-plane (see Bleistein and Handelsman [5]).

2. Application to the Euler–Poisson–Darboux equation. We consider the linear differential operator

$$L_\nu[f(r)] = \frac{\partial^2}{\partial r^2} f(r) + \frac{2\nu + 1}{r} \frac{\partial}{\partial r} f(r). \tag{2.1}$$

It is shown in Erdélyi [6] that if $\alpha > 0$, $f \in C^2[0, \infty)$, $r^{2\nu+1}f(r)$ is integrable at 0 and $r^{2\nu+1}f'(r) \to 0$ as $r \to 0$, then

$$I_{r^2}^{\nu;\alpha}(L_\nu[f(r)]) = L_{\nu+\alpha}[I_{r^2}^{\nu;\alpha} f(r)], \tag{2.2}$$

where

$$I_{r^2}^{\nu;\alpha} f = r^{-2(\nu+\alpha)} I_{r^2}^{\alpha}(r^{2\nu} f(r)).$$

The Euler–Poisson–Darboux equation for complex k, $\vec{x} \in \mathbb{R}^m$,

$$\begin{aligned} &\vec{\nabla} \circ \vec{\nabla} u = u_{tt} + \frac{k}{t} u_t = L_{(k-1)/2}[u], \qquad t > 0, \\ &u(\vec{x}, 0) = f_k(\vec{x}) \in C^2, \qquad u_t(\vec{x}, 0) = 0 \end{aligned} \tag{2.3}$$

was recently studied by Bresters [7] where a brief review of the literature can be found. We shall use the techniques of fractional integration to derive a solution for real values of $k > m - 1$, extend this representation to all values of k (excepting the negative odd integers) and use the result to give an asymptotic representation of the solution.

It is well known that the solution to problem (2.3) for $k = m - 1$ is given by

$$u(\vec{x}, t) = \tilde{M}(\vec{x}, t; f_{m-1}) = \frac{1}{w_m} \int_{|\beta|=1} f_{m-1}(\vec{x} + \vec{\beta} t)\, dw_m,$$

where $\tilde{M}$ is the spherical mean of $f(\vec{x})$ and w_m is the unit sphere surface area in m-dimensional space. Using (2.2) we see that for real k, $k > m - 1$,

$$I_{t^2}^{(m-2)/2,(k-m+1)/2} \tilde{M}(\vec{x}, t; f_{m-1}) \tag{2.4}$$

is a solution of (2.3) with initial data

$$I_{t^2}^{(m-2)/2,(k-m+1)/2}[f_{m-1}(\vec{x})] = \frac{\Gamma(m/2)}{\Gamma((k+1)/2)} f_{m-1}(\vec{x}).$$

Consequently, letting

$$f_{m-1}(\vec{x}) = \left(\Gamma\left(\frac{k+1}{2}\right) \Big/ \Gamma\left(\frac{m}{2}\right)\right) f_k(\vec{x}), \tag{2.5}$$

we have constructed a solution given by (2.4) and (2.5), which is known to be unique and is given by an absolutely convergent integral of the form (1.2) whenever $k > m - 1$. We note that the problem (2.3) is analytic in k and that our solution written as a convolution integral of the form (1.4),

$$u(\vec{x}, t) = \frac{\Gamma((k+1)/2)}{\Gamma(m/2)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma((m-z)/2) M[\tilde{M}(\vec{x}, t; f_k); z] t^{-z}\, dz}{\Gamma((1-z+k)/2)}, \tag{2.6}$$

is convergent, twice differentiable and analytic in k for $\operatorname{Re} k > m - 1$ and $0 < c < 1$.

In order to extend the representation (2.6) to other values of k we must impose growth conditions on $M[\tilde{M}(\vec{x}, t; f_k); 1 - x - iy]$ for $|y| \to \infty$, which will permit us to calculate two derivatives with respect to x and t of (2.6) and still maintain the convergence of the integral and thus the analytic dependence on k.

We assume that as $t \to 0^+$, $\tilde{M}$ has an asymptotic expansion. From Courant–Hilbert [8, p. 289] for $f(\vec{x}) \in C^{2r}$,

$$\tilde{M}(\vec{x}, t; f) - \Gamma\left(\frac{m}{2}\right) \sum_{n=0}^{r} \left(\frac{t}{2}\right)^{2n} \frac{(\vec{\nabla} \circ \vec{\nabla})^n f(\vec{x})}{n!\Gamma(n + m/2)} \sim \sum_{n=r+1}^{\infty} b_n(\vec{x}) t^{2n}. \tag{2.7}$$

We assume that the series can be differentiated termwise to yield an asymptotic representation for the derivatives of M with respect to t. Since

$$\frac{\Gamma((m - x - iy)/2)}{\Gamma((1 + k - x - iy)/2)} = O(|y|^{m - \operatorname{Re}(k+1)/2}),$$

we find from an argument paralleling that leading to (1.14) that if $f(\vec{x}) \in C^{r+3}(\vec{x})$ and has compact support, the representation of u and its second order derivatives given by (2.6) can be continued as analytic functions of k for $\operatorname{Re} k > m - 1 - (2r + 2)$ and all x. Consequently (2.6) furnishes a solution for $\operatorname{Re} k \neq -(2n + 1)$, $n = 0, 1, 2, \cdots$, provided $f(\vec{x})$ is sufficiently well-behaved.

For $t \to 0^+$ we have an immediate asymptotic representation for $u(x, t)$ using (2.2) and (1.13) which after some algebra reduces to

$$u(\vec{x}, t) = \Gamma\left(\frac{k+1}{2}\right) \sum_{n=0}^{q} \frac{b_n(\vec{x})\Gamma(n + m/2)}{\Gamma(((k+1)/2) + n)} t^{2n} + O(t^{2q+2}).$$

If $f(\vec{x}) \in C^{r+3}$, we have by (2.7) with $q = [\![r + 3/2]\!]$,

$$u(\vec{x}, t) = \Gamma\left(\frac{k+1}{2}\right) \sum_{n=0}^{q} \frac{1}{2^{2n} n!} \frac{(\vec{\nabla} \circ \vec{\nabla}) n f(\vec{x})}{\Gamma(((k+1)/2) + n)} t^{2n} + O(t^{2q+2}).$$

It is well known that for $k < 0$ the solution of (2.3) is not unique (see Bresters [7]), and that for $k = -1, -3, \cdots$ the solution (2.6) is singular, although it is still possible to recover an asymptotic expansion in this case by following a technique of Diaz and Weinberger [9]. These solutions were completely characterized by Blum [10].

We consider now expansions for $t \to \infty$. We assume that $f(\vec{x})$ has sufficient derivatives for the integral to converge for k of interest, and is of compact support. Consequently we have $\tilde{M}(\vec{x}, t; f_k) \sim 0$, and by (1.8),

$$u(\vec{x}, t) \sim \frac{\Gamma((k+1)/2)}{\Gamma(m/2)} \sum_{n=0}^{\infty} \frac{M[\tilde{M}(\vec{x}, t; f_k); 2n + m]}{n!\Gamma(((k - m + 1)/2) - n)} t^{-(2n+m)}.$$

It is notable that the asymptotic expansion which is valid only for $t \to +\infty$ may have odd powers of t, even though the solution is known to be even in t. Symmetry properties and solutions for negative t have been dealt with by Diaz and Ludford [11]. For $k \geqq 0$ the solution is known to be unique and we find for $k = m - 1, m - 3, m - 5, \cdots$ that $u(\bar{x}, t) \sim 0$, indicating the operation of a Huygen's principle (see Diaz and Weinberger [9]). For the particular case $m = 1$ and $k > 0$, if we assume that

$$f_k(x) \sim \sum_{n=0}^{\infty} a_n x^{-n},$$

then

$$\tilde{M}(x, t; f_k) = \tfrac{1}{2}[f_k(x - t) + f_k(x + t)] \sim a_0 + \sum_{m=0}^{\infty} d_m t^{-2m-2},$$

where

$$d_m(x) = \sum_{i=0}^{2m+1} \binom{2m+1}{i} a_{2m+2-i}(-x)^i$$

is a polynomial of degree $2m + 1$ in x.

Consequently, by (1.9) we have

$$u(x, t) \sim \frac{\Gamma((k+1)/2)}{\Gamma(\frac{1}{2})} \sum_{n=0}^{\infty} \frac{M[\frac{1}{2}(f_k(x+t) + f_k(x-t)); 2n+1](-1)^n}{n! t^{2n+1} \Gamma((k/2) - n)} + \sum_{n=0}^{\infty} \frac{d_n(x)\Gamma(\frac{1}{2} - n)}{t^{2n}\Gamma((k/2) + \frac{1}{2} - n)}.$$

3. Applications to axially symmetric generalized potentials.

(a) We consider for L_ν defined by (2.1) the equation

$$L_\nu u(r, y) + \frac{\partial^2 u(r, y)}{\partial y^2} = 0 \tag{3.1}$$

satisfied by a $2\nu + 3$-dimensional axially symmetric potential. This equation has many applications (see, for example, Weinstein [12]). In particular for $\nu = -\frac{1}{3}$ this elliptic equation is related to the Tricomi equations. For $\nu > 1$ it is known that solutions are analytic functions for $r \neq 0$ and are specified uniquely by the conditions $u(0^+, y) = g(y)$ and $(1/r)(\partial u/\partial r)$ bounded as $r \to 0^+$. For $\nu = -\frac{1}{2}$ the additional condition $(\partial u/\partial r)(0, y) = 0$ is necessary.

Erdélyi [6], [13] has shown that if $h(r, y)$ is an even harmonic function of r, then for $\nu > -\frac{1}{2}$,

$$u(r, y) = I_{r^2}^{-1/2, \nu + 1/2} h(r, y)$$

is a solution to (3.1) with boundary data $\Gamma(\frac{1}{2})h(0, y)/\Gamma(\nu + 1)$. The solution of boundary value problem (3.1) with the boundary condition $u(0, y) = g(y)$, an analytic function of y, is thus given by

$$u(r, y) = I_{r^2}^{-1/2, \nu + 1/2} \left\{ \frac{\Gamma(\nu + 1)}{\Gamma(\frac{1}{2})} \operatorname{Re}(g(y + ir)) \right\}, \tag{3.2}$$

which permits us to generate an asymptotic expansion.

It is known (Erdélyi [14]) that the singularities of the generalized axially symmetric potential u are dependent upon those of the harmonic potential h. The regions of regularity of u and h coincide, and will be determined by the analytic continuation of g. The nature of the singularity, however, depends on ν. We will assume that as $r \to \infty$,

$$\operatorname{Re}(g(y + ir)) \sim \sum_{m=0}^{\infty} d_m(y) r^{-2m};$$

consequently by (3.2), (1.8) and (2.2) we have

$$u(r, y) \sim \sum_{n=0}^{\infty} \frac{M[\operatorname{Re}(g(y + ir)); 2n + 1](-1)^n}{r^{2n+1} n! \Gamma(\nu - n + \frac{1}{2})} + \frac{d_n(y)\Gamma(\frac{1}{2} - n)}{r^{2n}\Gamma(\nu - n + 1)}.$$

Our results are not uniformly valid in y, but that is to be expected.

(b) Using Poisson's integral formula for symmetric harmonic potentials and fractional integral operators we can express axial values of $u(0, y)$ in terms of radial values $u(r, 0)$. Following Erdélyi [6], we have for bounded symmetric potentials continuous in the half-space $0 \leqq y$, $0 \leqq r$, for $\nu > -1$,

$$u(0, y) = \frac{\Gamma(\nu + \frac{3}{2})}{\Gamma(\frac{1}{2})\Gamma(\nu + 1)} \int_0^{\infty} (1 + t^2)^{-\nu - 3/2} t^{2\nu + 1} u(ty, 0)\, dt. \tag{3.3}$$

Although (3.3) is not of the form (1.2) the techniques of our method still apply. We assume that as $r \to \infty$,

$$u(r, 0) \sim \sum_{m=0}^{\infty} \gamma_m r^{-a_m}, \qquad \operatorname{Re}(a_m) \uparrow \infty, \quad \operatorname{Re} a_0 > -1,$$

and as $r \to 0^+$,

$$u(r, 0) \sim O(r^{-2\nu - 2}).$$

Following the techniques which lead to (1.9) we have

$$\begin{aligned} u(0, y) \sim & \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m + \nu + \frac{3}{2}) M[u(y, 0); 2(m + \nu + 1)]}{m! y^{2(m+\nu+1)} 2\Gamma(\frac{1}{2})\Gamma(\nu + 1)} \\ & + \sum_{m=0}^{\infty} \frac{\gamma_m \Gamma(\nu + 1 - a_m/2)\Gamma(\frac{1}{2} + a_m/2)}{y^{a_m} 2\Gamma(\frac{1}{2})\Gamma(\nu + 1)}, \end{aligned}$$

whenever $\nu + 1 - a_m/2$ is never zero or a negative integer.

The case $\nu = 0$ for three-dimensional axially symmetric potentials is of particular interest. If in addition we assume that $a_m = m$, we have by the techniques leading to (1.10),

$$\begin{aligned} u(0, y) \sim & \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma(m + \frac{3}{2}) \gamma_{2m+2}}{m! y^{2m+2} \Gamma(\frac{1}{2})} [\ln y + \psi(\tfrac{3}{2} + m)] \\ & + \sum_{m=0}^{\infty} \frac{\gamma_{2m+1} \Gamma(\frac{1}{2} - m) m!}{2\Gamma(\frac{1}{2}) y^{2m+1}} + \frac{\gamma_0}{2}, \end{aligned}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

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VARIATION DIMINISHING FOURIER–JACOBI TRANSFORMS*

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Abstract. Recently a convolution structure was developed for the Fourier–Jacobi transform by M. Flensted-Jensen and T. Koornwinder [3]. Here we use these results to prove theorems analogous to those of I. J. Schoenberg [13], [14], [15], [16] and I. I. Hirschman Jr. [6], [8] on variation diminishing convolution kernels. We show that a convolution kernel G is variation diminishing if and only if its Fourier–Jacobi transform

$$G\hat{}(\lambda) = K\, e^{-\gamma\lambda^2} \prod_k \left(1 + \frac{\rho^2 + \lambda^2}{b_k^2}\right)^{-1}, \qquad b_k > 0, \quad \Sigma b_k^2 < \infty, \quad \gamma > 0.$$

1. Introduction. Let $\phi(x)$ be a continuous function on an interval and let $V[\phi]$ denote the number of sign changes of ϕ in that interval. A function $G \in \mathscr{L}^1(-\infty, \infty)$ is said to be a variation diminishing *-kernel if $V[G*\phi] \leqq V[\phi]$ for every bounded continuous function on $(-\infty, \infty)$ where,

$$(G*\phi)(x) = \int_{-\infty}^{\infty} G(x - t)\phi(t)\, dt.$$

In 1950 I. J. Schoenberg [14] proved that G is a variation diminishing *-kernel if and only if

$$G\hat{}(\lambda) = \int_{-\infty}^{\infty} G(x)\, e^{-i\lambda x}\, dx = \left[e^{c\lambda^2 + ib\lambda} \prod_k \left(1 - \frac{i\lambda}{a_k}\right) e^{i\lambda/a_k}\right]^{-1},$$

a_k's real, $\sum a_k^{-2} < \infty$ and $b, c \geqq 0$.

In 1960–61 I. I. Hirschman Jr. wrote two papers [6], [8] from a series of four which established analogous results for convolution transforms associated with Hankel transforms and certain Sturm–Liouville differential equations. Generally, it was shown that, except for multiplicative constants, a function $\phi(t)$ is a variation diminishing convolution kernel if and only if its transform $\phi\hat{}(\lambda) = [e^{c\lambda} \prod_k (1 + \lambda/b_k)]^{-1}$, $b_k > 0, \sum b_k^{-1} < \infty$, $c > 0$. In the case of the Hankel transform we have

$$\phi\hat{}(\lambda) = \left[e^{c\lambda^2} \prod_k \left(1 + \frac{\lambda^2}{b_k^2}\right)\right]^{-1}, \qquad b_k > 0, \ \sum b_k^{-2} < \infty, \quad c > 0.$$

In this paper we prove analogous results for the convolution associated with the Fourier–Jacobi transform as defined by Flensted-Jensen and Koornwinder [3]. It will be shown that up to multiplicative constants a function $\phi(t)$ is a variation diminishing convolution kernel if and only if

$$\phi\hat{}(\lambda) = \left[e^{\gamma\lambda^2} \prod_k \left(1 + \frac{\rho^2 + \lambda^2}{b_k^2}\right)\right]^{-1}, \qquad b_k > 0, \ \sum b_k^{-2} < \infty, \quad \gamma > 0.$$

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The basic methods used to establish this result are adaptations of techniques developed by Hirschman in [6] and [8].

2. Fourier–Jacobi transforms and convolution. The Jacobi functions satisfy

$$\frac{1}{\Delta(t)}\frac{d}{dt}\left(\Delta(t)\frac{du(t)}{dt}\right) = \Lambda u(t) = -(\rho^2 + \lambda^2)u(t), \tag{2.1}$$

where

$$\Delta(t) = (e^t - e^{-t})^{2\alpha+1}(e^t + e^{-t})^{2\beta+1}, \qquad \rho = \alpha + \beta + 1 > 0.$$

Let $\phi_\lambda^{(\alpha,\beta)}(t) = \phi_\lambda(t)$ be a solution of (2.1) such that

$$\phi_\lambda(0) = 1, \qquad \phi'_\lambda(0) = 0.$$

Thus

$$\phi_\lambda(t) = F(\tfrac{1}{2}(\rho + i\lambda), \tfrac{1}{2}(\rho - i\lambda); 1 + \alpha; -(sht)^2). \tag{2.2}$$

One can write $\phi_\lambda(t) = R^{(\alpha,\beta)}_{(1/2)(i\lambda-\rho)}(ch2t)$, where $R_n^{\alpha,\beta}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$ and for n a nonnegative integer $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial. Let $\Phi_\lambda(t)$ be a Jacobi function of the second kind which is a solution of (2.1) and such that $\Phi_\lambda(t) = e^{(i\lambda-\rho)t}[1 + o(1)]$ as $t \to \infty$. Thus

$$\Phi_\lambda(t) = (e^t - e^{-t})^{i\lambda-\rho}F\left(\frac{\beta - \alpha + 1 - i\lambda}{2}, \frac{\rho - i\lambda}{2}; 1 - i\lambda; -\frac{1}{(sht)^2}\right). \tag{2.3}$$

One can show that

$$\phi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t), \tag{2.4}$$

where

$$c(\lambda) = \frac{2^{\rho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma\left(\frac{\rho+i\lambda}{2}\right)\Gamma\left(\frac{\alpha-\beta+1+i\lambda}{2}\right)}, \qquad c(\lambda) = O(|\lambda|^{-\alpha-1/2}). \tag{2.5}$$

Also in [2] it has been proved that for $\lambda = \xi + i\eta$

(2.6i) $\quad |\phi_\lambda(t)| \leqq \phi_{i\eta}(t) \quad$ for all $\lambda \in C$,

(2.6ii) $\quad |\phi_\lambda(t)| \leqq 1 \quad$ for all $|\eta| \leqq \rho$,

(2.6iii) $\quad |\phi_\lambda(t)| \leqq K(1 + t)\, e^{(|\eta|-\rho)t} \quad$ for all $\lambda \in C$.

It was proved in [2] that the mapping $f \to f^\wedge$ given by

$$f^\wedge(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty f(t)\phi_\lambda(t)\Delta(t)\,dt = \int_0^\infty f(t)\phi_\lambda(t)\,d\mu(t) \tag{2.7}$$

is a bijection between the space $\mathscr{D}$ consisting of even C^∞-functions of compact support and the space $\mathscr{H}_\infty$ of even, entire, rapidly decreasing functions of exponential type. The inverse mapping is given by

$$g^\vee(t) = \frac{1}{\sqrt{2\pi}}\int_0^\infty g(\lambda)\phi_\lambda(t)|c(\lambda)|^{-2}\,d\lambda = \int_0^\infty g(\lambda)\phi_\lambda(t)\,d\nu(\lambda). \tag{2.8}$$

Let us denote by $L^p(\mu)$ the space of those functions f on $(0, \infty)$ such that $\int_0^\infty |f|^p \, d\mu < \infty$, where μ is defined in (2.7) and $L^p(\nu)$ the corresponding space with ν defined by (2.8). $L(a, b)$ will denote the usual space with respect to Lebesgue measure on (a, b) where $-\infty \leqq a < b \leqq \infty$. Here we use $\mathscr{C}_0^\infty(a, b)$ to denote those C^∞ functions on (a, b) with compact support in (a, b).

If one considers the integrals in (2.7) and (2.8) to converge in $L^2(\nu)$ and $L^2(\mu)$ respectively, then Flensted-Jensen has shown [2, Thm. 3] that the mapping (2.7) is an isometry of $L^2(\mu)$ onto $L^2(\nu)$ with the inverse given by (2.8).

Here we define the convolution as in [3],

$$(f * g)(x) = \int_0^\infty \int_0^\infty f(t) f(s) K(x, s, t) \, d\mu(t) \, d\mu(s), \tag{2.9}$$

where

$$K(t_1, t_2, t_3) = \frac{2^{(1/2)-2\rho}\Gamma(\alpha + 1)(cht_1 cht_2 cht_3)^{\alpha-\beta-1}}{\Gamma(\alpha + \frac{1}{2})(sht_1 sht_2 sht_3)^{2\alpha}} \cdot F\left(\alpha + \beta, \alpha - \beta; \alpha + \tfrac{1}{2}; \frac{1 - B}{2}\right)$$

with

$$B = \frac{(cht_1)^2 + (cht_2)^2 + (cht_3)^2 - 1}{2cht_1 cht_2 cht_3}, \qquad |t_1 - t_2| < t_3 < t_1 + t_2,$$

and zero otherwise. The function $K(t_1, t_2, t_3)$ has the properties that

(i) $K(t_1, t_2, t_3)$ is symmetric in all three variables,

(ii) $K(t_1, t_2, t_3) \geqq 0$,

(iii) $\int_0^\infty K(t_1, t_2, t_3) \, d\mu(t_3) = 1$.

Also it has been shown that [3], [12]

$$\phi_\lambda(t_1)\phi_\lambda(t_2) = \int_0^\infty \phi_\lambda(t_3) K(t_1, t_2, t_3) \, d\mu(t_3). \tag{2.10}$$

Let $1 \leqq p < 2$ and $(1/p) + (1/q) = 1$. Define the strip

$$D_p = \{\lambda = \xi + i\eta \in C \mid |\eta| < ((2/p) - 1)\rho\}. \tag{2.11}$$

From [3] we have the following.

LEMMA 2.1. *Let $1 \leqq p < 2$, $(1/p) + (1/q) = 1$ and $f \in L^p(\mu)$. Then $f^\wedge(\lambda)$ is holomorphic in D_p, and for all $\lambda \in D_p$,*

$$|f^\wedge(\lambda)| \leqq \|f\|_p \|\phi_\lambda\|_q.$$

If $f \in L^1(\mu)$, $f^\wedge(\mu)$ is continuous in $\bar{D}_1$ and for all $\lambda \in \bar{D}_1$,

$$|f^\wedge(\lambda)| \leqq \|f\|_1.$$

Proof. See [3, p. 249].

THEOREM 2.2. *Let p, q, r* satisfy $(1/p) + (1/q) = 1 + (1/r)$, $1 \leqq p, q, r \leqq \infty$. *For $f \in L^p(\mu)$ and $g \in L^q(\mu)$, $f * g \in L^r(\mu)$ and*

$$\|f * g\|_r \leqq \|f\|_p \|g\|_q.$$

Proof. See [3, p. 258].

Remark 1. The two preceding results together with (2.6ii) imply that if $f, g \in L^1(\mu)$, then $(f * g)\hat{}(\lambda) = f\hat{}(\lambda)g\hat{}(\lambda)$.

LEMMA 2.3. *If $f \in L^1(\mu)$ and $f\hat{} \in L^1(\nu)$, then $(f\hat{})^\vee = f$ a.e.*

Proof. In [2] (Theorem 6 and Lemma 16) one finds a set of functions $\{v_\varepsilon\}_{\varepsilon > 0}$ which act as an approximate identity in $L^r(\mu)$, $0 < r < \infty$. Now supp $v_\varepsilon \subseteq [-\varepsilon, \varepsilon]$ so that $v_\varepsilon\hat{} \in H_\infty$ and $(v_\varepsilon\hat{})^\vee = v_\varepsilon$ by the remarks following (2.7). Thus

$$\begin{aligned}
(f * v_\varepsilon)(x) &= \int_0^\infty \int_0^\infty f(s)v_\varepsilon(t)K(x, s, t)\, d\mu(s)\, d\mu(t) \\
&= \int_0^\infty \int_0^\infty f(s)\left[\int_0^\infty v_\varepsilon\hat{}(\lambda)\phi_\lambda(x)\, d\nu(\lambda)\right]K(x, s, t)\, d\mu(s)\, d\mu(t) \\
&= \int_0^\infty f(s)\int_0^\infty v_\varepsilon\hat{}(\lambda)\int_0^\infty \phi_\lambda(t)K(x, s, t)\, d\mu(t)\, d\nu(\lambda)\, d\mu(s) \\
&= \int_0^\infty v_\varepsilon\hat{}(\lambda)\int_0^\infty \phi_\lambda(x)f(s)\phi_\lambda(s)\, d\mu(s)\, d\nu(\lambda) \\
&= \int_0^\infty v_\varepsilon\hat{}(\lambda)f\hat{}(\lambda)\phi_\lambda(x)\, d\nu(\lambda),
\end{aligned}$$

where the changes of order of integration follow from Fubini's theorem. Now $f * v_\varepsilon \to f$ in $L^1(\mu)$ so there exists v_{ε_n} such that $f * v_{\varepsilon_n} \to f$ a.e. By the Lebesgue limit theorem the right-hand side tends to $(f\hat{})^\vee(x)$.

LEMMA 2.4. *If $f \in \mathscr{C}_0^\infty(0, \infty)$, then $(1 + |\lambda|^n)f\hat{}(\lambda) \in L^1(\nu)$, $n = 0, 1, 2, \cdots$.*

Proof. This follows immediately from the remarks following formula (2.7), since f can be extended to be in $\mathscr{D}$.

LEMMA 2.5. *If $f \in \mathscr{D}$ and $g \in L^1(\mu)$, then*

$$f * g \in \mathscr{C}^{(n)}(0, \infty), \qquad n = 0, 1, 2, \cdots.$$

Proof. For $f \in \mathscr{D}$, $(1 + |\lambda|^n)f\hat{}(\lambda) \in L^1(\nu)$ by Lemma 2.4, and $g\hat{}(\lambda)$ is bounded by Lemma 2.1. Thus $(f * g)(x) = \int_0^\infty f\hat{}(\lambda)g\hat{}(\lambda)\phi_\lambda(x)\, d\nu(\lambda)$. Now

$$\left|\int_0^\infty f\hat{}(\lambda)g\hat{}(\lambda)\frac{d^n}{dx^n}\phi_\lambda(x)\, d\nu(\lambda)\right| \leqq K\int_0^\infty f\hat{}(\lambda)(1 + |\lambda|^n)(1 + x)\, e^{-\rho x}\, d\nu(\lambda).$$

Here we have used estimates from [2, Thm. 2]. Since the integral converges uniformly in x, $0 < x < \infty$, it follows that $f * g \in \mathscr{C}^{(n)}(0, \infty)$, $n = 0, 1, 2, \cdots$.

LEMMA 2.6. *If a is real, then $\phi_{ia}(t) > 0$, $0 < t < \infty$.*

Proof. For $\alpha > \beta > -\frac{1}{2}$, ([1], [11])

$$(2.12)\quad \phi_\lambda(t) = \int_{r=0}^1 \int_{\psi=0}^\pi [(cht)^2 + r^2(sht)^2 + rsh2t + \cos\psi]^{1/2(i\lambda - \rho)}\, dm(r, \psi),$$

where

$$dm(r, \psi) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha - \beta)\Gamma(\beta + \frac{1}{2})}(1 - r^2)^{\alpha - \beta - 1}r^{2\beta + 1}(\sin\psi)^{2\beta}\, d\psi\, dr.$$

It is easily shown that

$$\int_{r=0}^{1}\int_{\psi=0}^{\pi} dm(r,\psi) = 1. \tag{2.13}$$

Now $(cht)^2 + r^2(sht)^2 + rsh2t\cos\psi = (cht + rsht\, e^{i\psi})(cht + rsht\, e^{-i\psi}) \geqq 0$. Thus $e^{-2t} \leqq (cht + rsht\, e^{i\psi})(cht + rsht\, e^{-i\psi}) \leqq e^{2t}$, $0 \leqq t \leqq \infty$, $0 \leqq r \leqq 1$. Since $a + \rho$ is real, it follows from (2.12), the positivity of $dm(r\psi)$, and (2.13) that $e^{-(a+\rho)t} \leqq \phi_{ia}(t) \leqq e^{(a+\rho)t}$.

LEMMA 2.7. *If h is in the domain of Λ and $\gamma = \sqrt{p^2 + a^2}$, then*

$$(\Lambda - a^2)h(t) = \frac{1}{\Delta(t)\phi_{i\gamma}(t)}\frac{d}{dt}\left[\Delta(t)\phi_{i\gamma}^2(t)\frac{d}{dt}\left(\frac{h(t)}{\phi_{i\gamma}(t)}\right)\right] \tag{2.14}$$

$$= \frac{1}{\Delta(t)\Phi_{i\gamma}(t)}\frac{d}{dt}\left[\Delta(t)\Phi_{i\gamma}^2(t)\frac{d}{dt}\left(\frac{h(t)}{\Phi_{i\gamma}(t)}\right)\right]. \tag{2.15}$$

Proof. We use direct calculation and the fact that $\phi_{i\gamma}(t)$ and $\Phi_{i\gamma}(t)$ satisfy $\Lambda u = a^2 u$.

3. Elementary kernels. Suppose f is a real-valued function on an interval (a,b), $-\infty \leqq a < b \leqq \infty$. We say that f changes sign at least n times on (a,b) if there exist t_i, $i = 0, 1, 2, \cdots, n$; $a < t_0 < t_1 < t_2 < \cdots < t_n < b$ such that $f(t_{i-1})f(t_i) < 0$ for $i = 1, 2, 3, \cdots, n$. The function has n sign changes if it has at least n sign changes, but does not have at least $n + 1$ sign changes. We denote the number of changes of sign on (a,b) by $V_a^b[f]$. It is possible of course for $V_a^b[f] = \infty$.

DEFINITION. A convolution kernel h is variation diminishing if $h \in L^1(\mu)$ and $V_0^\infty[h * u] \leqq V_0^\infty[u]$ for every bounded continuous function u on $0 < x < \infty$. (Here we shall use $V = V_0^\infty$.)

LEMMA 3.1. *If*

$$g_a(t) = \sqrt{\frac{\pi}{2}}\frac{a^2}{\gamma c(-i\gamma)}\Phi_{i\gamma}(t), \qquad \gamma = \sqrt{\rho^2 + a^2}, \quad a > 0,$$

then $\hat{g_a}(\lambda) = (1 + (\rho^2 + \lambda^2)/a^2)^{-1}$. *Also* $g_a(t) \in L^1(\mu)$.

Proof. Using $\Lambda\Phi_{i\gamma}(t) = a^2\Phi_{i\gamma}$ and $\Lambda\phi_\lambda(t) = -(\rho^2 + \lambda^2)\phi_\lambda(t)$ it follows that

$$\frac{d}{dt}\{\Delta(t)W(\phi_\lambda, \Phi_{i\gamma})(t)\} = (a^2 + \rho^2 + \lambda^2)\Phi_{i\gamma}(t)\phi_\lambda(t)\,\Delta(t),$$

where W is the Wronskian. From (2.2), (2.3), (2.5) and

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(c-a)\Gamma(b)}(-z)^{-a}F\left(a, a+1-c; a+1-b; \frac{1}{z}\right) \tag{3.1}$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(c-b)\Gamma(a)}(-z)^{-b}F\left(b+1-c, b; b+1-a; \frac{1}{z}\right),$$

one obtains

$$(a^2 + \rho^2 + \lambda^2) \int_0^\infty \Phi_{i\gamma}(t)\phi_\lambda(t)\, \Delta(t)\, dt = 2\gamma c(-i\gamma)$$

or

$$\int_0^\infty g_a(t)\phi_\lambda(t)\, d\mu(t) = \left(1 + \frac{\rho^2 + \lambda^2}{a^2}\right)^{-1}.$$

By (3.1), $g_a(t)\Delta(t)$ is bounded at $t = 0$ and by (2.3), $g_a(t)\Delta(t) = \exp((\rho - \sqrt{\rho^2 + a^2})t) \cdot [1 + O(1)]$ and $g_a(t) \in L^1(\mu)$.

The remainder of this section is devoted to proving that $g_a(t)$ is a variation diminishing convolution kernel.

LEMMA 3.2. *If $f \in L^1(\mu)$ and $f^\wedge \in L^1(\nu)$ and if $h(x) = (g_a * f)(x)$, then $(1 - \Lambda/a^2)h(x) = f(x)$.*

Proof. (We follow the line of proof of Lemma 3d [6].) By the remark after Theorem 2.2, $h^\wedge(\lambda) = f^\wedge(\lambda)(1 + (\rho^2 + \lambda^2)/a^2)^{-1}$. Since $h^\wedge \in L^1(\nu)$ it follows from Lemma 2.3 that $h(x) = \int_0^\infty f^\wedge(\lambda)(1 + (\rho^2 + \lambda^2)/a^2)^{-1}\phi_\lambda(x)\, d\nu(\lambda)$. Since

$$\int_0^\infty f^\wedge(\lambda)\left(1 + \frac{\rho^2 + \lambda^2}{a^2}\right)^{-1}\left(1 - \frac{\Lambda}{a^2}\right)\phi_\lambda(x)\, d\nu(\lambda) = \int_0^\infty f^\wedge(\lambda)\phi_\lambda(x)\, d\nu(\lambda) = f(x)$$

converges uniformly in x it follows that $(1 - \Lambda/a^2)h(x) = f(x)$.

THEOREM 3.3. *For $a > 0$, $g_a(t)$ is a variation diminishing convolution kernel.*

Proof. Suppose $f \in \mathscr{C}_0^\infty(0, \infty)$. Then by Lemma 2.4, $f^\wedge \in L^1(\nu)$. If $h(x) = (g_a * f)(x)$, then by Lemma 3.2, $(1 - \Lambda/a^2)h(x) = f(x)$. From Theorem 2.2 and the fact that f is bounded and $g_a(t) \in L^1(\mu)$, we see that h is bounded. Since $f \in \mathscr{D}$ and $g_a(t) \in L^1(\mu)$, $h \in \mathscr{C}^{(n)}(0, \infty)$, $n = 0, 1, 2, \cdots$. Now it follows from (2.3) and (2.4) that $h(t)/\phi_{i\gamma}(t) \to 0$ as $t \to \infty$ and hence

$$V\left[\frac{d}{dt}\left(\frac{h(t)}{\phi_{i\gamma}(t)}\right)\right] \geqq V\left[\frac{h(t)}{\phi_{i\gamma}(t)}\right] = V[h]$$

since $\phi_{i\gamma} > 0$. Also one sees that

$$\Delta(t)\phi_{i\gamma}^2(t)\frac{d}{dt}\left[\frac{h(t)}{\phi_{i\gamma}(t)}\right] \to 0 \quad \text{as } t \to 0$$

and again

$$V\left\{\frac{d}{dt}\left[\Delta(t)\phi_{i\gamma}^2(t)\frac{d}{dt}\left(\frac{h(t)}{\phi_{i\gamma}(t)}\right)\right]\right\} \geqq V\left[\Delta(t)\phi_{i\gamma}^2(t)\frac{d}{dt}\left(\frac{h(t)}{\phi_{i\gamma}(t)}\right)\right]$$
$$= V\left[\frac{d}{dt}\left(\frac{h(t)}{\phi_{i\gamma}(t)}\right)\right] \geqq V[h].$$

From (2.9), Lemma 2.7 and Lemma 3.2, it follows that $V[f] \geqq V[h]$. If f is continuous and bounded on $(0, \infty)$, then there exists a sequence of functions $\{f_n(x)\}$, $0 < x < \infty$, such that

$$\text{(i) } f_n \in \mathscr{C}_0^\infty(0, \infty), \qquad \text{(ii) } \|f_n\|_\infty \leqq \|f\|_\infty,$$

$$\text{(iii) } \lim_{n\to\infty} f_n(x) = f(x), \quad 0 < x < \infty, \qquad \text{(iv) } V[f_n] \leqq V[f].$$

Let $h_n = g_a * f_n$ and by the Lebesgue limit theorem $h_n \to h$ on $(0, \infty)$. Thus $V[h_n] \leqq V[f_n] \leqq V[f]$ and by taking limits we get $V[h] \leqq V[f]$.

LEMMA 3.4. *For $a > 0$, $g_a(t) \geqq 0$.*

Proof. Hirschman [6] has proved that variation diminishing convolution kernels are always nonnegative or nonpositive. The same result is easily proved here with the appropriate modifications. Now since $g_a(t) > 0$ for sufficiently large t, by (2.3) it follows that $g_a(t) \geqq 0$.

4. Composite kernels.

LEMMA 4.1. *If G and H are variation diminishing convolution kernels, then so is $G * H$.*

Proof. See Hirschman [6].

THEOREM 4.2. *Let $c \geqq 0$ and $0 < a_1 \leqq a_2 \leqq a_3 \leqq \cdots$, where $\sum a_k^{-2} < \infty$,*

$$E(\lambda) = e^{c\lambda^2} \prod_k \left(1 + \frac{\rho^2 + \lambda^2}{a_k^2}\right).$$

Then $1/E(\lambda)$ is the Jacobi–Fourier transform of a variation diminishing convolution kernel $G(x)$. That is,

$$\int_0^\infty G(x)\phi_\lambda(x)\, d\mu(x) = \frac{1}{E(\lambda)}.$$

(There may be no a_k's, finitely many a_k's or infinitely many a_k's. We do exclude the case of $E(\lambda) \equiv 1$, i.e., no a_k's and $c = 0$.)

Proof. If $E(\lambda) = \prod_{k=1}^n (1 + (\rho^2 + \lambda^2)/a_k^2)$, set $G(x) = g_{a_1} * g_{a_2} * \cdots * g_{a_n} * (x)$ and apply Lemma 4.1, Theorem 3.3 and Remark 1. Next suppose $E(\lambda) = \prod_{k=1}^\infty (1 + (\rho^2 + \lambda^2)/a_k^2)$. Let $E_n = \prod_{k=1}^n (1 + (\rho^2 + \lambda^2)/a_k^2)$ and $G_n(x) = g_{a_1} * g_{a_2} * \cdots * g_{a_n}(x)$. Then

$$\int_0^\infty G_n(x)\phi_\lambda(x)\, d\mu(\lambda) = \frac{1}{E_n(\lambda)}. \tag{4.1}$$

For n sufficiently large $[E_n(\lambda)]^{-1} \in L^1(\nu)$ and we have

$$G_n(x) = \int_0^\infty \frac{\phi_\lambda(x)}{E_n(\lambda)}\, d\nu(\lambda).$$

If we set

$$G(x) = \int_0^\infty \frac{\phi_\lambda(x)}{E(\lambda)}\, d\nu(\lambda), \tag{4.2}$$

then

$$G(x) - G_n(x) = \int_0^\infty \left[\frac{1}{E(\lambda)} - \frac{1}{E_n(\lambda)}\right] \phi_\lambda(x)\, d\nu(\lambda).$$

For $n \geqq N$, $|(1/E(\lambda)) - (1/E_n(\lambda))| \leqq 2/E_N(\lambda)$ which for $N > 2\alpha + 2$ is in $L^1(\nu)$. Thus

$$|G(x) - G_n(x)| \leqq \int_0^\infty \left| \frac{1}{E(\lambda)} - \frac{1}{E_n(\lambda)} \right| d\nu(\lambda) \to 0$$

by the Lebesgue convergence theorem. Thus $\lim_{n\to\infty} G_n(x) = G(x)$ uniformly for $0 < x < \infty$.

From (2.2) one sees that $\phi_{i\rho}(x) \equiv 1$ and from (4.1) it follows that $\int_0^\infty G_n(x)\,d\mu(x) = 1$. Also since $g_{ai}(t) \geqq 0$ and $K(t_1, t_2, t_3) \geqq 0$ we have that $G_n(x) \geqq 0$ and $G(x) \geqq 0$. Applying Fatou's lemma we get $\int_0^\infty G(x)\,dx \leqq 1$ and $G \in L^1(\mu)$.

It is clear that $[E(\lambda)]^{-1} \in L^2(\nu)$ and hence $G(x) \in L^2(\mu)$. Also we have that $\lim_{R\to\infty} \int_0^R G(x)\phi_\lambda(x)\,d\mu(x) = [E(\lambda)]^{-1}$. However $\lim_{R\to\infty} \int_0^R G(x)\phi_\lambda(x)\,d\mu(x) = \hat{G}(\lambda)$ since $G \in L^1(\mu)$. Thus it follows that $\hat{G}(\lambda) = [E(\lambda)]^{-1}$ a.e. and by continuity, $\hat{G}(\lambda) = [E(\lambda)]^{-1} = \int_0^\infty G(x)\phi_\lambda(x)\,d\mu(x)$ [5, Thm. 2K]. Now if we let $\lambda \to i\rho$ we get $\int_0^\infty G(x)\,d\mu(x) = 1$. Since $\lim_{n\to\infty} G_n(x) = G(x)$, $\|G_n\|_1 = \|G\|_1$, we have $\|G_n - G\|_1 \to 0$. Consequently if f is continuous and bounded,

$$(G * f)(x) = \lim_{n\to\infty} (G_n * f)(x)$$

and

$$V[G * f] \leqq \lim_{n\to\infty} V[G_n * f] \leqq V[f]$$

and $G(x)$ is variation diminishing.

If $E(\lambda) = e^{c(\lambda^2+\rho^2)}$, then we set $E_n(\lambda) = (1 + c(\lambda^2 + \rho^2)/n)^n$ and proceed as before.

5. Variation diminishing kernels.

THEOREM 5.1. *If $G \in L^1(\mu)$ is a variation diminishing transform, then*

$$\hat{G}(\lambda) = K\, e^{-\gamma\lambda^2} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda^2 + \rho^2}{b_k^2}\right) b_k > 0, \qquad \sum \frac{1}{b_k^2} < \infty, \quad \gamma \geqq 0.$$

Proof. Let $G \in L^1(\mu)$ be a variation diminishing transform, i.e., $V[G * u] \leqq V[u]$ for all continuous and bounded functions on $(0, \infty)$. Let $G_1 = G * h_1$, where $\hat{h}_1(\lambda) = e^{-\lambda^2}$. Now h_1 is variation diminishing so that G_1 is also variation diminishing and $\hat{G}_1(\lambda) = \hat{G}(\lambda)\,e^{-\lambda^2}$ and $\hat{G}_1(\lambda) \in L^1(\nu)$, since $\hat{G}(\lambda)$ is bounded.

Now for $u \in \mathscr{C}_0^\infty$,

$$\begin{aligned}
(G_1 * u)(x) &= \int_0^\infty \int_0^\infty G_1(y)u(t)K(x, y, t)\,d\mu(x)\,d\mu(y) \\
&= \frac{1}{(2\pi)} \int_0^\infty u(t)\Delta(t) \int_0^\infty \hat{G}_1(\lambda) \int_0^\infty \phi_\lambda(y)K(x, y, t)\Delta(y)\,dy\,d\nu(\lambda)\,dt \\
&= \frac{1}{2\pi} \int_0^\infty u(t)\Delta(t) \int_0^\infty \hat{G}_1(\lambda)\phi_\lambda(x)\phi_\lambda(t)\,d\nu(\lambda)\,dt.
\end{aligned}$$

Let $H(x, t) = (\Delta(t)/2\pi) \int_0^\infty \hat{G}_1(\lambda)\phi_\lambda(t)\phi_\lambda(x)\,d\nu(\lambda)$ so that

$$(G_1 * u)(x) = \int_0^\infty H(x, t)u(t)\,dt.$$

Now $\phi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t)$ and $\Phi_\lambda(t) = e^{(i\lambda-\rho)t}[1 + e^{-2t}\Theta(\lambda, t)]$, where $|\Theta(\lambda, t)| \leqq M$ for $t \in [c, \infty)$, $c > 0$ and $\lambda \in \{\lambda \in C | \lambda = \xi + i\eta, \eta \geqq -\varepsilon|\xi|, \varepsilon > 0\}$.

$$H(x + r, t + r) = \frac{\Delta(t + r)}{2\pi} \int_0^\infty \hat{G}_1(\lambda)\phi_\lambda(x + r)\phi_\lambda(t + r)|c(\lambda)|^{-2}\, d\lambda.$$

Thus

$$\begin{aligned}
&\Delta(t + r)\phi_\lambda(t + r)\phi_\lambda(x + r)[c(\lambda)c(-\lambda)]^{-1} \\
&\quad = \left[\frac{c(\lambda)}{c(-\lambda)}\right] e^{-i\lambda(x+t+2r)}\, e^{-\rho(x-t)}[1 + S(\lambda, x, t, r)] \\
&\qquad + 2\cos\lambda(x - t)\, e^{-\rho(x-t)}[1 + S(\lambda, x, t, r)] \\
&\qquad + \frac{c(-\lambda)}{c(\lambda)} e^{-i\lambda(x+t+2r)}\, e^{-\rho(x-t)}[1 + S(\lambda, x, t, r)].
\end{aligned}$$

One finds that $S(\lambda, x, t, r) = e^{-2r}O(1)$ and $|c(\lambda)/c(-\lambda)| \leqq M_1$ and $|c(-\lambda)/c(\lambda)| \leqq M_2$ for $\eta \geqq -\varepsilon|\xi|$. Thus

$$H(x + r, t + r) = e^{-\rho(x-t)}[J_1 + J_2 + J_3 + J_4],$$

where

$$\begin{aligned}
J_1 &= \frac{1}{2\pi}\int_0^\infty \hat{G}_1(\lambda)\, e^{i\lambda(x+t+2r)} \frac{c(\lambda)}{c(-\lambda)}\, d\lambda, \\
J_2 &= \frac{1}{\pi}\int_0^\infty \cos\lambda(x - t)\hat{G}_1(\lambda)\, d\lambda, \\
J_3 &= \frac{1}{2\pi}\int_0^\infty \hat{G}_1(\lambda)\, e^{-i\lambda(x+t+2r)} \frac{c(-\lambda)}{c(\lambda)}\, d\lambda, \\
J_4 &= \frac{1}{2\pi}\int_0^\infty \hat{G}_1(\lambda) R(\lambda, x, t, r)\, d\lambda.
\end{aligned}$$

Here $|R(\lambda, x, t, r)| \leqq M$ and $\lim_{r\to\infty} R(\lambda, x, t, r) = 0$. J_1 and $J_3 \to 0$ as $r \to \infty$ by the Riemann–Lebesgue lemma. $J_4 \to 0$ as $r \to \infty$ by the Lebesgue convergence theorem and $J_2 = L(x - t) = (1/\pi)\int_0^\infty \cos\lambda(x - t)\hat{G}_1(\lambda)\, d\lambda$. Thus we see that $H(x + r, t + r) \to L(x - t)\, e^{-\rho(x-t)}$ as $r \to \infty$.

Now since G_1 is variation diminishing,

$$V\left[\int_0^\infty H(x, t)u(t)\, dt\right] \leqq V[u] \quad \text{for all } u \in \mathscr{C}_0^\infty(-\infty, \infty). \tag{5.1}$$

Let $v(t) \in \mathscr{C}_0^\infty(-\infty, \infty)$. Then $e^{-\rho(t-r)}v(t - r) \in \mathscr{C}_0^\infty(-\infty, \infty)$ and (4.1) becomes

$$V_r\left[\int_{-r}^\infty H(x + r, t + r)\, e^{-\rho t}v(t)\, dt\right] \leqq V_r[v(t)\, e^{-\rho t}] = V_r[v(t)],$$

where $V_r[v]$ denotes the number of variations on $(-r, \infty)$. Using the Lebesgue limit theorem as $r \to \infty$ we get

$$V_\infty[e^{-\rho x}(L * v)(t)] = V_\infty[(L * v)(t)] \leqq V_\infty[v(t)]. \tag{5.2}$$

Using an approximation argument, one shows that (5.2) holds for all bounded continuous functions on $(-\infty, \infty)$. Thus $L(x)$ is a variation diminishing transform and by Schoenberg, $\hat{G}_1(\lambda) = K e^{-\gamma\lambda^2} \prod_{k=0}^{\infty} (1 + \lambda^2/a_k^2)^{-1}$. Since $G_1 \in L^1(\mu)$, $\hat{G}_1(\lambda)$ is continuous and bounded in $|\eta| \leqq \rho$ by Lemma 2.1. Thus $a_k^2 > \rho^2$. Let $a_k^2 = b_k^2 + \rho^2$, $b_k > 0$. Then

$$
\begin{aligned}
\hat{G}(\lambda) &= K e^{-\gamma'\lambda^2} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda^2}{b_k^2 + \rho^2}\right)^{-1} \\
&= K\left[\prod_{k=0}^{\infty} \left(1 + \frac{\rho^2}{a_k^2}\right)\right] e^{-\gamma'\lambda^2} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda^2 + \rho^2}{b_k^2}\right)^{-1} \\
&= K' e^{-\gamma'\lambda^2} \prod_{k=0}^{\infty} \left(1 + \frac{\lambda^2 + \rho^2}{b_k^2}\right)^{-1}, \qquad b_k > 0, \quad \sum \frac{1}{b_k^2} < \infty, \quad \gamma' \geqq 0.
\end{aligned}
$$

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ASYMPTOTIC NATURE OF NONOSCILLATORY SOLUTIONS OF nTH ORDER RETARDED DIFFERENTIAL EQUATIONS*

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Abstract. The equation

$$r(t)y^{(n)}(t) + r'(t)y^{(n-1)}(t) + a(t)y_\tau(t) = f(t)$$

is studied for its nonoscillation behavior. Under certain conditions, it is shown that if $y(t)$ is a nonoscillatory solution, then $y^{(n-2)}(t) \to 0$ as $t \to \infty$. The theorem is then applied to the case of 2nd order and 3rd order equations.

1. Introduction. Our main concern in this paper is to study the asymptotic nature of nonoscillatory solutions of the retarded nth order differential equation

$$r(t)y^{(n)}(t) + r'(t)^{(n-1)}y(t) + a(t)y_\tau(t) = f(t), \qquad n \geqq 2, \tag{1}$$

where

$$y_\tau(t) \equiv y(t - \tau(t)); \qquad \left(y^{(i)}(t) \equiv \frac{d^i}{dt^i}y(t)\right),$$

$a(t)$, $f(t)$, $\tau(t)$ and $r(t)$ are continuous on the whole real line R. $r(t)$, $\tau(t)$ and $a(t)$ are to be nonnegative with $\tau(t)$ bounded above by a positive constant M. $r(t)$ and $\tau(t)$ are further assumed to be continuously differentiable on R.

In what follows, we are only going to consider continuous solutions of (1) which are extendable on some positive half-line $[T, \infty)$, $T > 0$. The term "solutions" will apply only to such solutions of (1).

DEFINITION. We call a function on $[T, \infty)$ *oscillatory* if it has arbitrarily large zeros. Otherwise we call it *nonoscillatory*.

Equations of type (1) with *bounded* $r(t)$ are prototypes of the ones associated with variable mass problems. What we have here is a set of preliminary results both for bounded $r(t)$ and unbounded $r(t)$. Known results of Hammett [7] are greatly generalized.

Our main result is Theorem 1 which essentially states that for unbounded $r(t)$ any nonoscillatory solution $y(t)$ of equation (1), $y^{(n-2)}(t)$ tends to a finite limit as $t \to \infty$. The conditions assumed for this derivation are mild and practical. In Theorems 2 and 3, application of this result is shown. The paper contains examples to show the applicability of Theorems 2 and 3. In § 4 we consider the case when $r(t)$ is bounded.

Recently Hammett [7] proved such a result about the nonoscillatory solutions of

$$(r(t)y'(t))' + p(t)g(y(t)) = f(t), \tag{2}$$

where it was assumed that $r(t)$ and $p(t)$ are positively bounded away from zero. Burton and Grimmer [5] modified Hammett's results. Hammett's method was

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based on a theorem of Bhatia [2] which does not apply to retarded equations. In fact Travis [9] showed that the equation

$$y''(t) + \frac{\sin t}{2 - \sin t} y(t - \pi) = 0 \tag{3}$$

has the nonoscillatory solution $2 + \sin t$ even though

$$\int^{\infty} (\sin t)/(2 - \sin t)\, dt = \infty .$$

But according to Bhatia's theorem, all solutions of the equation

$$y''(t) + \frac{\sin t}{2 - \sin t} y(t) = 0 \tag{4}$$

are oscillatory.

In view of these observations, equation (1) deserves a special treatment to contain the delay term and higher order derivative in order to prove our main result.

2. Main results.

THEOREM 1. *Suppose*

(i) $r(t) > 0, \int^{\infty} (1/r(t))\, dt < \infty$,

(ii) *for sequences* $\{a_n\}$, $\{b_n\}$, $a_n \to \infty$, $b_n \to \infty$, $b_n > a_n$ *for all* n *and* $\lim_{n\to\infty} (b_n - a_n) = \infty$, *let* $\lim_{n\to\infty} \int_{a_n}^{b_n} a(t)\, dt = \infty$.

(iii) $\int^{\infty} |f(t)|\, dt < \infty$.

Let $y(t)$ *be a nonoscillatory solution of* (1). *Then* $y^{(n-2)}(t)$ *tends to a finite limit as* $t \to \infty$.

Proof. Let T be a large enough positive number so that for $t \geqq T$, $y(t)$ and $y_\tau(t)$ assume a constant sign. Suppose $y(t)$ and $y_\tau(t)$ become positive for $t \geqq T$. The case when $y(t)$ and $y_\tau(t)$ assume a negative sign can be handled in an identical manner.

Rewriting (1) as

$$(r(t)y^{(n-1)}(t))' + a(t)y_\tau(t) = f(t) \tag{5}$$

and integrating both sides of (5) over $[T, t]$ we have

$$r(t)y^{(n-1)}(t) - r(T)y^{(n-1)}(T) + \int_T^t a(s)y_\tau(s)\, ds \leqq \int_T^t |f(s)|\, ds. \tag{6}$$

Now as $t \to \infty$, the right-hand side of inequality (6) is bounded owing to (iii). Two cases arise.

Case 1.

$$\int_T^{\infty} a(s)y_\tau(s)\, ds = \infty . \tag{7}$$

(6) and (7) imply

$$r(t)y^{(n-1)}(t) \to -\infty \quad \text{as } t \to \infty .$$

Since $r(t) > 0$, it means $y^{(n-1)}(t)$ eventually assumes a negative sign. But then $y^{(n-2)}(t)$ is monotonic and decreasing. This in turn implies that $y^{(n-2)}(t)$ eventually assumes a constant sign. Since $y(t) \geqq 0$ for $t \geqq T$, $y^{(n-2)}(t)$ must assume a positive sign. In fact if $y^{(n-2)}(t)$ and $y^{(n-1)}(t)$ are both negative, $y(t)$ will eventually become negative, which is a contradiction. Hence $y(^{n-2})(t)$ tends to a finite limit as $t \to \infty$.

Case 2.

$$\int_T^\infty a(s)y_\tau(s)\,ds < \infty. \tag{8}$$

Now

$$\int_T^\infty a(s)\,ds \geqq \lim_{N\to\infty} \int_{T+N}^{T+2N} a(s)\,ds = \infty \tag{9}$$

by condition (ii). From (8) and (9), we get

$$\liminf_{t\to\infty} y_\tau(t) = \liminf_{t\to\infty} y(t) = 0. \tag{10}$$

Now either $y^{(n-2)}(t)$ is oscillatory or it assumes a constant sign. In either case we will show that $y^{(n-2)}(t)$ tends to a finite limit as $t \to \infty$. Suppose first that $y^{(n-2)}(t)$ assumes a constant sign. This implies

$$y^{(i)}(t), \qquad i = 0, 1, 2, \cdots, (n-3),$$

are monotonic eventually. Let

$$\liminf_{t\to\infty} |y^{(n-2)}(t)| = \alpha. \tag{11}$$

Then $\alpha = 0$. If $\alpha > 0$, then $y(t) \to \infty$ as $t \to \infty$, a contradiction to (10). Thus we have here that $y^{(n-2)}(t) \geqq 0$ and

$$\liminf_{t\to\infty} y^{(n-2)}(t) = \liminf_{t\to\infty} |y^{(n-2)}(t)| = 0. \tag{12}$$

If $y^{(n-2)}(t)$ is oscillatory, then we already have

$$\liminf_{t\to\infty} |y^{(n-2)}(t)| = 0. \tag{13}$$

Suppose now

$$\limsup_{t\to\infty} |y^{(n-2)}(t)| > d > 0. \tag{14}$$

(13) and (14) imply that there exists a sequence of points $\{t_k\}_{k=0}^\infty$ in $[T, \infty)$ such that $\int_{t_0}^\infty |f(t)|\,dt < \varepsilon$ for some arbitrarily small $\varepsilon > 0$ and

(iv) $t_k \to \infty$ as $k \to \infty$, $t_{k+1} > t_k$,

(v) $|y^{(n-2)}(t_k)| < d/4$ for all $k \geqq 0$,

(vi) $d_k \geqq 3d/4 > 0$, where d_k is a true maxima of $|y^{(n-2)}(t)|$ in $[t_{k-1}, t_k]$. Let $z_k \in [t_{k-1}, t_k]$ such that $d_k = |y^{(n-2)}(z_k)|$.

(vii) Let (a_k, b_k) be the largest open interval containing z_k such that $|y^{(n-2)}(t)| > d_k/2$ for all t in this interval. Note that

$$|y^{(n-2)}(a_k)| = |y^{(n-2)}(b_k)| = d_k/2 \quad \text{for } k \geqq 1. \tag{15}$$

The choice of a_k and b_k implies that

$$|y^{(n-2)}(t)| > d_k/2 \quad \text{in } (a_k, b_k).$$

Now

$$y^{(n-2)}(z_k) = y^{(n-2)}(a_k) + \int_{a_k}^{z_k} y^{(n-1)}(t)\,dt$$

from which we have

$$|y^{(n-2)}(z_k)| \leqq |y^{(n-2)}(a_k)| + \int_{a_k}^{z_k} |y^{(n-1)}(t)|\,dt. \tag{16}$$

From (15) and (16),

$$d_k/2 \leqq \int_{a_k}^{z_k} |y^{(n-1)}(t)|\,dt. \tag{17}$$

Also

$$-y^{(n-2)}(z_k) = -y^{(n-2)}(b_k) + \int_{z_k}^{b_k} y^{(n-1)}(t)\,dt$$

leads to

$$|y^{(n-2)}(z_k)| \leqq |y^{(n-2)}(b_k)| + \int_{z_k}^{b_k} |y^{(n-1)}(t)|\,dt. \tag{18}$$

Again by (15) and (18) we have

$$d_k/2 \leqq \int_{z_k}^{b_k} |y^{(n-1)}(t)|\,dt. \tag{19}$$

Adding equations (17) and (19) we have

$$d_k \leqq \int_{a_k}^{b_k} |y^{(n-1)}(t)|\,dt. \tag{20}$$

Squaring both sides of (20) we get

$$\begin{aligned} d_k^2 &\leqq \left[\int_{a_k}^{b_k} |y^{(n-1)}(t)|\,dt\right]^2 \\ &= \left[\int_{a_k}^{b_k} \frac{1}{(r(t))^{1/2}}(r)^{1/2}|y^{(n-1)}(t)|^{1/2}\cdot|y^{(n-1)}(t)|^{1/2}\,dt\right] \\ &\leqq \int_{a_k}^{b_k} \frac{1}{r(t)}\,dt\cdot\int_{a_k}^{b_k} \{ry^{(n-1)}(t)\}\cdot y^{(n-1)}(t)\,dt \end{aligned}$$

by Schwarz's inequality. Thus

$$\left(d_k^2 \Big/ \int_{a_k}^{b_k} \frac{1}{r}\,dt\right) \leqq \int_{a_k}^{b_k} \{ry^{(n-1)}\}y^{(n-1)}\,dt. \tag{21}$$

Integrating the right-hand side of (21) by parts we have

$$
(22)\qquad d_k^2 \Big/ \int_{a_k}^{b_k} \frac{1}{r}\,dt \leqq [y^{(n-2)}(b_k)r(b_k)y^{(n-1)}(b_k) - y^{(n-2)}(a_k)r(a_k)y^{(n-1)}(a_k)] - \int_{a_k}^{b_k} (r(t)y^{(n-1)}(t))'y^{(n-2)}(t)\,dt.
$$

If $y^{(n-2)}(t) > 0$ in $[t_{k-1}, t_k]$, then the choice of a_k and b_k in $[t_{k-1}, t_k]$ implies $y^{(n-1)}(b_k) \leqq 0$ and $y^{(n-1)}(a_k) \geqq 0$. Similarly if $y^{(n-2)}(t) < 0$ in $[t_{k-1}, t_k]$, then the choice of a_k and b_k in $[t_{k-1}, t_k]$ implies $y^{(n-1)}(b_k) \geqq 0$ and $y^{(n-1)}(a_k) \leqq 0$. Thus in any case we have the following inequality for the first term on the right of (22), namely,

$$
(23)\qquad y^{(n-2)}(b_k)r(b_k)y^{(n-1)}(b_k) - y^{(n-2)}(a_k)r(a_k)y^{(n-1)}(a_k) \leqq 0.
$$

From (22) and (23) we have

$$
(24)\qquad d_k^2 \Big/ \int_{a_k}^{b_k} \frac{1}{r}\,dt \leqq - \int_{a_k}^{b_k} (r(t)y^{(n-1)}(t))'y^{(n-2)}(t)\,dt.
$$

Making use of (5) in (24) we get

$$
(25)\qquad \begin{aligned} d_k^2 \Big/ \int_{a_k}^{b_k} \frac{1}{r}(t)\,dt &\leqq \int_{a_k}^{b_k} y^{(n-2)}(t)a(t)y_\tau(t)\,dt - \int_{a_k}^{b_k} y^{(n-2)}(t)f(t)\,dt \\ &\leqq \int_{a_k}^{b_k} |y^{(n-2)}(t)|a(t)y_\tau(t)\,dt + \int_{a_k}^{b_k} |y^{(n-2)}(t)|\,|f(t)|\,dt. \end{aligned}
$$

Since $|y^{(n-2)}(t)| \leqq d_k$ in $[a_k, b_k]$, we have

$$
(26)\qquad d_k \Big/ \int_{a_k}^{b_k} \frac{1}{r}(t)\,dt \leqq \int_{a_k}^{b_k} a(t)y_\tau(t)\,dt + \int_{a_k}^{b_k} |f(t)|\,dt.
$$

Now

$$
\int_T^\infty a(t)y_\tau(t)\,dt \geqq \sum_{k=1}^\infty \int_{a_k}^{b_k} a(t)y_\tau(t)\,dt
$$

from which

$$
(27)\qquad \int_T^\infty a(t)y_\tau(t)\,dt \geqq \sum_{k=1}^\infty \frac{d_k}{\int_{a_k}^{b_k}(1/r(t))\,dt} - \varepsilon \geqq \frac{3d}{4}\sum_{k=1}^\infty \frac{1}{\int_{a_k}^{b_k}(1/r(t))\,dt} - \varepsilon
$$

on using (26) and the fact that

$$
\varepsilon > \int_{t_0}^\infty |f(t)|\,dt \geqq \sum_{k=1}^\infty \int_{a_k}^{b_k} |f(t)|\,dt.
$$

Now $\lim_{k\to\infty} \int_{a_k}^{b_k} 1/r(t)\,dt = 0$ by condition (i) since $a_k \to \infty$, $b_k \to \infty$. But then the right-hand side of (27) tends to ∞ as $t \to \infty$ since $d > 0$. This is a contradiction to (8). Thus as long as $\limsup_{t\to\infty} |y^{(n-2)}(t)|$ remains greater than any positive number d, we will encounter the above contradiction. Hence

$$
\limsup_{t\to\infty} |y^{(n-2)}(t)| = 0
$$

and the theorem is proved.

3. Applications.

THEOREM 2. *Under the conditions of Theorem* 1, *all nonoscillatory solutions of*

$$(r(t)y'(t))' + a(t)y_\tau(t) = f(t) \tag{28}$$

tend to zero as $t \to \infty$.

Proof. From Theorem 1, if $y(t)$ is a nonoscillatory solution of (28), then taking $n = 2$, we find that $y(t)$ tends to a finite limit as $t \to \infty$. Now integrating (28) over $[T, t]$, where $y_\tau(t) > 0$ for $t \geqq T$ (as before) we have

$$r(t)y'(t) - r(T)y'(T) + \int_T^t a(t)y_\tau(t)\,dt \leqq \int_T^t |f(t)|\,dt.$$

From Case 1 of the proof of Theorem 1, if

$$\int_T^\infty a(t)y_\tau(t)\,dt = \infty,$$

then $y'(t) < 0$ and $y(t) \to 0$ as $t \to \infty$.

From Case 2 of the proof of Theorem 1, if

$$\int_T^\infty a(t)y_\tau(t)\,dt < \infty,$$

then

$$\liminf_{t\to\infty} y_\tau(t) = \liminf_{t\to\infty} y(t) = 0 \Rightarrow y(t) \to 0 \quad \text{as } t \to \infty.$$

THEOREM 3. *In addition to the conditions of Theorem* 1, *suppose* $r(t)$ *is bounded away from zero*, $0 \leqq \tau'(t) \leqq L < 1$, *and*

$$\lim_{t\to\infty} \left[\left(\int^t a(s)\,ds\right)\Big/ r(t)\right] = \infty. \tag{29}$$

Then all nonoscillatory solutions of

$$(r(t)y''(t))' + a(t)y_\tau(t) = f(t) \tag{30}$$

tend to zero as $t \to \infty$.

Proof. Let $y(t)$ be a nonoscillatory solution of (30). As in the proof of Theorem 1, let T be large enough so that for $t \geqq T$, $y(t)$ and $y_\tau(t)$ are nonnegative (without any loss). Integrating both sides of (30) over $[T, t]$ we get

$$r(t)y''(t) - r(T)y''(T) + \int_T^t a(s)y_\tau(s)\,ds \leqq \int_T^t |f(s)|\,ds. \tag{31}$$

Suppose first that

$$\lim_{t\to\infty} \int_T^t a(s)y_\tau(s)\,ds = \infty.$$

Since the right-hand side of (31) is bounded as $t \to \infty$, we have

$$r(t)y''(t) \to -\infty$$

as $t \to \infty$. Since $r(t) > 0$, $y''(t) \leqq 0$ eventually. Let $T_1 > T$ be large enough so that for $t \geqq T_1$, $y''(t) \leqq 0$. Thus $y(t)$ is monotonic. Also $y'(t) \geqq 0$ eventually, because otherwise $y(t)$ will become negative. Thus $y(t)$ is increasing. Dividing (31) by $r(t)$, we have

$$y''(t) - \frac{r(T)y''(T)}{r(t)} + \frac{y_\tau(T)\int_T^t a(s)\,ds}{r(t)} \leqq \frac{\int_T^t |f(s)|\,ds}{r(t)}. \tag{32}$$

Since $r(t)$ is bounded away from zero, the right-hand side of (32) remains bounded, while on the left,

$$\lim_{t\to\infty} \frac{\int_T^t a(s)\,ds}{r(t)} = \infty .$$

Thus $y''(t) \to -\infty$ as $t \to \infty$. But this forces $y(t)$ to be negative, which is a contradiction. Hence we must have

$$\lim_{t\to\infty} \int_T^t a(s)y_\tau(s)\,ds < \infty . \tag{33}$$

As in the proof of Theorem 1,

$$\int_T^\infty a(s)\,ds = \infty ,$$

hence we have from (33),

$$\liminf_{t\to\infty} y_\tau(t) = 0. \tag{34}$$

Suppose to the contrary that

$$\limsup_{t\to\infty} y_\tau(t) > p > 0. \tag{35}$$

In a manner of Hammett [7], there exists a sequence of numbers $\{S_n\}$, $n \geqq 0$, with the following properties:

(A) $\lim_{n\to\infty} S_n = \infty$, $S_n \geqq T$ for all n.
(B) For each n, $y_\tau(S_n) > p$.
(C) For each $n \geqq 1$, there exists a number S_n' such that $S_{n-1} < S_n' < S_n$ and $y_\tau(S_n') < p/2$.

Let (α_n, β_n) be the largest open interval containing S_n. Note that

$$y_\tau(\alpha_n) = y_\tau(\beta_n) = p/2$$

for $n \geqq 1$. Now in $[\alpha_n, S_n]$, there exists a number $S_n'' \in (\alpha_n, S_n)$ such that

$$\{1 - \tau'(S_n'')\}y_\tau'(S_n'') = \frac{y_\tau(S_n) - y_\tau(\alpha_n)}{S_n - \alpha_n} > \frac{p - p/2}{\beta_n - \alpha_n} = \frac{p}{2(\beta_n - \alpha_n)}. \tag{36}$$

But by Case 2 of the proof of Theorem 1 applied to (30), we have $y_\tau'(t) \to 0$ as $t \to \infty$. Since $\tau'(t) < L$, it follows from (36) that

$$\lim_{n\to\infty} (\beta_n - \alpha_n) = \infty$$

since $p > 0$.

Also because of the way in which α_n and β_n were chosen, we have

$$y_\tau(t) \geqq p/2 > 0$$

on $[\alpha_n, \beta_n]$. Now from (33),

$$\begin{aligned} \infty &> \int_T^\infty a(s) y_\tau(s)\, ds \\ &\geqq \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} a(s) y_\tau(s)\, ds \\ &\geqq \frac{p}{2} \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} a(s)\, ds = \infty . \end{aligned}$$

This is the required contradiction and the proof is complete. Hence $y(t) \to 0$ as $t \to \infty$.

To satisfy Theorem 3, consider the equation

$$(e^{t/4} y''(t))' + e^{(t/2)-\pi} y(t-\pi) = -3/4\, e^{-3t/4} + e^{-t/2}. \tag{37}$$

Here

$$r(t) \equiv e^{t/4}, \qquad a(t) = e^{(t/2)-\pi}, \qquad f(t) = -3/4\, e^{-3t/4} + e^{-t/2},$$

$$\frac{\int_T^t a(t)}{r(t)} = \frac{2(e^{(t/2)-\pi} - e^{(T/2)-\pi})}{e^{t/4}} \to \infty \quad \text{as } t \to \infty .$$

All conditions of Theorem 3 are satisfied. Hence all nonoscillatory solutions of (37) tend to zero as $t \to \infty$. In fact $y(t) = e^{-t}$ is a nonoscillatory solution of (37).

As an example of Theorem 2, we consider the equation

$$(e^t y'(t))' + e^{(t/2)-\pi} y(t-\pi) = e^{-t/2}. \tag{38}$$

All conditions of Theorem 2 are satisfied. Thus all nonoscillatory solutions of (38) approach zero as $t \to \infty$. The function e^{-t} is again a solution of (38).

COROLLARY 1. *Under the conditions of Theorem* 3, *if $y(t)$ is a nonoscillatory solution of* (5), *then $y^{(n-2)}(t) \to 0$ as $t \to \infty$.*

Proof. We find that Case 1 of the proof of Theorem 1 does not materialize. The rest of the proof is the same as that of Theorem 1 in Case 2.

Remark. In the beginning it was assumed for convenience that $a(t)$, $f(t)$ and $r(t)$ were to be continuous on the whole real line R. In fact these functions need be continuous only on some positive half-line $[T_0, \infty]$. Since $\tau(t)$ is bounded, all the results remain valid.

The following example shows that it may not be possible to weaken condition (29) of Theorem 3 if all other conditions are satisfied. Consider the equation

$$(4t^3 y''(t))' + \frac{3}{2} y(t) = \frac{45}{2t^2}, \qquad t > 0. \tag{39}$$

This equation has

$$y(t) = t^{1/2} - 1/t^2$$

as a nonoscillatory solution approaching ∞ as $t \to \infty$. All conditions are satisfied except condition (29).

4. Bounded $r(t)$.

THEOREM 4. *Suppose $n = 2$ and conditions* (ii) *and* (iii) *of Theorem* 1 *hold. Further suppose that $r(t)$ is a positive bounded function that satisfies the following*:

$$\lim_{k\to\infty} \int_{a_k}^{b_k} \frac{1}{r(t)}\,dt = \infty, \quad \text{then } (b_k - a_k) \to \infty \text{ as } k \to \infty. \tag{40}$$

Suppose $0 \leqq \tau'(t) \leqq L < 1$. *Let $y(t)$ be a nonoscillatory solution of* (1). *Then*

$$y(t) \to 0 \quad \text{as } t \to \infty.$$

Proof. We proceed as in Theorem 1 and arrive at Case 1. Since $r(t)$ is bounded we have

$$r(t)y'(t) \to -\infty \Rightarrow y'(t) \to -\infty \Rightarrow y(t) \to -\infty,$$

which is a contradiction. Hence we must have Case 2. Following the proof further we arrive at conclusion (26).

Now $d_k > 3d/4 > 0$ and the right-hand side of (26) approaches zero as $k \to \infty$ due to condition (iii) and conclusion (8), we must have

$$\lim_{k\to\infty} \int_{a_k}^{b_k} \frac{1}{r(t)}\,dt = \infty \tag{41}$$

which from (40) gives

$$(b_k - a_k) \to \infty \quad \text{as } k \to \infty. \tag{42}$$

Let

$$g(t) = t - \tau(t).$$

Then $g(t)$ is increasing. Also let

$$g(\alpha_k) = a_k \quad \text{and} \quad g(\beta_k) = b_k.$$

Now

$$\begin{aligned}(b_k - a_k) &= \frac{g(\beta_k) - g(\alpha_k)}{\beta_k - \alpha_k}(\beta_k - \alpha_k)\\ &= g'(\delta_k)(\beta_k - \alpha_k)\\ &\leqq (\beta_k - \alpha_k)L_0 \quad \text{for some } L_0 > 0\end{aligned}$$

since $0 \leqq \tau'(t) < L < 1$. Thus

$$\beta_k - \alpha_k \to \infty \quad \text{as } k \to \infty. \tag{43}$$

Again, since $g(t)$ is increasing,

$$g(t) \in [a_k, b_k] \quad \text{implies } t \in [\alpha_k, \beta_k]$$

and

$$|y(g(t))| \geqq d_k/2 \quad \text{for } t \in [\alpha_k, \beta_k].$$

Hence

$$y_\tau(t) \geqq d_k/2 \geqq 3d/8 \tag{44}$$

for $t \in [\alpha_k, \beta_k]$. From (8) we have

$$\begin{aligned}\infty &> \int_T^\infty a(t)y_\tau(t)\,dt \\ &> \int_{\alpha_k}^{\beta_k} a(t)y_\tau(t)\,dt \\ &> \frac{3d}{8}\int_{\alpha_k}^{\beta_k} a(t)\,dt \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

This contradiction completes the proof.

THEOREM 5. *Let $n = 3$ and conditions of Theorem 2 hold. Then nonoscillatory solutions of equation*

$$(r(t)y''(t))' + a(t)y_\tau(t) = f(t)$$

approach zero.

Proof. Let $y(t)$ be a nonoscillatory solution of (1). We follow the proof of Theorem 4 and arrive at

$$\lim_{k\to\infty}(b_k - a_k) = \infty, \qquad \lim_{k\to\infty}(\beta_k - \alpha_k) = \infty. \tag{45}$$

Here we know that

$$|y'(t)| \geqq d_k/2, \qquad t \in [a_k, b_k], \tag{46}$$

$$|y_\tau'(t)| \geqq d_k/2, \qquad t \in [\alpha_k, \beta_k]. \tag{47}$$

In view of (8), (45) and condition (ii), let k_1 be large enough to insure that

$$\int_{\alpha_{k_1}}^{(\alpha_{k_1}+\beta_{k_1})/2} a(t)\,dt > 1, \tag{48}$$

$$\int_{\alpha_{k_1}}^{\beta_{k_1}} a(t)y_\tau(t)\,dt < 1 \tag{49}$$

and

$$\beta_{k_1} - \alpha_{k_1} \geqq \frac{16}{3d(1-L)}. \tag{50}$$

Case 1. $y_\tau'(t) < 0$ in $[\alpha_{k_1}, \beta_{k_1}]$.

Here

$$y_\tau'(t) < -d_k/2 \leqq -3d/8. \tag{51}$$

We now consider the t-interval

$$[\tfrac{1}{2}\alpha_{k_1} + \tfrac{1}{2}\beta_{k_1}, \beta_{k_1}].$$

By the mean value theorem, there exists $\delta_{k_1} \in (\frac{1}{2}\alpha_{k_1} + \frac{1}{2}\beta_{k_1}, \beta_{k_1})$ such that

$$y_\tau(\tfrac{1}{2}\alpha_{k_1} + \tfrac{1}{2}\beta_{k_1}) + (t - \tfrac{1}{2}\alpha_{k_1} - \tfrac{1}{2}\beta_{k_1})y_\tau'(\delta_{k_1})(1 - \tau'(\delta_{k_1})) = y_\tau(t) > 0. \tag{52}$$

Taking $t = \beta_{k_1}$ we have

$$y_\tau(\tfrac{1}{2}\alpha_{k_1} + \tfrac{1}{2}\beta_{k_1}) > (3d/8)(1 - L)(\tfrac{1}{2}\beta_{k_1} - \tfrac{1}{2}\alpha_{k_1}) \tag{53}$$

since $0 \leqq \tau'(t) < L$.

From (50) and (53) we have

$$y_\tau(\tfrac{1}{2}\alpha_{k_1} + \tfrac{1}{2}\beta_{k_1}) > 1 \tag{54}$$

and since $y_\tau'(t) < 0$ in $[\alpha_{k_1}, \beta_{k_1}]$, from (54) we have

$$y_\tau(t) > 1, \qquad t \in [\alpha_{k_1}, \tfrac{1}{2}\alpha_{k_1} + \tfrac{1}{2}\beta_{k_1}]. \tag{55}$$

From (54) and (55) we have

$$\begin{aligned}\int_{\alpha_{k_1}}^{\beta_{k_1}} a(t)y_\tau(t)\,dt &\geqq \int_{\alpha_{k_1}}^{(\beta_{k_1}+\alpha_{k_1})/2} a(t)y_\tau(t)\,dt \\ &\geqq \int_{\alpha_{k_1}}^{(\beta_{k_1}+\alpha_{k_1})/2} a(t)\,dt \\ &\geqq 1, \quad \text{from (48)}.\end{aligned}$$

But this is a contradiction to (49), and hence we must have the following.

Case 2.

$$y_\tau'(t) > 0, \qquad t \in [\alpha_{k_1}, \beta_{k_1}].$$

A similar analysis as in Case 1 yields a contradiction. Hence

$$y'(t) \to 0 \quad \text{as } t \to \infty.$$

The proof now is the same as that of Theorem 3 from conclusion (34) onward.

The proof of Theorem 5 is now complete.

Remark. Condition (40) on $r(t)$ is automatically satisfied when $r(t) \geqq \gamma > 0$. Thus Hammett's results are generalized.

Remark. The hypothesis of boundedness of $r(t)$ was used in Theorems 4 and 5 only to eliminate Case 1 of the proof of Theorem 1. Thus in the statements of Theorem 4 and Theorem 5, if we eliminate the boundedness requirement on $r(t)$, the nonoscillatory solutions of the corresponding equations approach *finite* limits.

For example, the equation

$$(ty'(t))' + y(t - \pi) = \frac{1}{(t-\pi)^2} + \frac{4}{t^3}, \qquad t > \pi, \tag{56}$$

has

$$y(t) = 1/t^2$$

as a nonoscillatory solution approaching a *finite* limit as $t \to \infty$. This is not covered by Theorem 2. But Theorem 4 applies, since if

$$\int_{a_k}^{b_k} \frac{1}{r(t)}\,dt = (\ln b_k - \ln a_k) \to \infty \quad \text{as } k \to \infty,$$

then $(b_k - a_k) \to \infty$ as $k \to \infty$.

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INTEGRAL OPERATORS AND THE NONCHARACTERISTIC CAUCHY PROBLEM FOR PARABOLIC EQUATIONS*

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Abstract. In this paper integral operators for parabolic equations in three space variables are constructed and then used to construct a singular solution for these equations. Using this singular solution, an integral representation for the solution to a noncharacteristic surface Cauchy problem is found.

1. Introduction. We are interested in deriving an integral representation for solutions to the noncharacteristic Cauchy problem for the partial differential operator

$$l[u] \equiv \Delta_3 u + a(x_1\ x_2, x_3, t)u - b(x_1, x_2, x_3, t)u_t. \tag{1.1}$$

The coefficients in (1.1) are assumed to be entire functions of the (complex) variables x_1, x_2, x_3 and t, and $\Delta_3 u$ denotes $\sum_{i=1}^{3} \partial^2 u/\partial x_i^2$. Such a problem arises when one considers various inverse single phase free boundary problems. For a more thorough discussion of these matters the reader is referred to [2], [3], [6], [11] and [14].

To get the desired integral representation one first constructs a particular singular solution for the equation $l[u] = 0$ (cf. § 4). An expression involving the Cauchy data and the singular solution is then integrated over a region in $\mathbb{C}^4$ [cf. (5.10)]. This technique is a generalization of the methods of Colton ([3] and [5]) and Hill ([11] and [12]) whose work was for parabolic equations of 1- and 2-space variables. In § 2 Bergman type integral operators are constructed which map analytic functions of 3-complex variables onto solutions of (1.1). For a thorough treatment of integral operators for elliptic equations see [1], [7], [10] and [15]. For integral operators for parabolic equations the reader is referred to [1] and [4]. In § 3 inversion formulas for the operators constructed in § 2 are given. These formulas are then used in § 4 to construct the desired singular solution.

The results contained in this paper are part of the author's Ph.D. thesis written at Indiana University, Bloomington, Indiana, under the direction of Professor David Colton, whose advice and encouragement were invaluable.

2. Integral operators. In this section integral operators are constructed which map analytic functions of three complex variables onto analytic solutions of $l[u] = 0$. Inversion formulas (which are given in § 3) will enable us to conclude that this mapping is in fact onto.

We first introduce the variables z, z^*, x and t defined by

$$x_1 = z - z^*, \quad x_2 = \frac{z + z^*}{i}, \quad x_3 = x, \quad t = t. \tag{2.1}$$

We note that x_1 and x_2 will be real if and only if $z^* = -\bar{z}$ ($\bar{z}$ denotes the complex

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conjugate of z). This change of variable transforms (1.1) into

$$L[U] \equiv U_{xx} - U_{zz^*} + A(z, z^*, x, t)U - B(z, z^*, x, t)U_t, \tag{2.2}$$

where $A(z, z^*, x, t) = a(x_1, x_2, x_3, t)$, $B(z, z^*, x, t) = b(x_1, x_2, x_3, t)$ and $U(z, z^*, x, t) = u(x_1, x_2, x_3, t)$. Motivated by the work of Colton [4], we look for a solution of (2.2) of the form

$$\begin{aligned} U(z, z^*, x, t) &= \mathscr{E}^{(1)}[f](z, z^*, x, t) \\ &\equiv \int_{|\tau - t| = \delta} \int_{|\zeta| = 1} \int_{\gamma} E^{(1)}(z, z^*, x, t, s, \zeta, \tau) \\ &\qquad \cdot f(\mu[1 - s^2], \zeta, \tau) \frac{ds}{\sqrt{1 - s^2}} \frac{d\zeta}{\zeta} d\tau, \end{aligned} \tag{2.3}$$

where $\mu = x + \zeta z + \zeta^{-1} z^*$, and γ is a path from -1 to 1 not passing through the origin. Let $D_\varepsilon = \{\zeta : 1 - \varepsilon < |\zeta| < 1 + \varepsilon\}$, $0 < \varepsilon < 1$, D_γ be some domain in $\mathbb{C}$ containing γ, and $G = \mathbb{C}^3 \times D_\gamma \times D_\varepsilon \times \{\mathbb{C}^2 | (\tau = t)\}$.

THEOREM 2.1. *Let $E^{(1)} = E^{(1)}(z, z^*, x, t, s, \zeta, \tau)$ be an analytic function in G, and suppose E satisfies the partial differential equation*

$$\begin{aligned} &E_{xx}^{(1)} - E_{zz^*}^{(1)} + A(z, z^*, x, t)E^{(1)} - B(z, z^*, x, t)E_t^{(1)} \\ &\quad - \frac{1 - s^2}{2\mu s \zeta} E_{zs}^{(1)} + \frac{1}{2\mu s^2 \zeta} E_z^{(1)} - \frac{\zeta(1 - s^2)}{2\mu s} E_{z^*s}^{(1)} \\ &\quad + \frac{\zeta}{2\mu s^2} E_{z^*}^{(1)} + \frac{1 - s^2}{\mu s} E_{xs}^{(1)} - \frac{1}{\mu s^2} E_x^{(1)} = 0. \end{aligned} \tag{2.4}$$

If $f(\mu, \zeta, \tau)$ is an analytic function of three complex variables in a domain $D_1 \times D_2 \times D_3$ such that $\{\zeta : |\zeta| = 1\} \subset D_2$, then $\mathscr{E}^{(1)}[f](z, z^*, x, t)$, as defined by (2.3), *is an analytic function of its* (*complex*) *variables and satisfies $L[\mathscr{E}^{(1)}[f]] = 0$ in the domain $H_1 \times H_2 \times H_3 \times D_3$, where H_i, $i = 1, 2, 3$, have the property that if $(z, z^*, x) \in H_1 \times H_2 \times H_3$, then $(x + \zeta z + \zeta^{-1} z^*)(1 - s^2) \in D_1$ for all $|\zeta| = 1$ and $s \in \gamma$.*

Proof. The fact that $\mathscr{E}^{(1)}[f](z, z^*, x, t)$ is an analytic function of its arguments is obvious. The conditions on the domains ensure that the integrations over the respective paths make sense. Thus all that remains is to verify that $L[\mathscr{E}^{(1)}[f]] = 0$. Substituting (2.3) into (2.2) gives

$$\begin{aligned} &L[\mathscr{E}^{(1)}[f]] \\ &\quad = \int_{|\tau - t| = \delta} \int_{|\zeta| = 1} \int_{\gamma} \{[E_{xx}^{(1)} f + 2E_x^{(1)}(1 - s^2) f_1 + E^{(1)}(1 - s^2)^2 f_{11}] \\ &\qquad + [E_{zz^*}^{(1)} f + E_z^{(1)} \zeta^{-1}(1 - s^2) f_1 + E_{z^*}^{(1)} \zeta (1 - s^2) f_1 + E^{(1)}(1 - s^2)^2 f_{11}] \\ &\qquad + A(z, z^*, x, t)E^{(1)} f - B(z, z^*, x, t)E_t^{(1)} f\} \frac{ds}{\sqrt{1 - s^2}} \frac{d\zeta}{\zeta} d\tau, \end{aligned} \tag{2.5}$$

where $f_1 = \partial f/\partial\mu$ and $f_{11} = \partial^2 f/\partial\mu^2$. But $\partial f/\partial\mu = (-1/2s\mu)(\partial f/\partial s)$, and integrating by parts shows that if $E^{(1)}$ satisfies (2.4), then (2.5) is identically zero.

We now wish to show that an $E^{(1)}$ function satisfying the conditions of Theorem 2.1 exists. To this end we introduce the following variables:

$$\xi_1 = 2\zeta z, \qquad \xi_2 = x + 2\zeta z, \qquad \xi_3 = x + 2\zeta^{-1}z^*, \qquad \mu = \tfrac{1}{2}(\xi_2 + \xi_3),$$

$$\begin{aligned} \tilde{E} &= \tilde{E}(\xi_1, \xi_2, \xi_3, t; s, \zeta, \tau) = E^{(1)}(z, z^*, x, t; s, \zeta, \tau),\\ \tilde{A} &= \tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta) = A(z, z^*, x, t),\\ \tilde{B} &= \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta) = B(z, z^*, x, t). \end{aligned} \tag{2.6}$$

Equation (2.4) is transformed to

$$\begin{aligned} &\mu s(\tilde{E}_{22} - 2\tilde{E}_{23} + \tilde{E}_{33} - 4\tilde{E}_{13} + \tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta)\tilde{E}\\ &\qquad - \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta)\tilde{E}_t) - (1 - s^2)\tilde{E}_{1s} + (1/s)\tilde{E}_1 = 0. \end{aligned} \tag{2.7}$$

We now look for a solution to (2.7) of the form

$$\tilde{E}(\xi_1, \xi_2, \xi_3, t; s, \zeta, \tau) = \frac{1}{\tau - t} + \sum_{n=1}^{\infty} s^{2n} p^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau), \tag{2.8}$$

where $\tilde{E}(0, \xi_2, \xi_3, t; s, \zeta, \tau) = 1/(\tau - t)$, and the $p^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau)$, $n = 1, 2, \cdots$, are to be determined. Substituting (2.8) into the differential equation (2.7) leads to the recursive relation

$$p_1^{(1)} = \frac{\tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta)}{\tau - t} - \frac{\tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta)}{(\tau - t)^2}, \tag{2.9a}$$

$$\begin{aligned} p_1^{(n+1)} = \frac{1}{2n+1}\{&p_{22}^{(n)} - 2p_{23}^{(n)} + p_{33}^{(n)} - 4p_{13}^{(n)} + \tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta)p^{(n)}\\ &- \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta)p_t^{(n)}\} \end{aligned} \tag{2.9b}$$

and

$$p^{(n)}(0, \xi_2, \xi_3, t; \zeta) = 0, \qquad n = 1, 2, \cdots, \tag{2.9c}$$

where $p_i^{(n)} = \partial p^{(n)}/\partial\xi_i$ and $p_{ij}^{(n)} = \partial^2 p^{(n)}/(\partial\xi_i\partial\xi_j)$. Now let $Q^{(n)} = Q^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta)$ be defined by

$$Q^{(n)} = (\tau - t)^{n+1}p^{(n)}, \qquad n = 1, 2, \cdots. \tag{2.10}$$

An easy calculation shows that the $p^{(n)}$ satisfy (2.9) if and only if the $Q^{(n)}$ satisfy

$$Q_1^{(1)} = (\tau - t)\tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta) - \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta), \tag{2.11a}$$

$$\begin{aligned} Q_1^{(n+1)} = \frac{\tau - t}{2n+1}\{&Q_{22}^{(n)} - 2Q_{23}^{(n)} + Q_{33}^{(n)} - 4Q_{13}^{(n)} + \tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta)Q^{(n)}\\ &- \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta)Q_t^{(n)} - \frac{n+1}{\tau - t}B(\xi_1, \xi_2, \xi_3, t; \zeta)Q^{(n)}\} \end{aligned} \tag{2.11b}$$

and

$$Q^{(n)}(0, \xi_2, \xi_3, t; \zeta) = 0, \qquad n = 1, 2, \cdots. \tag{2.11c}$$

In what follows, the concept of a dominate is needed. We give a definition and refer the reader to [1] and [10] for further details.

DEFINITION 2.1. Let

$$f(z_1, z_2, \cdots, z_k; \zeta) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} a_{n_1 \cdots n_k}(\zeta) z_1^{n_1} \cdots z_k^{n_k}$$

and

$$g(z_1, \cdots, z_k) = \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} b_{n_1 \cdots n_k} z_1^{n_1} \cdots z_k^{n_k}.$$

We will say g is a dominate of f (denoted by $f(z_1, \cdots, z_k; \zeta) \ll g(z_1, \cdots, z_k)$ or $f \ll g$), if $0 \leqq |a_{n_1, \cdots, n_k}(\zeta)| \leqq b_{n_1, \cdots, n_k}$, $n = 0, 1, \cdots$, $i = 1, 2, \cdots, k$, and the inequalities hold for all relevant values of the parameter ζ.

LEMMA 2.1. *Let $\tilde{A} = \tilde{A}(\xi_1, \xi_2, \xi_3, t; \zeta)$ and $\tilde{B} = \tilde{B}(\xi_1, \xi_2, \xi_3, t; \zeta)$ be entire functions in τ and ξ_i, $i = 1, 2, 3$, and analytic in ζ for $\zeta \in \bar{D}_\varepsilon$. Then there exists a sequence of functions $Q^{(n)} = Q^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau)$, $n = 1, 2, \cdots$, which are entire in t, τ and ξ_i, $i = 1, 2, 3$, and analytic in ζ for ζ in $\bar{D}_\varepsilon$ such that the $Q^{(n)}$ satisfy* (2.11). *Moreover for each pair δ_1, δ_2 of positive constants there exists a positive constant $M = M(\delta_1, \delta_2)$ such that*

$$\begin{aligned} &Q^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau) \\ (2.12) \qquad &\ll M\left(26\frac{\delta_2}{\delta_1}\right)^n \left\{\left(1 - \frac{\xi_1}{\delta_1}\right)\left(1 - \frac{\xi_2}{\delta_1}\right)\left(1 - \frac{\xi_3}{\delta_1}\right)\left(1 - \frac{t}{\delta_2}\right)\left(1 - \frac{\tau}{\delta_2}\right)\right\}^{-3n} \end{aligned}$$

for $|\xi_i| < \delta_1$, $i = 1, 2, 3$, $|t| < \delta_2$, $|\tau| < \delta_2$, and $\zeta \in \bar{D}_\varepsilon$, $n = 1, 2, \cdots$.

Proof. Let $C = C(\delta_1, \delta_2)$ be a positive constant such that

$$(2.13) \qquad \tilde{A}, \tilde{B} \ll C\left\{\left(1 - \frac{\xi_1}{\delta_1}\right)\left(1 - \frac{\xi_2}{\delta_1}\right)\left(1 - \frac{\xi_3}{\delta_1}\right)\left(1 - \frac{t}{\delta_2}\right)\right\}^{-1}$$

for $|\xi_i| < \delta_1$, $i = 1, 2, 3$, $|t| < \delta_2$, and ζ in $\bar{D}_\varepsilon$. Let $N \geqq 2(\delta_1)^2 C$. Set

$$(2.14) \quad Q^{(1)}(\xi_1, \xi_2, \xi_3, t; \zeta) = \int_0^{\xi_1} [(\tau - t)\tilde{A}(\eta, \xi_2, \xi_3, t; \zeta) - \tilde{B}(\eta, \xi_2, \xi_3, t; \zeta)]\, d\eta$$

and

$$Q^{(n+1)}(\xi_1, \xi_2, \xi_3, t; \zeta) = \int_0^{\xi_1} H^{(n)}(\eta, \xi_2, \xi_3, t; \zeta, \tau)\, d\eta, \qquad n = 1, 2, \cdots,$$

where $H^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau)$ is the right-hand side of (2.11b). We see that each $Q^{(n)}$ has the required domain of analyticity and clearly (2.11) is satisfied. Since each $Q^{(n)}$ is analytic we may find an $M = M(\delta_1, \delta_2)$ such that (2.12) holds with this M for $k \leqq N$. We now assume that (2.12) holds for $n = k$ and show that it holds for $n = k + 1$. Using the standard properties of dominates and the fact that

$$(2.15) \qquad \tau, t \ll \delta_2\left\{\left(1 - \frac{t}{\delta_2}\right)\left(1 - \frac{\tau}{\delta_2}\right)\right\}^{-1}$$

if $|t|$, $|\tau| < \delta_2$, we find that

$$
\begin{aligned}
Q_1^{(k+1)} \ll M\left(26\frac{\delta_2}{\delta_1}\right)^k\frac{\delta_2}{\delta_1}\left\{\frac{16(3k+1)^2}{(2k+1)\delta_1} + C\left(\frac{(7k+1)\delta_1}{(2k+1)\delta_2} + \frac{2\delta_1}{2k+1}\right)\right\} \\
\cdot\left\{\left(1-\frac{\xi_1}{\delta_1}\right)\left(1-\frac{\xi_2}{\delta_1}\right)\left(1-\frac{\xi_3}{\delta_1}\right)\left(1-\frac{t}{\delta_2}\right)\left(1-\frac{\tau}{\delta_2}\right)\right\}^{-3(k+1)},
\end{aligned}
\tag{2.16}
$$

and hence

$$
\begin{aligned}
Q^{(k+1)} \ll M\left(26\frac{\delta_2}{\delta_1}\right)^k\frac{\delta_2}{\delta_1}\frac{\delta_1}{3(k+1)}\left\{\frac{16(3k+1)^2}{(2k+1)\delta_1} + C\left(\frac{(7k+1)\delta_1}{(2k+1)\delta_2} + \frac{2\delta_1}{2k+1}\right)\right\} \\
\cdot\left\{\left(1-\frac{\xi_1}{\delta_1}\right)\left(1-\frac{\xi_2}{\delta_1}\right)\left(1-\frac{\xi_3}{\delta_1}\right)\left(1-\frac{t}{\delta_2}\right)\left(1-\frac{\tau}{\delta_2}\right)\right\}^{-3(k+1)}
\end{aligned}
\tag{2.17}
$$

Without loss of generality we may assume that $\delta_2 \geqq 1$ and $k \geqq N \geqq 2C(\delta_1)^2$. We thus have

$$
Q^{(k+1)} \ll M\left(26\frac{\delta_2}{\delta_1}\right)^{k+1}\left\{\left(1-\frac{\xi_1}{\delta_1}\right)\left(1-\frac{\xi_2}{\delta_1}\right)\left(1-\frac{\xi_3}{\delta_1}\right)\left(1-\frac{t}{\delta_2}\right)\left(1-\frac{\tau}{\delta_2}\right)\right\}^{-3(k+1)}
\tag{2.18}
$$

It now follows by induction that (2.12) holds for all n.

LEMMA 2.2. *Let $\tilde{A}$ and $\tilde{B}$ be as in Lemma 2.1. Let $p^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau)$ satisfy (2.9). Then the series $\sum_{n=1}^{\infty} s^{2n}\mu^n p^{(n)}$ converges absolutely and uniformly on compact subsets of $G = \mathbb{C}^4 \times \bar{D}_\varepsilon \times \{\mathbb{C}^2|(\tau = t)\}$, where $\mathbb{C}^4$ is the space of four complex variables (ξ_1, ξ_2, ξ_3, s) and $\bar{D}_\varepsilon = \{\zeta : 1 - \varepsilon \leqq |\zeta| \leqq 1 + \varepsilon\}$.*

Proof. Let K be a compact subset of G. Then there exist positive constants θ, κ_1 and δ_2 such that

$$
K \subset \left\{(\xi_1, \xi_2, \xi_3, t; s, \zeta, \tau) : \begin{array}{c} |\xi_i| \leqq \kappa_1, i = 1, 2, 3, |s| \leqq \delta_2 \\ \theta \leqq |\tau - t|, |t|, |\tau| \leqq \delta_2, \zeta \in \bar{D}_\varepsilon \end{array}\right\}.
\tag{2.19}
$$

Pick α, $0 < \alpha < 1$, such that $\alpha/(1-\alpha)^9 < \frac{1}{2}\theta/(52(4\delta_2)^3)$, and δ_1 such that $\kappa_1 < \alpha\delta_1$. Let $Q^{(n)}$ be the functions constructed in Lemma 2.1. Then $p^{(n)} = (\tau - t)^{-n-1}Q^{(n)}$ and, using (2.12) and (2.19), we see that for all points of K we have

$$
\begin{aligned}
&|p^{(n)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau)| \\
&\leqq \frac{M}{\theta^{n+1}}\left(26\frac{2\delta_2}{\delta_1}\right)^n\{(1-\alpha)^3(\tfrac{1}{2})^2\}^{-3n}, \qquad n = 1, 2, \cdots.
\end{aligned}
\tag{2.20}
$$

We therefore have

$$
\begin{aligned}
\sum_{n=1}^{\infty} |s|^{2n}|\mu|^n|p^{(n)}| &\leqq \sum_{n=1}^{\infty} \delta_2^{2n}(\alpha\delta_1)^n\frac{M}{\theta^{n+1}}\left(26\frac{2\delta_2}{\delta_1}\right)^n\{(1-\alpha)^3(\tfrac{1}{2})^2\}^{-3n} \\
&\leqq \frac{M}{\theta}.
\end{aligned}
\tag{2.21}
$$

Lemma 2.2 and the preceding discussion now imply the following theorem.

THEOREM 2.2. *There exists a function $E^{(1)} = E^{(1)}(z, z^*, x, t; s, \zeta, \tau)$ which is entire in z, z^*, x, s, analytic in ζ for ζ in $\bar{D}_\varepsilon$, and analytic in t and τ if $t \neq \tau$, such that $E^{(1)}$ satisfies (2.4) and $E^{(1)}(0, z^*, x, t; s, \zeta, \tau) = 1/(\tau - t)$.*

The integral operator (2.3) can be used to map Goursat data, prescribed on the $z = 0$ hyperplane, onto a solution $U(z, z^*, x, t)$ of $L[U] = 0$ such that $U(0, z^*, x, t)$ assumes the prescribed data (cf. Theorem 3.1). In order to handle data given on the $z^* = 0$ hyperplane, we will need a second integral operator. Its construction will only be outlined. Define $\mathscr{E}^{(2)}[f](z, z^*, x, t)$ by

$$\mathscr{E}^{(2)}[f](z, z^*, x, t)$$

$$= \int_{|\tau - t| = \delta} \int_{|\zeta| = 1} \int_{\gamma} E^{(2)}(z, z^*, x, t; s, \zeta, \tau) f(\hat{\mu}(1 - s^2), \zeta, \tau) \frac{ds}{\sqrt{1 - s^2}} \frac{d\zeta}{\zeta} d\tau, \tag{2.22}$$

where $\hat{\mu} = x + \zeta^{-1} z + \zeta z^*$, and $f(\hat{\mu}, \zeta, \tau)$ is an analytic function of three complex variables (we note the positions of z and z^* with respect to the parameter variable have been interchanged). If the function $E^{(2)}(z, z^*, x, t; s, \zeta, \tau)$ satisfies the partial differential equation

$$\begin{aligned} &E^{(2)}_{xx} - E^{(2)}_{zz^*} + A(z, z^*, x, t)E^{(2)} - B(z, z^*, x, t)E^{(2)}_t - \frac{\zeta(1 - s^2)}{2\hat{\mu}s} E^{(2)}_{zs} \\ &+ \frac{\zeta}{2\hat{\mu}s^2} E^{(2)}_z - \frac{1 - s^2}{2\zeta\hat{\mu}s^2} E^{(2)}_{z^*s} + \frac{1}{2\zeta\hat{\mu}s^2} E^{(2)}_{z^*} \\ &+ \frac{1 - s^2}{\hat{\mu}s} E^{(2)}_{xs} - \frac{1}{\hat{\mu}s^2} E^{(2)}_x = 0, \end{aligned} \tag{2.23}$$

then $\mathscr{E}^{(2)}[f](z, z^*, x, t)$ is an analytic solution of $L[U] = 0$ (cf. Theorem 2.1). We next introduce the variables η_i, $i = 1, 2, 3$, defined by

$$\eta_1 = 2\zeta z^*, \qquad \eta_2 = x + 2\zeta z^*, \qquad \eta_3 = x + 2\zeta^{-1} z, \tag{2.24}$$

and then define the following functions:

$$\begin{aligned} &\tilde{E}^{(2)} = \tilde{E}^{(2)}(\eta_1, \eta_2, \eta_3, t; s, \zeta, \tau) = E^{(2)}(z, z^*, x, t; s, \zeta, \tau), \\ &\hat{A} = \hat{A}(\eta_1, \eta_2, \eta_3, t; \zeta) = A(z, z^* x, t), \\ &\hat{B} = \hat{B}(\eta_1, \eta_2, \eta_3, t; \zeta) = B(z, z^*, x, t). \end{aligned} \tag{2.25}$$

It is readily verified that (2.23) is transformed into (2.7) with μ, $\tilde{A}$, $\tilde{B}$, and $\tilde{E}$ replaced by $\hat{\mu}$, $\hat{A}$, $\hat{B}$ and $\tilde{E}^{(2)}$, respectively. We again look for a solution to this transformed equation of the form (2.8). From the above it is clear that Theorem 2.2 is true with $E^{(1)}$ replaced by $E^{(2)}$ and $E^{(1)}(0, z^*, x, t; s, \zeta, \tau) = 1/(\tau - t)$ replaced by $E^{(2)}(z, 0, x, t; s, \zeta, \tau) = 1/(\tau - t)$.

We remark that the construction of these integral operators does not depend in an essential manner on the fact that the coefficients in (1.1) are entire. If we weaken this requirement and only demand that the coefficients be analytic in some ball in $\mathbb{C}^4$, then we may proceed as before. In this case the $E^{(1)}$ and $E^{(2)}$ functions will not be entire in the z, z^*, x and s variables.

3. Goursat problems in the complex domain. In this section we wish to construct solutions of $L[U] = 0$ which assume prescribed values on the complex hyperplanes $z = 0$ and $z^* = 0$; that is, we wish to solve the following Goursat problem:

$$\begin{aligned} &L[U] \equiv U_{xx} - U_{zz^*} + A(z, z^*, x, t)U - B(z, z^*, x, t)U_t = 0, \\ &U(0, z^*, x, t) = F(0, z^*, x, t), \qquad U(z, 0, x, t) = F(z, 0, x, t), \end{aligned} \tag{3.1}$$

where $F(z, z^*, x, t)$ is entire in z, z^* and x, and is analytic in t except possibly at $t = 0$, where there may be an essential singularity. The following lemma will be needed in our construction of the solution to (3.1).

LEMMA 3.1. *Let $E^{(1)}$ and $E^{(2)}$ be the functions constructed in* § 2. *Then $E^{(1)}(z, 0, x, t; s, \zeta, \tau)$ and $E^{(2)}(0, z^*, x, t; s, \zeta, \tau)$ can be analytically continued as functions of ζ to $|\zeta| \leqq 1$ and*

$$E^{(1)}(z, 0, x, t; s, 0, \tau) = E^{(2)}(0, z^*, x, t; s, 0, \tau) = 1/(\tau - t). \tag{3.2}$$

Proof. This will only be shown for $E^{(1)}$ as the proof for $E^{(2)}$ is similar. Since

$$E^{(1)}(z, 0, x, t; s, \zeta, \tau) = \frac{1}{\tau - t} + \sum_{n=1}^{\infty} s^{2n}(x + \zeta z)^n p^{(n)}(2\zeta z, x + 2\zeta z, x, t; \zeta, \tau), \tag{3.3}$$

it will suffice to show that $p^{(n)}(2\zeta z, x + 2\zeta z, x, t; \zeta, \tau)$ is analytic in ζ for $|\zeta| \leqq 1$ and that $p^{(n)}(0, x, x, t; 0, \tau) = 0$, $n = 1, 2, \cdots$. The fact that the series still converges absolutely and uniformly for $|\zeta| \leqq 1$ follows by observing that the dominant estimates that were derived for $p^{(n)}$ when $\zeta \in \bar{D}_\varepsilon$, will be unchanged when $|\zeta| \leqq 1$. From (2.9) we have

$$p^{(1)}(\xi_1, \xi_2, \xi_3, t; \zeta, \tau) = \int_0^{\xi_1} \left[\frac{\tilde{A}(\eta, \xi_2, \xi_3, t; \zeta)}{\tau - t} - \frac{\tilde{B}(\eta, \xi_2, \xi_3, t; \zeta)}{(\tau - t)^2} \right] d\eta. \tag{3.4}$$

Thus

$$\begin{aligned} &p^{(1)}(2\zeta z, x + 2\zeta z, x, t; \zeta, \tau) \\ &\quad = \int_0^{2\zeta z} \left[\frac{\tilde{A}(\eta, x + 2\zeta z, x, t; \zeta)}{\tau - t} - \frac{\tilde{B}(\eta, x + 2\zeta z, x, t; \zeta)}{(\tau - t)^2} \right] d\eta \\ &\quad = \int_0^{2\zeta z} \left[\frac{A(\eta/(2\zeta), (\zeta/2)(\eta - 2\zeta z), x + 2\zeta z - \eta, t)}{\tau - t} \right. \\ &\qquad\qquad \left. - \frac{B(\eta/(2\zeta), (\zeta/2)(\eta - 2\zeta z), x + 2\zeta z - \eta, t)}{(\tau - t)^2} \right] d\eta \\ &\quad = 2\zeta \int_0^{z} \left[\frac{A(\lambda, (\zeta/2)(2\zeta\lambda - 2\zeta z), x + 2\zeta z - 2\zeta\lambda, t)}{\tau - t} \right. \\ &\qquad\qquad \left. - \frac{B(\lambda, (\zeta/2)(2\zeta\lambda - 2\zeta z), x + 2\zeta z - 2\zeta\lambda, t)}{(\tau - t)^2} \right] d\lambda. \end{aligned} \tag{3.5}$$

From (3.5) it is clear that $p^{(1)}(2\zeta z, x + 2\zeta z, x, t; \zeta, \tau)$ is analytic in ζ for $|\zeta| \leqq 1$ and that $p^{(1)}(0, x, x, t; 0, \tau) = 0$. The proofs that $p^{(n)}$, $n > 1$, have the same properties are similar.

In the next three lemmas we will show how the data function $F(z, z^*, x, t)$ appearing in (3.1) can be used to define other analytic functions which will then be mapped by the operators $\mathscr{E}^{(1)}$ and $\mathscr{E}^{(2)}$ onto the solution of (3.1). We will need the following formulas:

$$\tag{3.6} \begin{aligned} \int_\gamma (1 - s^2)^n \frac{ds}{\sqrt{1 - s^2}} &= \frac{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(n + 1)}, \\ -\frac{1}{2\pi} \int_\gamma (1 - s^2)^n \frac{ds}{s^2} &= \frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})}, \end{aligned}$$

where $\Gamma(x)$ is the gamma function and γ is a curve lying in the complex plane not passing through the origin (cf. [8]). Since $F(z, z^*, x, t)$ is analytic in $\mathbb{C}^3 \times \{\mathbb{C}^1|0\}$ we can write F in the form

$$\tag{3.7} F(z, z^*, x, t) = \sum_{l,m,n=0}^{\infty} \sum_{p=-\infty}^{\infty} a_{lmnp} z^l z^{*m} x^n t^p.$$

LEMMA 3.2. *Let*

$$\tag{3.8} f(\mu, t) = \frac{1}{8\pi^3} \int_\gamma F(0, 0, \mu(1 - s^2), t) \frac{ds}{s^2}.$$

Then

$$\tag{3.9} \mathscr{E}^{(1)}[f](0, z^*, x, t) = \mathscr{E}^{(2)}[f](z, 0, x, t) = F(0, 0, x, t).$$

Proof. $F(0, 0, x, t) = \sum_{n=0}^{\infty} \sum_{p=-\infty}^{\infty} b_{np} x^n t^p$, where $b_{np} = a_{00np}$, $n = 0, 1, \cdots$, $p = 0, \pm 1, \pm 2, \cdots$. Using (3.6) we have

$$\tag{3.10} f(\mu, t) = \frac{-1}{4\pi^2} \sum_{n,p} b_{np} \frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})} \mu^n t^p.$$

Since $E(0, z^*, x, t; s, \zeta, \tau) = 1/(\tau - t)$, we have

$\mathscr{E}[f](0, z^*, x, t)$

$$\tag{3.11} \begin{aligned} &= \int_{|\tau - t| = \delta} \int_{|\zeta| = 1} \int_\gamma \frac{f([x + \zeta^{-1} z^*](1 - s^2), \tau)}{\tau - t} \frac{ds}{\sqrt{1 - s^2}} \frac{d\zeta}{\zeta} d\tau \\ &= \frac{1}{2\pi i} \int_{|\zeta| = 1} \int_\gamma \sum_{n,p} b_{np} \frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})} (x + \zeta^{-1} z^*)^n (1 - s^2)^n t^p \\ &\quad \cdot \frac{ds}{\sqrt{1 - s^2}} \frac{d\zeta}{\zeta} = F(0, 0, x, t). \end{aligned}$$

Using Lemma 3.1 and integrating with respect to ζ first, one may similarly show that

$$\mathscr{E}^{(2)}[f](z, 0, x, t) = F(0, 0, x, t).$$

The following lemmas will be given without proofs as they involve calculations similar to those in Lemma 3.2. Let $\tilde{F}(z, z^*, x, t)$ be defined by

$$\tilde{F}(z, z^*, x, t) = F(z, z^*, x, t) - F(0, 0, x, t). \tag{3.12}$$

LEMMA 3.3. *Let*

$$G(z^*, x, t) = z^* \int_0^1 \frac{\partial \tilde{F}}{\partial z^*}(0, \theta z^*, [1 - \theta]x, t)\, d\theta \tag{3.13a}$$

and

$$g(\mu, \zeta, t) = \frac{1}{8\pi^3} \int_\gamma G([1 - s^2]\mu\zeta, [1 - s^2]\mu, t)\frac{ds}{s^2}. \tag{3.13b}$$

Then

$$\mathscr{E}^{(1)}[g](0, z^*, x, t) = \tilde{F}(0, z^*, x, t) \tag{3.14a}$$

and

$$\mathscr{E}^{(1)}[g](z, 0, x, t) = 0. \tag{3.14b}$$

LEMMA 3.4. *Let*

$$H(z, x, t) = z \int_0^1 \frac{\partial F}{\partial z}(\theta z, 0, [1 - \theta]x, t)\, d\theta \tag{3.15a}$$

and

$$h(\mu, \zeta, t) = \frac{1}{8\pi^2} \int_\gamma H([1 - s^2]\mu\zeta, [1 - s^2]\mu, t)\frac{ds}{s^2}. \tag{3.15b}$$

Then

$$\mathscr{E}^{(2)}[h](0, z^*, x, t) = 0 \tag{3.16a}$$

and

$$\mathscr{E}^{(2)}[h](z, 0, x, t) = \tilde{F}(z, 0, x, t). \tag{3.16b}$$

THEOREM 3.1. *Let $F(z, z^*, x, t)$ be analytic in $\mathbb{C}^3 \times \{\mathbb{C}^1|(0)\}$. Let $f(\mu, t)$, $g(\mu, \zeta, t)$ and $h(\mu, \zeta, t)$ be the functions defined in* (3.8), (3.13b) *and* (3.15b), *respectively. Then the solution to the Goursat problem* (3.1) *is given by*

$$\begin{aligned} U(z, z^*, x, t) = {} & \mathscr{E}^{(1)}[f](z, z^*, x, t) + \mathscr{E}^{(1)}[g](z, z^*, x, t) \\ & + \mathscr{E}^{(2)}[h](z, z^*, x, t) \end{aligned} \tag{3.17}$$

for (z, z^*, x, t) in $\mathbb{C}^3 \times \{\mathbb{C}^1|(0)\}$.

Proof. $U(z, z^*, x, t)$ as defined by (3.17) is certainly a solution of the partial differential equation $L[U] = 0$. That it assumes the given data is assured by Lemmas 3.2, 3.3 and 3.4.

We remark that one really only needs the data function to be analytic (that is, not necessarily entire) in the space variables as in the above lemmas no essential use has been made of the fact that F is entire in those variables.

It is now a direct consequence of Hormander's generalized Cauchy–Kowalewski theorem (cf. [13]) that every analytic solution of $L[U] = 0$, and hence $l[u] = 0$, has a representation of the form (3.17).

4. Singular solutions. In this section we will construct a singular solution to equation (1.1) which is a generalization to three space variables of the singular solutions constructed by Hill [11] for parabolic equations in one space variable and by Colton [5] and Hill [12] for parabolic equations in two space variables. The essential distinction between these new singular solutions and the usual fundamental solution for parabolic equations is that they assume prescribed Goursat data on intersecting complex hyperplanes and, in the case of an odd space dimension, have an isolated essential singularity in the complex t-plane instead of the usual branch point singularity.

Following the method of Hadamard [9], we look for a solution to the formal adjoint of (1.1) of the form

$$v(\mathbf{x}, t; \boldsymbol{\xi}, t_0) = \Gamma^{-1/2} \sum_{j=0}^{\infty} u^{(j)}(\mathbf{x}, t; \boldsymbol{\xi}, t_0)\Gamma^j + w(\mathbf{x}, t; \boldsymbol{\xi}, t_0), \tag{4.1}$$

where $(\mathbf{x}, t; \boldsymbol{\xi}, t_0) = (x_1, x_2, x_3, t; \xi_1, \xi_2, \xi_3, t_0)$, $\Gamma = \sum_{i=1}^{3} (x_i - \xi_i)^2$, $\Gamma^{1/2}$ denotes the positive square root of Γ, and the $u^{(j)}$, $j = 0, 1, \cdots$, and w are entire functions of the space variables x_i, $i = 1, 2, 3$, with a pole-like (or essential singularity) in time at $t = t_0$. If we define

$$\begin{aligned} V = V(z, z^*, x, t; z_0, z_0^*, x_0, t_0) &= v(\mathbf{x}, t; \boldsymbol{\xi}, t_0), \\ R^2(z, z^*, x, z_0, z_0^*, x_0) &= \Gamma(\mathbf{x}; \boldsymbol{\xi}) \\ &= (x - x_0)^2 - 4(z - z_0)(z^* - z_0^*) \end{aligned} \tag{4.2}$$

and

$$U^{(j)}(z, z^*, x, t; z_0, z_0^*, x_0, t_0) = u^{(j)}(\mathbf{x}, t; \boldsymbol{\xi}, t_0),$$

[cf. (2.1)], then we want V to satisfy the following conditions for each value of the parameter variables z_0, z_0^*, x_0, t_0:

$$M[V] \equiv V_{xx} - V_{zz^*} + A(z, z^*, x, t)V + (B(z, z^*, x, t)V)_t = 0, \tag{4.3a}$$

$$V_z|_{z^* = z_0^*} = V_{z^*}|_{z = z_0} = 0, \tag{4.3b}$$

$$V(z_0, z_0^*, x, t; z_0, z_0^*, x_0, t_0) = \frac{1}{(x - x_0)(t - t_0)} + H(z_0, z_0^*, x, t; z_0, z_0^*, x_0, t_0), \tag{4.3c}$$

where $H(z_0, z_0^*, x, t; z_0, z_0^*, x_0, t_0)$ is an entire function of x and has an essential singularity at $t = t_0$. (The reasons for (4.3b and c) will become apparent in § 5.) We now proceed with the construction of v. Let $m[u]$ be defined by

$$m[u] \equiv \Delta_3 u + a(x_1, x_2, x_3, t)u + (b(x_1, x_2, x_3, t)u)_t; \tag{4.4}$$

that is, m is the formal adjoint of the operator defined in (1.1). We wish to determine the $u^{(j)}$, $j = 0, 1, \cdots$, so that

$$m\left[\Gamma^{-1/2}\sum_{j=0}^{\infty} u^{(j)}\Gamma^{j}\right] = 0. \tag{4.5}$$

Let u be an arbitrary analytic function. Then

$$\begin{aligned} m[u\Gamma^{j-(1/2)}] = m[u]\Gamma^{j-(1/2)} &+ 4(j-\tfrac{1}{2})\sum_{i=1}^{3}\Gamma^{j-(3/2)}(x_i-\xi_i)\frac{\partial u}{\partial x_i} \\ &+ 4j(j-\tfrac{1}{2})u\Gamma^{j-(3/2)}. \end{aligned} \tag{4.6}$$

If r is a parameter which indicates the radial direction from the point (ξ_1, ξ_2, ξ_3) to the point (x_1, x_2, x_3), then

$$\sum_{i=1}^{3}(x_i-\xi_i)\frac{\partial u}{\partial x_i} = r\frac{\partial u}{\partial r}. \tag{4.7}$$

From (4.6) and (4.7) we have

$$\begin{aligned} m&\left[\sum_{j=0}^{\infty} u^{(j)}\Gamma^{j-(1/2)}\right] \\ &= \sum_{j=0}^{\infty}\left\{m[u^{(j)}] + 4(j+\tfrac{1}{2})\left(r\frac{\partial u^{(j+1)}}{\partial r} + (j+1)u^{(j+1)}\right)\right\}\Gamma^{j-(1/2)} \\ &\quad - 2r\frac{\partial u^{(0)}}{\partial r}\Gamma^{-3/2}. \end{aligned} \tag{4.8}$$

Thus, if we wish (4.5) to hold, we must have

$$\begin{aligned} &r\frac{\partial u^{(0)}}{\partial r} = 0, \\ &r\frac{\partial u^{(j+1)}}{\partial r} + (j+1)u^{(j+1)} = -\frac{1}{4(j+\frac{1}{2})}m[u^{(j)}], \qquad j = 0, 1, \cdots. \end{aligned} \tag{4.9}$$

Remembering (4.3c), we define

$$\begin{aligned} u^{(0)} &= \frac{1}{t-t_0}, \\ u^{(j+1)} &= \frac{r^{-j-1}}{4(j+\frac{1}{2})}\int_0^r s^j m[u^{(j)}](s, \theta, \phi)\, ds, \end{aligned} \tag{4.10}$$

where r, θ, ϕ are the spherical coordinates of the point (x_1, x_2, x_3) with respect to the point (ξ_1, ξ_2, ξ_3). We note that not only are the $u^{(j)}$, $j = 0, 1, \cdots$, entire functions of x_i, $i = 1, 2, 3$, but they are also entire functions of the parameter variables ξ_i, $i = 1, 2, 3$.

Let us assume for the moment that the series in (4.1) has been shown to converge. Making the change of variables (2.1), we have

$$M\left[\frac{1}{R}\sum_{j=0}^{\infty} U^{(j)}R^{2j}\right] = 0. \tag{4.11}$$

We note that

$$R^2|_{z^*=z_0^*} = R^2|_{z=z_0} = (x - x_0)^2. \tag{4.12}$$

This, along with $U^{(0)} = 1/(t - t_0)$, gives us

$$\begin{aligned} V_z|_{z^*=z_0^*} &= \sum_{j=1}^{\infty} U_z^{(j)}|_{z^*=z_0^*}(x - x_0)^{2j-1} + W_z|_{z^*=z_0^*}, \\ V_{z^*}|_{z=z_0} &= \sum_{j=1}^{\infty} U_{z^*}^{(j)}|_{z=z_0}(x - x_0)^{2j-1} + W_{z^*}|_{z=z_0}. \end{aligned} \tag{4.13}$$

Hence V satisfies (4.3a and b) if W satisfies

$$\begin{aligned} &M[W] = 0, \\ &W|_{z=z_0} = -\sum_{j=1}^{\infty} U^{(j)}|_{z=z_0}(x - x_0)^{2j-1}, \\ &W|_{z^*=z_0^*} = -\sum_{j=1}^{\infty} U^{(j)}|_{z^*=z_0^*}(x - x_0)^{2j-1}. \end{aligned} \tag{4.14}$$

We now apply Theorem 1.1, after linearly translating the point (z_0, z_0^*) to the origin, to infer the existence of W and, furthermore, obtain an integral representation for this solution to the adjoint equation. Since the condition (4.3c) is automatically satisfied once we have chosen $U^{(0)} = 1/(t - t_0)$, the only detail left to verify is the convergence of the series $\sum_{j=0}^{\infty} u^{(j)}\Gamma^j$. Without loss of generality set ξ_i, $i = 1, 2, 3$, and t_0 equal to zero. Define the functions $Q^{(j)}(x_1, x_2, x_3, t)$, $j = 0, 1, \cdots$, by

$$Q^{(j)} = t^{j+1}u^{(j)}. \tag{4.15}$$

Then we have

$$\begin{aligned} Q^{(0)} &= 1, \\ Q^{(j+1)} &= -\frac{r^{-j-1}}{4(j + \frac{1}{2})} \int_0^r s^j\{t\Delta_3 Q^{(j)} + [td(x_1, x_2, x_3, t) \\ &\qquad - (j + 1)b(x_1, x_2, x_3, t)Q^{(j)} + tb(x_1, x_2, x_3, t)Q_t^{(j)}]\}\, ds, \end{aligned} \tag{4.16}$$

where $d(x_1, x_2, x_3, t) = a(x_1, x_2, x_3, t) + b_t(x_1, x_2, x_3, t)$. The following two lemmas are stated without proofs since their proofs are essentially the same as those for Lemmas 2.1 and 2.2.

LEMMA 4.1. *If $a(x_1, x_2, x_3, t)$, $b(x_1, x_2, x_3, t)$ and $d(x_1, x_2, x_3, t)$ are entire functions of x_i, $i = 1, 2, 3$, and t, and $Q^{(j)}$, $j = 0, 1, \cdots$, are defined by* (4.16), *then for every pair δ_1 and δ_2 of arbitrarily large positive constants there exists a positive constant $N = N(\delta_1, \delta_2)$ such that*

$$Q^{(j)} \ll N\left(\frac{31\delta_2}{4\delta_1}\right)^j \left\{\left(1 - \frac{x_1}{\delta_1}\right)\left(1 - \frac{x_2}{\delta_1}\right)\left(1 - \frac{x_3}{\delta_1}\right)\left(1 - \frac{t}{\delta_2}\right)\right\}^{-3j} \tag{4.17}$$

LEMMA 4.2. *If $a(x_1, x_2, x_3, t)$ and $b(x_1, x_2, x_3, t)$ are entire functions of x_i, $i = 1, 2, 3$, and t, then the series $\sum_{j=0}^{\infty} u^{(j)}\Gamma^j$, where the $u^{(i)}$ are defined by* (4.8) *and $\Gamma = \sum_{i=1}^{3} (x_i - \xi_i)^2$, converges absolutely and uniformly on all compact subsets of $\mathbb{C}^3 \times \{\mathbb{C}|(0)\}$.*

The above discussion is summarized in the following theorem.

THEOREM 4.1. *Let $A(z, z^*, x, t)$ and $B(z, z^*, x, t)$ be entire functions of their independent* (*complex*) *variables. Then there exists a function $V = V(z, z^*, x, t; z_0, z_0^*, x_0, t_0)$ such that*

$$V = \frac{1}{R} \sum_{j=0}^{\infty} U^{(j)} R^{2j} + W, \tag{4.18}$$

where $R^2 = (x - x_0)^2 - 4(z - z_0)(z^* - z_0^*)$ and the series in (4.18) *converges uniformly and absolutely on compact subsets of $\mathbb{C}^6 \times \{\mathbb{C}^2|(t = t_0)\}$. The functions $U^{(j)}$, $j = 0, 1, \cdots$, and W are entire in z, z_0, z^*, z_0^*, x, x_0 and analytic in t and t_0 except when $t = t_0$. Moreover V satisfies the singular Goursat problem* (4.3).

5. The noncharacteristic Cauchy problem. In this section we construct an integral representation for the solution to the following Cauchy problem:

$$\begin{aligned} &\Delta_3 u + a(x_1, x_2, x_3, t)u - b(x_1, x_2, x_3, t)u_t = 0, \\ &u|_s = f(x_1, x_2, x_3, t), \qquad u_n|_s = g(x_1, x_2, x_3, t), \end{aligned} \tag{5.1}$$

where S is a noncharacteristic surface (cf. [13]) and n is a normal to this surface. We will assume that the surface S is the zero set of a real-valued analytic function $F(x_1, x_2, x_3, t)$. The Cauchy data is assumed to be analytic in some ball in $\mathbb{C}^4$ and the coefficients appearing in the partial differential equation are again assumed to be entire functions.

As in the preceding sections we make the change of variables (2.1). The surface S is then analytically continued to the surface $\tilde{S}$, which is realized as the zero set of the function $\tilde{F}(z, z^*, x, t) = F(z - z^*, (z + z^*)/i, x, t)$. Let (z_0, z_0^*, x_0, t_0) be some point in $\mathbb{C}^4$ which does not lie on $\tilde{S}$. We let $I(x, t)$ be the point (z_0, z_0^*, x, t). Let $P(x, t) = (p(z_0^*, x, t), z_0^*, x, t)$ and $Q(x, t) = (z_0, q(z_0, x, t), x, t)$ be points that lie on the surface $\tilde{S}$. The fact that analytic functions $p(z_0^*, x, t)$ and $q(z_0, x, t)$ exist such that $P(x, t)$ and $Q(x, t)$ lie on $\tilde{S}$, for (z_0, z_0^*, x_0, t_0) sufficiently close to S and (x, t) close to (x_0, t_0), follows from S being noncharacteristic and the implicit function theorem. Let $C_1(x, t)$ and $C_2(x, t)$ denote, respectively, the straight lines joining the points $I(x, t)$ and $P(x, t)$ and the points $Q(x, t)$ and $I(x, t)$. Let $C_3(x, t)$ be a path in $\tilde{S}$ joining $P(x, t)$ to $Q(x, t)$ and let $D(x, t)$ be the two-dimensional region whose boundary consists of the curves $C_i(x, t)$, $i = 1, 2, 3$. The region $D(x, t)$ is sketched in Fig. 5.1.

Let $\Omega_1 = \{x : |x - x_0| = \delta_1\}$ and let $\Omega_2 = \{t : |t - t_0| = \delta_2\}$, where δ_1 and δ_2 are positive constants (a restriction on δ_1 will be made later). Define the region G contained in $\mathbb{C}^4$ by

$$G = \bigcup_{(x,t) \in \Omega_1 \times \Omega_2} D(x, t). \tag{5.2}$$

Let L be the partial differential operator defined in (2.2), M the formal adjoint of L, and let $V = V(z, z^*, x, t; z_0, z_0^*, x_0, t_0)$ be the singular solution constructed in §4. Letting $U = U(z, z^*, x, t)$ be any analytic solution of $L[U] = 0$ in some

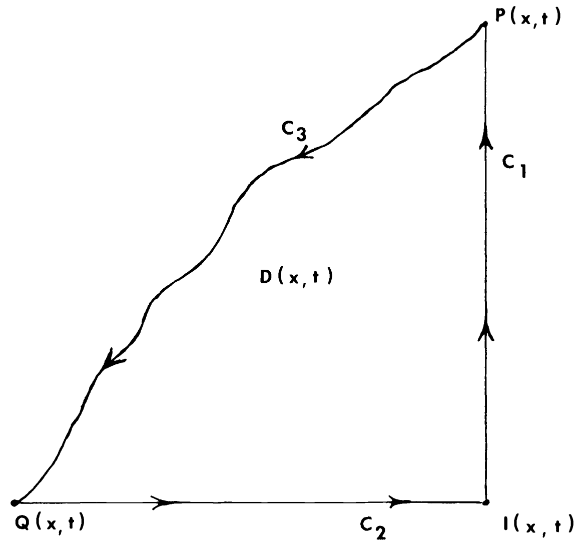

FIG. 5.1

region containing G, we integrate the expression $VL[U] - UM[V]$ over the region G. Since $M[V] = L[U] = 0$, we have

$$
\begin{aligned}
0 &= \int_G \{VL[U] - UM[V]\}\, dz\, dz^*\, dx\, dt \\
&= \int_G \{(VU_x - V_xU)_x - (VBU)_t\}\, dz\, dz^*\, dx\, dt \qquad (5.3)\\
&\quad + \tfrac{1}{2}\int_G \{(V_{z^*}U - VU_{z^*})_z + (V_zU - VU_z)_{z^*}\}\, dz\, dz^*\, dx\, dt.
\end{aligned}
$$

If $\tilde{\delta}_1$ is picked so that $\tilde{\delta}_1 > 2\sup_{(z,z^*,x_0,t_0)\in D(x_0,t_0)}\{(|z - z_0|\,|z^* - z_0^*|)^{1/2}\}$, then the function $\sqrt{(x - t_0)^2 - 4(z - z_0)(z^* - z_0^*)} = R$ is analytic and nonzero for $|x - x_0| = \delta_1 \geqq \tilde{\delta}_1$ and $|z - z_0|\,|z^* - z_0^*| < \tilde{\delta}_1^2/4$. (The implications of such a $\tilde{\delta}_1$ existing for the region G will be discussed later.) Since the function V can be written as in (4.18), we see that $(VU_x - UV_x)$ has a Laurent expansion in $x - x_0$. Thus $(VU_x - UV_x)_x$ has a Laurent expansion which does not contain the term $(x - x_0)^{-1}$. We may therefore conclude that

$$\int_G (VU_x - UV_x)_x\, dz\, dz^*\, dx\, dt = 0. \qquad (5.4)$$

Similarly we may conclude that

$$\int_G (VBU)_t\, dz\, dz^*\, dx\, dt = 0. \qquad (5.5)$$

Thus (5.3) reduces to

$$
\begin{aligned}
0 &= \frac{1}{2}\int_{\Omega_1\times\Omega_2}\int_{D(x,t)} \{(V_{z^*}U - VU_{z^*})_z + (V_zU - UV_z)_{z^*}\}\,dz\,dz^*\,dx\,dt \\
&= \frac{1}{2}\int_{\Omega_1\times\Omega_2}\int_{\partial D(x,t)} \{(V_{z^*}U - VU_{z^*})\,dz^* - (V_zU - VU_z)\,dz\}\,dx\,dt
\end{aligned}
\tag{5.6}
$$

from the divergence theorem. Here $\partial D(x, t)$ denotes the boundary of $D(x, t)$, which equals $C_1(x, t) + C_2(x, t) + C_3(x, t)$. Since $dz^* = 0$ on $C_1(x, t)$ and $V_z = 0$ when $z^* = z_0^*$, we have

$$
\begin{aligned}
&\int_{C_1(x,t)} \{(V_{z^*}U - VU_{z^*})\,dz^* - (V_zU - VU_z)\,dz\} \\
&\qquad = -\int_{C_1(x,t)} (V_zU - VU_z)\,dz = \int_{C_1(x,t)} (VU)_z\,dz = VU|_{I(x,t)}^{P(x,t)}.
\end{aligned}
\tag{5.7}
$$

Similarly we have

$$
\int_{C_2(x,t)} \{(V_{z^*}U - VU_{z^*})\,dz^* - (V_zU - VU_z)\,dz\} = -VU|_{Q(x,t)}^{I(x,t)}.
\tag{5.8}
$$

Combining (5.6), (5.7) and (5.8) we have

$$
\begin{aligned}
0 &= \frac{1}{2}\int_{\Omega_1\times\Omega_2}\left[\int_{C_3(x,t)} (V_{z^*}U - VU_{z^*})\,dz^* - (V_zU - VU_z)\,dz\right]dx\,dt \\
&\quad + \frac{1}{2}\int_{\Omega_1\times\Omega_2} \{V(P(x, t))U(P(x, t)) + V(Q(x, t))U(Q(x, t))\}\,dx\,dt \\
&\quad - \int_{\Omega_1\times\Omega_2} V(I(x, t))U(I(x, t))\,dx\,dt.
\end{aligned}
\tag{5.9}
$$

Using (4.3c) we evaluate the last term on the right-hand side of (5.9) and get

$$
\begin{aligned}
&U(z_0, z_0^*, x_0, t_0) \\
&\quad = -\frac{1}{8\pi^2}\int_{\Omega_1\times\Omega_2}\left[\int_{C_3(x,t)} (V_{z^*}U - VU_{z^*})\,dz^* - (V_zU - VU_z)\,dz\right]dx\,dt \\
&\qquad + \int_{\Omega_1\times\Omega_2} [V(P)U(P) + V(Q)U(Q)]\,dx\,dt.
\end{aligned}
\tag{5.10}
$$

Formula (5.10) is a local representation of the solution U in terms of its Cauchy data on the surface $\tilde{S}$. This representation gives us a means of analytically continuing this local solution to a global solution.

In our derivation of (5.10) we assumed that

$$R = \sqrt{(x - x_0)^2 - 4(z - z_0)(z^* - z_0^*)}$$

was analytic and nonzero throughout G, that is, $|z - z_0|\,|z^* - z_0^*| < \tilde{\delta}_1^2/4$ for all $(z, z^*, x, t) \in G$, where $|x - x_0| = \delta_1 \geqq \tilde{\delta}_1$. What this means in terms of the surface G is that (z_0, z_0^*, x_0, t) may have to lie close to S in order that $\tilde{\delta}_1$ may exist.

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GENERALIZED COMPLETELY CONVEX FUNCTIONS AND STURM–LIOUVILLE OPERATORS*

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Abstract. A generalization of the class of completely convex functions, those functions in $C^\infty[0, 1]$ whose even derivatives alternate in sign, is developed by means of the Sturm–Liouville differential operator $Lf = -(Pf')' + Qf$. The functions obtained are called *LB-positive* functions. A series expansion associated with the operator is introduced and used to represent *LB*-positive functions. For special choices of P and Q, the representation reduces to the classical relationship between completely convex functions and Lidstone series.

1. Introduction. An infinitely differentiable function f defined on the interval $[0, 1]$ is said to be *completely convex* provided that

$$(-1)^n f^{(2n)}(x) \geqq 0, \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, 2, \cdots.$$

This class of functions was introduced by D. V. Widder in 1940 [9]. The most familiar completely convex function is $\sin \pi x$, and every completely convex function is the restriction to $[0, 1]$ of an entire function of exponential type at most π.

There is a close connection between completely convex functions and *Lidstone* series. A Lidstone series has the form

$$\sum_{n=0}^{\infty} \{a_n A_n(z) + b_n A_n(1 - z)\},$$

where $\{a_n\}$ and $\{b_n\}$ are complex sequences and $A_n(z)$ is the unique polynomial of degree $2n + 1$ determined by the conditions $A_0(z) = z$, $A_n(0) = A_n(1) = 0$, and $A_n''(z) = A_{n-1}(z)$, for $n = 1, 2, 3, \cdots$. A function $f \in C^\infty[0, 1]$ is completely convex if and only if

$$f(x) = C \sin \pi x + \sum_{n=0}^{\infty} \{a_n A_n(x) + b_n A_n(1 - x)\}, \tag{1.1}$$

where C is a nonnegative constant and $a_n \geqq 0$, $b_n \geqq 0$, $n = 0, 1, 2, \cdots$. The expansion (1.1) is a result of Widder's characterization [10] of "minimal" completely convex functions and R. P. Boas' [2] extension of Widder's result to completely convex functions. In [4] the present authors studied various problems concerning convergence of Lidstone series.

In this paper, we employ the Sturm–Liouville differential operator to develop a generalization of completely convex functions. We shall introduce a series expansion associated with the operator and obtain a representation theorem analogous to (1.1). The series expansion reduces to the Lidstone series in a special case. Together with the representation theorem, our principal results include a characterization of the class of generalized completely convex functions and several theorems regarding convergence of the series expansion.

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Let P be a positive, continuously differentiable function defined on the interval $[a, b]$ and let Q be a real continuous function on $[a, b]$. Let L denote the Sturm–Liouville operator

$$Lf = -(Pf')' + Qf$$

and let B_a and B_b denote the endpoint linear forms

$$B_a f = \alpha f(a) + \alpha' f'(a),$$
$$B_b f = \beta f(b) + \beta' f'(b),$$

where α, α', β, β' are real numbers such that $|\alpha| + |\alpha'| > 0$ and $|\beta| + |\beta'| > 0$. The eigenvalue problem

$$Lf = \lambda f, \qquad B_a f = B_b f = 0, \tag{1.2}$$

is known as a regular Sturm–Liouville system. It is well known that the operator L is self-adjoint, that the eigenvalues of (1.2) are real and comprise a countable collection with no limit point, and that the associated normalized eigenfunctions of (1.2) form a complete orthonormal set.

We shall require only one hypothesis regarding the system (1.2). We suppose that P, Q, B_a and B_b are such that

(1.3) the eigenvalues of (1.2) are all positive.

Further, we shall assume that the signs on the constants α, α', β, β' are normalized so that

$$\begin{aligned} &\alpha' \leqq 0 \quad \text{and} \quad \alpha > 0 \quad \text{if } \alpha' = 0, \\ &\beta' \geqq 0 \quad \text{and} \quad \beta > 0 \quad \text{if } \beta' = 0. \end{aligned} \tag{1.4}$$

If the linear forms B_a and B_b are given, then (1.4) either holds or can be brought about by multiplying one or both of the equations $B_a f = 0$ and $B_b f = 0$ by -1. Such an operation affects neither the eigenvalues nor solutions of (1.2). Using the Prüfer substitution method [1] it is not difficult to show that the normalization (1.4) implies that any solution f of the homogeneous equation $Lf = 0$ which satisfies either $B_a f = 0$ and $B_b f > 0$ or $B_a f > 0$ and $B_b f = 0$ is nonnegative in the interval $[a, b]$.

For each nonnegative integer n, let $LB^n[a, b]$ denote the set of all functions f on $[a, b]$ such that $(L^k f)(x)$, $B_a L^k f$ and $B_b L^k f$ are all defined for $a \leqq x \leqq b$ and $0 \leqq k \leqq n$. Here, L^k is the kth iterant of L. Now let $LB^\infty[a, b] = \bigcap_{n=0}^{\infty} LB^n[a, b]$.

DEFINITION. Let $f \in LB^\infty[a, b]$. We say that f is *LB-positive* provided that

(i) $(L^k f)(x) \geqq 0$, $a \leqq x \leqq b$, $k = 0, 1, 2, \cdots$;
(ii) $(B_a L^k f) \geqq 0$, $k = 0, 1, 2, \cdots$;
(iii) $(B_b L^k f) \geqq 0$, $k = 0, 1, 2, \cdots$.

Thus if $P = 1$, $Q = 0$, and $\alpha' = \beta' = 0$, then the LB-positive functions are precisely the completely convex functions on $[a, b]$. The selection $P = 1$, $Q = 0$ and $\alpha = \beta' = 0$ leads to the similar class of functions studied by S. Pethe and A. Sharma [7].

The eigenvalues of (1.2) will be denoted by $\{\lambda_k\}_{k=0}^{\infty}$, with $0 < \lambda_0 < \lambda_1 < \cdots$ and $\lim_{k\to\infty} \lambda_k = +\infty$, and the corresponding eigenfunctions by $y_0, y_1, y_2, \cdots$.

Thus, for each k, one has

$$Ly_k = \lambda_k y_k, \qquad B_a y_k = B_b y_k = 0. \tag{1.5}$$

We recall [1] that each y_k has precisely k zeros in the open interval (a, b), and that we are therefore free to choose y_0 so that

$$y_0(x) \geqq 0, \qquad a \leqq x \leqq b.$$

Further, we assume that the y_k are normalized so that

$$\|y_k\|_2 = \left\{ \int_a^b [y_k(x)]^2 \, dx \right\}^{1/2} = 1, \qquad k = 0, 1, 2, \cdots.$$

Let $G(x, t)$, $a \leqq x$, $t \leqq b$, denote the Green's function for the operator L. Recall that G is continuous and that $(\partial/\partial x)G$ exists and is continuous on each of the triangles $a \leqq x \leqq t \leqq b$ and $a \leqq t \leqq x \leqq b$. Further, as a function of x, $G(x, t)$ satisfies

$$B_a G = B_b G = 0. \tag{1.6}$$

It can also be shown [6] that the conditions $P(x) > 0$ and $\lambda_0 > 0$ imply

$$G(x, t) \geqq 0, \qquad a \leqq x, t \leqq b. \tag{1.7}$$

Let p_0 and p_1 be linearly independent solutions of the homogeneous equation $Ly = 0$ which satisfy

$$\begin{aligned} B_a p_0 &= 0, & B_b p_0 &= 1, \\ B_a p_1 &= 1, & B_b p_1 &= 0. \end{aligned} \tag{1.8}$$

From our remarks following the normalization (1.4), we have

$$p_0(x) \geqq 0 \quad \text{and} \quad p_1(x) \geqq 0, \qquad a \leqq x \leqq b. \tag{1.9}$$

Observe also that any solution to the homogeneous equation is a linear combination of p_0 and p_1. Define the operator $\mathscr{G}$ by

$$(\mathscr{G}f)(x) = \int_a^b G(x, t) f(t) \, dt$$

for $f \in C[a, b]$, and let $\mathscr{G}^n$ denote the nth iterant of $\mathscr{G}$. Thus $\mathscr{G}^0$ is the identity map, $\mathscr{G}^1 = \mathscr{G}$, etc. Finally, define the sequence $\{p_k\}_{k=0}^{\infty}$ by

$$\begin{aligned} p_{2k} &= \mathscr{G}^k p_0, \\ p_{2k+1} &= \mathscr{G}^k p_1, \qquad k = 0, 1, 2, \cdots. \end{aligned} \tag{1.10}$$

We can now introduce our series expansion. With each $f \in LB^\infty[a, b]$ let us associate the infinite series

$$f(x) \sim \sum_{n=0}^{\infty} \{(B_b L^n f) p_{2n}(x) + (B_a L^n f) p_{2n+1}(x)\}. \tag{1.11}$$

We shall term a series of the form (1.11) an *LB-series*. We note that when $P = 1$, $Q = 0$ and $\alpha' = \beta' = 0$, the LB-series is a Lidstone series. It is the question of when the LB-series in (1.11) converges that we study first.

We can now state our principal results.

THEOREM 1.1. *Let $f \in LB^{\infty}[a, b]$. Then f is LB-positive if and only if*

$$f(x) = Cy_0(x) + \sum_{n=0}^{\infty} \{a_n p_{2n}(x) + b_n p_{2n+1}(x)\}, \tag{1.12}$$

where $C \geqq 0$, $a_n \geqq 0$ and $b_n \geqq 0$, $n = 0, 1, 2, \cdots$.

We shall prove that whenever (1.12) holds, $a_n = B_b L^n f$ and $b_n = B_a L^n f$, $n = 0, 1, 2, \cdots$. Moreover, the convergence in (1.12) is uniform in $[a, b]$.

As a corollary to Theorem 1.1, it will follow that if f is LB-positive, then the sequence $\{(L^n f)(x)/\lambda_0^n\}_{n=0}^{\infty}$ is uniformly bounded in $[a, b]$. The following theorem characterizes, via endpoint conditions, those functions f with $\{L^n f/\lambda_0^n\}$ uniformly bounded which are also LB-positive.

Let us denote by $\bar{B}_a$ the endpoint linear form

$$\bar{B}_a f = -\alpha' f(a) + \alpha f'(a),$$

and observe that $\bar{B}_a y_0 \neq 0$.

THEOREM 1.2. *Let $f \in LB^{\infty}[a, b]$. Then f is LB-positive if and only if*

(i) $B_a L^n f \geqq 0$ *and* $B_b L^n f \geqq 0$, $n = 0, 1, 2, \cdots$;

(ii) *the series* $\displaystyle\sum_{n=0}^{\infty} \frac{(B_a L^n f) + (B_b L^n f)}{\lambda_0^n}$ *converges*;

(iii) $\displaystyle\lim_{n \to \infty} \frac{(\bar{B}_a L^n f)}{(\bar{B}_a y_0)\lambda_0^n}$ *exists and is nonnegative*;

(iv) *the sequence $\{L^n f/\lambda_0^n\}$ is uniformly bounded in $[a, b]$.*

Finally, our techniques yield the following generalization of Schoenberg's classical result [3], [8] on Lidstone series.

THEOREM 1.3. *Let $f \in LB^{\infty}[a, b]$ satisfy $B_a L^n f = B_b L^n f = 0$, $n = 0, 1, 2, \cdots$, and suppose there exists a positive constant M such that*

$$\limsup_{n \to \infty} |(L^n f)(x)|^{1/n} < M, \qquad a \leqq x \leqq b.$$

Then

$$f(x) = \sum_{\lambda_k < M} (f, y_k) y_k(x), \qquad a \leqq x \leqq b.$$

The symbol $(\cdot, \cdot)$ indicates the usual inner product

$$(u, v) = \int_a^b u(t) v(t)\, dt.$$

2. Asymptotic estimates. In this section we derive our LB-series (1.11) and some asymptotic estimates on the sequence $\{p_n\}_0^{\infty}$.

For a function $\varphi \in C[a, b]$, the unique solution to the boundary value problem

$$Ly = \varphi, \qquad B_a y = B_b y = 0 \tag{2.1}$$

is given by

$$y(x) = (\mathcal{G}\varphi)(x) = \int_a^b G(x, t)\varphi(t)\,dt. \tag{2.2}$$

Replacing y in (2.1) by its determination $y = \mathcal{G}\varphi$, we arrive at the identity

$$L\mathcal{G}\varphi = \varphi, \tag{2.3}$$

valid for all $\varphi \in C[a, b]$. Now let $f \in LB^1[a, b]$. Then (2.3) implies $L(f - \mathcal{G}Lf) = Lf - Lf = 0$, and hence

$$f - \mathcal{G}Lf = c_0 p_0 + c_1 p_1,$$

where c_0 and c_1 are constants. Further, (1.6) and (1.8) give

$$B_a f = B_a(f - \mathcal{G}Lf) = B_a(c_0 p_0 + c_1 p_1) = c_1,$$

and

$$B_b f = B_b(f - \mathcal{G}Lf) = B_b(c_0 p_0 + c_1 p_1) = c_0.$$

We are therefore led to the identity

$$f = (B_b f)p_0 + (B_a f)p_1 + \mathcal{G}Lf \tag{2.4}$$

for all $f \in LB^1[a, b]$. For a function $f \in LB^2[a, b]$, we can apply the preceding argument to the function Lf to obtain

$$Lf = (B_b Lf)p_0 + (B_a Lf)p_1 + \mathcal{G}L^2 f,$$

and hence, by (1.10),

$$\mathcal{G}Lf = (B_b Lf)p_2 + (B_a Lf)p_3 + \mathcal{G}^2 L^2 f.$$

Substitution of this equation into (2.4) gives the identity

$$f = (B_b f)p_0 + (B_a f)p_1 + (B_b Lf)p_2 + (B_a Lf)p_3 + \mathcal{G}^2 L^2 f.$$

By repeated application of this procedure one sees that

$$f = \sum_{k=0}^{n-1} \{(B_b L^k f)p_{2k} + (B_a L^k f)p_{2k+1}\} + \mathcal{G}^n L^n f, \tag{2.5}$$

for each positive integer n and each $f \in LB^n[a, b]$. In particular, if $f \in LB^1[a, b]$ and $B_a f = B_b f = 0$, then (2.5) yields the important identity

$$f = \mathcal{G}Lf \quad (B_a f = B_b f = 0). \tag{2.6}$$

To acquire asymptotic estimates on the sequence $\{p_n\}_0^\infty$, we use the eigenfunction expansion [5]

$$\mathcal{G}\varphi = \sum_{k=0}^{\infty} \lambda_k^{-1}(\varphi, y_k)y_k, \tag{2.7}$$

which is valid for all $\varphi \in C[a, b]$, the convergence being uniform in $[a, b]$. Since $\mathcal{G}y_k = \lambda_k^{-1}y_k$ and $\mathcal{G}$ is continuous, then

$$\mathcal{G}^n\varphi = \sum_{k=0}^{\infty} \lambda_k^{-n}(\varphi, y_k)y_k, \qquad n = 1, 2, 3, \cdots, \tag{2.8}$$

with uniform convergence in $[a,b]$. Combining (2.8) with (1.10) we now have

$$(2.9)\qquad \begin{aligned} p_{2n} &= \sum_{k=0}^{\infty} \lambda_k^{-n}(p_0, y_k)y_k, \\ p_{2n+1} &= \sum_{k=0}^{\infty} \lambda_k^{-n}(p_1, y_k)y_k, \qquad n = 1, 2, 3, \cdots. \end{aligned}$$

LEMMA 2.1. *The sequences* $\{(p_0, y_k)\}_{k=0}^{\infty}$ *and* $\{(p_1, y_k)\}_{k=0}^{\infty}$ *are bounded, and the sequence of functions* $\{\lambda_k^{-1}y_k(x)\}_{k=0}^{\infty}$ *is uniformly bounded in* $[a,b]$.

Proof. From the Schwarz inequality,

$$|(p_0, y_k)| \leqq \|p_0\|_2\|y_k\|_2 = \|p_0\|_2,$$

and a similar result holds for (p_1, y_k). Applying the Schwarz inequality to $\lambda_k^{-1}y_k = \mathscr{G}y_k$, we have

$$\begin{aligned} |\lambda_k^{-1}y_k(x)| &= \left|\int_a^b G(x,t)y_k(t)\,dt\right| \\ &\leqq (b-a)^{1/2} \max_{a \leqq x,t \leqq b} |G(x,t)|, \end{aligned}$$

and the result follows.

The following lemma contains the necessary asymptotic bounds.

LEMMA 2.2. *There exist positive constants* K_0 *and* K_1 *such that*

$$(2.10)\qquad \begin{aligned} |p_{2n}(x) - \lambda_0^{-n}(p_0, y_0)y_0(x)| &\leqq K_0\lambda_1^{-n}, \\ |p_{2n+1}(x) - \lambda_0^{-n}(p_1, y_0)y_0(x)| &\leqq K_1\lambda_1^{-n}, \end{aligned}$$

for $a \leqq x \leqq b$ *and* $n = 1, 2, 3, \cdots$.

Proof. Let n be a positive integer. By (2.9) and Lemma 2.1, there exists a constant $K > 0$, independent of n, such that

$$\begin{aligned} |p_{2n}(x) - \lambda_0^{-n}(p_0, y_0)y_0(x)| &\leqq \sum_{k=1}^{\infty} |\lambda_k^{-n}(p_0, y_k)y_k(x)| \\ &\leqq K \sum_{k=1}^{\infty} \lambda_k^{-(n-1)} = \lambda_1^{-n}K\lambda_1 \sum_{k=1}^{\infty} (\lambda_1/\lambda_k)^{n-1}, \end{aligned}$$

for $a \leqq x \leqq b$. The first inequality in (2.10) follows from noting that $\sum_{k=1}^{\infty} (\lambda_1/\lambda_k)^{n-1}$ is convergent for $n \geqq 3$. The second inequality in (2.10) follows similarly.

If we let I denote the identity function on $[a,b]$, then (2.8) implies

$$\mathscr{G}^n I = \sum_{k=0}^{\infty} \lambda_k^{-n}(I, y_k)y_k, \qquad n = 1, 2, 3, \cdots.$$

Combining this with (2.10), one sees that

$$(2.11)\qquad \begin{aligned} 0 &\leqq p_{2n}(x) \leqq M_0\lambda_0^{-n}(p_0, y_0), \\ 0 &\leqq p_{2n+1}(x) \leqq M_1\lambda_0^{-n}(p_1, y_0), \\ 0 &\leqq (\mathscr{G}^n I)(x) \leqq M_2\lambda_0^{-n} \end{aligned}$$

for $a \leqq x \leqq b$, $n = 0, 1, 2, \cdots$, and for suitable constants M_0, M_1 and M_2.

3. Expansion theorems.

THEOREM 3.1. *Suppose that* $f \in LB^\infty[a, b]$ *and that the sequence* $\{(L^n f)(x)/\lambda_0^n\}_{n=0}^\infty$ *converges uniformly to* 0 *in* $[a, b]$. *Then*

$$f(x) = \sum_{k=0}^\infty \{(B_b L^k f) p_{2k}(x) + (B_a L^k f) p_{2k+1}(x)\} \tag{3.1}$$

with uniform convergence in $[a, b]$.

Proof. By (2.5) we have

$$f - \sum_{k=0}^{n-1} \{(B_b L^k f) p_{2k} + (B_a L^k f) p_{2k+1}\} = \mathscr{G}^n L^n f, \qquad n = 1, 2, 3, \cdots, \tag{3.2}$$

and it is therefore sufficient to prove that the sequence $\{\mathscr{G}^n L^n f\}_{n=1}^\infty$ converges uniformly to 0. By (2.8),

$$\mathscr{G}^n L^n f = \sum_{k=0}^\infty \lambda_k^{-n} (L^n f, y_k) y_k, \qquad n = 1, 2, 3, \cdots.$$

If we write $(L^n f)(x) = \lambda_0^n \varepsilon_n(x)$, $a \leqq x \leqq b$, $n = 0, 1, 2, \cdots$, then the functions ε_n satisfy $\lim_{n \to \infty} \|\varepsilon_n\|_2 = 0$, and Schwarz' inequality yields

$$|(L^n f, y_k)| \leqq \lambda_0^n \|\varepsilon_n\|_2.$$

By Lemma 2.1, there exists a constant $K > 0$ such that

$$|(\mathscr{G}^n L^n f)(x)| \leqq \|\varepsilon_n\|_2 K \sum_{k=0}^\infty (\lambda_0/\lambda_k)^{n-1}, \qquad a \leqq x \leqq b,$$

for all $n \geqq 3$, and the desired result follows.

The expansion (3.1) is an example of a two-point expansion, or "grouped" series expansion. Note that for LB-positive functions possessing the expansion (3.1), all the terms in the series are nonnegative. Hence the series converges in the ordinary sense; i.e., if f is LB-positive with $(L^n f/\lambda_0^n) \to 0$, $n \to \infty$, and if $h_{2k} = B_b L^k f$, $h_{2k+1} = B_a L^k f$, $k = 0, 1, 2, \cdots$, then

$$f(x) = \sum_{k=0}^\infty h_k p_k(x),$$

uniformly in $[a, b]$.

We shall next consider *necessary* conditions for convergence of LB-series. First, we note that the bounds (2.10) imply

$$\begin{aligned} &\lim_{n \to \infty} [\lambda_0^n p_{2n}(x)/y_0(x)] = (p_0, y_0), \\ &\lim_{n \to \infty} [\lambda_0^n p_{2n+1}(x)/y_0(x)] = (p_1, y_0) \end{aligned} \tag{3.3}$$

for all x in (a, b).

THEOREM 3.2. *Let* $\{h_k\}_0^\infty$ *be a real sequence and suppose that the series*

$$\sum_{k=0}^\infty h_k p_k(x) \tag{3.4}$$

converges at a point x_0, $a < x_0 < b$. *Then* (3.4) *converges uniformly in* $[a, b]$ *to a function* $F(x)$, *and*

$$(3.5) \qquad (L^n F)(x) = \sum_{k=2n}^{\infty} h_k p_{k-2n}(x), \qquad a \leqq x \leqq b, \quad n = 0, 1, 2, \cdots.$$

Moreover, the series $\sum_{k=0}^{\infty} \alpha_k$, *where*

$$\alpha_{2k} = \lambda_0^{-k}(p_0, y_0)h_{2k}, \qquad \alpha_{2k+1} = \lambda_0^{-k}(p_1, y_0)h_{2k+1}, \quad k = 0, 1, 2, \cdots,$$

is convergent.

Proof. Following Widder[10], we observe first that the convergence of (3.4) at x_0 implies

$$(3.6) \qquad \lim_{k \to \infty} h_k p_k(x_0) = 0.$$

By (3.3), there exists a constant $\gamma > 0$ such that

$$\lambda_0^k p_{2k}(x_0) \geqq \gamma \quad \text{and} \quad \lambda_0^k p_{2k+1}(x_0) \geqq \gamma$$

for k sufficiently large. Combining this with (3.6), we see that

$$(3.7) \qquad |h_{2k}| \leqq \gamma_0 \lambda_0^k \quad \text{and} \quad |h_{2k+1}| \leqq \gamma_1 \lambda_0^k, \qquad k = 0, 1, 2, \cdots,$$

for certain constants γ_0 and γ_1. Thus, noting that $\lambda_0 < \lambda_1$, it follows from (2.10) that the series

$$(3.8) \qquad \sum_{k=0}^{\infty} \left\{ h_{2k}\left[p_{2k}(x_0) - \frac{(p_0, y_0)y_0(x_0)}{\lambda_0^k} \right] + h_{2k+1}\left[p_{2k+1}(x_0) - \frac{(p_1, y_0)y_0(x_0)}{\lambda_0^k} \right] \right\}$$

converges *absolutely*. Then subtraction of (3.8) from (3.4), evaluated at x_0, results in the convergent series

$$(3.9) \qquad y_0(x_0) \sum_{k=0}^{\infty} \alpha_k = y_0(x_0) \sum_{k=0}^{\infty} \left[\frac{(p_0, y_0)h_{2k}}{\lambda_0^k} + \frac{(p_1, y_0)h_{2k+1}}{\lambda_0^k} \right].$$

The convergence of $\sum_0^{\infty} \alpha_k$ follows from noting $y_0(x_0) \neq 0$.

Observe now that the bounds (2.10) hold uniformly in $[a, b]$. Therefore, (3.8) remains uniformly convergent when x_0 is replaced by the variable x, $a \leqq x \leqq b$. Obviously, the same is true of (3.9). Thus the sum of (3.8) and (3.9), the series (3.4), converges uniformly in $[a, b]$.

To prove (3.5), note first that (2.10), (3.7) and the convergence of (3.9) imply convergence of each of the series

$$\sum_{k=0}^{\infty} \{h_{2k+2n}[p_{2k} - \lambda_0^{-k}(p_0, y_0)y_0] + h_{2k+2n+1}[p_{2k+1} - \lambda_0^{-k}(p_1, y_0)y_0]\}$$

and

$$y_0 \sum_{k=0}^{\infty} [\lambda_0^{-k}(p_0, y_0)h_{2k+2n} + \lambda_0^{-k}(p_1, y_0)h_{2k+2n+1}], \qquad n = 0, 1, 2, \cdots,$$

uniformly in $[a, b]$. Hence their sum, the right side of (3.5), converges uniformly in $[a, b]$ for $n = 0, 1, 2, \cdots$. Applying the operator $\mathscr{G}^n$ termwise to the right side of (3.5), we have

$$\mathscr{G}^n\left[\sum_{k=2n}^{\infty} h_k p_{k-2n}\right] = \sum_{k=2n}^{\infty} h_k p_k = F - \sum_{k=0}^{2n-1} h_k p_k.$$

Hence

$$L^n\mathscr{G}^n\left[\sum_{k=2n}^{\infty} h_k p_{k-2n}\right] = L^nF - \sum_{k=0}^{2n-1} h_k L^n p_k = L^nF,$$

and equation (3.5) now follows from (2.3). This completes the proof of the theorem.

If we differentiate the equation

$$y_k(x) = \lambda_k \int_a^b G(x, t) y_k(t)\, dt$$

with respect to x, there follows

$$y_k'(x) = \lambda_k \int_a^b \frac{x}{\partial x} G(x, t) y_k(t)\, dt.$$

Using the continuity properties of $(\partial/\partial x)G$ and Schwarz' inequality, one sees that

$$|y_k'(x)| \leqq K\lambda_k, \qquad a \leqq x \leqq b, \quad k = 0, 1, 2, \cdots,$$

where K is a constant independent of x and k. Therefore the series

$$\sum_{k=0}^{\infty} \lambda_k^{-n}(p_0, y_k) y_k'(x)$$

converges uniformly in $[a, b]$ for $n \geqq 3$. Taking into account (2.9), one has

$$p_{2n}'(x) = \sum_{k=0}^{\infty} \lambda_k^{-n}(p_0, y_k) y_k'(x), \qquad a \leqq x \leqq b, \quad n \geqq 3, \tag{3.10}$$

and similarly,

$$p_{2n+1}'(x) = \sum_{k=0}^{\infty} \lambda_k^{-n}(p_1, y_k) y_k'(x), \qquad a \leqq x \leqq b, \quad n \geqq 3. \tag{3.11}$$

In analogy to Lemma 2.2, one can now prove the following.

LEMMA 3.1. *There exist positive constants K_0' and K_1' such that*

$$\begin{aligned} |p_{2n}'(x) - \lambda_0^{-n}(p, y_0) y_0'(x)| &\leqq K_0' \lambda_1^{-n}, \\ |p_{2n+1}'(x) - \lambda_0^{-n}(p_1, y_0) y_0'(x)| &\leqq K_1' \lambda_1^{-n}, \end{aligned} \tag{3.12}$$

for $a \leqq x \leqq b$ and $n = 0, 1, 2, \cdots$.

THEOREM 3.3. *Suppose that*

$$f(x) = \sum_{k=0}^{\infty} h_k p_k(x), \qquad a \leqq x \leqq b.$$

Then $h_{2k} = B_b L^k f$ and $h_{2k+1} = B_a L^k f$, $k = 0, 1, 2, \cdots$.

Proof. We proceed as in Theorem 3.2. First observe that (3.7) holds for the sequence $\{h_k\}$. By (3.12), the series

$$\sum_{k=0}^{\infty} \{h_{2k+2n}[p'_{2k} - \lambda_0^{-k}(p_0, y_0)y'_0] + h_{2k+2n+1}[p'_{2k+1} - \lambda_0^{-k}(p_1, y_0)y'_0]\} \tag{3.13}$$

converges absolutely and uniformly in $[a, b]$ for $n = 0, 1, 2, \cdots$. Moreover, the series

$$y'_0 \sum_{k=0}^{\infty} [\lambda_0^{-k}(p_0, y_0)h_{2k+2n} + \lambda_0^{-k}(p_1, y_0)h_{2k+2n+1}] \tag{3.14}$$

converges everywhere in $[a, b]$. The sum of (3.13) and (3.14) therefore converges; that is,

$$\sum_{k=2n}^{\infty} h_k p'_{k-2n}(x)$$

converges uniformly in $[a, b]$, $n = 0, 1, 2, \cdots$. By (3.5),

$$(L^n f)'(x) = \sum_{k=2n}^{\infty} h_k p'_{k-2n}(x), \qquad a \leqq x \leqq b, \quad n = 0, 1, 2, \cdots.$$

It follows that the functionals B_a and B_b can be applied termwise to the right side of (3.5). This results in

$$B_a L^n f = \sum_{k=2n}^{\infty} h_k B_a p_{k-2n},$$

$$B_b L^n f = \sum_{k=2n}^{\infty} h_k B_b p_{k-2n}, \qquad n = 0, 1, 2, \cdots.$$

Now (1.10) and (1.6) imply $B_a p_n = B_b p_n = 0$, for $n \geqq 2$. By (1.8), then,

$$B_a L^n f = h_{2n+1} \quad \text{and} \quad B_b L^n f = h_{2n}, \qquad n = 0, 1, 2, \cdots,$$

and this completes the proof.

4. Principal results. In this section we prove the results stated in § 1. We begin with a lemma concerning the eigenfunctions $\{y_k\}_0^{\infty}$.

LEMMA 4.1. *There exists a sequence of positive constants* $\{C_k\}_{k=0}^{\infty}$ *such that*

$$|y_k(x)| \leqq C_k y_0(x), \qquad a \leqq x \leqq b, \quad k = 0, 1, 2, \cdots. \tag{4.1}$$

Proof. Let k be a fixed nonnegative integer. Since y_0 can vanish only at $x = a$ and $x = b$, it is enough to prove that the quotient y_k/y_0 is bounded in a neighborhood of $x = a$ and also in a neighborhood of $x = b$. Consider the behavior at $x = a$. If $y_0(a) \neq 0$, the result is trivial. If $y_0(a) = 0$, then $y'_0(a) \neq 0$, as otherwise $y_0 \equiv 0$. Then the endpoint condition $\alpha y_0(a) + \alpha' y'_0(a) = 0$ implies $\alpha' = 0$. Consequently $y_k(a) = 0$, and so

$$\lim_{x \to a} \frac{y_k(x)}{y_0(x)} = \frac{y'_k(a)}{y'_0(a)}.$$

Thus y_k/y_0 is bounded in a neighborhood of $x = a$. The argument for the endpoint $x = b$ is similar.

We shall also need the eigenfunction expansion [5]

$$f(x) = \sum_{k=0}^{\infty} (f, y_k) y_k(x), \qquad a \leqq x \leqq b, \tag{4.2}$$

valid for all f such that $B_a f = B_b f = 0$ and $f \in C^2[a, b]$, and the self-adjointness property

$$(Lf, g) = (f, Lg), \tag{4.3}$$

which holds whenever $B_a f = B_b f = B_a g = B_b g = 0$.

Proof of Theorem 1.1. Suppose that f is LB-positive. For each positive integer n, let

$$S_n = \sum_{k=0}^{n-1} \{(B_b L^k f) p_{2k} + (B_a L^k f) p_{2k+1}\},$$

$$R_n = \mathscr{G}^n L^n f.$$

Then by (2.5), $f = S_n + R_n$, $n = 1, 2, 3, \cdots$. From (1.9) and our assumption on f, there follows $f(x) \geqq S_n(x) \geqq 0$ for each n and x. Since the Green's function is nonnegative, $f(x) \geqq R_n(x) \geqq 0$. Observing that the sequences $\{S_n(x)\}_{n=1}^{\infty}$ and $\{R_n(x)\}_{n=1}^{\infty}$ are, respectively, increasing and decreasing, let us write

$$R(x) = \lim_{n \to \infty} R_n(x), \qquad a \leqq x \leqq b,$$

so that

$$f(x) = R(x) + \sum_{k=0}^{\infty} \{(B_b L^k f) p_{2k}(x) + (B_a L^k f) p_{2k+1}(x)\}. \tag{4.4}$$

Now Theorem 3.2 implies

$$L^n R = L^n f - \sum_{k=n}^{\infty} \{(B_b L^k f) p_{2k-2n} + (B_a L^k f) p_{2k-2n+1}\} \tag{4.5}$$

for $n = 0, 1, 2, \cdots$. It follows from Theorem 3.3 that

$$B_a L^n R = B_a L^n f - B_a L^n f = 0$$

and, similarly,

$$B_b L^n R = 0, \qquad n = 0, 1, 2, \cdots.$$

We will now show that $(R, y_k) = 0$ for $k > 0$. Identity (2.5), applied to $L^n f$, yields

$$L^n f = \sum_{k=0}^{m-1} \{(B_b L^{k+n} f) p_{2k} + (B_a L^{k+n} f) p_{2k+1}\} + \mathscr{G}^m L^{m+n} f$$

for all $n \geqq 0$ and $m \geqq 1$. In particular,

$$L^n f \geqq \sum_{k=0}^{m-1} \{(B_b L^{k+n} f) p_{2k} + (B_a L^{k+n} f) p_{2k+1}\}, \quad n = 0, 1, 2, \cdots$$

Letting $m \to \infty$ and noting (4.5), we find

$$(L^n R)(x) \geqq 0, \qquad a \leqq x \leqq b, \quad n = 0, 1, 2, \cdots.$$

Now let j be a positive integer. From (1.5) and (4.3),

$$|(R, y_j)| = |\lambda_j^{-n}(R, L^n y_j)| = \lambda_j^{-n}|(L^n R, y_j)|,$$

for $n = 0, 1, 2, \cdots$. From (4.1),

$$\begin{aligned}|(L^n R, y_j)| &\leqq \int_a^b |(L^n R)(x) y_j(x)|\, dx \leqq C_j \int_a^b |(L^n R)(x) y_0(x)|\, dx \\ &= C_j \int_a^b (L^n R)(x) y_0(x)\, dx = C_j (L^n R, y_0) \\ &= C_j (R, L^n y_0) = C_j \lambda_0^n (R, y_0), \qquad n = 0, 1, 2, \cdots.\end{aligned}$$

Therefore,

$$|(R, y_j)| \leqq (\lambda_0/\lambda_j)^n C_j (R, y_0), \qquad n = 0, 1, 2, \cdots. \tag{4.6}$$

Letting $n \to \infty$ in (4.6) results in $(R, y_j) = 0$. By the completeness property of the eigenfunctions $\{y_k\}_0^\infty$, there follows

$$R(x) = (R, y_0) y_0(x), \qquad a \leqq x \leqq b,$$

which, in view of (4.4), completes the proof of the necessity.

For the sufficiency, let F be defined by

$$F(x) = C y_0(x) + \sum_{k=0}^{\infty} \{a_k p_{2k}(x) + b_k p_{2k+1}(x)\},$$

with $C \geqq 0$, $a_k \geqq 0$ and $b_k \geqq 0$, $k = 0, 1, 2, \cdots$. By Theorem 3.2,

$$L^n F = C \lambda_0^n y_0 + \sum_{k=n}^{\infty} \{a_k p_{2k-2n} + b_k p_{2k-2n+1}\},$$

so that $(L^n F)(x) \geqq 0$, $a \leqq x \leqq b$, $n = 0, 1, 2, \cdots$. In view of Theorem 3.3,

$$B_a L^n F = b_n \geqq 0 \quad \text{and} \quad B_b L^n F = a_n \geqq 0, \qquad n = 0, 1, 2, \cdots,$$

and this completes the proof of the theorem.

COROLLARY 4.1. *If f is LB-positive, then the sequence $\{(L^n f)(x)/\lambda_0^n\}_{n=0}^\infty$ is uniformly bounded in $[a, b]$.*

Proof. Because of Theorem 1.1, f has the representation

$$f(x) = C y_0(x) + \sum_{k=0}^{\infty} h_k p_k(x)$$

with $C \geqq 0$ and $h_k \geqq 0$, $k = 0, 1, 2, \cdots$, and by Theorem 3.2,

$$L^n f = C \lambda_0^n y_0 + \sum_{k=2n}^{\infty} h_k p_{k-2n}, \qquad n = 0, 1, 2, \cdots.$$

By (2.11),

$$\lambda_0^k p_{2k}(x) \leqq M_0(p_0, y_0) \quad \text{and} \quad \lambda_0^k p_{2k+1}(x) \leqq M_1(p_1, y_0)$$

for $a \leqq x \leqq b$ and $k = 0, 1, 2, \cdots$. Therefore,

$$(L^n f)(x) = C\lambda_0^n y_0(x) + \lambda_0^n \sum_{k=0}^{\infty} \left\{ \frac{h_{2k+2n}}{\lambda_0^{k+n}} p_{2k}(x)\lambda_0^k + \frac{h_{2k+2n+1}}{\lambda_0^{k+n}} p_{2k+1}(x)\lambda_0^k \right\}$$

$$\leqq \lambda_0^n \left\{ C y_0(x) + \max(M_0, M_1) \sum_{k=2n}^{\infty} \alpha_k \right\}, \qquad n = 0, 1, 2, \cdots,$$

where $\{\alpha_k\}$ is defined as in Theorem 3.2. Thus

$$\limsup_{n \to \infty} \frac{(L^n f)(x)}{\lambda_0^n} \leqq C y_0(x), \qquad a \leqq x \leqq b,$$

and this completes the proof.

Proof of Theorem 1.2. Let f be LB-positive. Then (i) is trivial, (ii) is a consequence of Theorem 3.2, and (iv) follows from Corollary 4.1. To prove (iii) we first write

$$f(x) = C y_0(x) + \sum_{k=0}^{\infty} h_k p_k(x)$$

with $C \geqq 0$ and $h_k \geqq 0$, $k = 0, 1, 2, \cdots$, and conclude

$$(L^n f)(x) = C\lambda_0^n y_0(x) + \sum_{k=2n}^{\infty} h_k p_{k-2n}(x), \qquad n = 0, 1, 2, \cdots.$$

As in the proof of Theorem 3.3, we apply the functional $\bar{B}_a$ to both sides of the above equation to obtain

$$\begin{aligned} \bar{B}_a L^n f &= C\lambda_0^n \bar{B}_a y_0 - \alpha' \sum_{k=2n}^{\infty} h_k p_{k-2n}(a) + \alpha \sum_{k=2n}^{\infty} h_k p'_{k-2n}(a) \\ &= C\lambda_0^n \bar{B}_a y_0 \\ &\quad - \alpha'\lambda_0^n \sum_{k=0}^{\infty} \left\{ \frac{h_{2k+2n}}{\lambda_0^{k+n}} p_{2k}(a)\lambda_0^k + \frac{h_{2k+2n+1}}{\lambda_0^{k+n}} p_{2k+1}(a)\lambda_0^k \right\} \\ &\quad + \alpha\lambda_0^n \sum_{k=0}^{\infty} \left\{ \frac{h_{2k+2n}}{\lambda_0^{k+n}} p'_{2k}(a)\lambda_0^k + \frac{h_{2k+2n+1}}{\lambda_0^{k+n}} p'_{2k+1}(a)\lambda_0^k \right\} \end{aligned} \tag{4.7}$$

for $n = 0, 1, 2, \cdots$. Using (2.11) and (3.12), one can easily show that, for some constant $K > 0$,

$$\left| \frac{\bar{B}_a L^n f}{\lambda_0^n \bar{B}_a y_0} - C \right| \leqq K(|\alpha| + |\alpha'|) \sum_{k=2n}^{\infty} \alpha_k, \tag{4.8}$$

and so (iii) follows from the convergence of $\sum_0^\infty \alpha_k$. This completes the proof of necessity.

Now suppose (i)–(iv) hold. Using (2.11) and the convergence of the series in (ii), one can easily prove that the series

$$S(x) = \sum_{k=0}^{\infty} \{(B_b L^k f) p_{2k}(x) + (B_a L^k f) p_{2k+1}(x)\}$$

converges uniformly in $[a, b]$. Let $R(x) = f(x) - S(x)$. By Theorem 3.2 and Theorem 3.3, $B_a L^k R = B_b L^k R = 0$, $k = 0, 1, 2, \cdots$. We shall prove that $(R, y_k) = 0$ for $k \geqq 1$. First, note that $S(x)$ is LB-positive. By the Corollary, the sequence $\{(L^n S)(x)/\lambda_0^n\}_{n=0}^{\infty}$ is uniformly bounded in $[a, b]$. Noting hypothesis (iv), we find that the sequence $\{(L^n R)(x)/\lambda_0^n\}_{n=0}^{\infty}$ is likewise uniformly bounded. Then for $k \geqq 1$,

$$\begin{aligned} |(R, y_k)| &= \lambda_k^{-n} |(L^n R, y_k)| \\ &\leqq \lambda_k^{-n} \int_a^b |(L^n R)(x) y_k(x)|\, dx \\ &\leqq K(\lambda_0/\lambda_k)^n, \qquad\qquad n = 0, 1, 2, \cdots, \end{aligned}$$

for some appropriate constant $K > 0$. Letting $n \to \infty$ results in $(R, y_k) = 0$. Since $\{y_k\}_0^{\infty}$ is a complete orthonormal set, then

$$R(x) = (R, y_0) y_0(x), \qquad a \leqq x \leqq b,$$

and hence

$$f(x) = (R, y_0) y_0(x) + \sum_{k=0}^{\infty} \{(B_b L^k f) p_{2k}(x) + (B_a L^k f) p_{2k+1}(x)\}.$$

By Theorem 1, it is therefore sufficient to show that $(R, y_0) \geqq 0$. For this, we employ hypothesis (iv). In fact, proceeding as in (4.7) and (4.8), one sees that

$$(R, y_0) = \lim_{n \to \infty} \frac{\bar{B}_a L^n f}{\lambda_0^n \bar{B}_a y_0} \geqq 0,$$

and the proof is complete.

Proof of Theorem 3. By (4.2), f admits the eigenfunction expansion

$$f(x) = \sum_{k=0}^{\infty} (f, y_k) y_k(x), \qquad a \leqq x \leqq b.$$

Choose k so that $\lambda_k \geqq M$, let $\varepsilon > 0$ satisfy

$$M - \varepsilon > \limsup_{n \to \infty} |(L^n f)(x)|^{1/n}, \qquad a \leqq x \leqq b,$$

and let N be a positive integer such that $n \geqq N$ implies $|(L^n f)(x)| \leqq (M - \varepsilon)^n$, $a \leqq x \leqq b$. Then if $n \geqq N$,

$$\begin{aligned} |(f, y_k)| &= |\lambda_k^{-n}(f, L^n y_k)| = |\lambda_k^{-n}(L^n f, y_k)| \\ &\leqq \lambda_k^{-n} \int_a^b |(L^n f)(x) y_k(x)|\, dx \\ &\leqq \left(\frac{M - \varepsilon}{\lambda_k}\right)^n \int_a^b |y_k(x)|\, dx. \end{aligned}$$

Letting $n \to \infty$, we obtain $(f, y_k) = 0$, whenever $\lambda_k \geqq M$. Thus

$$f(x) = \sum_{\lambda_k < M} (f, y_k) y_k(x),$$

which is the desired result.

5. Applications. In this section we illustrate our methods by considering the special case of the harmonic oscillator $L_0 y = -y''$, with general, separated end-point conditions. Thus, we consider the eigenvalue problem

$$y'' + \lambda y = 0, \qquad \alpha y(a) + \alpha' y'(a) = \beta y(b) + \beta' y'(b) = 0, \tag{5.1}$$

where α, α', β, β' are prescribed by (1.3) and (1.4). We begin by observing that, for this operator, the p_k are polynomials and the eigenfunctions y_k are entire functions of exponential type $\sqrt{\lambda_k}$.

THEOREM 5.1. *A function f on $[a, b]$ has an L_0B-series expansion*

$$f(x) = \sum_{k=0}^{\infty} h_k p_k(x), \qquad a \leqq x \leqq b, \tag{5.2}$$

with uniform convergence on $[a, b]$, if and only if f is the restriction to $[a, b]$ of an entire function F satisfying

$$\lim_{n \to \infty} (\sqrt{\lambda_0})^{-n} F^{(n)}(0) = 0. \tag{5.3}$$

Proof. For the sufficiency, let us write

$$|F^{(n)}(0)| = \varepsilon_n (\sqrt{\lambda_0})^n, \qquad n = 0, 1, 2, \cdots,$$

where $\lim_{n \to \infty} \varepsilon_n = 0$. Then

$$|f^{(2n)}(x)| = \left| \sum_{k=0}^{\infty} F^{(2n+k)}(0) \frac{x^k}{k!} \right|$$

$$\leqq \lambda_0^n \sum_{k=0}^{\infty} \varepsilon_{2n+k} \frac{|x\sqrt{\lambda_0}|^k}{k!} \leqq \lambda_0^n (\max_{j \geqq 2n} \varepsilon_j)(\exp |x\sqrt{\lambda_0}|)$$

for $a \leqq x \leqq b$ and $n = 0, 1, 2, \cdots$. Since $(L_0^n f)(x) = (-1)^n f^{(2n)}(x)$, $n = 0, 1, 2, \cdots$, Theorem 3.1 implies

$$F(x) = \sum_{k=0}^{\infty} [(B_b L^k f) p_{2k}(x) + (B_a L^k f) p_{2k+1}(x)], \qquad a \leqq x \leqq b.$$

The representation (5.2) will follow if we show that

$$\lim_{k \to \infty} (B_b L^k f) p_{2k}(x) = 0$$

uniformly in $[a, b]$. Note first that

$$\lim_{k \to \infty} (\lambda_0^{-k} B_b L^k f) = \lim_{k \to \infty} (-1)^k \lambda_0^{-k} [\beta f^{(2k)}(b) + \beta' f^{(2k+1)}(b)] = 0$$

by the hypothesis on F. Then, by (2.11),

$$\lim_{k \to \infty} (B_b L^k f) p_{2k}(x) = \lim_{k \to \infty} (\lambda_0^k p_{2k}(x)) \frac{B_b L^k f}{\lambda_0^k} = 0$$

uniformly in $[a, b]$, and this completes the proof of sufficiency.

In the other direction, let

$$f(x) = \sum_{k=0}^{\infty} h_k p_k(x), \qquad a \leqq x \leqq b.$$

By Theorem 3.2 and the proof of Theorem 3.3, we have

$$(5.4)\qquad (-1)^n f^{(2n+j)}(x) = \sum_{k=0}^{\infty} h_{k+2n} p_k^{(j)}(x), \qquad a \leqq x \leqq b,$$

for $n = 0, 1, 2, \cdots$, and $j = 0, 1$. Define the sequence of functions $\{g_k\}_{k=0}^{\infty}$ by

$$g_{2k}(x) = \frac{\lambda_0^k p_{2k}(x)}{(p_0, y_0)} - y_0(x),$$

$$g_{2k+1}(x) = \frac{\lambda_0^k p_{2k+1}(x)}{(p_1, y_0)} - y_0(x), \qquad k = 0, 1, 2, \cdots.$$

Then (5.4) may be written

$$(-1)^n f^{(2n+j)}(x) = \lambda_0^n \left\{ y_0^{(j)}(x) \sum_{k=2n}^{\infty} \alpha_k + \sum_{k=2n}^{\infty} \alpha_k g_{2n+k}^{(j)}(x) \right\}$$

for $n = 0, 1, 2, \cdots$, and $j = 0, 1$ and where $\{\alpha_k\}_{k=0}^{\infty}$ is defined as in Theorem 3.2. The asymptotic estimates (2.10) and (3.12) imply

$$|g_k^{(j)}(x)| \leqq K(\lambda_0/\lambda_1)^k, \qquad k = 0, 1, 2, \cdots, \quad j = 0, 1,$$

where $K > 0$ is a constant, and this together with the convergence of $\sum_{k=0}^{\infty} \alpha_k$ yields

$$(5.5)\qquad |f^{(k)}(x)| \leqq M(\sqrt{\lambda_0})^k, \qquad a \leqq x \leqq b, \quad k = 0, 1, 2, \cdots,$$

where $M > 0$ is a constant. Therefore f admits the Taylor series expansion

$$(5.6)\qquad f(x) = \sum_{k=0}^{\infty} f^{(k)}(x_0)\frac{(x - x_0)^k}{k!}$$

about any point x_0, $a < x_0 < b$. The bound (5.5) further implies that the series (5.6) converges at all points in the complex plane, uniformly on compact sets, and that its sum, say $F(z)$, satisfies

$$\lim (\sqrt{\lambda_0})^{-n} F^{(n)}(0) = 0.$$

Since $F = f$ on $[a, b]$, the proof is complete.

The boundary value problem

$$y'' + \lambda y = 0, \qquad y(1) = y'(0) = 0$$

has eigenvalues and eigenfunctions

$$\lambda_k = \left(\frac{(2k + 1)\pi}{2}\right)^2, \qquad y_k(x) = \sqrt{2}\cos\frac{(2k + 1)\pi x}{2}, \qquad k = 0, 1, 2, \cdots$$

and leads, via identity (2.5), to the modified Abel series

$$(5.7)\qquad f(x) = \sum_{k=0}^{\infty} [f^{(2k)}(1) q_{2k}(x) + f^{(2k+1)}(0) q_{2k+1}(x)].$$

Pethe and Sharma [7] have shown that any entire function f of exponential type less than $\pi/2$ possesses a series expansion (5.7). Theorem 5.1 yields the improved result that $\lim_{n\to\infty} (\pi/2)^{-n} f^{(n)}(0) = 0$ is necessary and sufficient for f to admit a modified Abel series expansion.

The following theorem, a trivial consequence of Theorem 1 and Theorem 5.1, generalizes the result of Widder [9] which relates completely convex functions to entire functions of finite exponential type.

THEOREM 5.2. *Let $f \in C^\infty[a,b]$ and suppose*

$$(-1)f^{(2n)}(x) \geqq 0, \quad (-1)^n B_a f^{2n} \geqq 0, \quad (-1)^n B_b f^{2n} \geqq 0$$

for $a \leqq x \leqq b$ and $n = 0, 1, 2, \cdots$. Then f is the restriction to $[a,b]$ of an entire function of exponential type not exceeding $\sqrt{\lambda_0}$.

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ON THE ERROR IN THE PADÉ APPROXIMANTS FOR A FORM OF THE INCOMPLETE GAMMA FUNCTION INCLUDING THE EXPONENTIAL FUNCTION*

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Abstract. Closed form expressions for all entries of the Padé matrix table and their errors are derived for the incomplete gamma function

$$H(c, z) = cz^{-c}e^{-z}\int_0^z e^t t^{c-1}\,dt, \qquad R(c) > 0, \qquad H(0, z) = e^{-z}.$$

Asymptotic estimates for the error are developed. Let (μ, ν) be a position in the Padé matrix table where μ and ν are the degrees of the denominator and numerator polynomials, respectively, which define the Padé approximant. Our error representations hold for c and z fixed, c not a negative integer, with μ or ν or both μ and ν approaching infinity. Under these conditions the Padé approximants converge along all rows, columns and diagonals. The asymptotic representations are remarkable in that they give very easy to apply and very realistic estimates when the parameters μ and ν are rather small. In the case $c = 0$, that is, the exponential function, uniform asymptotic estimates are developed for the main diagonal and first and second subdiagonal entries of the Padé matrix.

1. Introduction. In [1, vol. 2, pp. 189–194], we developed asymptotic estimates of the main diagonal and first subdiagonal Padé approximations for the incomplete gamma function

$$(1) \qquad H(c, z) = cz^{-c}e^{-z}\int_0^z e^t t^{c-1}\,dt, \qquad R(c) > 0,$$

$$(2) \qquad H(c, z) = {}_1F_1(1; c + 1; -z); \qquad H(0, z) = e^{-z}.$$

Here and throughout we make free use of the notation and definitions given in [1]. It is also convenient to use the notation

$$(3) \qquad {}_1F_1^m(a; c; z) = \sum_{k=0}^{m} \frac{(a)_k z^k}{(c)_k k!}.$$

The asymptotic estimates noted above show that the Padé approximants converge to $H(c, z)$ on all compact subsets of the complex plane. In the present paper, the analysis is extended to cover all positions of the Padé matrix table for $H(c, z)$. Let (μ, ν) be a position in the Padé matrix table where μ and ν are the degrees of the denominator and numerator polynomials, respectively, which define the Padé approximant. Our error representations hold for c and z fixed with μ or ν or both μ and ν approaching infinity. The asymptotic representations are remarkable in that they give very easy to apply and very realistic estimates when the parameters μ and ν are rather small. For the case $c = 0$, that is, the exponential function, we show that for $\mu \geqq \nu$, the error can be expressed as the ratio of two series, each composed of $\mu + 1 - \nu$ terms. The numerator and denominator of this ratio involve the Bessel functions $K_r(z)$ and $I_r(z)$, respectively. For the situations

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$\mu = \nu + s, s = 0, 1, 2$, which corresponds to the main diagonal, and first and second subdiagonals of the Padé matrix, we make use of uniform asymptotic expansions for the Bessel functions to derive like representations for the error in the Padé approximations. Further comments on this and other matters are considered later.

Several authors have examined the convergence of the Padé approximations for e^z which can lie anywhere in the Padé matrix. The analysis is based on establishing bounds for the error. Underhill and Wragg [2] and Saff [3] show that all entries of the Padé table for e^z converge in $|z| < b$, b fixed but arbitrary. The former authors also show that the derivatives of the Padé approximants for e^z converge to this function in the domain cited. Such approximants are called derivative convergent. Wragg and Davies [4] show that all row and column entries of the Padé table for e^z as well as the $(n, n - a)$, $a = 0, \pm 1$ Padé approximants are derivative convergent. Ehle [5] proves that entries on the 1st and 2nd subdiagonals of the Padé table for e^{-z} are analytic and bounded by one in the entire right half-plane. Saff and Varga [6] study the convergence of particular Padé approximants to e^{-z} on certain unbounded sets in the complex plane. Such results are needed in the evaluation of e^A where A is a matrix and the application of e^A to the solution of differential equations. Varga [7] proved that Padé approximants to e^z also converge to e^A when A is Hermitian. In that paper and also in Varga [8], the main emphasis is on application of Padé approximations for e^A to solve parabolic partial differential equations. Fair and Luke [9] showed that the $(n - a, n)$, $a = 0, 1$ Padé approximants for e^A converge when A is a bounded linear operator on a Banach space. A priori estimates of the error are developed. Wragg and Davies [4] use their bounds for the error in the Padé approximations to e^z to get bounds for like approximations to e^A.

In virtue of the data presented in this paper, the results of Fair and Luke [9] readily apply for all Padé approximants to e^A. Furthermore, in practice these data should be sufficient even for large z in view of the multiplicative property of the exponential function.

2. Padé approximations for $H(c, z)$. Let $U_{\mu,\nu}(z)$ and $V_{\mu,\nu}(z)$ be polynomials in z of degree ν and μ, respectively, $U_{\mu,\nu}(z)/V_{\mu,\nu}(z)$ be the Padé approximant of order $\mu + \nu + 1$ to $H(c, z)$, and $R_{\mu,\nu}(z)$ the error in this approximation. Thus

$$H(c, z) = \{U_{\mu,\nu}(z)/V_{\mu,\nu}(z)\} + R_{\mu,\nu}(z), \tag{4}$$

and from [10] we deduce that

$$V_{\mu,\nu}(z) = \sum_{k=0}^{\mu} \frac{(-\mu)_k z^k}{(-c-\mu-\nu)_k k!} = {}_1F_1^{\mu}\left(\begin{matrix} -\mu \\ -c-\mu-\nu \end{matrix} \middle| z\right), \tag{5}$$

$$U_{\mu,\nu}(z) = \sum_{k=0}^{\nu} b_k z^k, \tag{6}$$

$$b_k = \frac{(-)^k}{(c+1)_k} \sum_{r=0}^{k} \frac{(-\mu)_r(-c-k)_r}{(c-\mu-\nu)_r} = \frac{(-)^k}{(c+1)_k}\, {}_2F_1^{k}\left(\begin{matrix} -\mu, -c-k \\ -c-\mu-\nu \end{matrix} \middle| 1\right),$$

$$\mu \geqq \nu, \tag{7}$$

$$b_k = \frac{(-)^k(-\mu)_k}{(-c-\mu-\nu)_k k!}\, {}_3F_2\left(\begin{matrix}-k, c+1+\mu+\nu-k, 1\\ 1+\mu-k, c+1\end{matrix}\,\middle|\, 1\right), \qquad \mu \geqq \nu.$$

If $\mu < \nu$, then the above expressions for b_k are valid for $0 \leqq k \leqq \mu - 1$, and

$$\text{(8)} \qquad b_k = \frac{(-)^k(\mu+\nu-k)!}{(c+1)_k(c+1+\nu)_\mu(\nu-k)!}, \qquad \mu \leqq k \leqq \nu.$$

It is proved in [10] that the Padé table for $H(c, z)$ is normal provided c is not a negative integer. The error can be expressed in the form

$$\text{(9)} \qquad R_{\mu,\nu}(z) = \frac{(-)^{\nu+1}\mu! z^{\mu+\nu+1}}{(c+1)_{\mu+\nu+1}(c+1+\nu)_\mu} \frac{{}_1F_1\left(\begin{matrix}\mu+1\\ c+\mu+\nu+2\end{matrix}\,\middle|\, -z\right)}{{}_1F_1^\mu\left(\begin{matrix}-\mu\\ -c-\mu-\nu\end{matrix}\,\middle|\, z\right)}$$

$$\text{(10)} \qquad = \frac{(-)^{\nu+1}\mu! z^{\mu+\nu+1}e^{-z}}{(c+1)_{\mu+\nu+1}(c+1+\nu)_\mu} \frac{{}_1F_1\left(\begin{matrix}c+\nu+1\\ c+\mu+\nu+2\end{matrix}\,\middle|\, z\right)}{{}_1F_1^\mu\left(\begin{matrix}-\mu\\ -c-\mu-\nu\end{matrix}\,\middle|\, z\right)}.$$

If $c = 0$, the numerator and denominator polynomials of the Padé approximant are simply related. Thus

$$\text{(11)} \qquad U_{\mu,\nu}(z) = V_{\nu,\mu}(-z).$$

3. Asymptotic analysis of the error. As a preliminary to the error analysis, we have need for some expansions of the confluent hypergeometric form which do not seem to have been previously given in the literature. We have

$$\text{(12)} \qquad {}_1F_1(a; c; z) = e^{az/c} \sum_{k=0}^{\infty} g_k z^k, \qquad g_0 = 1,$$

$$\text{(13)} \qquad g_1 = 0, \qquad g_2 = \frac{a(c-a)}{2c^2(c+1)}, \qquad g_3 = \frac{a(c-a)(c-2a)}{3c^3(c+1)(c+2)},$$

$$\text{(14)} \qquad g_{k+1} = \frac{[a(c-a)g_{k-1} + kc(c-2a)g_k]}{c^2(k+1)(k+c)}.$$

This expansion is readily derived as follows. Starting with the known differential equation for ${}_1F_1$, we easily get the differential equation for $e^{-az/c}\,{}_1F_1$, and by use of Frobenius' method, the above series expansion follows. We remark that the expansion in (12) converges for all z. Further, with

$$g_k = h_k/\{(c)_k c^{k-1} k!\},$$

then

$$\text{(15)} \qquad h_{k+1} = k[a(c-a)(c+k-1)h_{k-1} + (c-2a)h_k].$$

For fixed α, β, γ and δ, $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \beta$, $2\alpha \neq \beta$, let $a = \alpha n + \gamma$ and $c = \beta n + \delta$. Then h_{2k} and h_{2k+1} are polynomials in n of degree $3k - 1$ and $3k$, respectively. So as $n \to \infty$,

$$g_k = O(n^{-b}), \qquad b = [\tfrac{1}{2}k + \tfrac{1}{2}], \quad k > 1, \quad k \text{ fixed}, \tag{16}$$

where $[m]$ is the largest integer contained in m.

Again, let a and c be as above except that $\alpha = \beta$. Then both h_{2k} and h_{2k+1} are polynomials in n of degree $3k - 1$, whence as $n \to \infty$,

$$g_k = O(n^{-k}), \qquad k > 1, \quad k \text{ fixed}. \tag{17}$$

A useful expansion, valid when c is large and $(\frac{1}{2}c - a)$ is bounded is

$${}_1F_1(a; c; z) = e^{z/2} \sum_{n=0}^{N-1} d_n(z)/u^n + O(u^{-N}), \tag{18}$$

where

$$d_0(z) = 1, \qquad u = \tfrac{1}{2}(c - 1), \tag{19}$$

and the d_n's can be generated by

$$d_{n+1}(z) = -\tfrac{1}{2}z d_n'(z) + \frac{1}{2}\int_0^z (\tfrac{1}{4}t - p)\, d_n(t)\, dt, \qquad p = \tfrac{1}{2}c - a. \tag{20}$$

In illustration,

$$d_1(z) = \frac{z^2}{16} - \frac{pz}{2}, \qquad d_2(z) = \frac{z^4}{512} - \frac{pz^3}{32} + \frac{(2p^2 - 1)z^2}{16} + \frac{pz}{4}. \tag{21}$$

In (18), the order term is uniform in z for $|R(z)| \leqq b$, b fixed but arbitrary. For further details on (18)–(21), see [1, vol. 1, p. 133].

If a is bounded, z is fixed and c is large, then it is known that

$$\begin{gathered} {}_1F_1^g(a; c; z) = \sum_{k=0}^{m-1} \frac{(a)_k z^k}{(c)_k k!} + O(c^{-m}), \\ m \leqq g \quad \text{or} \quad m < g \quad \text{if } g \to \infty. \end{gathered} \tag{22}$$

Application of the appropriate results above shows that for z and c fixed, the Padé approximants converge along all rows, columns and diagonals of the Padé matrix. We make no attempt to give a single formula to cover the remainder for all values of μ and ν. Instead, we propose to give asymptotic formulas for five stripes of the Padé matrix and their neighborhoods including the first row ($\mu = 0$), the main diagonal ($\mu = \nu$) and the first column ($\nu = 0$). The other two stripes can be roughly characterized as lying midway between the $\mu = 0$ and $\mu = \nu$ stripes and midway between the $\mu = \nu$ and $\nu = 0$ stripes. In each case we suppose that

$$\mu + \nu = 2n + O(1), \tag{23}$$

where n is a large positive integer. In the following discussion s and t are fixed integers with respect to n. The five stripes or cases are as follows.

Case 1. The first row and its neighborhood. Thus $\mu = s$, $\nu = 2n - t$.

Case 2. $\mu = r + s$, $\nu = 3r - t$, $r = \frac{1}{2}n$ or $\frac{1}{2}n + \frac{1}{2}$ according as n is even or odd, respectively.

Case 3. The main diagonal and its neighborhood. Thus $\mu = n + s$, $\nu = n - t$.

Case 4. $\mu = 3r + s$, $\nu = r - t$, r as in Case 2.

Case 5. The first column and its neighborhood. Thus $\mu = 2n - t$, $\nu = s$.

We now develop the representations for each of these cases.

Case 1. $\mu = s$, $\nu = 2n - t$. From (11), (12) and (22), we get

$$R_{\mu,\nu}(z) = \frac{(-)^{t+1}t!\Gamma(c+1)\Gamma(N-s)z^{N-c}e^{-z}e^{(N-s)z/(N+1)}}{\Gamma(N)\Gamma(N+1)} \tag{24}$$
$$\frac{\left[1 + \dfrac{(N-s)(s+1)z^2}{2(N+1)^2(N+2)} + O(n^{-3})\right]}{\left[1 + \dfrac{sz}{N-1} + \dfrac{s(s-1)z^2}{2(N-1)(N-2)} + O(n^{-3})\right]},$$
$$N = 2n + c + 1 + s - t.$$

Case 2. $\mu = r + s$, $\nu = 3r - t$, $r = \frac{1}{2}n$ or $\frac{1}{2}n + \frac{1}{2}$ according as n is even or odd, respectively. From (11) and (12), we have

$$R_{\mu,\nu}(z) = \frac{(-)^{3r+1+t}(r+s)!\Gamma(c+1)\Gamma(N-r-s)z^{N-c}e^{-z}}{\Gamma(N)\Gamma(N+1)}$$
$$\cdot \exp\left\{\frac{(N-r-s)z}{N+1} - \frac{(r+s)z}{N-1}\right\}\cdot \tag{25}$$
$$\cdot\left[1 + \frac{(N-r-s)(r+1+s)z^2}{2(N+1)^2(N+2)} + \frac{(r+s)(N-1-r-s)z^2}{2(N-1)^2(N-2)} + O(n^{-2})\right],$$

with N as in (24) if n is even, and N replaced by $N + 2$ if n is odd.

Case 3. $\mu = n + s$, $\nu = n - t$. From (11) and (18), we have

$$R_{\mu,\nu}(z) = \frac{(-)^{n+1+t}(n+s)!\Gamma(c+1)\Gamma(n+c+1-t)}{\Gamma(N)\Gamma(N+1)} \cdot z^{N-c}e^{-z}\exp(z[z - 4(s+t-c)]/(4N))[1 + O(n^{-3})], \tag{26}$$

with N as in (24). Here we have employed the fact that

$$\frac{\sum_{k=0}^{\infty} d_k/u^k}{\sum_{k=0}^{\infty}(-)^k d_k/u^k} = 1 + \frac{2d_1}{u} + \frac{2d_1^2}{u^2} + O(u^{-3}) = e^{2d_1/u}[1 + O(u^{-3})], \tag{27}$$

where it is understood that if each (at least formal) infinite sum is truncated after m terms, the remainder is $O(u^{-m})$.

Case 4. $\mu = 3r + s$, $\nu = r - t$, $r = \frac{1}{2}n$ or $\frac{1}{2}n + \frac{1}{2}$ according as n is even or odd, respectively. From (11) and (12), we have

$$R_{\mu,\nu}(z) = \frac{(-)^{r+1+t}(3r+s)!\Gamma(c+1)\Gamma(c+1+r-t)z^{N-c}e^{-z}}{\Gamma(N)\Gamma(N+1)}$$

$$\cdot \exp\left\{\frac{(r+1+c-t)z}{N+1} - \frac{(3r+s)z}{N-1}\right\} \tag{28}$$

$$\cdot\left[1 + \frac{(r+1+c-t)(3r+1+s)z^2}{2(N+1)^2(N+2)} + \frac{(r+c-t)(3r+s)z^2}{2(N-1)^2(N-2)} + O(n^{-2})\right],$$

with N as in (24) if n is even, and N replaced by $N+2$ if N is odd.

Case 5. $\mu = 2n - t$, $\nu = s$. Again we use (11), but this time it is convenient to employ the Kummer transformation formula

$$ {}_1F_1^{\mu}\left(\begin{matrix}-\mu \\ -c-\mu-\nu\end{matrix}\middle| z\right) = e^z\, {}_1F_1\left(\begin{matrix}-c-\nu \\ -c-\mu-\nu\end{matrix}\middle| -z\right). \tag{29}$$

Then with the aid of (22), we have

$$R_{\mu,\nu}(z) = \frac{(-)^{t+1}(2n-t)!\Gamma(c+1)\Gamma(c+1+s)z^{N-c}e^{-2z}}{\Gamma(N+1)\Gamma(N)}$$

$$\cdot\frac{\left[1 + \frac{(c+t+1)z}{N+1} + \frac{(c+t+1)(c+t+2)z^2}{2(N+1)(N+2)} + O(n^{-3})\right]}{\left[1 - \frac{(c+t)z}{N-1} + \frac{(c+t)(c+t-1)z^2}{2(N-1)N} + O(n^{-3})\right]}, \tag{30}$$

with N as in (24)

It is of interest to compare the remainders for five stripes, one from each of the cases with $s = t = 0$. Thus

$$\frac{R_{n,n}(z)}{R_{0,2n}(z)} = \frac{(-)^n(n\pi)^{1/2}e^{-z}}{2^{2n+c}}[1 + O(n^{-3})], \tag{31}$$

which shows the superiority of the main diagonal Padé approximation over the truncated Taylor series expansion of essentially the same number of terms. Actually, in the computation of the main diagonal approximation for e^{-z}, considerable economy can be achieved since $V_{n,n}(-z) = U_{n,n}(z)$. Thus if we write $V_{n,n}(z) = M_n(z^2) + zN_n(z^2)$, then the approximation only necessitates the evaluation of essentially $(n+1)$ terms. For further details, see Luke [1, vol. 2, pp. 192–194]. A more realistic approach is to compare $R_{n,n}(z)$ with $R_{0,n}(z)$ for $c = 0$ with $s = t = 0$, and so find

$$\frac{R_{n,n}(z)}{R_{0,n}(z)} = \frac{n\pi(-)^n z^n e^{-z}}{2^{4n+1}n!}[1 + O(n^{-1})], \tag{32}$$

which manifests the striking superiority of the main diagonal approximation.

We also have

$$\frac{R_{n,n}(z)}{R_{r,3r}(z)} = \frac{(-)^r 2^{c+1}(16/27)^r e^{-z/2}}{3^{c+(1/2)}}[1 + O(n^{-1})], \tag{33}$$

$$\frac{R_{n,n}(z)}{R_{3r,r}(z)} = \frac{(-)^r 2^{c+1}(16/27)^r e^{z/2}}{3^{1/2}}[1 + O(n^{-1})], \tag{34}$$

where n is even, $r = \frac{1}{2}n$, and

$$\frac{R_{n,n}(z)}{R_{2n,0}(z)} = \frac{(-)^n n^{c+(1/2)}\pi^{1/2} e^{z}}{\Gamma(c+1)2^{2n}}[1 + O(n^{-1})]. \tag{35}$$

Clearly, the main diagonal approximation is superior to all other Padé approximations.

We conclude with some numerical examples (see Table 1) which manifest the striking realism of our error estimates. We take

$$\mu + \nu = 8, \qquad \mu = 0(2)8, \qquad z = 2, \qquad c = 0.$$

In Table 1 a number in parentheses following a base number indicates the power of 10 by which the base number is to be multiplied. Also in the evaluation of the estimated errors, the order terms are, of course, omitted.

TABLE 1

μ	ν	$U_{\mu,\nu}(z)/V_{\mu,\nu}(z)$	$R_{\mu,\nu}(z)$ (True)	$R_{\mu,\nu}(z)$ (Estimated) from Eq.	
0	8	0.13650 794	$-0.117(-2)$	$-0.117(-2)$	(24)
2	6	0.13535 353	$-0.183(-4)$	$-0.183(-4)$	(25)
4	4	0.13533 834	$-0.306(-5)$	$-0.305(-5)$	(26)
6	2	0.13533 835	$-0.307(-5)$	$-0.303(-5)$	(28)
8	0	0.13536 742	$-0.321(-4)$	$-0.320(-4)$	(30)

4. Uniform error approximations for the exponential function. The error formulas of the last section hold for all z in the complex plane $|z| \leqq R$, R fixed, and with μ or ν or both μ and ν sufficiently large. On the pragmatic side, these tools should be sufficient for virtually all z. For if $z = \rho\, e^{i\theta}$, $R < \rho < 2R$, we should apply the multiplicative properties of the exponential function and write $\exp(-z) = \exp(-[\rho - R]\, e^{i\theta}) \exp(-R\, e^{i\theta})$. Subsequent actions are obvious. Further, in view of the above and the remarks surrounding (32), the main diagonal Padé approximation to e^{-z} is vastly superior to the best Chebyshev approximation to e^{-x} over the range $0 \leqq x < \infty$. For some work on best Chebyshev approximation of the kind just described, see Cody, Meinardus and Varga [11], Schönhage [12] and Newman [13]. See also the references [3], [6] already noted.

The above comments notwithstanding, we will develop uniform asymptotic estimates for the error in the Padé approximations to e^{-z} when $\mu = \nu + r$,

$r = 0, 1, 2$ for $|\arg z| < \pi/2$. Extension of the data to cover the sector $|\arg z| = \pi/2$ is indicated. Some comments on getting results for all $\mu \geqq \nu$ are offered. It would seem that the discussion excludes the error in Padé approximations for e^z when $R(z) > 0$. However, the analysis lends itself to this situation in view of known formulas for analytic continuation of the Bessel functions. A simple approach is to note from (4) that

$$e^z = \frac{V_{\mu,\nu}(z)}{U_{\mu,\nu}(z)} + S_{\mu,\nu}(z), \qquad S_{\mu,\nu}(z) \sim -\left\{\frac{V_{\mu,\nu}(z)}{U_{\mu,\nu}(z)}\right\}^2 R_{\mu,\nu}(z), \qquad S_{\mu,\nu}(z) \sim -e^{2z}R_{\mu,\nu}(z). \tag{36}$$

We now consider uniform asymptotic estimates of the error, and in particular derive same for the main diagonal, first and second subdiagonal approximations to e^{-z} with $\arg z$ as indicated. The ideas are as follows. We show that for $\mu \geqq \nu$, the error can be written as a ratio of two functions, where the numerator and denominator can be expressed as a finite sum involving the modified Bessel functions $I_m(z)$ and $K_m(z)$, respectively. We then apply the known uniform asymptotic developments for these functions due to Olver [14] and so obtain our desired estimates. The general subdiagonal case does not appear tractable by this approach. A better idea is to seek asymptotic developments of the pertinent confluent hypergeometric functions which enter the error by use of the differential equations satisfied by these functions. We have made some progress in this direction. The analysis is not complete and we defer further discussion to a future paper.

From (5) and (29) with $c = 0$, [1, vol. 2, p. 48, eq. (8)], and the definition of $K_\nu(z)$ in terms of $I_\nu(z)$ and $I_{-\nu}(z)$, we have for $\mu \geqq \nu$,

$$V_{\mu,\nu}(z) = \frac{\mu!(z/\pi)^{1/2}\, e^{z/2}}{(2\mu)!} z^\mu \sum_{k=0}^{\mu-\nu} \frac{(-)^k(2\mu+1-2k)(-2\mu-1)_k(\nu-\mu)_k}{(2\mu+1)k!(-\mu-\nu)_k} \cdot K_{\mu+(1/2)-k}(z/2). \tag{37}$$

Also from [1, vol. 2, p. 48, eq. (8)] with $\mu \geqq \nu$, we have

$${}_1F_1\left(\begin{matrix}\mu+1\\ \mu+\nu+2\end{matrix}\;\middle|\; -z\right) = \left(\frac{\pi}{z}\right)^{1/2}\left(\frac{4}{z}\right)^{\nu} e^{z/2}\left(\frac{3}{2}\right)_\nu \cdot \sum_{k=0}^{\mu-\nu} \frac{(k+\nu+\frac{1}{2})(2\nu+1)_k(\nu-\mu)_k}{(\nu+\frac{1}{2})k!(\mu+\nu+2)_k} I_{k+\nu+(1/2)}(z/2). \tag{38}$$

Thus for the main diagonal, first and second subdiagonals, we have the respective representations

$$R_{n,n}(z) = \frac{(-)^{n+1}\pi\, e^{-z} I_{n+(1/2)}(w)}{K_{n+(1/2)}(w)}, \tag{39}$$

$$R_{n,n-1}(z) = \frac{(-1)^n\pi\, e^{-z}[(1+q/w)I_q(w) - I_q'(w)]}{[(1+qw^{-1})K_q(w) - K_q'(w)]}, \qquad q = n - \tfrac{1}{2}, \qquad z = 2w, \tag{40}$$

$$
\begin{aligned}
&R_{n+1,n-1}(z) = \frac{(-)^n \pi\, e^{-z}[AI_q(w) - BI'_q(w)]}{[AK_q(w) - BK'_q(w)]}, \\
&A = 1 + v + (2q+1)v^2/2q, \quad B = (A-1)v, \quad v = q/w,
\end{aligned} \tag{41}
$$

where q and w are as in (40). To get (40) and (41), we have employed well-known difference-differential properties of the Bessel functions.

To get uniform asymptotic estimates for (39)–(41), we employ uniform asymptotic expansions for the Bessel functions and their derivatives given in [14]. The pertinent expansions are as follows. Let

$$
\zeta = u^{-1} + \ln\left(\frac{uz}{1+u}\right), \qquad u = (1+z^2)^{-1/2}, \tag{42}
$$

$$
\begin{aligned}
&U_{s+1} = \tfrac{1}{2}u^2(1-u^2)\frac{dU_s}{du} + \frac{1}{8}\int_0^u (1-5u^2)U_s\,du, \\
&U_0 = 1, \qquad U_1 = \frac{3u - 5u^3}{24}, \qquad U_2 = \frac{9u^2}{128} - \frac{77u^4}{192} + \frac{385u^6}{1152}, \text{ etc.},
\end{aligned} \tag{43}
$$

$$
\begin{aligned}
&V_s = U_s - u(1-u^2)\left(\frac{1}{2}U_{s-1} + u\frac{dU_{s-1}}{du}\right), \\
&V_0 = 1, \qquad V_1 = \frac{(-9u + 7u^3)}{24}, \qquad V_2 = -\frac{15u^2}{128} + \frac{99u^4}{192} - \frac{455u^6}{1152}, \text{ etc.}
\end{aligned} \tag{44}
$$

$$
K_\nu(\nu z) \sim \left(\frac{\pi u}{2\nu}\right)^{1/2} e^{-\nu\zeta} \sum_{s=0}^{\infty} \frac{(-)^s U_s}{\nu^s}, \tag{45}
$$

$$
I_\nu(\nu z) \sim \left(\frac{u}{2\pi\nu}\right)^{1/2} e^{\nu\zeta} \sum_{s=0}^{\infty} \frac{U_s}{\nu^s}, \tag{46}
$$

$$
K'_\nu(\nu z) \sim -z^{-1}\left(\frac{\pi}{2\nu u}\right)^{1/2} e^{-\nu\zeta} \sum_{s=0}^{\infty} \frac{(-)^s V_s}{\nu^s}, \tag{47}
$$

$$
I'_\nu(\nu z) \sim z^{-1}(2\pi\nu u)^{1/2} e^{\nu\zeta} \sum_{s=0}^{\infty} \frac{V_s}{\nu^s}. \tag{48}
$$

The above expansions are valid for $|\nu| \to \infty$, $|\arg \nu| < \pi/2$ uniformly with respect to z in $|\arg z| \leqq \frac{1}{2}\pi - \varepsilon$, $\varepsilon > 0$.

Using these results, we find as follows:

$$
\begin{aligned}
&R_{n,n}(z) \sim (-)^{n+1} e^{-z} e^{2\nu\zeta} e^{2U_1/\nu}[1 + O(\nu^{-3})], \\
&\nu = n + \tfrac{1}{2}, \quad z = 2\nu x, \quad \zeta = u^{-1} + \ln\left(\frac{ux}{1+u}\right), \quad u = (1+x^2)^{-1/2}, \\
&U_1 = \frac{3u - 5u^3}{24}.
\end{aligned} \tag{49}
$$

If x is large, then

$$
2\nu\zeta - z = -\frac{\nu}{x}\left[1 - \frac{1}{12x^2} + O(x^{-3})\right], \tag{50}
$$

and

$$(51)\quad R_{n,n}(z) \sim (-)^{n+1} \exp\left[-\frac{v}{x}\left(1 - \frac{1}{12x^2} + O(x^{-3})\right)\right] \exp\left[\frac{2U_1}{v}\right][1 + O(v^{-3})].$$

To illustrate, let $n = 4$, $z = 9$ whence $v = 9/2$ and $x = 1$. The true value of $R_{4,4}(z)$ is -0.01503. From (49) with order terms neglected, we get $R_{4,4}(z) \sim -0.01474$.

$$(52)\quad R_{n,n-1}(z) \sim \frac{(-)^n e^{2v\zeta - z}\left[\left(1 + \frac{1}{x}\right)\sum_{k=0}^{\infty}\frac{U_k}{v^k} - \frac{(1+x^2)^{1/2}}{x}\sum_{k=0}^{\infty}\frac{V_k}{v^k}\right]}{\left(1 + \frac{1}{x}\right)\sum_{k=0}^{\infty}\frac{(-)^k U_k}{v^k} + \frac{(1+x^2)^{1/2}}{x}\sum_{k=0}^{\infty}\frac{(-)^k V_k}{v^k}},$$

$$v = n - \tfrac{1}{2},\ z = 2vx,$$

where ζ, U_s and V_s are as in (42)–(44) with z replaced by x. If now we take x large, then

$$(53)\quad R_{n,n-1}(z) = \frac{(-)^n \exp\left[-\frac{v}{x}\left(1 - \frac{1}{12x^2} + O(x^{-3})\right)\right]}{2x + 1 + O(x^{-1})},$$

$$(54)\quad zR_{n,n-1}(z) \sim (-)^n n, \qquad |z| \to \infty, \qquad |\arg z| \leqq \tfrac{1}{2}\pi - \varepsilon, \qquad \varepsilon > 0.$$

In illustration, let $n = 4$, $z = 4$. Use (52) excluding all U_k, V_k terms, $k > 1$, and get 0.0008533. The true error is 0.00086 59.

$$R_{n+1,n-1}(z) \sim (-)^n e^{2v\zeta - z}$$
$$\cdot\frac{\left[1 + \frac{U_1}{v} + O(v^{-2}) - \left(1 + \frac{2v+1}{2vx}\right)\frac{(1+x^2)^{1/2}(1-u)}{x}\left\{1 + \frac{W_1}{v} + O(v^{-2})\right\}\right]}{1 - \frac{U_1}{v} + O(v^{-2}) + \left(1 + \frac{2v+1}{2vx}\right)\frac{(1+x^2)^{1/2}(1+u)}{x}\left\{1 - \frac{Q_1}{v} + O(v^{-2})\right\}},$$

$$(55)\ U_1 = \frac{3u - u^3}{24}, \qquad W_1 = \frac{u}{24}(5u^2 + 12u + 9), \qquad Q_1 = \frac{-u}{24}(5u^2 - 12u + 9),$$

with ζ, v and z as in (52). Also

$$(56)\ z^2 R_{n+1,n-1}(z) \sim (-)^n n(n+1), \qquad |z| \to \infty, \qquad |\arg z| \leqq \pi/2 - \varepsilon, \qquad \varepsilon > 0.$$

The above work is based on representations for e^{-z}. We can also replace z by $z\,e^{-i\pi/2}$ and so obtain representations for e^{iz}. To evaluate $R_{\mu,v}(z)$, we need the connecting relations

$$(57)\qquad I_v(z\,e^{-i\pi/2}) = e^{-iv\pi/2} J_v(z),$$

$$(58)\qquad K_v(z\,e^{-i\pi/2}) = \tfrac{1}{2}\pi\, e^{iv\pi/2} H_v^{(1)}(z).$$

Thus, for example,

$$(59)\qquad R_{n,n}(z\,e^{-i\pi/2}) = \frac{2\,e^{iz} J_{n+(1/2)}(z/2)}{H^{(1)}_{n+(1/2)}(z/2)}.$$

Uniform asymptotic estimates for (59) can also be obtained from appropriate data in [14]. The results are more complicated than (49), and we omit details.

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DUALITY METHODS FOR LINEAR VARIATIONAL PROBLEMS IN L^∞*

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Abstract. Minimization problems of the form $\inf\{\|Lu\|_{L^\infty}: u \in U\}$ are considered where L is a linear operator and U is a convex subset of some Hilbert space determined by a finite number of linear functional constraints. The usual variational techniques which provide an Euler equation for such problems in L^p for $p < \infty$ are not applicable in L^∞; the solution is obtained and necessary conditions that it must satisfy are found by dualizing the problem to an appropriate extremal problem in a subspace of L^1. Several applications are given where L is a linear differential operator acting on a Sobolev space.

Introduction. The purpose of this short note is to show that some elementary functional analysis can be used to provide a unified setting for the solutions to certain variational problems in L^∞. Since the usual direct variational methods which provide an Euler equation for such problems in L^p for $1 \leqq p < \infty$ are not applicable in the L^∞ setting, we show that the L^∞ problem can be dualized to an appropriate extremal problem in a subspace of L^1. It is a consequence of our investigations that spline functions arise in many instances as solutions to both the basic problem and to the dual problem. Our main theorem is in § 1; several applications to theorems both new and old are in § 2.

1. Main existence theorem. In this section, we give the main existence theorem which also provides a necessary condition that the solution must satisfy. It is technically easier to deal with the case where the operator is viewed as starting in a Hilbert space and ending in $L^2(\Omega)$, and indeed this will always be the case in our applications. Throughout, Ω is a bounded domain in some Euclidean space, and the L^p-spaces are taken with respect to Lebesgue measure.

THEOREM 1.1. *Let H be a Hilbert space and L a bounded linear operator from H into $L^2(\Omega)$ whose range has finite codimension in $L^2(\Omega)$. Let N be the nullspace of L, let $l_1, \cdots, l_m$ be continuous linear functionals on N, let $F_1, \cdots, F_m$ be $L^1(\Omega)$ functions and $W = \{x \in H : Lx \in L^\infty(\Omega)\}$. For $1 \leqq j \leqq m$, let L_j be the linear functional on W given by $L_j(x) = \int_\Omega F_j Lx + l_j(Px)$, where P is the orthogonal projection of H on N. Let Λ be a closed convex set in $\mathbb{R}^m$ and let*

$$U = \{x \in W : \{L_j x\}_1^m \in \Lambda\}.$$

Consider the minimization problem

$$(*) \qquad \alpha = \inf\{\|Lx\|_\infty : x \in U\}.$$

This minimization problem has a solution. If Lu_0 is a solution to (*) *and if $U_0 = \{x \in W : L_j x = 0$ for $1 \leqq j \leqq m\}$, then there is a $g \in L^1(\Omega)$ with $\|g\|_1 = 1, 0 = \int gLv$ for all $v \in U_0$ and $\int_\Omega gLu_0 = \alpha$; consequently, $gLu_0 \geqq 0$ a.e. and $|Lu_0| = \alpha$ where $g \neq 0$.*

Proof. Let $H = N \oplus H'$ be the direct sum decomposition of H. L is 1–1 on H', and the range of L on H' coincides with the range of L on H. Further, $x \in W$

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if and only if the projection of x onto H' is in W. The range of L is closed in L^2 since it has finite codimension, and hence the range of L on W is closed in L^∞ and has finite codimension. Suppose $x_k \in U$ and $\|Lx_k\|_\infty \to \alpha$; $x_k = h_k + n_k$ where $n_k \in N$ and $h_k \in H'$. The open mapping theorem implies that $\|h_k\|_H \leqq C$ for all k. Let $T: N \to \mathbb{R}^m$ by $Tn = (l_1(n), \cdots, l_m(n))$, and let N_0 be the kernel of T. Then N/N_0 is finite-dimensional, and since $\|Tn_k\|$ is bounded, there are elements n'_k of N with $n_k - n'_k \in N_0$ and $\|n'_k\|_H \leqq C'$ for all k. Let $x'_k = h_k + n'_k$; then $\|x'_k\|_H \leqq C + C'$ for all k and $Lx'_k = Lx_k$. Some subsequence of $\{x'_k\}$, again denoted by $\{x'_k\}$, converges weakly to an element x of H. Since x lies in the norm closure of the convex hull of $\{x'_k\}_{k \geqq k_0}$ for all k_0, we may assume that x'_k converges in the norm of H to x. Hence, $Lx'_k \to Lx$ in $L^2(\Omega)$. A further subsequence converges a.e. to Lx; hence, $Lx \in L^\infty$, and since $\|Lx'_k\|_\infty \leqq \alpha + \varepsilon$ for $k \geqq k_1(\varepsilon)$, we find that $x \in U$ and $\|Lx\|_\infty \leqq \alpha$. Thus x is a solution of (*).

We next assert that LU_0 is a weak* closed in L^∞. To prove this, it suffices to prove that the unit ball of LU_0 is weak* sequentially closed. If $x_n \in U_0$ and $\|Lx_n\|_\infty \leqq 1$ for all n and $Lx_n \to^* u$, $u \in L^\infty$, then, as in the first paragraph, we may assume that $\|x_n\|_H \leqq C$ for all n. Thus we may also assume that $\{x_n\}$ converges weakly to an element x of H. There is a sequence $\{y_n\}$ of convex combinations of $\{x_n\}$ such that $Ly_n \to u$ in $L^2(\Omega)$ and $Ly_n \to u$ boundedly almost everywhere and $\{y_n\}$ converges weakly in H to some y. A convex combination of the y_n gives a sequence $\{z_n\}$ of elements of U_0 such that $Lz_n \to u$ both in L^2 and boundedly and also $z_n \to y$ in the norm of H. Hence, $y \in U_0$ and $Ly = u$, which shows LU_0 is weak* closed.

Finally, we note that W/U_0 is finite-dimensional and hence so is LW/LU_0; also L^∞/LW is finite-dimensional. Thus L^∞/LU_0 is finite-dimensional, and hence $S = \{g \in L^1 : \int_\Omega gLv = 0 \text{ for all } v \in U_0\}$ is also finite-dimensional since the dual space of S is just L^∞/LU_0. Consequently, if u_0 is a solution to (*), then

$$\begin{aligned}
\alpha = \|Lu_0\|_\infty &= \inf\{\|Lu_0 + Lv\|_\infty : v \in U_0\} \\
&= \text{norm of the coset } Lu_0 + LU_0 \text{ in } L^\infty/LU_0 \\
&= \text{norm of } Lu_0 \text{ as a linear functional on } S \\
&= \sup\left\{\left|\int_\Omega gLu_0\right| : g \in S \text{ and } \|g\|_1 \leqq 1\right\}.
\end{aligned}$$

This supremum is actually a maximum because S is finite-dimensional, and the theorem is proved.

COROLLARY 1.2. *Suppose that L maps H onto $L^2(\Omega)$ and that $l_1, \cdots, l_m$ do not appear in the definition of $L_1, \cdots, L_m$. Then the function g in the conclusion of Theorem 1.1 has the form $\sum_1^m c_j F_j$ for some choice of scalars $c_1, \cdots, c_m$.*

Proof. We know that g lies in S, as do $F_1, \cdots, F_m$. If $h \in L^\infty$ and $\int hF_j = 0$ for $1 \leqq j \leqq m$, then because L is onto, $h = Lx$ for some $x \in H$, and thus $x \in U_0$ so that $h \in LU_0$. Hence, $\int hs = 0$ for all $s \in S$ so that S is just the linear span of $F_1, \cdots, F_m$.

Remarks.

1. There is an alternative method of proof of Theorem 1.1 which makes use of a lemma of I. Singer on the extension of linear functionals defined on a finite-dimensional subspace; it is, however, no shorter nor any more elementary than the proof presented here.

2. The theorem is also valid if the variation is taken in L^p for $1 < p < \infty$ although, of course, L^1 must be replaced by L^q where $1/p + 1/q = 1$. In this setting, however, the direct method yields the conclusion with less work.

3. Suppose H is a Sobolev space and L is a linear differential operator, as will be the case in our applications. If the linear functionals are evaluations of $f \in H$ or of the derivatives of f at points $p_1, \cdots, p_m$ in Ω, then roughly speaking, g will satisfy the equation $L^*g = 0$ on $\Omega - \{p_1, \cdots, p_m\}$ where L^* is the formal adjoint of L. If L^* is, in fact, a well-defined differential operator with smooth coefficients, then g will be nonzero and continuous on "most" of Ω, so that Lu_0 will be constant on a large open set in Ω. Thus the smoothness of the coefficients of L will, in turn, imply smoothness for u_0 on this set. We shall see in specific instances in § 2 how this general principle applies.

2. Applications.

2.1. Functions of several variables.

Example 1. *Elliptic operators.* Let Ω be a bounded domain in $\mathbb{R}^r$ and L an elliptic differential operator of order $2K$ acting on $H^{2K} \cap H_0^K$ for which the Fredholm alternative holds; see [2] for conditions which imply this. Let $F_1, \cdots, F_m$ be L^1 functions on Ω which are continuous except for finitely many points and which are linearly independent over any set of positive measure, and let $l_1, \cdots, l_m$ be zero. Suppose first that 0 is not an eigenvalue of L; then L maps $H^{2K} \cap H_0^K$ both 1–1 and onto $L^2(\Omega)$, and according to Corollary 1.2, the solution Lu_0 to (*) must, therefore, have constant modulus α a.e. on Ω. This is the conclusion in [1]. Suppose now that 0 is an eigenvalue of L; then by Theorem 1.1, a solution to (*) exists and g satisfies

$$0 = \int_\Omega gLv \quad \text{for all } v \in U_0.$$

To illustrate this case, suppose L has real analytic coefficients and is uniformly strongly elliptic in Ω. There are a finite number of functions $f_1, \cdots, f_r$ in $L^2(\Omega)$ which are a basis for the orthogonal complement of the range of L, and hence these functions must be real analytic in Ω. S is the linear span of $f_1, \cdots, f_r$ and $F_1, \cdots, F_m$, and hence any function in S is real analytic in Ω except for possibly a finite number of points. Thus g cannot vanish on any set of positive measure, so that Lu_0 is either α or $-\alpha$ in Ω with the possible exception of a closed set of measure zero.

For example, take Ω to be the open unit disc $\{x^2 + y^2 < 1\}$, let $p_1, \cdots, p_m$ be points of Ω, take $U = \{f \in H^2 \cap H_0^1 : f(p_i) \in I_i,\ i = 1, \cdots, m\}$ and take L to be the Laplacian. Then L maps $H^2 \cap H_0^1$ both 1–1 and onto $L^2(\Omega)$ and F_j is the Green's function with pole at p_j for $1 \leqq j \leqq m$. Hence g is harmonic on $\Omega' = \Omega - \{p_1, \cdots, p_m\}$ and $|\Delta u_0| \equiv \alpha$ on Ω. Hence $L^*g = 0$ on Ω' and u_0 is real analytic on Ω' except possibly on the set where g is 0.

Example 2. *Product operators.* Let p and q be positive integers and let L be defined by

$$Lu = \frac{\partial^p}{\partial x^p}\left(\frac{\partial^q u}{\partial y^q}\right)$$

for u in the Sobolev space of order $p + q$. For simplicity, we will take H to be the orthogonal complement in this Sobolev space of the nullspace of L. We also take Ω to be the rectangle $a \leqq x \leqq b$, $c \leqq y \leqq d$. Let points $p_1, \cdots, p_r$ in Ω be given with $p_1 = (a, c)$, let $w_1, \cdots, w_r$ be given numbers and let $U = \{f \in W : f(p_i) = w_i\}$. (Since $p + q \geqq 2$, the functions in the Sobolev space are continuous on Ω.) L maps H onto $L^2(\Omega)$ and W onto $L^\infty(\Omega)$; if we take $L_i f = f(p_i)$ for $i = 1, \cdots, r$, then

$$L_i f = c_i \int_\Omega (Lf)(x)F(x, p_i)\,dx, \qquad 1 \leqq i \leqq r,$$

where $F(x, s) = c(x_1 - s_1)_+^{p-1}(x_2 - s_2)_+^{q-1}$ and c_i is a constant. According to Corollary 1.2, the function g is in the linear span of $F_1, \cdots, F_r$ so that g is a "piecewise" polynomial in the variables x_1, x_2, and consequently, there is at least one grid rectangle determined by the points $p_1, \cdots, p_r$ on which the zero set of g consists of at most a finite number of line segments. Hence $|Lu_0| = \alpha$ a.e. on this rectangle and, of course, on any other rectangle where g is not identically zero.

2.2. Functions of a single variable. Let $[a, b]$ be an interval; let $H^{n,p}$ be those functions f on $[a, b]$ for which $f^{(n-1)}$ is absolutely continuous and $f^{(n)}$ lies in L^p, $1 \leqq p \leqq \infty$. $H_0^{n,p}$ consists of those functions f in $H^{n,p}$ which also satisfy $f^{(\nu)}(a) = 0$, $\nu = 0, \cdots, n - 1$.

Example 3. The first application is a very brief proof of a theorem of R. Louboutin [4]; also see I. J. Schoenberg [5].

THEOREM. *Let* $U = \{f \in H_0^{n,\infty}(-1, 1) : f^{(\nu)}(1) = 0,\ 1 \leqq \nu \leqq n - 1,\ f(1) = 1\}$ *and*

$$\beta = \inf\{\|f^{(n)}\|_\infty : f \in U\}.$$

Then $\beta = (n - 1)!2^{n-2}$, *and there is a unique extremal function* f_0 *which satisfies* $|f_0^{(n)}| \equiv \beta$ *and* $f_0^{(n)}T_{n-1} \geqq 0$ *where* T_{n-1} *is the Chebyshev polynomial of the second kind of degree* $n - 1$.

Proof. Take $H = H_0^{n,2}$ and $L = D^n$. Then $W = H_0^{n,\infty}$ and L maps H onto L^2 and W onto L^∞ and, further, L is 1-1 on H. The functionals $L_1, \cdots, L_n$ are

$$L_j f = f^{(j)}(1) = c_j \int_{-1}^{1} (Lf)(t)(1 - t)^{n-j-1}\,dt, \qquad 0 \leqq j \leqq n - 1,$$

where c_j is a constant so that F_j is a constant multiple of $(1 - t)^{n-j-1}$, $0 \leqq j \leqq n - 1$, and we are in the setting of Corollary 1.2. Hence g is a polynomial of degree $n - 1$ (or less). Clearly then, $|f_0^{(n)}| = \beta$ a.e., and if f_1 is another solution, then so is $1/2(f_0 + f_1)$ so that $f_0^{(n)} = f_1^{(n)}$ and thus $f_0 \equiv f_1$. Thus the solution is unique. If $f \in U$ and p is any polynomial of degree $n - 1$, then integration by parts yields

$$\int_{-1}^{1} f^{(n)}(t)p(t)\,dt = (-1)^{n-1}p^{(n-1)}(1).$$

Thus

$$\beta = \max\left\{\left|\int_{-1}^{1} f_0^{(n)}(t)p(t)\,dt\right| : p \text{ has degree } n - 1 \text{ and } \|p\|_1 \leqq 1\right\}$$
$$= \max\{|p^{(n-1)}(1)| : p \text{ has degree } n - 1 \text{ and } \|p\|_1 \leqq 1\}.$$

It follows that p is (a constant multiple of) the Chebyshev polynomial of degree $n - 1$ of the second kind. Hence, $\beta = (n - 1)!2^{n-2}$, and the theorem is proved.

Example 4. The next application is a theorem of G. Glaeser [3].

THEOREM. *Let* $\alpha_0, \cdots, \alpha_{n-1}$ *and* $\beta_0, \cdots, \beta_{n-1}$ *be given real numbers and let*

$$U = \{f \in H^{n,\infty}(a,b): f^{(\nu)}(a) = \alpha_\nu, f^{(\nu)}(b) = \beta_\nu, \nu = 0, \cdots, n-1\}$$

and

$$\alpha = \inf\{\|f^{(n)}\|_\infty : f \in U\}.$$

Then there is a unique solution f_0 *of this minimization problem*; $|f_0^{(n)}| = \alpha$ a.e., *and* $f_0^{(n)}$ *has at most* $n-1$ *sign changes in* (a, b).

Proof. Here again we can take $H = H^{n,2}$, $L = D^n$ and $W = H^{n,\infty}$. Also $L_j(f) = f^{(j)}(a)$ for $j = 0, \cdots, n-1$ can be expressed as a linear functional on the nullspace of D^n since

$$f(x) = \int_a^x (Lf)(t)\frac{(x-t)^{n-1}}{(n-1)!}dt + p(x), \qquad a \leqq x \leqq b,$$

where p is a uniquely determined polynomial of degree $n-1$. Further, $L_{j+n}(f) = f^{(j)}(b)$ for $j = 0, \cdots, n-1$ also has the desired form.

Suppose that $f \in U_0$; then $f^{(\nu)}(a) = 0$ for $\nu = 0, \cdots, n-1$ so that the polynomial part of f is zero. Hence S is just the linear span of the functions $(b-x)^i$, $i = 0, \cdots, n-1$, and so g is a polynomial of degree $n-1$ or less and thus has no more than $n-1$ zeros in $[a, b]$ (g cannot be identically zero since $\|g\|_1 = 1$). It follows immediately (as in Example 3 above) that f_0 is unique and is a perfect spline with no more than $n-1$ knots in (a, b).

It is obvious that if the n conditions $f^{(\nu)}(a)$, $\nu = 0, \cdots, n-1$ are prescribed and if r conditions, $r \leqq n$, involving derivatives of f at b of order k or less are also prescribed, then the solution to the minimization problem exists and, because g will be a polynomial of degree k, this solution will be unique and is perfect spline with no more than k knots

Example 5. Since Examples 3 and 4 above represent only the special choice of $L = D^n$ and the special choice of $\{L_i\}$ as evaluation of f or its derivatives at points of $[a, b]$, there is considerable latitude remaining for other applications of Theorem 1.1 and Corollary 1.2.

Let E be a closed subset of $[a, b]$ of positive measure and let $L_1 f = \int_E f(x)\,dx$, $Lf = f^{(n)}$, and $H = H_0^{n,2}(a, b)$. Let $k(x, t) = ((n-1)!)^{-1}(x-t)_+^{n-1}$; then $f(x) = \int_a^b k(x, t)Lf(t)\,dt$ for all $f \in H$, and hence

$$L_1 f = \int_a^b (Lf)(t)F_1(t)\,dt,$$

where $F_1(t) = \int_E k(x, t)\,dx$. Note that $F_1 \geqq 0$ and vanish for $t \geqq t_0$, where t_0 is the smallest number such that $[t_0, b] \cap E$ has measure 0, and that $F_1 > 0$ for $t < t_0$. Hence the solution u_0 of the minimization problem

$$\alpha = \inf\left\{\|f^{(n)}\|_\infty : \int_E f(x)\,dx = 1\right\}$$

satisfies $D^n u_0 = \alpha$ on $[a, t_0)$ and hence is a polynomial of degree n on this interval. In particular, if $t_0 = b$, then u_0 is a polynomial.

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ASYMPTOTIC BEHAVIOR AND LOWER BOUNDS FOR SEMILINEAR WAVE EQUATIONS IN HILBERT SPACE WITH APPLICATIONS*

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Abstract. In this paper the asymptotic behavior of abstract wave equations of the form

$$\frac{d^2u}{dt^2} + A(t)u = \mathscr{F}(t, u, u_t) \tag{1}$$

is discussed. The $A(t)$ are nonnegative symmetric operators defined on a dense subdomain D of a Hilbert space H and $u:(0, \infty) \to D$ is a twice strongly continuously differentiable solution to (1). Under certain broad conditions on A and $\mathscr{F}$, it is shown that the energy

$$\mathscr{E}(t, u) \equiv \|u(t)\|^2 + \|u_t(t)\|^2 + \langle u(t), A(t)u(t)\rangle$$

of the solution satisfies

$$\mathscr{E}(t, u) \geqq KE(\tau, u)c^{-\gamma f(t)}$$

for some positive constants K, γ unless $u \equiv 0$. Here $f(t) = t^c$ for some $c \geqq 0$ or $f(t) = \ln(t)$. Using these abstract results, lower bounds are obtained for solutions to the classical equations of linear elasticity with time dependent elasticities, for solutions to the Euler–Poisson–Darboux equation and for a nonlinear equation of motion for a transversely vibrating plate undergoing longitudinal stress.

1. Introduction. This paper discusses the behavior of solutions of an abstract wave equation

$$u_{tt} + A(t)u = \mathscr{F}(t, u, u_t), \tag{1.1}$$

where the $A(t)$ are symmetric nonnegative operators defined on a dense subspace D of a Hilbert space H. Under certain broad conditions on A and $\mathscr{F}$, we establish lower bounds for the energy

$$\mathscr{E}(T, u) = \|u(T)\|^2 + \|u_t(T)\|^2 + \langle u(T), A(T)u(T)\rangle$$

of the solution at time T. This bound has the form

$$\mathscr{E}(T, u) \geqq K\mathscr{E}(\tau, u)\, e^{-\gamma f(T)} \quad \text{for } T > \tau, \tag{1.2}$$

where f depends on the properties of A and $\mathscr{F}$.

We assume that the right side of (1.1) satisfies a Lipschitz condition

$$\|\mathscr{F}(t, x, y)\|^2 \leqq k_1^4 g(t)^4\|x\|^2 + k_2^2 g(t)^2\{\|y\|^2 + \langle x, Ax\rangle\}, \tag{1.3}$$

for all x, y in D. The coefficient $g(t)$ must satisfy certain technical conditions; examples of such functions g are t^{-1} and t^{c-1} for $c \geqq 1$. We also assume that the

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family $A(t)$ has a strong derivative $\dot{A}(t)$ such that

$$\langle x, \dot{A}(t)x\rangle \geqq -\tfrac{1}{2}\beta g(t)\langle x, A(t)x\rangle$$

for $x \in D$.[1] The function f of (1.2) will then be an antiderivative of g.

In view of the assumption (1.3), it is convenient to consider directly solutions of the differential inequality

$$\|u_{tt} + A(t)u\|^2 \leqq k_1^4 g(t)^4\|u\|^2 + k_2^2 g(t)^2\{\|u_t\|^2 + \langle u, Au\rangle\}. \tag{1.4}$$

In § 2 we establish the lower bound (1.2) by means of a weighted energy estimate. Specifically we have the following results:

(i) If u satisfies (1.4) with $g(t) = t^{-1}$, then there are positive numbers K and γ such that $\mathscr{E}(T, u) \geqq K\mathscr{E}(\tau, u)T^{-\gamma}$ for $T > \tau$.

(ii) If u satisfies (1.4) with $g(t) = t^{c-1}$ for $c \geqq 1$, then there are numbers K and γ such that $\mathscr{E}(T, u) \geqq K\mathscr{E}(\tau, u)\, e^{-\gamma T^c}$ for $T > \tau$.

Section 3 contains several physical examples.

The operator $Lu \equiv u_{tt} + A(t)u$ can be realized as a partial differential operator acting on functions $u = u(\mathbf{x}, t)$ in a cylinder $\Omega \times (0, \infty)$ in $(\mathbf{x}, t)$-space. The abstract formulation of the problem allows several directions of generality. The $u(\mathbf{x}, t)$ can be vector-valued as well as real-valued; this allows application to elasticity. The operators $A(t)$ need not be second order; this allows application to the vibration of a clamped plate. By including $\|u\|^2$ terms in (1.4) and $\mathscr{E}(t, u)$, we do not need to assume a bound of the form

$$\|u\|^2 \leqq a(t)\langle u, A(t)u\rangle \tag{1.5}$$

for $u \in D$. Such a bound arises from a boundary condition $u = 0$ on $\partial\Omega \times \mathbb{R}$ when Ω is bounded and $A(t)$ is second order.

Protter in [10], [11] has studied the special case of (1.4) where the $A(t)$ are second order operators in a bounded domain Ω in $\mathbb{R}^n$, and the $u(\cdot, t)$ must lie in a subspace of the Sobolev space $W_2^1(\Omega)$ where (1.5) holds. By using weighted energy estimates, Protter showed that nonzero solutions could not decay arbitrarily fast. Murray in [5], [6] adapted his methods to treat hyperbolic inequalities in unbounded regions, giving lower bounds for the energy in rapidly expanding domains of x-space. The assumption of rapid expansion took the place of a boundary condition. This paper applies the method of [5], [6] to the abstract problem (1.4). Although it does not sharpen Protter's results, it does extend them.

Using similar methods, Murray and Protter in [7] gave the lower bounds for vector-valued solutions of inequalities which are formally similar to (1.4) but in which the $A(t)$ are independent of t and not necessarily symmetric or definite.

In [8], Ogawa found explicit lower bounds for solutions of (1.4), (1.5) by studying the logarithmic derivative of the more classical energy

$$E(t, u) = \langle u_t, u_t\rangle + \langle u, A(t)u\rangle.$$

Under assumption (1.5), our method reproduces Ogawa's result. If (1.5) fails, then $\mathscr{E}$ and E are not equivalent and our method produces a sharper bound than does the natural adaptation of Ogawa's analysis.

[1] Actually, a slightly weaker assumption will suffice (see § 2).

Levine in [3] extended Ogawa's arguments and results to certain abstract inequalities of the form

$$\|Pu_{tt} + Mu_t + Nu\| \leqq \Phi(t)\{\langle u_t, Pu_t\rangle + \langle u, N_1 u\rangle\}^{1/2},$$

where the operators P, M, N depend on t, and N_1 is the symmetric part of N. These results, like those of [8], use an energy which is not equivalent with $\mathscr{E}$ unless (1.5) holds.

2. The abstract theorem. Let H be a real Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. Let D be a dense linear subspace of H. We use $\mathscr{V}$ to denote the class of twice strongly differentiable functions $v:[0,\infty) \to D$ such that v_t also takes values in D, while v_{tt} may take values in H.

We consider an abstract wave operator

$$Lv \equiv -A(t)\bullet - v_{tt}$$

defined on functions v in $\mathscr{V}$. For each $t \geqq 0$, $A(t)$ is a linear operator from D into H such that

(A-I) $A(t)$ is symmetric on D and $\langle x, A(t)x\rangle \geqq 0$ for all $x \in D$, and

(A-IIa) if $v \in \mathscr{V}$, the derivative $d/dt\,\langle v(t), A(t)v(t)\rangle$ exists.

We use Q_A to denote the quantity

$$Q_A(v,v) \equiv \frac{d}{dt}\langle v(t), A(t)v(t)\rangle - 2\langle v_t(t), A(t)v(t)\rangle.$$

The next hypothesis is a weakened version of the bound on $A(t)$ mentioned in the Introduction:

There is a positive constant β_0 and a positive C^2-function $g = g(t)$ defined for $0 < t < \infty$, such that for $v \in \mathscr{V}$,

(A-IIb) $Q_A(v,v) \geqq -\frac{1}{2}\beta_0 g(t)\langle v(t), A(t)v(t)\rangle$ for $t > 0$.

We will use certain technical hypotheses about this function g. These were used by Murray [5] in her study of hyperbolic inequalities. Examples of such functions are $g(t) = t^{-1}$ and $g(t) = t^{c-1}$ for $c \geqq 1$.

(G-I) If $\delta > 0$, then $\lim_{t\to+\infty}\int_\delta^t g(s)\,ds = +\infty$.

(G-II) If $t > 0$, then $g'(t) \geqq -g^2(t)$. Further, for each $t_0 > 0$, there is an $\alpha_1 = \alpha_1(t_0)$ such that

$$g'(t) < \alpha g^2(t) \quad \text{for } t \geqq t_0, \quad \alpha \geqq \alpha_1(t_0).$$

(G-III) For every $\beta \geqq 3$ and every $t_0 > 0$, there is an $\alpha_2 = \alpha_2(\beta, t_0)$ such that

$$\alpha\beta g^3(t) + (2\alpha - \beta)g'(t)g(t) - g''(t) \geqq \alpha g^3(t)$$

for all $\alpha \geqq \alpha_2$, $t \geqq t_0$.

(G-IV) There are numbers $t_1 > 0$ and $\mu \geqq 1$ such that

$$g^2(t) \leqq \mu \exp\left(\int_{t_1}^t g(s)\,ds\right) \quad \text{for } t \geqq t_1.$$

Two "energies" are useful to measure the size of functions in the class $\mathscr{V}$:

$$E(t, v) \equiv \|v_t(t)\|^2 + \langle v(t), A(t)v(t)\rangle,$$
$$\mathscr{E}(t, v) \equiv \|v(t)\|^2 + E(t, v).$$

For the rest of this section, let t_0 be fixed so that $0 < t_0 < t_1$. We define

$$f(t) = \int_{t_0}^{t} g(s)\, ds \quad \text{for } t > t_0.$$

Under these assumptions, we begin a series of lemmas leading to the basic weighted energy estimate for functions w in $\mathscr{V}$. Applying this estimate to solutions of (1.4), we will obtain the desired lower bounds (1.2).

LEMMA 2.1. *Suppose $v \in \mathscr{V}$; $\beta \geqq \beta_0$ and $T > \tau \geqq t_0$. Then*

$$\begin{aligned} &2\int_{\tau}^{T} e^{\beta f(t)}\langle Lv, v_t\rangle\, dt + e^{\beta f(T)}E(T, v) \\ &\qquad \geqq \tfrac{1}{2}\beta \int_{\tau}^{T} g(t)\, e^{\beta f(t)}E(t, v)\, dt + e^{\beta f(\tau)}E(\tau, v). \end{aligned} \tag{2.1}$$

Proof. Suppressing the t arguments in v and A, we find

$$2\langle Lv, v_t\rangle = Q_A(v, v) - \frac{d}{dt}E(t, v).$$

Multiplying this by $e^{\beta f(t)}$ and integrating the result by parts, we obtain

$$\begin{aligned} 2\int_{\tau}^{T} e^{\beta f(t)}\langle Lv, v_t\rangle\, dt &= \int_{\tau}^{T} e^{\beta f(t)}[Q_A(v, v) + \beta g(t)E(t, v)]\, dt \\ &\quad - \int_{\tau}^{T} \frac{d}{dt}[e^{\beta f(t)}E(t, v)]\, dt. \end{aligned}$$

The result now follows from an application of (A-IIb).

LEMMA 2.2. *Suppose $w \in \mathscr{V}$ and $v(t) = e^{\alpha f(t)}w(t)$. If $\beta > \max\{\beta_0, 2\}$, $T > \tau \geqq t_0$ and $\alpha > \alpha_2(\beta, t_0)$, then*

$$\begin{aligned} &\int_{\tau}^{T} e^{(\beta + 2\alpha)f(t)}g^{-1}(t)\|Lw\|^2\, dt + 2\alpha e^{\beta f(T)}E(T, v) \\ &\qquad \geqq 2\alpha^3 \int_{\tau}^{T} e^{(\beta + 2\alpha)f(t)}g^3(t)\|w\|^2\, dt \\ &\qquad\quad - 2\alpha^2 \int_{\tau}^{T} \frac{d}{dt}\{e^{\beta f(t)}[\alpha g^2(t) - g'(t)]\, \|v\|^2\}\, dt. \end{aligned} \tag{2.2}$$

Proof. The auxiliary function v is also in $\mathscr{V}$. An elementary calculation shows that

$$e^{\alpha f(t)}Lw = \{Lv + \alpha[g'(t) - \alpha g^2(t)]v\} + 2\alpha g(t)v_t.$$

Since $\|x + y\|^2 \geqq 2\langle x, y\rangle$ for all x, y in H, we find

$$e^{2\alpha f(t)}\|Lw\|^2 \geqq 4\alpha g(t)\langle Lv, v_t\rangle + 4\alpha^2 g(t)[g'(t) - \alpha g^2(t)]\langle v, v_t\rangle.$$

Multiplying by $g^{-1}(t)\,e^{\beta f(t)}$ and integrating, we find that

$$\begin{aligned}
&\int_\tau^T e^{(\beta+2\alpha)f(t)}g^{-1}(t)\|Lw\|^2\,dt \\
&\qquad \geqq 4\alpha\int_\tau^T e^{\beta f(t)}\langle Lv, v_t\rangle\,dt \\
&\qquad\quad + 2\alpha^2\int_\tau^T e^{\beta f(t)}[g'(t)-\alpha g^2(t)]\frac{d}{dt}(\|v\|^2)\,dt.
\end{aligned} \tag{2.3}$$

Because $\beta > \beta_0$, we can apply Lemma 2.1 to the first term on the right. Thus, since the left-hand side of (2.1) is nonnegative, we have

$$4\alpha\int_\tau^T e^{\beta f(t)}\langle Lv, v_t\rangle\,dt \geqq -2\alpha e^{\beta f(T)}E(T, v). \tag{2.4}$$

The other term on the right of (2.3) can be integrated by parts. Using the identity

$$-\frac{d}{dt}\{e^{\beta f(t)}[g'(t)-\alpha g^2(t)]\} = e^{\beta f(t)}[\alpha\beta g^3 + (2\alpha-\beta)g'g - g'']$$

and hypothesis (G-III), one finds

$$\begin{aligned}
&2\alpha^2\int_\tau^T e^{\beta f(t)}[g'-\alpha g^2]\frac{d}{dt}(\|v\|^2)\,dt \\
&\qquad \geqq 2\alpha^2\int_\tau^T \frac{d}{dt}\{e^{\beta f(t)}[g'-\alpha g^2]\,\|v\|^2\}\,dt \\
&\qquad\quad + 2\alpha^3\int_\tau^T e^{\beta f(t)}g^3(t)\,\|v\|^2\,dt.
\end{aligned} \tag{2.5}$$

We obtain (2.2) from (2.3) using (2.4) and (2.5).

We now rewrite (2.2) in terms of w only, weakening it somewhat. We have

$$\begin{aligned}
E(T, v) &= e^{2\alpha f(T)}[\|w_t + \alpha g w\|^2 + \langle w, Aw\rangle] \\
&\leqq 2e^{2\alpha f(T)}[E(T, w) + \alpha^2 g^2(T)\|w(T)\|^2].
\end{aligned}$$

Using this bound in (2.2), we obtain an estimate for $\|w\|^2$ in terms of $\|Lw\|^2$, namely, Lemma 2.3.

LEMMA 2.3. *Suppose* $w \in \mathscr{V}$ *and* $\beta > \max\{\beta_0, 2\}$. *If* $\alpha > \alpha_2(\beta, t_0)$ *and* $T > \tau > t_0$, *then*

$$\begin{aligned}
&\int_\tau^T e^{(\beta+2\alpha)f(t)}g^{-1}(t)\|Lw\|^2\,dt \\
&\qquad + e^{(\beta+2\alpha)f(T)}[4\alpha E(T, w) + \{6\alpha^3 g^2(T) - 2\alpha^2 g'(T)\}\,\|w(T)\|^2] \\
&\geqq 2\alpha^3\int_\tau^T e^{(\beta+2\alpha)f(t)}g^3(t)\|w\|^2\,dt \\
&\qquad + 2\alpha^2\,e^{(\beta+2\alpha)f(\tau)}\{\alpha g(\tau)^2 - g'(\tau)\}\,\|w(\tau)\|^2.
\end{aligned} \tag{2.6}$$

The next step is to bound $E(t, w)$ in terms of $\|Lw\|^2$.

LEMMA 2.4. *Suppose $w \in \mathscr{V}$ and $\beta > \max\{\beta_0, 2\}$. If $\alpha > 0$ and $T > \tau > t_0$, then*

$$\begin{aligned} &\int_\tau^T e^{(\beta+2\alpha)f(t)} g^{-1}(t)\|Lw\|^2\,dt + e^{(\beta+2\alpha)f(T)}E(T, w) \\ &\qquad \geqq \alpha \int_\tau^T e^{(\beta+2\alpha)f(t)} g(t) E(t, w)\,dt + e^{(\beta+2\alpha)f(\tau)}E(\tau, w). \end{aligned} \tag{2.7}$$

Proof. Since $(\beta + 2\alpha) > \beta_0$, we can apply Lemma 2.1 to w, with $(\beta + 2\alpha)$ playing the role of β. Thus

$$\begin{aligned} &2\int_\tau^T e^{(\beta+2\alpha)f(t)}\langle Lw, w_t\rangle\,dt + e^{(\beta+2\alpha)f(T)}E(T, w) \\ &\qquad \geqq \tfrac{1}{2}(\beta + 2\alpha)\int_\tau^T g(t)\,e^{(\beta+2\alpha)f(t)}E(t, w)\,dt + e^{(\beta+2\alpha)f(\tau)}E(\tau, w). \end{aligned}$$

But

$$\begin{aligned} 2\langle Lw, w_t\rangle &\leqq g^{-1}(t)\|Lw\|^2 + g(t)\|w_t\|^2 \\ &\leqq g^{-1}(t)\|Lw\|^2 + g(t)E(t, w). \end{aligned}$$

Combining this with the preceding inequality, we get

$$\begin{aligned} &\int_\tau^T e^{(\beta+2\alpha)f(t)} g^{-1}(t)\|Lw\|^2\,dt + e^{(\beta+2\alpha)f(T)}E(T, w) \\ &\qquad \geqq (\tfrac{1}{2}\beta + \alpha - 1)\int_\tau^T e^{(\beta+2\alpha)f(t)} g(t)E(t, w)\,dt + e^{(\beta+2\alpha)f(\tau)}E(\tau, w). \end{aligned}$$

Since $\beta > 2$, we get $(\frac{1}{2}\beta + \alpha - 1) > \alpha$, and (2.7) follows. Lemmas 2.3 and 2.4 combine to produce the basic energy estimate.

LEMMA 2.5. *Suppose w is in the class $\mathscr{V}$. Assume $\beta > \max\{\beta_0, 2\}$ and $\alpha \geqq \max\{\alpha_1(t_0), \alpha_2(\beta, t_0)\}$. If $T > \max\{\tau, t_1\} \geqq t_0$, then there are positive constants C and c such that*

$$\begin{aligned} &2\int_\tau^T e^{(\beta+2\alpha)f(t)} g(t)^{-1}\|Lw\|^2\,dt + C\,e^{(1+\beta+2\alpha)f(T)}\mathscr{E}(T, w) \\ &\qquad \geqq \alpha\int_\tau^T e^{(\beta+2\alpha)f(t)} g(t)E(t, w)\,dt + 2\alpha^3\int_\tau^T e^{(\beta+2\alpha)f(t)} g^3(t)\|w\|^2\,dt \\ &\qquad\quad + c\,e^{(\beta+2\alpha)f(\tau)}\mathscr{E}(\tau, w). \end{aligned} \tag{2.8}$$

Proof. By adding (2.6) and (2.7), we get

$$\begin{aligned} &2\int_\tau^T e^{(\beta+2\alpha)f(t)} g^{-1}(t)\|Lw\|^2\,dt + e^{(\beta+2\alpha)f(T)}H(T) \\ &\qquad \geqq \alpha\int_\tau^T e^{(\beta+2\alpha)f(t)} g(t)E(t, w)\,dt + 2\alpha^3\int_\tau^T e^{(\beta+2\alpha)f(t)} g^3(t)\|w\|^2\,dt \\ &\qquad\quad + e^{(\beta+2\alpha)f(\tau)}h(\tau), \end{aligned}$$

where

$$H(T) \equiv (4\alpha + 1)E(T, w) + \{6\alpha^3 g^2(T) - 2\alpha^2 g'(T)\} \|w(T)\|^2$$

and

$$h(\tau) \equiv E(\tau, w) + 2\alpha^2\{\alpha g^2(\tau) - g'(\tau)\} \|w(\tau)\|^2.$$

It remains to estimate $H(T)$ and $h(\tau)$ appropriately. Because of (G-II), we have $-g'(T) \leqq g^2(T)$, and, therefore,

$$6\alpha^3 g^2(T) - 2\alpha^2 g'(T) \leqq 2\alpha^2(3\alpha + 1)g^2(T).$$

But since $t_0 < t_1 < T$, assumption (G-IV) gives us

$$g^2(T) \leqq \mu \exp \int_{t_1}^{T} g(s)\, ds = \mu\, e^{-f(t_1)}\, e^{f(T)} < \mu\, e^{f(T)}.$$

By defining

$$C(\alpha, \mu) = \mu \max\{(4\alpha + 1), 2\alpha^2(3\alpha + 1)\},$$

we get

$$H(T) \leqq C(\alpha, \mu)\, e^{f(T)} \mathscr{E}(T, w).$$

If we set

$$c(\alpha, \tau) = \min\{1, 2\alpha^2[\alpha g^2(\tau) - g'(\tau)]\},$$

then (G-II) assures us that $c(\alpha, \tau) > 0$ and

$$h(\tau) \geqq c(\alpha, \tau)\mathscr{E}(\tau, w).$$

The lemma follows with $C = C(\alpha, \mu)$ and $c = c(\alpha, \tau)$. We can now establish Theorem 2.1.

THEOREM 2.1. *Let $u:(0, \infty) \to D$ be a strong solution on $(0, \infty)$ to the following differential inequality on $(0, \infty)$ such that u_t, the strong derivative of u, takes values in D:*

$$\left\| \frac{d^2u}{dt^2} + A(t)u \right\|^2 \leqq k_1^4 g^4(t)\|u(t)\|^2 + k_2^2 g^2(t)\left\{\langle u(t), A(t)u(t)\rangle + \left\|\frac{du}{dt}\right\|^2\right\} \tag{2.9}$$

(Here k_1 and k_2 are nonnegative constants.)

Then u cannot decay faster than $e^{-\rho f(t)}$ for every $\rho > 0$ in the sense that

$$\liminf_{t \to +\infty} e^{\rho f(t)} \mathscr{E}(t, u) > 0$$

unless $u \equiv 0$. (u decays faster than $e^{-\rho f(t)}$, by definition, if $\liminf_{t\to+\infty} e^{\rho f(t)}\mathscr{E}(t, u) = 0$.)

Proof. The proof follows Murray [5], [6]. Let $u = w$ in (2.8) and replace $\|Lu\|^2$ by its upper bound in (2.9). There results, for any $t_0 > 0$, any $\beta > \max\{2, \beta_0\}$, any $\alpha > \max\{\alpha_1(t_0), \alpha_2(t_0, \beta)\}$ and any T, τ with $T \geqq \max\{\tau, t_0, t_1\}$ and $\tau \geqq t_0$,

$$C(\alpha, \mu)\mathscr{E}(T, u)\, e^{(1+2\alpha+\beta)f(T)}$$
$$\geqq (\alpha - 2k_2^2)\int_\tau^T e^{(\alpha+2\beta)f(t)} g(t)E(t, u)\, dt$$
$$+ 2(\alpha^3 - k_1^4)\int_\tau^T e^{(\alpha+2\beta)f(t)} g(t)^3 \|u(t)\|^2\, dt$$
$$+ c(\alpha, \tau)\, e^{(2\alpha+\beta)f(\tau)}\mathscr{E}(\tau, u).$$

Now if we choose $\alpha > \max\{\alpha_1(t_0), \alpha_2(t_0, \beta),\ 2k_2^2, (k_1)^{4/3}\}$, we see that for $T \geqq \max\{\tau, t_0, t_1\} \geqq \tau \geqq t_0$ and some positive K depending only upon α, β, t_0, μ and τ, we have

$$\mathscr{E}(T, u) \geqq K(\alpha, \beta, t_0, \mu, \tau)\mathscr{E}(\tau, u)\, e^{-(1+2\alpha+\beta)f(T)}. \tag{2.10}$$

Now suppose the theorem fails so that u decays faster than $e^{-\rho f(t)}$ for every $\rho > 0$ and that $u \not\equiv 0$ on $(0, \infty)$. Then, for some $\tau > 0$, we have $u(\tau) \neq 0, \mathscr{E}(\tau, u) > 0$. Let $\rho = 1 + 2\alpha + \beta$ where α and β are chosen as above and $t_0 \in (0, \tau)$ is fixed. Then

$$0 = \liminf_{T\to\infty} e^{(1+2\alpha+\beta)f(T)}\mathscr{E}(T, u) \geqq K(\alpha, \beta, t_0, \mu, \tau)\mathscr{E}(\tau, u) > 0,$$

which is a contradiction. Therefore, $u \equiv 0$ on $(0, \infty)$.

Remark 2.1. Let, for each $t \in (0, \infty)$, $B(t)$ be a bounded linear operator and suppose that for some constant k and all $t \in (0, \infty)$ $\sup_{x\in H, \|x\|=1} \|B(t)x\| \equiv |||B(t)||| \leqq kg(t)$. Then Theorem 2.1 applies to solutions of differential inequalities of the form

$$\|u_{tt} + B(t)u_t + A(t)u\|^2 \leqq k_1^4 g^4(t)\|u\|^2 + k_2^2 g^2(t)E(t, u), \tag{2.11}$$

where A and g are as in Theorem 2.1. This follows because solutions of (2.11) are likewise solutions of an inequality of the form (2.9) since

$$\|u_{tt} + A(t)u\|^2 \leqq (\|u_{tt} + B(t)u_t + A(t)u\| + \|B(t)u_t\|)^2$$
$$\leqq 2[k_1^4 g^4(t)\|u\|^2 + k_2^2 g^2(t)E(t, u)] + 2\|B(t)u_t\|^2$$
$$\leqq 2k_1^4 g^4(t)\|u\|^2 + (2k_2^2 + 2k^2)g^2(t)E(t, u),$$

which is of the same form as (2.9).

Remark 2.2. It is possible to extend Theorem 2.1 to the case in which the domain of $A(t)$, D, depends on time. The only difference is that we require solutions to (2.9) to have the property that for each $t \in (0, \infty)$, $u(t)$ and $u_t(t)$ belong to $D(t)$, the domain of $A(t)$ and that the a priori estimates obtained in Lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 hold for vector-valued functions with this property.

3. Examples. In this section, we give three examples to illustrate the scope of Theorem 2.1 and the wide variety of physical problems to which it may be applied. The list of examples is in no way intended to be exhaustive or complete.

Example I. Let $B \subseteq R^3$ be a bounded domain with boundary ∂B smooth enough to admit of applications of the divergence theorem. Let B be filled with an anisotropic elastic material. Let $\mathbf{u} = (u_1, u_2, u_3)$ denote the displacement vector, $\nabla_x \mathbf{u}$ be the 3×3 matrix $[u_{i,j}]$, and $u_{i,t} = \partial u_i/\partial t$. The equations of motion are assumed to take the following form in $B \times (0, \infty)$,

$$\rho(\mathbf{x})u_{i,tt} = (c_{ijkl}(\mathbf{x}, t)u_{k,l})_{,j} + \mathscr{F}_i(\mathbf{x}, t, \mathbf{u}, \nabla\mathbf{u}, \mathbf{u}_t), \qquad i = 1, 2, 3, \tag{3.1}$$

where $\rho > 0$ is the density of the material and the c_{ijkl} are the (possibly time-dependent) elasticities.

In practice, one prescribes

$$u_i(\mathbf{x}, 0), u_{i,t}(\mathbf{x}, 0)$$

on $\bar{B}$ and requires that the solutions satisfy

$$B_i(\mathbf{u})(\mathbf{x}, t) = 0, \qquad (\mathbf{x}, t) \in \partial B \times [0, \infty), \quad i = 1, 2, 3, \tag{3.2}$$

where the B_i are linear boundary conditions (displacement, traction or possibly mixed). (See [2].)[2]

The system (3.1) (3.2) may be written more compactly as

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho(\mathbf{x})} \frac{\partial}{\partial x_j}\left(C_{jl} \frac{\partial \mathbf{u}}{\partial x_l}\right) + \frac{1}{\rho(\mathbf{x})} \mathscr{F}(\mathbf{x}, t, \mathbf{u}, \nabla\mathbf{u}, \mathbf{u}_t), \\ \mathscr{B}\mathbf{u}(\mathbf{x}, t) = 0 \quad \text{on } \partial B \times (0, \infty), \end{gathered} \tag{3.3}$$

where $u = \operatorname{col}(u_1, u_2, u_3)$, $\mathscr{F} = \operatorname{col}(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$, $(C_{jl})_{ik} = c_{ijkl}$, $i, j, k, l = 1, 2, 3$ and $\nabla\mathbf{u} = (u_{i,j})_{3 \times 3}$.

In order to recast (3.2) into a Cauchy problem for solutions to a differential inequality of the form (2.9) in a Hilbert space, it will be necessary to impose certain additional conditions on the matrix functions C_{jl}, the nonlinearity $\mathscr{F}$ and the density $\rho(\mathbf{x})$. We suppose the following:

(ρ-1) There are constants ρ_M and ρ_m such that

$$\rho_M \geqq \rho(\mathbf{x}) \geqq \rho_m > 0$$

for all $\mathbf{x} \in \bar{B}$. ρ is a continuous function.

(C-1) $C_{jl} = C_{lj}^T$, $l, j = 1, 2, 3$ and all $(\mathbf{x}, t) \in \bar{B} \times (0, \infty)$. That is, $c_{ijkl} = c_{klij}$ (see [2]) and these functions are at least continuously differentiable in $\bar{B} \times (0, \infty)$ in both $\mathbf{x}$ and t.

(C-2) The $C_{ij}(x, t)$ satisfy, for all $\xi = (\xi_{ij})_{n \times n}$, $c_M \xi_{ij}\xi_{ij} \geqq c_{ijkl}\xi_{ij}\xi_{kl} \geqq c_m \xi_{ij}\xi_{ij}$ uniformly in $\bar{B} \times (0, \infty)$ for some positive constants $c_M c_m$.

(C-3) For all indices i, j, k, l, uniformly in $\mathbf{x}$,

$$\left| \frac{\partial c_{ijkl}}{\partial t} \right| \leqq c g(t),$$

where g is a function satisfying (G-I) through (G-IV).

(F-1) The nonlinearities, $\mathscr{F}_i$, are Lipschitz continuous at $\mathbf{u} = \mathbf{0}$; i.e., there are nonnegative continuous functions $k_1(\mathbf{x}, t)$ and $k_2(\mathbf{x}, t)$ such that

$$|\mathscr{F}_i(\mathbf{x}, t, \boldsymbol{\xi}, (\xi_{ij}), \boldsymbol{\eta}|^2 \leqq k_1(\mathbf{x}, t)\xi_i\xi_i + k_2(\mathbf{x}, t)(\eta_i\eta_i + \xi_{ij}\xi_{ij})$$

[2] For this example, we shall assume that on the part of the lateral boundary, $\partial B \times (0, \infty)$, where the tractions are specified, the elasticities, c_{ijkl}, do not depend upon t. However, this assumption may be relaxed in view of Remark 2.1.

for all vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ in R^3 and all tensors $(\xi_{ij})_{3\times 3}$ of rank 2 and all points $(\mathbf{x}, t) \in B \times (0, \infty)$.

(F-2) The functions k_1 and k_2 satisfy

$$k_1(\mathbf{x}, t) \leqq k_1^4 g^4(t), \qquad k_2(\mathbf{x}, t) \leqq k_2^2 g^2(t)$$

for some constants k_1 and k_2 and the function g used in (C-3) above.

Let $H_1 = \{f : B \to R | \int_B f^2(\mathbf{x})\rho(\mathbf{x})\, d\mathbf{x} < \infty\}$ and $H = H_1 \oplus H_1 \oplus H_1 = \{(f_1, f_2, f_3) | f_1 \in H_1$ for $i = 1, 2, 3\}$. We make H into a Hilbert space with the scalar product

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_B f_i(\mathbf{x}) g_i(\mathbf{x}) \rho(\mathbf{x})\, d\mathbf{x}.$$

For D, we take

$$\{f \in H | \mathbf{f} \in C^2(B), \mathscr{B}\mathbf{f} = 0 \text{ and } A(t)\mathbf{f} \in H\},$$

where

$$A(t)\mathbf{f} = \frac{1}{\rho(\mathbf{x})} \frac{\partial}{\partial x_j}\left[(C_{jl}(\mathbf{x}, t) \frac{\partial \mathbf{f}}{\partial \mathbf{x}_l}\right].$$

Then, taking into account the boundary conditions and (C-1), one easily sees that

$$\langle \mathbf{f}, A(t)\mathbf{g} \rangle = \int_B c_{ijkl}(\mathbf{x}, t) f_{i,j} g_{k,l}\, d\mathbf{x} = \langle A(t)\mathbf{f}, \mathbf{g} \rangle.$$

From (C-2) it follows that $\langle f, A(t)f \rangle \geqq 0$ for all $f \in D$. Thus (A-1) holds. It is easily seen that (A-II) will hold if we can establish that for all $f \in D$ and all $\beta \geqq \beta_0 > 0$,

$$\begin{aligned} Q_A(\mathbf{f}, \mathbf{f}) &\equiv \int c_{ijkl,t}(\mathbf{x}, t) f_{i,j} f_{k,l}\, d\mathbf{x} \\ &\geqq -\tfrac{1}{2}\beta g(t) \int_B c_{ijkl}(\mathbf{x}, t) f_{i,j} f_{k,l}\, dx. \end{aligned} \tag{3.4}$$

However, we have that

$$\begin{aligned} \int_B c_{ijkl,t} f_{i,j} f_{k,l}\, dx &\geqq -\int_B |c_{ijkl,t}|\, |f_{i,j}|\, |f_{k,l}|\, dx \\ &\geqq -\int_B \frac{c}{2} g(t)[f_{i,j} f_{i,j} + f_{k,l} f_{k,l}]\, dx \\ &\geqq -9cg(t) \int_B f_{i,j} f_{i,j}\, dx \\ &\geqq -9cc_m^{-1} g(t) \int_B c_{ijkl} f_{i,j} f_{k,l}\, dx \\ &\geqq -\tfrac{1}{2}\beta g(t) \langle \mathbf{f}, A(t)\mathbf{f} \rangle \end{aligned}$$

for all $\beta \geqq \beta_0 \equiv 18cc_m^{-1}$. Thus (3.4) and consequently (A-II) holds.

Now suppose $u:\bar{B} \times (0, \infty) \to D$ is a solution to (3.3). Then we have that

$$\begin{aligned}\|\mathbf{u}_{tt} + A(t)\mathbf{u}\|^2 &= \sum_{i=1}^{3} \int_B \rho(\mathbf{x})\left[u_{i,tt} - \frac{1}{\rho(x)}(c_{ijkl}u_{k,l})_{,j}\right]^2 d\mathbf{x} \\ &= \int_B \rho^{-1}(\mathbf{x})\mathscr{F}_i\mathscr{F}_i\, d\mathbf{x} \\ &\leqq \int_B \rho_m^{-1}\{k_1(\mathbf{x}, t)u_iu_i + k_2(\mathbf{x}, t)[u_{i,t}u_{i,t} + u_{i,j}u_{i,j}]\}\, d\mathbf{x}\end{aligned}$$

Thus

$$\|\mathbf{u}_{tt} + A(t)\mathbf{u}\|^2 \leqq K_1^4 g^4(t)\|\mathbf{u}\|^2 + K_2^2 g^2(t)\{\langle\mathbf{u}_t, \mathbf{u}_t\rangle + \langle\mathbf{u}, A(t)\mathbf{u}\rangle\}, \tag{3.5}$$

where $K_1^4 = \rho_m^{-1}k_1^4$ and $K_2^2 = k_2^2 \cdot \max(1, c_m^{-1})$.

It therefore follows that $\tilde{u}$ satisfies a differential inequality of the form (2.9) subject to conditions (A-1) and (A-II). Then (2.10) holds where

$$\mathscr{E}(t, \mathbf{u}) = \int_B \{\rho(\mathbf{x})[u_i(\mathbf{x}, t)u_i(\mathbf{x}, t) + u_{i,t}(\mathbf{x}, t)u_{i,t}(\mathbf{x}, t)] + c_{ijkl}(\mathbf{x}, t)u_{i,j}u_{k,l}\}\, d\mathbf{x}. \tag{3.6}$$

Consequently, unless $\mathbf{u} \equiv \mathbf{0}$, there is $\rho_0 > 0$ such that

$$\liminf_{t \to +\infty}\left[\exp\left(\rho_0 \int_\tau^t g(\eta)\, d\eta\right)\mathscr{E}(t, \mathbf{u})\right] > 0 \tag{3.7}$$

unless $\mathbf{u} \equiv \mathbf{C}$ in $B \times (0, \infty)$.

Remark 3.1. Since we assumed that the c_{ijkl}'s are bounded above, that is,

$$c_{ijkl}(\mathbf{x}, t)\xi_{ij}\xi_{kl} \leqq c_M\xi_{ij}\xi_{ij}$$

for some positive constant c_M, we have that (2.11) and (3.7) hold with $\mathscr{E}(\mathbf{t}, \mathbf{u})$ replaced by

$$\mathscr{E}_1(t, \mathbf{u}) \equiv \int_B [u_i(\mathbf{x}, t)u_i(\mathbf{x}, t) + u_{i,t}(\mathbf{x}, t)u_{i,t}(\mathbf{x}, t) + u_{i,j}(\mathbf{x}, t)u_{i,j}(\mathbf{x}, t)]\, d\mathbf{x}. \tag{3.8}$$

Thus, if $g(t) = 1$ and $\mathscr{E}_1$ decays faster than $e^{-\rho t}$ for every $\rho > 0$, then $\mathbf{u} = \mathbf{0}$; if $g(t) = 1/t$ and $\mathscr{E}_1$ decays faster than $t^{-\rho}$ for every $\rho > 0$, then $\mathbf{u} \equiv \mathbf{0}$; and if $g(t) = t^{c-1}$ for some $c > 1$, $\mathscr{E}_1$ cannot decay faster than $e^{-\rho t^c}$ for every $\rho > 0$ unless $\mathbf{u} \equiv \mathbf{0}$.

Remark 3.2. If the tractions are specified on the entire lateral boundary of $B \times (0, \infty)$, then one cannot bound $\int_B u_iu_i\, dx$ above by the operator $A(t)$ with a Poincaré type inequality of the form $\langle u(t), A(t)u(t)\rangle \geqq k\langle u(t), u(t)\rangle$. See the remarks in the Introduction.

Example II. Here we consider an application of Theorem (2.1) as applied to (2.11) in Remark 2.1. This time we let $B \subseteq R^n$ be a bounded domain and let ∂B be smooth enough to admit of applications of the divergence theorem. We could consider the Euler–Poisson–Darboux equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{t}\frac{\partial u}{\partial t} - \Delta_n u = 0, \qquad -\infty < k < \infty, \tag{3.9}$$

in $B \times (0, \infty)$ where u is a real-valued function of $\mathbf{x}$ and t vanishing on $\partial B \times (0, \infty)$ and $\Delta_n u = u_{,ii}$.

However, we consider a somewhat more general version of (3.9), namely,

$$\frac{\partial^2 u}{\partial t^2} + k(t)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{x}, t)\frac{\partial u}{\partial x_j}\right) = \mathscr{F}(\mathbf{x}, t, u, u_t, \nabla u), \tag{3.10}$$

$$(\mathbf{x}, t) \in B \times (0, \infty),$$

where $k(t)$ is a continuous function on $(0, \infty)$, $\nabla u = (u_{x_1}, \cdots, u_{x_n})$ and the matrix $[a_{ij}(\mathbf{x}, t)]_{m\times n}$ with real, continuously differentiable entries satisfies the condition of uniform strong ellipticity, namely,

$$a_{ij}(\mathbf{x}, t)\xi_i\xi_j \geqq a_0\xi_i\xi_i > 0 \tag{a-1}$$

for all $\boldsymbol{\xi} \in R^n$, $(\mathbf{x}, t) \in B \times (0, \infty)$ and some constant $a_0 > 0$. We assume that

$$\left|\frac{\partial a_{ij}}{dt}(\mathbf{x}, t)\right| \leqq k_1 g(t), \qquad i, j = 1, 2, \cdots, n, \tag{a-2}$$

for $(\mathbf{x}, t) \in B \times (0, \infty)$ and some constant k_1 where g satisfies conditions (G-I) through (G-IV) of Theorem 2.1. We will also suppose that

$$|k(t)| \leqq kg(t), \qquad t \in (0, \infty),$$

for some constant k and

$$|\mathscr{F}(\mathbf{x}, t, z_1, z_2, \boldsymbol{\xi})|^2 \leqq k_1^4 g(t)^4 z_1^2 + k_2^2 g(t)^2\{z_2^2 + \xi_i\xi_i\} \tag{F-1}$$

on $B \times (0, \infty)$ for some constants k_1, k_2.

We now restrict our class of solutions to (3.10) to those satisfying

$$u(\mathbf{x}, t) = 0, \qquad (\mathbf{x}, t) \in B \times (0, \infty). \tag{3.11}$$

Then we can easily show that any such solution satisfies a differential inequality of the form (2.9) where $\|f\| = (\int_B |f(\mathbf{x})|^2\, d\mathbf{x})^{1/2}$, $H = \mathscr{L}^2(B)$, $D = \{f \in H \mid f \in C^2$, $f = 0$ on ∂B, and $(A(t)f)(\mathbf{x}) = (a_{ij}(\mathbf{x}, t)f_{,i}(\mathbf{x}))_{,j} \in H\}$. Use of (a-1) and (a-2) permits one to verify that (A-I) and (A-II) hold.

Thus with

$$\mathscr{E}(t, u) = \int_B [u^2(\mathbf{x}, t) + u_{,t}^2(\mathbf{x}, t) + a_{ij}(\mathbf{x}, t)u_{,i}u_{,j}]\, dx,$$

we see that Theorem 2.1 holds and

$$\liminf_{t\to\infty} e^{\rho f(t)}\mathscr{E}(t, u) > 0 \tag{3.12}$$

for some $\rho > 0$ unless $u \equiv 0$.

In particular, solutions to (3.9) satisfy, for all t sufficiently large and some $c > 0$ and $\rho > 0$,

$$\mathscr{E}(t, u) \geqq ct^{-\rho}, \qquad t \to +\infty, \tag{3.13}$$

unless $u \equiv 0$.

Remark 3.3. Using the results of [10], one can also obtain (3.13) for solutions to (3.9). In fact, using the results of Murray [5], one can even obtain this result for domains which are "expanding faster than light." However, our results apply also to higher order problems as we see from the next example.

Example III. In this example we consider, in n dimensions, an analogue of the equation of motion of a clamped vibrating plate under normal loads. Let $[(a_{ij}(\mathbf{x})]_{n\times n}$ be, at each point $\mathbf{x}$ in a bounded subset B of R^n, a symmetric matrix with C^3 entries. Let ∂B be smooth enough to admit of as many applications of the divergence theorem as needed. We wish to consider the problem

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial x_i}\left\{a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}\left[a(\mathbf{x})\frac{\partial}{\partial x_k}\left(a_{kl}(\mathbf{x})\frac{\partial u(\mathbf{x}, t)}{\partial x_l}\right)\right]\right\} \\ &\qquad = \mathscr{F}(\mathbf{x}, t, u, u_t, \mathbf{D}u, D^2 u) \quad \text{in } B \times (0, \infty), \\ &u(\mathbf{x}, t) = a_{ij}(\mathbf{x})n_i\frac{\partial u}{\partial x_j} = 0, \qquad (\mathbf{x}, t) \in \partial B \times (0, \infty), \end{aligned} \tag{3.14}$$

where $a(\mathbf{x})$ is a given positive C^2-function in B, $\mathbf{n} = (n_1, \cdots, n_n)$ is the exterior normal to ∂B and where $\mathbf{Du} = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$, $D^2u = [\partial^2 u/\partial x_i\, \partial x_j]_{n\times n}$. If $\mathscr{F} \equiv 0$, $n = 2$, $a_{ij} = \delta_{ij}$ and $a \equiv 1$, (3.14) is the classical initial-boundary value problem for the transverse vibrations of a clamped plate. See [4], for example.

In order to apply our theory directly to (3.14), we shall let $H = \mathscr{L}^2(B)$ as before and

$$D = \{f \in H | f \in C^4(B),\ Af \in H \text{ and } f = a_{ij}(\mathbf{x})n_i f_{,j} = 0 \text{ on } \partial B\},$$

where

$$Af(\mathbf{x}) \equiv \frac{\partial}{\partial x_i}\left\{a_{ij}(\mathbf{x})\frac{\partial}{\partial x_j}\left[a(\mathbf{x})\frac{\partial}{\partial x_k}\left(a_{kl}(\mathbf{x})\frac{\partial f(\mathbf{x})}{\partial x_l}\right)\right]\right\}.$$

A is symmetric on D; in fact,

$$\langle Af, g\rangle = \int_B a(\mathbf{x})(a_{ij}g_{,i})_{,j}(a_{kl}f_{,k})_{,l}\, d\mathbf{x}.$$

Moreover, since $a > 0$ at each point of B, $\langle Af, f\rangle \geqq 0$ for all $f \in B$.

We shall assume that for every function $u(\mathbf{x}, t)$ such that $u(\cdot, t) \in D$, $\mathscr{F}$ satisfies

$$\begin{aligned} \int_B |\mathscr{F}(\mathbf{x}, t, u, u_t, Du, D^2u)|^2\, d\mathbf{x} &\leqq k_1^4 g^4(t) \int_B u^2(\mathbf{x}, t)\, d\mathbf{x} \\ &\quad + k_2^2 g^2(t) \int_B \left[\left(\frac{\partial u}{\partial t}\right)^2 + a(\mathbf{x})[(a_{ij}u_{,i})_{,j}]^2\right] d\mathbf{x} \end{aligned} \tag{3.15}$$

for some constants k_1 and k_2 independent of the choice of $u(\mathbf{x}, t)$.

Thus from Theorem 2.1, we have

$$\mathscr{E}(t, u) \equiv \int_B [u^2 + u_t^2 + a(\mathbf{x})[(a_{ij}u_{,i})_{,j}]^2]\, dx$$

cannot decay faster than $e^{-\rho f(t)}$ for every $\rho > 0$ unless $u \equiv 0$.

Remark 3.4. The function $a(\mathbf{x})$ can be time-dependent if we assume that

$$\frac{\partial a}{\partial t}(\mathbf{x}, t) \geqq -kg(t)a(\mathbf{x}, t)$$

for some constant $k > 0$. Then $\langle Af, f\rangle \geqq -kg(t)\langle A(t)f, f\rangle$ so that (A-II) holds

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SUMMABILITY METHODS OBTAINED BY SUBSTITUTION OF POWER SERIES*

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Abstract. The Euler summability method is obtained by substituting into the series $\sum a_n x^n$ the expression $x = f(y) = y/(1 - qy)$, thereby obtaining a series $\sum b_n y^n$. In this paper we consider more general expressions for $f(y)$, $f(y) = \sum f_i y_i$. A class of sequences naturally associated with such methods is given by $\sigma^*: \{s_n\}$ is in σ^* if $\overline{\lim}_n \sqrt[n]{|a_n|} < 1$ where $a_n = s_n - s_{n-1}$. We obtain both matrix conditions (for an arbitrary summability method) and functional conditions (for the series substitution method) for regularity with respect to σ^*. Also, we show that if $f_i \geqq 0$ the method is regular in the usual sense. Examples, including several from the theory of stability of numerical processes, are given of summability methods regular with respect to σ^*.

1. Introduction. The Euler summability method and its generalization [3, p. 178] is obtained by substituting into the series $\sum a_n x^n$ the expression $x = y/(1 - qy) = f(y)$, thereby obtaining a series $\sum b_n y^n$. In this paper we consider the more general expressions, $f(y) = \sum f_i y^i$. One result we find is that for $f_i \geqq 0$ (for all i) the summability method is regular (Theorem 4).

Major emphasis, however, is on a particular class of sequences which we denote by σ^*: $\{s_n\}$ is in σ^* if $\overline{\lim}_n \sqrt[n]{|a_n|} < 1$, where $a_n = s_n - s_{n-1}$ and $s_0 = 0$. Section 2 gives matrix conditions (in the standard form) for regularity of a summability method with respect to σ^*. (The main result appears as Theorem 1.) Section 3 discusses summability methods obtained by series substitution. Theorem 2 gives functional conditions for regularity with respect to σ^* of such methods; the primary condition is

$$|f(y)| \leqq 1 \quad \text{for } |y| = 1. \tag{1}$$

A relationship between this condition and the matrix form of the series substitution method is then obtained (Theorem 3).

It is of interest to note that (1) is the condition associated with stability of numerical processes (via Fourier series techniques). Whether this relationship between summability methods and stability of numerical processes will prove profitable to numerical analysts remains to be seen, but in the meantime we can draw on established results of numerical analysis to obtain examples of summability methods regular with respect to σ^*. Such examples, as well as several others, are given in § 4.

2. σ^*-regular transformations. Given a sequence $\{s_n\}$ and an infinite matrix $C = (c_{ij})$, the sequence $\{t_m\}$ is defined by

$$t_m = \sum_{i=1}^{\infty} c_{mi} s_i. \tag{2}$$

C is a σ^*-regular transformation if $\lim_{m\to\infty} t_m = \lim_{n\to\infty} s_n$ for all $\{s_n\}$ in σ^*. We will use the notation $\bar{c}_{mi} = \sum_{j=i}^{\infty} c_{mj}$ [6, p. 39].

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THEOREM 1. *C is σ^*-regular if and only if the following conditions are satisfied*:

(a) *Given any $\varepsilon > 0$, there exists an m_ε such that for $m > m_\varepsilon$, $|\bar{c}_{mn}| < (1+\varepsilon)^n$, for all n.*

(b) $\lim_{m\to\infty} \bar{c}_{mn} = 1$ *for all n.*

Proof. The proof follows the standard format for such results ([3, p. 44], [5, p. 8], [6, p. 11]). To show sufficiency of the conditions, let $\sigma = \{s_n\}$ be in σ^*, with $\overline{\lim}_n \sqrt[n]{|a_n|} = \delta < 1$ and $s = \lim_{n\to\infty} s_n$. From condition (b) it follows that

$$\lim_{n\to\infty} \bar{c}_{mn} = 0 \quad \text{for every } m. \tag{3}$$

Note that $\sum_{i=1}^N c_{mi}s_i = \sum_{i=1}^N \bar{c}_{mi}a_i - \bar{c}_{m,N+1}s_N$. Since $\sum |a_i| < \infty$, it follows from (3) that t_m exists and that

$$t_m = \sum_{i=1}^{\infty} a_i\bar{c}_{mi}. \tag{4}$$

Now, $t_m - s_N = \sum_{i=1}^N a_i(\bar{c}_{mi} - 1) + \sum_{i=N+1}^{\infty} a_i\bar{c}_{mi}$. From condition (a), for m sufficiently large, N can be chosen so large that the second sum is arbitrarily small. From condition (b), for fixed N, m can be chosen so large that the first sum is arbitrarily small. Thus, C is σ^*-regular.

The necessity of condition (b) is clear. It remains to establish the necessity of condition (a). Suppose (a) is false. Then there exists $\varepsilon > 0$ and two integer sequences, $\{m_i\}$ and $\{n_i\}$, where $\{m_i\}$ is strictly increasing, such that $|\bar{c}_{m_i,n_i}| \geqq (1+\varepsilon)^{n_i}$. The sequence $\{n_i\}$ is unbounded, for otherwise condition (b) would be violated. Then for every $\tilde{m}$ and $\tilde{n}$ there exists $m > \tilde{m}$ and $n > \tilde{n}$ for which

$$|\bar{c}_{mn}| > (1+\varepsilon)^n. \tag{5}$$

Two sequences $\{m_k\}$, $\{n_k\}$ will now be defined inductively. Choose $m_0 = n_0 = 1$. Suppose m_{k-1} and n_{k-1} have been chosen. By condition (b) choose $m_k' > m_{k-1}$ such that

$$m > m_k' \Rightarrow |\bar{c}_{mj}| < 1 + \varepsilon \quad \text{for } 1 \leqq j \leqq n_{k-1}. \tag{6}$$

By (5) choose $m_k > m_k'$ and $n_k' > n_{k-1}$ such that

$$|\bar{c}_{m_k,n_k'}| \geqq (1+\varepsilon)^{n_k'}. \tag{7}$$

By (3) choose $n_k > n_k'$ such that

$$j > n_k \Rightarrow |\bar{c}_{m_k,j}| < \varepsilon. \tag{8}$$

Choose δ in $(0, 1)$ such that $(1+\varepsilon)\delta > 1$. Define σ in σ^* by $a_1 = \delta$ and $a_i = \delta^i \operatorname{sgn} \bar{c}_{m_k,i}$ for $n_{k-1} < i \leqq n_k$.

Now, from (4), (7), (6) and (8), and the choice of σ, we obtain for $k \geqq 1$,

$$\begin{aligned} |t_{m_k}| &\geqq \sum_{i=n_{k-1}+1}^{n_k} \delta^i|\bar{c}_{m_k,i}| - \sum_{i=1}^{n_{k-1}} \delta^i|\bar{c}_{m_k,i}| - \sum_{i=n_k+1}^{\infty} \delta^i|\bar{c}_{m_k,i}| \\ &\geqq \delta^{n_k'}(1+\varepsilon)^{n_k'} - \sum_{i=1}^{n_{k-1}} \delta^i(1+\varepsilon) - \sum_{i=n_k+1}^{\infty} \delta^i\varepsilon. \end{aligned}$$

Thus, by the choice of δ, $\lim_{k\to\infty} |t_{m_k}| = \infty$, or C is not σ^*-regular.

3. Transformations obtained by substitution of power series. Let

(9)
$$g(x) = \sum_{i=1}^{\infty} a_i x^i$$

and

(10)
$$x = f(u) = \sum_{i=1}^{\infty} f_i u^i \quad \text{with } f(1) = 1.$$

Then

(11)
$$g(x) = h(u) = \sum_{i=1}^{\infty} b_i u^i.$$

The summability method based on f is denoted by C_f and is defined as follows. Let $\sigma = \{s_n\}$, where $s_n = \sum_{i=1}^{n} a_i$. Then $C_f\sigma = \{t_n\}$, where $t_n = \sum_{i=1}^{n} b_i$ and the b_i are defined by (9), (10) and (11).

(If the radius of convergence of the series in (9) is zero, then the above can be considered a formal manipulation.)

THEOREM 2. *Suppose*

(12)
$$\overline{\lim_n} \sqrt[n]{|f_n|} < 1.$$

Then C_f is a σ^*-regular transformation if and only if

(13)
$$\max_{|u|=1} |f(u)| = 1.$$

Proof. By (12), $f(u)$ is analytic in some region $|u| < R$, where $R > 1$. Suppose (13) is satisfied. If σ is in σ^*, then the $g(x)$ of (9) is analytic for $|x| < 1 + \varepsilon$, where $\varepsilon > 0$. But then, by (13), $h(u)$ is analytic for $|u| < 1 + \delta$ for some $\delta > 0$. In particular, $h(1) = g(1)$, or C_f is a σ^*-regular transformation.

Suppose (13) is not satisfied. Then there is a point u^* such that $f(u^*) = x^*$, where $|u^*| < 1$ and $|x^*| > 1$. Let

$$g(x) = \frac{x}{1 - x/x^*} = \sum_{i=1}^{\infty} a_i x^i.$$

The corresponding sequence $\sigma = \{s_n\}$ is in σ^*. Since $h(u)$ has a pole at $u = u^*$, the radius of convergence of the series (11) is $|u^*|$ (<1). Therefore, the series cannot converge at $u = 1$, or $C_f\sigma$ does not exist.

In order to consider the summability properties of C_f we need to find its matrix representation. We introduce the notations

(14)
$$f(u) = \sum_{i=1}^{\infty} F_{i1} u^i \quad \text{with } \sum_{i=1}^{\infty} F_{i1} = 1,$$

(15)
$$[f(u)]^m = \sum_{i=m}^{\infty} F_{im} u^i,$$

and

(16)
$$\beta_{ni} = \sum_{j=i}^{n} F_{ji}.$$

Then $h(u) = \sum_{i=1}^{\infty} a_i [f(u)]^i = \sum_{j=1}^{\infty} b_j u^j$, where $b_j = \sum_{i=1}^{j} a_i F_{ji}$ [2, p. 138]. Also

$$t_n = \sum_{j=1}^{n} b_j = \sum_{i=1}^{n} a_i \beta_{ni} = \sum_{i=1}^{n-1} s_i(\beta_{ni} - \beta_{n,i+1}) + \beta_{nn} s_n.$$

Thus, the matrix C of the C_f transformation takes the form

$$c_{mn} = \begin{cases} \beta_{mn} - \beta_{m,n+1}, & m > n, \\ \beta_{mm}, & m = n, \\ 0, & m < n. \end{cases} \tag{17}$$

Since $\bar{c}_{mn} = \beta_{mn}$, the matrix conditions under which C_f is a σ^*-regular transformation are obtained directly from Theorem 1. The result is as follows.

THEOREM 3. *C_f is a σ^*-regular transformation if and only if the following conditions are satisfied*:

(a) *For any $\varepsilon > 0$ there exists an m_ε such that for $m > m_\varepsilon$,*

$$|\beta_{mn}| < (1 + \varepsilon)^n \quad \textit{for all } n. \tag{18}$$

(b)

$$\lim_{m \to \infty} \beta_{mn} = 1 \quad \textit{for all } n. \tag{19}$$

COROLLARY. If $\overline{\lim}_n \sqrt[n]{|F_{n1}|} < 1$, *then conditions* (13) *and* (18) *are equivalent in the sense that either is a necessary and sufficient condition in order that C_f be a σ^*-regular transformation.*

Proof. We need only note that if $\overline{\lim}_n \sqrt[n]{|F_{n1}|} < 1$, then (19) is satisfied.

Remark. If $f(u)$ has a radius of convergence greater than 1, then for any fixed n, $\lim_{m\to\infty} \beta_{mn} = 1$. Thus given any N and $\varepsilon > 0$ there exists an m_ε such that for $m > m_\varepsilon$, $|\beta_{mn}| < (1 + \varepsilon)^n$ for $1 \leqq n \leqq N$. In effect, C_f is a σ^*-regular transformation if and only if the Taylor series expansions for $[f(u)]^m$ converge "uniformly", in the sense of condition (18). Furthermore, this is possible if and only if $|f(u)| \leqq 1$ for $|u| = 1$.

Our final result concerns conditions under which C_f is a regular transformation.

THEOREM 4. *If $F_{i1} \geqq 0$ for all i, the corresponding C_f transformation is regular.*

Proof. We need to establish condition (b) of Theorem 1 plus the following condition which replaces condition (a) [5, p. 8]:

$$\sum_{n=1}^{\infty} |c_{mn}| < H, \tag{20}$$

where H is a constant independent of m. Our earlier discussion for condition (b) still applies. To establish (20), we first note that from multiplication of power series,

$$\sum_{j=n+1}^{m} F_{j,n+1} = \sum_{j=n}^{m-1} F_{jn}\left(\sum_{i=1}^{m-j} F_{i1}\right).$$

Then using (16),

$$\beta_{mn} - \beta_{m,n+1} = F_{mn} + \sum_{j=n}^{m-1} F_{jn}\left(1 - \sum_{i=1}^{m-j} F_{i1}\right) \geqq 0.$$

From (17) it now follows that $c_{mn} \geqq 0$. Thus

$$\sum_{n=1}^{\infty} |c_{mn}| = \sum_{n=1}^{\infty} c_{mn} = \beta_{m1}.$$

Since $\lim_{m\to\infty} \beta_{m1} = 1$, condition (20) is satisfied.

4. Examples of σ^*-regular transformations. We consider now several examples relating to the preceding theory.

(a) The Euler (E, q) method [3, p. 178] employs the transformation $x = u/(1 - qu)$. Normalizing so that $f(1) = 1$, we obtain

$$x = f(u) = \frac{1}{1+q}\left[\frac{u}{1 - [q/(1+q)]u}\right] = \frac{u}{1+q}\sum_{i=0}^{\infty}\left(\frac{q}{1+q}\right)^i u^i.$$

Theorem 4 now gives the well-known result that the (E, q) method is regular for $q \geqq 0$.

(b) Poincaré [7], in establishing an existence theorem for a system of differential equations, employed the transformation $u = (e^{\rho x} - 1)/(e^{\rho x} + 1)$, where $\rho > 0$. Inverting we find that

$$x = \frac{1}{\rho}\ln\left(\frac{1+u}{1-u}\right) = \sum_{i=1}^{\infty} a_i u^i = g(u),$$

where

$$a_i = \begin{cases} 0 & \text{if } i \text{ is even}, \\ 2/(\rho i) & \text{if } i \text{ is odd}. \end{cases}$$

Noting that $g(u^*) = 1$, where $u^* = (e^{\rho} - 1)/(e^{\rho} + 1) > 0$, we obtain $x = f(u) = \sum_{i=1}^{\infty} \tilde{a}_i u^i$. Since $\tilde{a}_i = (u^*)^i a_i \geqq 0$, we have from Theorem 4 that C_f is a regular transformation.

(c) We consider $f(u)$ to be a first, second, or third degree polynomial and look for conditions under which C_f is a σ^*-regular transformation. In these cases condition (13) seems simpler to verify than condition (18). For example, consider $f(u) = 2u - u^2$. Since $f(-1) = -3$, condition (13) does not hold and C_f is not σ^*-regular. However, it is not so obvious that C_f does not satisfy condition (18).

(i) For first degree polynomials, since we require $f(0) = 0$ and $f(1) = 1$, $f(u) = u$ is the only possibility. C_f is, of course, regular.

(ii) For second degree polynomials we need to consider $f(u) = u(u + \alpha)/(1 + \alpha)$. Condition (13) is satisfied only if α is real and positive. In this case, by Theorem 4, C_f is regular.

(iii) For third degree polynomials we consider $f(u) = u(u^2 + \alpha u + \beta)/(1 + \alpha + \beta)$. For α and β real it can be verified that condition (13) holds only if α and β satisfy one of the following sets of constraints: (*j*) $\alpha \geqq 0$ and $\beta \geqq 0$, (*jj*) $-1 < \beta < 0$ and $\alpha \geqq 4|\beta|/(1 - |\beta|)$, (*jjj*) $\beta < -1$ and $\alpha \leqq -4|\beta|/(|\beta| - 1)$. In the case of set (j) C_f is regular, while in the other two cases C_f is σ^*-regular.

(d) We consider next several examples based on numerical methods for solving partial differential equations. We use the notation $v_j^n = v(n\Delta t, j\Delta x)$.

(i) The heat conduction equation, $\partial v/\partial t = \partial^2 v/\partial x^2$, can be approximated by the difference equation [8, p. 93],

$$v_j^{n+1} - v_j^n = p(v_{j+1}^n - 2v_j^n + v_{j-1}^n), \tag{21}$$

where $p = \Delta t/(\Delta x)^2 \geqq 0$. To study stability of (21) one can substitute the Fourier term $v_j^n = \xi^n u^j$, where $u = e^{i\theta}$, $i = \sqrt{-1}$. This gives $\xi = (1 - 2p) + p(u + u^{-1})$. For stability it is required that $|\xi| \leqq 1$ for all θ. This requirement is equivalent to $|f(u)| \leqq 1$ for $|u| = 1$, where $f(u) = u[p + (1 - 2p)u + pu^2]$. Thus we see that (21) is stable if and only if C_f is σ^*-regular. From example (c.iii), C_f is σ^*-regular only if $1 - 2p \geqq 0$. This is the well-known result for stability of (21).

(ii) Another approximation to the heat conduction equation is obtained by applying a 4th-order Runga–Kutta formula to $\partial v/\partial t$ [9]. This takes the form

$$v_j^{n+1} - v_j^n = \tfrac{1}{6}(k_{1j} + 2k_{2j} + 2k_{3j} + k_{4j}), \tag{22}$$

where

$$\begin{aligned} k_{1j} &= p(\delta^2 v)_j^n, \\ k_{2j} &= k_{1j} + \frac{p}{2}(\delta^2 k_{1j})_j^n, \\ k_{3j} &= k_{1j} + \frac{p}{2}(\delta^2 k_{2j})_j^n, \\ k_{4j} &= k_{1j} + p(\delta^2 k_{3j})_j^n \end{aligned}$$

and

$$p = \Delta t/(\Delta x)^2, \qquad (\delta^2 g)_j^n = g_{j-1}^n - 2g_j^n + g_{j+1}^n.$$

After expansion, this leads to the function

$$f(u) = u[a_4 + a_3 u + a_2 u^2 + a_1 u^3 + a_0 u^4 + a_1 u^5 + a_2 u^6 + a_3 u^7 + a_4 u^8],$$

where

$$\begin{aligned} a_0 &= 1 - 2a_1 - 2a_2 - 2a_3 - 2a_4, \\ a_1 &= p - (p^2/6)(12 - 15p + 14p^2), \\ a_2 &= (p^2/6)(7p^2 - 6p + 3), \\ a_3 &= (p^3/6)(1 - 2p), \\ a_4 &= p^4/24. \end{aligned}$$

The difference equation (22) can be shown to be stable only for $p \leqq p^*$, where $p^* \cong 0.69632$. (This is somewhat tedious to establish. One can show that all coefficients are positive for $0 \leqq p \leqq \frac{1}{2}$, which establishes stability in this range. The upper bound p^* is obtained by working with the polynomial, in p, resulting from setting $u = -1$. One then needs to show that $u = -1$ gives the worst case for $\frac{1}{2} \leqq p \leqq p^*$.) Thus, $f(u)$ is a σ^*-regular transformation for $0 \leqq p \leqq p^*$.

(iii) The hyperbolic equation $\partial v/\partial t = \partial v/\partial x$ can be approximated by the difference equation [4],

$$v_j^{n+1} - v_j^n = \frac{p}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{p^2}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n), \tag{23}$$

where $p = \Delta t/\Delta x \geqq 0$. Proceeding as before, we obtain $f(u) = u[(p/2)(p-1) + (1-p^2)u + (p/2)(p+1)u^2]$. Since (23) is known to be stable for $0 \leqq p \leqq 1$, we can conclude that C_f is σ^*-regular for $0 \leqq p \leqq 1$. (This result can also be verified from example (c. iii) for $\alpha = 2(1-p)/p$ and $\beta = (p-1)/(p+1)$. For $p > 1$, since $\beta > 0$ and $\alpha < 0$, C_f is not σ^*-regular. For $p \leqq 1$, since $-1 \leqq \beta \leqq 0$ and $\alpha = 4|\beta|/(1-|\beta|)$, C_f is σ^*-regular.)

(iv) Our final examples will indicate that a stable difference equation need not produce a σ^*-regular transformation. Let us approximate $\partial v/\partial t = \partial^2 v/\partial x^2$ by [8, p. 93],

$$v_j^{n+1} - v_j^n = p(v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}), \tag{24}$$

where $p = \Delta t/\Delta x^2 \geqq 0$. This leads to $f(u) = u/(u - p(1-u)^2)$. The approximation (24) is stable for all values of p. Thus, C_f would be σ^*-regular if $f(u)$ were analytic for $|u| \leqq 1 + \delta$, for some $\delta > 0$. However, $f(u)$ fails to be analytic at

$$u = \frac{(2p+1) - \sqrt{4p+1}}{2p},$$

which is inside the unit circle.

The Dufort–Frankel approximation to $\partial v/\partial t = \partial^2 v/\partial x^2$ is given by ([1], [8, p. 83])

$$v_j^{n+1} = \frac{p}{1+p}(v_{j-1}^n + v_{j+1}^n) + \frac{1-p}{1+p}v_j^{n-1}, \tag{25}$$

where $p = 2\Delta t/\Delta x^2 \geqq 0$. The approximation (25) is stable for all p. As above, this leads to

$$f(u) = \frac{pu}{2(1+p)}\left[1 + u^2 + \sqrt{(1-u^2)^2 + \frac{4u^2}{p^2}}\right].$$

For $p \geqq 1$, $f(u)$ fails to be analytic at points on the unit circle, while for $p < 1$, $f(u)$ fails to be analytic at points inside the unit circle. Thus, C_f will not be a σ^*-regular transformation.

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POSITIVE INTEGRALS OF BESSEL FUNCTIONS*

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Abstract. It is shown that some new and some already known positivity results for integrals of Bessel functions and for generalized hypergeometric functions can be easily obtained by writing the integrals and functions either as a sum of squares of Bessel functions with positive coefficients or as a fractional integral of such a sum. In particular, this method is used to prove that

$$\int_0^x (x-t)^{\lambda} t^{\lambda+1/2} J_\alpha(t)\,dt > 0, \qquad 0 \leqq \lambda \leqq \alpha - 1/2, \quad \alpha > 1/2, \quad x > 0,$$

and to give a simple proof of Steinig's recent result that the Lommel function $s_{\mu,\nu}(x) > 0$ for $x > 0$ if $\mu = 1/2$ and $-1/2 < \nu < 1/2$, or if $\mu > 1/2$ and $-\mu \leqq \nu \leqq \mu$.

1. Introduction. Over the years methods have had to be developed to prove the positivity of various integrals of Bessel functions. Several excellent examples of this are contained in the listed references. In [16] Cooke used intricate properties of Bessel functions and Lommel functions and a method devised by Watson to prove that

$$\int_0^x J_\alpha(t)\,dt > 0, \qquad \alpha > -1, \tag{1.1}$$

if x is the second positive zero of the Bessel function $J_\alpha(t)$. This result, along with his results in [14], enabled Cooke to complete the proof of his well-known result that the graph of $J_\alpha(x)$, $x > 0$, consists of waves whose areas form a strictly decreasing sequence when $\alpha > -1$. Thus (1.1) holds for all $x > 0$.

Steinig [37] employed an oscillation theorem of Makai [29] (also see [27], [35], [38] and [39]) for second order differential equations and two fractional integrals to prove that the Lommel function

$$\begin{aligned} s_{\mu,\nu}(x) &= 2^{(\mu-\nu-1)/2}\Gamma((\mu-\nu+1)/2)x^{-\nu} \\ &\quad \cdot \int_0^x (x^2-t^2)^{(\mu+\nu-1)/2} t^{(\nu-\mu+1)/2} J_{(\mu-\nu+1)/2}(t)\,dt, \end{aligned} \tag{1.2}$$

where $\mu + \nu > -1$, is (strictly) positive for $x > 0$ if $\mu = 1/2$ and $-1/2 < \nu < 1/2$, or if $\mu > 1/2$ and $-\mu \leqq \nu \leqq \mu$; which extends some earlier results of Cooke [15]. Throughout this paper it will be assumed that $x > 0$.

Askey and Pollard [8] used properties of completely monotonic functions to prove that $x^{-2c}(x^2+1)^{-c}$ is completely monotonic for $c > 0$, which by a theorem of Bernstein is equivalent (see [1] and [40, p. 161]) to

$$\int_0^x (x-t)^{2\alpha} t^\alpha J_\alpha(t)\,dt \geqq 0, \qquad \alpha > -1/2. \tag{1.3}$$

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The positivity of this integral as well as the inequality

$$(1.4)\qquad \int_0^x (x-t)^{\alpha+3/2}t^{\alpha+1}J_\alpha(t)\,dt > 0, \qquad \alpha > -1/2,$$

which implies the complete monotonicity of $x^{-c}(x^2+1)^{-c}$ for $c \geqq 1$, were proved recently by Fields and Ismail [20] (also see [19]) by applying an asymptotic argument of Darboux type to an integral representation for a ${}_1F_2$. Both (1.3) and (1.4) were motivated by the observations in Askey [1] concerning positivity of the Cesàro means of Jacobi series.

A limiting case of a conjecture in Askey and Gasper [7] concerning positive sums of Jacobi polynomials suggested the inequality

$$(1.5)\qquad \int_0^x (x-t)^\lambda t^{\lambda+1/2}J_\alpha(t)\,dt > 0, \qquad 0 \leqq \lambda \leqq \alpha - 1/2, \quad \alpha > 1/2.$$

Note that, by an integration by parts,

$$(1.6)\qquad \int_0^x (x-t)^{\alpha+3/2}t^{\alpha+1}J_\alpha(t)\,dt = \left(\alpha+\frac{3}{2}\right)\int_0^x (x-t)^{\alpha+1/2}t^{\alpha+1}J_{\alpha+1}(t)\,dt,$$

so the case $\lambda = \alpha - 1/2$ of (1.5) is equivalent to (1.4). The case $\lambda = 0$ of (1.5) is a well-known result of Makai [29]. Since the above-mentioned methods for proving positivity did not seem to be particularly applicable to the intermediate cases $0 < \lambda < \alpha - 1/2$ of (1.5), and since when $\lambda = 0$ and $\alpha = 1/2$ the integral in (1.5) is a positive multiple of a square of a Bessel function, explicitly

$$(1.7)\qquad \int_0^x t^{1/2}J_{1/2}(t)\,dt = x\left(\frac{\pi}{2}\right)^{1/2} J_{1/2}^2\left(\frac{x}{2}\right),$$

the author attempted to try to prove (1.5) by writing the integral as a sum of squares of Bessel functions with coefficients which are positive. This attempt succeeded and, as we shall show, it turned out that one could easily prove the positivity of the integrals in (1.1), (1.3), (1.4), (1.5) and of certain generalized hypergeometric functions by showing that the integrals and functions have series expansions of the form

$$(1.8)\qquad x^\gamma \sum_{n=0}^\infty a_n J_{n+\nu}^2(x/2),$$

with $a_0 > 0$, $a_1 \geqq 0$, $a_2 > 0$ and $a_n \geqq 0$ for $n = 3, 4, \cdots$. We shall also use expansions of the form (1.8) and fractional integrals to give a simple proof of Steinig's above-mentioned positivity results for the Lommel function $s_{\mu,\nu}(x)$ and to obtain some new positivity results for generalized hypergeometric functions.

Before considering our expansions of the form (1.8) we shall point out some additional applications for positive integrals of Bessel functions. Inequality (1.1) and the case $\lambda = 0$ of (1.5) were used by Lorch and Szego [25], [26] to prove an inequality involving integrals of Bessel functions which arose in Wilf's work [41] on the stability of least square smoothing. Askey [3] used the case $\alpha = 1/2$ of (1.4) in an equivalent form to prove a sufficient condition for a function to be the characteristic function of a unimodal distribution, while (1.1) and some other special cases of (1.4) are the main tools that were needed in his paper on radial characteristic functions [2].

In [42] Williamson shows that

$$\int_0^1 (1-t)^2 t \sin xt\, dt > 0, \qquad x > 0, \tag{1.9}$$

which is equivalent to the case $\alpha = 1/2$ of (1.4) since $\sin t = (\pi t/2)^{1/2} J_{1/2}(t)$, and then uses this inequality to give a shorter proof of Royall's result [33] that the Laplace transform $f(s)$ of a three-times monotone function is univalent for $\operatorname{Re}(s) > 0$. Hille's paper [22] considers the distribution of the zeros of generalized hypergeometric functions and in it he observes that if $p < q$ and $\varepsilon > 0$, then all but a finite number of the zeros of

$${}_pF_q(z) = {}_pF_q[a_1, \cdots, a_p; b_1, \cdots, b_q; z]$$

are located in the sector $\pi - \varepsilon < \arg z < \pi + \varepsilon$. Since, by (2.2) and (2.20) below, each of the above integrals is a positive multiple of a ${}_2F_3(-x^2/4)$ function with positive parameters (in which case ${}_2F_3(z)$ has infinitely many zeros), Hille's observation helps explain why it is so difficult to prove the positivity of the above integrals by using asymptotic estimates (cf. [20]).

In [24] Kuttner considers the Riesz $(R, n^\lambda, \varkappa)$ means

$$G(\lambda, \varkappa; u, \theta) = \tfrac{1}{2} + \sum_{1 \leqq n \leqq u} \left(1 - \frac{n^\lambda}{u^\lambda}\right)^\varkappa \cos n\theta \tag{1.10}$$

for the series $\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots$, and shows that a necessary and sufficient condition for $G(\lambda, \varkappa; u, \theta) \geqq 0$ for all u, θ is that

$$\int_0^1 (1 - t^\lambda)^\varkappa \cos xt\, dt \geqq 0, \qquad x > 0, \tag{1.11}$$

which is equivalent to

$$\int_0^x (x^\lambda - t^\lambda)^\varkappa t^{1/2} J_{-1/2}(t)\, dt \geqq 0, \qquad x > 0. \tag{1.12}$$

Note that the case $\lambda = \varkappa = 1$ of (1.12) follows from (1.4) by letting $\alpha \to -1/2$.

2. Integrals of Bessel functions. First observe that using either

$$J_\alpha(t) = \frac{(t/2)^\alpha}{\Gamma(\alpha+1)}\, {}_0F_1[\alpha + 1; -t^2/4], \qquad [17, 7.2(3)] \tag{2.1}$$

or [18, 13.1(56)] we have

$$\begin{aligned} \int_0^x (x-t)^\lambda t^\mu J_\alpha(t)\, dt &= \frac{\Gamma(\lambda+1)\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\lambda+\mu+2)} 2^{-\alpha} x^{\alpha+\lambda+\mu+1} \\ &\quad \cdot {}_2F_3\left[\begin{matrix} (\alpha+\mu+1)/2, (\alpha+\mu+2)/2; -x^2/4 \\ \alpha+1, (\alpha+\lambda+\mu+2)/2, (\alpha+\lambda+\mu+3)/2 \end{matrix}\right], \end{aligned} \tag{2.2}$$

when $\alpha + \mu > -1, \lambda > -1$. Then, using

$$z^{2\nu} = \frac{\Gamma^2(\nu+1) 2^{2\nu+1}}{\Gamma(2\nu+1)} \sum_{n=0}^\infty \frac{(n+\nu)\Gamma(n+2\nu)}{n!} J_{n+\nu}^2(z), \quad [39, 5.5(1)] \tag{2.3}$$

with $z = x/2$, we obtain the expansion

$$\begin{aligned}(2.4)\quad &\int_0^x (x-t)^\lambda t^\mu J_\alpha(t)\,dt\\ &= \frac{\Gamma(\lambda+1)\Gamma(\alpha+\mu+1)\Gamma^2(\nu+1)}{\Gamma(\alpha+1)\Gamma(\alpha+\lambda+\mu+2)} 2^{4\nu-\alpha} x^{\alpha+\lambda+\mu+1-2\nu}\\ &\cdot \sum_{n=0}^{\infty} {}_5F_4\left[\begin{matrix} -n, n+2\nu, \nu+1, (\alpha+\mu+1)/2, (\alpha+\mu+2)/2; \\ \nu+1/2, \alpha+1, (\alpha+\lambda+\mu+2)/2, (\alpha+\lambda+\mu+3)/2\end{matrix}\right]\\ &\cdot \frac{(2\nu+1)_n}{n!}\frac{2n+2\nu}{n+2\nu} J^2_{n+\nu}\left(\frac{x}{2}\right),\end{aligned}$$

when $\alpha+\mu > -1$, $\lambda > -1$, and $2\nu \neq -1, 2, \cdots$. Here $(\nu)_n = \Gamma(n+\nu)/\Gamma(\nu)$, and the factor $(2n+2\nu)/(n+2\nu)$ must be replaced by 1 when $n = 0$.

To prove inequality (1.5) we set $\mu = \lambda + 1/2$ in (2.4). Then the above ${}_5F_4$ becomes

$$(2.5)\qquad {}_5F_4\left[\begin{matrix} -n, n+2\nu, \nu+1, (\alpha+\lambda+3/2)/2, (\alpha+\lambda+5/2)/2; \\ \nu+1/2, \alpha+1, (\alpha+2\lambda+5/2)/2, (\alpha+2\lambda+7/2)/2\end{matrix}\right].$$

Since it is not at all obvious when these ${}_5F_4$'s are positive, we now set $\nu = (\alpha + \lambda + 1/2)/2$ so that the ${}_5F_4$ in (2.5) reduces to the Saalschützian ${}_4F_3$ series

$$(2.6)\qquad {}_4F_3\left[\begin{matrix} -n, n+\alpha+\lambda+1/2, (\alpha+\lambda+5/2)/2, (\alpha+\lambda+5/2)/2; \\ \alpha+1, (\alpha+2\lambda+5/2)/2, (\alpha+2\lambda+7/2)/2\end{matrix}\right].$$

When $\lambda = \alpha - 1/2$ or $\lambda = 0$ this ${}_4F_3$ reduces to a ${}_3F_2$ series which can be summed by Saalschütz's formula [10, 2.2(1)]. It then follows from (2.4) that

$$\begin{aligned}(2.7)\quad &\int_0^x (x-t)^{\alpha-1/2} t^\alpha J_\alpha(t)\,dt\\ &= \frac{\Gamma(\alpha+1/2)\Gamma(2\alpha+1)\Gamma(\alpha+1)}{\Gamma(3\alpha+3/2)} 2^{3\alpha} x^{\alpha+1/2}\\ &\cdot \sum_{n=0}^{\infty} \frac{((2\alpha+1)/4)_n((2\alpha-1)/4)_n}{((6\alpha+3)/4)_n((6\alpha+5)/4)_n}\frac{(2\alpha+1)_n}{n!}\frac{2n+2\alpha}{n+2\alpha} J^2_{n+\alpha}\left(\frac{x}{2}\right), \qquad \alpha > -1/2,\end{aligned}$$

$$\begin{aligned}(2.8)\quad &\int_0^x t^{1/2} J_\alpha(t)\,dt\\ &= \frac{\Gamma^2((2\alpha+5)/4)}{(\alpha+3/2)\Gamma(\alpha+1)} 2^{\alpha+1} x\\ &\cdot \sum_{n=0}^{\infty} \frac{(1/2)_n((2\alpha-1)/4)_n}{(\alpha+1)_n((2\alpha+7)/4)_n}\frac{(\alpha+3/2)_n}{n!}\frac{2n+\alpha+1/2}{n+\alpha+1/2} J^2_{n+(2\alpha+1)/4}\left(\frac{x}{2}\right),\\ &\qquad \alpha > -\frac{3}{2}.\end{aligned}$$

Since the positive zeros of $J_\nu(x)$ are interlaced [39, p. 479] with those of $J_{\nu+1}(x)$, it is obvious from (2.7) and (2.8) that inequality (1.5) holds for $\alpha > 1/2$ when

$\lambda = 0$ or $\lambda = \alpha - 1/2$. Thus (1.4) follows directly from (2.7) and (1.6).

Even though we cannot sum the ${}_4F_3$ in (2.6) when $0 < \lambda < \alpha - 1/2$, we can use Whipple's transformation formula [10, 4.3(4)] to show that this ${}_4F_3$ equals

$$\frac{(\lambda + \frac{1}{2})_n((2\alpha - 2\lambda - 1)/4)_n}{(\alpha + 1)_n((2\alpha + 3\lambda + 7)/4)_n} \tag{2.9}$$
$$\cdot {}_7F_6\left[\begin{matrix} \frac{2\alpha + 6\lambda + 3}{4}, \frac{2\alpha + 6\lambda + 11}{8}, \frac{\lambda + 1}{2}, \frac{\lambda}{2}, \frac{2\alpha + 2\lambda + 5}{4}, n + \alpha + \lambda + \frac{1}{2}, -n; \\ \frac{2\alpha + 6\lambda + 3}{8}, \lambda + \frac{2\alpha + 5}{4}, \lambda + \frac{2\alpha + 7}{4}, \lambda + \frac{1}{2}, \frac{5 + 2\lambda - 2\alpha}{4} - n, n + \frac{2\alpha + 6\lambda + 7}{4} \end{matrix}\right],$$

which is clearly positive when $0 < \lambda < \alpha - 1/2$, since each of the $n + 1$ terms in this ${}_7F_6$ series is positive. This shows that if $0 < \lambda < \alpha - 1/2$ and $\nu = (\alpha + \lambda + 1/2)/2$ then the integral in (1.5) has an expansion of the form (1.8) with positive coefficients, which completes our proof of (1.5).

In the same fashion it can be shown that

$$\int_0^x (x - t)^\lambda t^{\lambda - 1/2} J_\alpha(t)\, dt > 0, \qquad 1 \leqq \lambda \leqq \alpha + 3/2, \quad \alpha > -1/2. \tag{2.10}$$

This gives (1.4) when $\lambda = \alpha + 3/2$. The restriction $\lambda \leqq \alpha + 3/2$ in (2.10) cannot be relaxed, for Askey [4] has shown that if $\varepsilon > 0$ and $\lambda > -1$, then the inequality

$$\int_0^x (x - t)^\lambda t^{\alpha + \varepsilon + 1} J_\alpha(t)\, dt \geqq 0$$

fails for some $x > 0$. Also, since

$$\int_0^x (x - t) t^{1/2} J_{-1/2}(t)\, dt = x\left(\frac{\pi}{2}\right)^{1/2} J_{1/2}^2\left(\frac{x}{2}\right),$$

the inequality (2.10) fails for infinitely many x when $\alpha = -1/2$, $\lambda = 1$. To prove (1.3) we set $\lambda = 2\alpha$ and $\mu = \alpha$ in (2.4) and observe that if $\nu = 2\alpha + 1/2$, then the ${}_5F_4$ in (2.4) reduces to a ${}_3F_2$ which can be summed by Watson's formula [10, 3.3(1)] to give

$$\begin{aligned} \int_0^x (x - t)^{2\alpha} t^\alpha J_\alpha(t)\, dt = {} & \frac{\pi\Gamma(4\alpha + 2)2^{-\alpha}}{\Gamma(\alpha + 1)} \\ & \cdot \sum_{n=0}^{\infty} \frac{(1/2)_n(\alpha + 1/2)_n}{(\alpha + 1)_n(2\alpha + 1)_n} \frac{(4\alpha + 2)_{2n}}{(2n)!} \frac{4n + 4\alpha + 1}{2n + 4\alpha + 1} \\ & \cdot J_{2n + 2\alpha + 1/2}^2\left(\frac{x}{2}\right), \end{aligned} \tag{2.11}$$

for $\alpha > -1/2$. The positivity of the integral in (2.11) for $\alpha > -1/2$ then follows from the positivity of the coefficients in (2.11) and the fact that the positive zeros of $J_\nu(x)$ are interlaced [39, p. 480] with those of $J_{\nu+2}(x)$. Similarly, to prove Cooke's inequality (1.1) it suffices to notice that from (2.4) and Watson's formula [10, 3.3(1)] we have

$$\begin{aligned}\int_0^x J_\alpha(t)\,dt &= \frac{\Gamma^2((\alpha+3)/2)2^{\alpha+2}}{\Gamma(\alpha+2)} \\ &\quad\cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{2})_n}{((\alpha+2)/2)_n((\alpha+2)/2)_n} \frac{(\alpha+2)_{2n}}{(2n)!} \frac{4n+\alpha+1}{2n+\alpha+1} \\ &\quad\cdot J^2_{2n+(\alpha+1)/2}\left(\frac{x}{2}\right) > 0, \qquad \alpha > -1. \end{aligned} \tag{2.12}$$

Steinig [36] has pointed out that (1.1) also follows directly from Bailey's [13, (3.4)] identity

$$\int_0^{2x} J_{2\alpha}(t)\,dt = 2x \int_0^{\pi/2} J_\alpha^2(x \sin\theta)\sin\theta\,d\theta, \qquad \alpha > -\tfrac{1}{2}. \tag{2.13}$$

In § 3 we shall derive the identity

$$\begin{aligned}&\int_0^x (x-t)^{2\alpha} t^\alpha J_\alpha(t)\,dt \\ &= 2^{3\alpha+1}\Gamma(\alpha+1)\int_0^{x/2} ((x/2)^2 - t^2)^{\alpha-1/2} t J_\alpha^2(t)\,dt, \qquad \alpha > -\tfrac{1}{2},\end{aligned} \tag{2.14}$$

which gives yet another proof of the positivity of the integrals in (1.3). So far I have not been able to obtain such a representation in terms of an integral of a square of a Bessel function for the integrals in (1.4) and (1.5). However, in this respect it is of interest to note that from Askey [3] we do have an integral representation of the form

$$\int_0^x (x-t)^2 t^{3/2} J_{1/2}(t)\,dt = 4\left(\frac{2}{\pi}\right)^{1/2} \int_0^x (1-\cos t)(1-\cos(x-t))\,dt, \tag{2.15}$$

which gives (1.4) for the special case $\alpha = 1/2$. A limiting case of Conjecture 4 in Askey and Gasper [7] is the conjecture that

$$\int_0^x (x-t)^{\alpha+2\mu-1/2} t^{\alpha+\mu} J_\alpha(t)\,dt \geqq 0 \tag{2.16}$$

when $0 \leqq \mu \leqq 1$ and $\alpha + \mu \geqq 1/2$. Since this integral reduces to the integral (1.4) when $\mu = 1$ and to the case $\lambda = \alpha - 1/2$ of the integral (1.5) when $\mu = 0$, we have already proved (2.16) for $\mu = 0, \alpha \geqq 1/2$ and for $\mu = 1, \alpha \geqq -1/2$, which are the lower and upper boundary lines of the set $S = \{(\alpha,\mu): 0 \leqq \mu \leqq 1, \alpha + \mu \geqq 1/2\}$. Replacing λ and μ in (2.4) by $\alpha + 2\mu - 1/2$ and $\alpha + \mu$ and setting

$\nu = \alpha + \mu$, so that the ${}_5F_4$ in (2.4) becomes the Saalschützian series

$$(2.17)\qquad {}_5F_4\left[\begin{matrix} -n, n+2\alpha+2\mu, \alpha+\mu+1, \dfrac{2\alpha+\mu+1}{2}, \dfrac{2\alpha+\mu+2}{2}; \\ \alpha+\mu+\dfrac{1}{2}, \alpha+1, \dfrac{3\alpha+3\mu+3/2}{2}, \dfrac{3\alpha+3\mu+5/2}{2} \end{matrix}\right],$$

we find that in order to prove (2.16) for any (α, μ) in S it suffices to prove the non-negativity of (2.17) for each n. So far, in addition to the cases $\mu = 0$ and $\mu = 1$ discussed above the only other case I have been able to handle is the case $0 < \mu < 1$, $\alpha + \mu = 1/2$, which is the remaining part of the boundary of S. In this case the series (2.17) reduces to the ${}_4F_3$ series

$$(2.18)\qquad {}_4F_3\left[\begin{matrix} -n, n+1, (2-\mu)/2, (3-\mu)/2; \\ 1, 3/2-\mu, 2 \end{matrix}\right],$$

which, by the limiting case $d \to a + 1$ of Whipple's formula [10, 4.3(4)], is equal to

$$(2.19)\qquad \begin{aligned} &\frac{\mu(1-\mu)(3-\mu)(5-\mu)}{2(3-2\mu)(2n+\mu-1)(2n+3-\mu)} \frac{((1+\mu)/2)_n}{((3-\mu)/2)_n} \\ &\cdot {}_7F_6\left[\begin{matrix} \dfrac{3-\mu}{2}, \dfrac{9-\mu}{4}, \dfrac{3-\mu}{2}, \dfrac{2+\mu}{2}, \dfrac{5-\mu}{2}, n+2, 1-n; \\ \dfrac{5-\mu}{4}, 2, \dfrac{5-2\mu}{2}, 2, \dfrac{3-\mu}{2}-n, \dfrac{5-\mu}{2}+n \end{matrix}\right] \end{aligned}$$

for $n = 1, 2, \cdots$. From this it is obvious that the series in (2.18) are positive for $0 < \mu < 1$, which implies the positivity of the integrals (2.16) for $0 < \mu < 1$, $\alpha + \mu = 1/2$. A reasonable conjecture is that the ${}_5F_4$ series in (2.17) are also positive for (α, μ) in the interior of S (which would then imply the positivity of the integrals (2.16) for $0 < \mu < 1, \alpha + \mu > 1/2$), but another method will have to be used to prove this since Whipple's formula [10, 4.3(4)] is not applicable to (2.17) in this case. Some other methods for proving positivity of generalized hypergeometric series will be mentioned in § 3.

By using the formula [18, 13.1(65)]

$$(2.20)\qquad \begin{aligned} &\int_0^x (x^2-t^2)^\lambda t^\mu J_\alpha(t)\,dt \\ &= \frac{\Gamma(\lambda+1)\Gamma((\alpha+\mu+1)/2)x^{\alpha+\mu+2\lambda+1}}{\Gamma(\alpha+1)\Gamma((\alpha+\mu+2\lambda+3)/2)2^{\alpha+1}}\, {}_1F_2\left[\begin{matrix} \dfrac{\alpha+\mu+1}{2}; & -\dfrac{x^2}{4} \\ \alpha+1, & \dfrac{\alpha+\mu+2\lambda+3}{2} \end{matrix}\right], \\ &\qquad\qquad \alpha+\mu > -1, \qquad \lambda > -1, \end{aligned}$$

and (2.3) one can also obtain expansions of the type (1.8) for the above integrals and then use the expansions to determine values of λ, μ, α for which these integrals are positive. However, since many other integrals (and functions [17], [18], [28]) are multiples of ${}_pF_q$ functions, rather than considering each type of integral separately it is clearly preferable to consider the general case of expansions of the form (1.8) for ${}_pF_q$ functions, as we shall do in the next section. In particular, from formulas (3.2) and (3.3) below it is easily seen that the integral (2.20) is positive in the following four cases:

(i) $\lambda = \alpha - 1/2 > -1, \alpha > \mu, \alpha + \mu > -1$,
(ii) $\mu = \lambda + 1/2 > 0, \alpha > \lambda + 1/2$,
(iii) $\alpha > -1, \lambda > -1/2, \mu = 0$,
(iv) $2\lambda = \alpha + \mu - 1, \alpha > \mu, \alpha + \mu > -1$.

3. ${}_pF_q$ functions. One can use either (2.3) or [28, 9.1(13)] to find that

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; -x^2y \\ b_1, \cdots, b_q \end{matrix}\right] &= \Gamma^2(v+1)\left(\frac{2}{x}\right)^{2v} \sum_{n=0}^{\infty} \frac{(2v+1)_n}{n!} \frac{2n+2v}{n+2v} J_{n+v}^2(x) \\
&\cdot {}_{p+3}F_{q+1}\left[\begin{matrix} -n, n+2v, v+1, a_1, \cdots, a_p; y \\ v+1/2, b_1, \cdots, b_q \end{matrix}\right]
\end{aligned} \tag{3.1}
$$

when $2v$ is not a negative integer and either $p \leqq q$, or $p = q + 1$ and $|x^2y| < 1$. As usual, it is assumed that no denominator parameter b_j is a negative integer or zero.

Setting $p = 2, q = 3, a_1 = a, a_2 = v + 1/2, b_1 = b, b_2 = v + 1, b_3 = 2v + a + 1 - b, y = 1$, and using Saalschütz's formula, we obtain

$$
\begin{aligned}
&{}_2F_3[a, v + 1/2; b, v + 1, 2v + 1 + a - b; -x^2] \\
&= \Gamma^2(v+1)\left(\frac{2}{x}\right)^{2v} \sum_{n=0}^{\infty} \frac{(b-a)_n(2v+1-b)_n}{(b)_n(2v+1+a-b)_n} \frac{(2v+1)_n}{n!} \frac{2n+2v}{n+2v} J_{n+v}^2(x).
\end{aligned} \tag{3.2}
$$

Since the coefficients in the expansion (3.2) are positive for $v > -1/2$ under each of the following conditions,

(i) $0 < b < 2v + 1, 0 < b - a < 2v + 1$,
(ii) $-j - 1 \leqq 2v + 1 - b < -j, -j - 1 < 2v + 1 + a - b < -j$,
(iii) $-j - 1 \leqq 2v + 1 - b < -j, -j - 1 \leqq b - a < -j$,
(iv) $-j - 1 < 2v + 1 + a - b < -j, -j - 1 < b < -j$,
(v) $-j - 1 \leqq b - a < -j, -j - 1 < b < -j$,

where j is any nonnegative integer, it is obvious that the ${}_2F_3$ in (3.2) is positive when $v > -1/2$ and one of the above conditions is satisfied. In particular, formula (2.2) and the case $a = v + 1$ of (3.2) gives (1.5) when $\lambda = 0$ or $\lambda = \alpha - 1/2$.

When $p = 1, q = 2, a_1 = a, b_1 = 2a, b_2 = v + 1$ and $y = 1$, application of Watson's formula [10, 3.3(1)] to (3.1) gives (also see [28, 9.4.7(16)])

$$
\begin{aligned}
&{}_1F_2[a; 2a, v + 1; -x^2] \\
&= \Gamma^2(v+1)\left(\frac{2}{x}\right)^{2v} \sum_{n=0}^{\infty} \frac{(v+1/2-a)_n}{(a+1/2)_n} \frac{(v+1)_n}{n!} \frac{2n+v}{n+v} J_{2n+v}^2(x), \\
&> 0, \qquad v + 1/2 > a > -1/2,
\end{aligned} \tag{3.3}
$$

where for $a = 0$ the right-hand side of (3.3) is the expansion of

$$
\lim_{a \to 0} {}_1F_2[a; 2a, v + 1; -x^2] = 1 - \frac{x^2}{2(v+1)} {}_1F_2[1; 2, v + 2; -x^2]. \tag{3.4}
$$

Inequalities (1.1) and (1.3) follow from (2.2) and (3.3). Also, Theorem 1 in Fields and Ismail [20] follows directly from (3.2) and (3.3).

We will now show how (3.2) leads to a simple proof of the previously mentioned result of Steinig concerning the positivity of the Lommel function $s_{\mu,\nu}(x)$. Steinig's first step in [37] was to use Makai's oscillation theorem for second order differential equations to prove that

$$s_{1/2,\nu}(x) > 0, \qquad x > 0, \quad -1/2 < \nu < 1/2. \tag{3.5}$$

To prove this by the use of expansions of the form (1.8), it suffices to observe that from (1.2), (2.20) and (3.2) we have

$$s_{1/2,\nu}(x) = \frac{4\pi x^{1/2}}{9-4\nu^2}\sum_{n=0}^{\infty}\frac{((1-2\nu)/4)_n((1+2\nu)/4)_n}{((7-2\nu)/4)_n((7+2\nu)/4)_n}(2n+1)J^2_{n+1/2}\left(\frac{x}{2}\right). \tag{3.6}$$

Then, as in [37], it follows by using the fractional integrals on page 127 of [37] (or use (3.15) below) that $s_{\mu,\nu}(x) > 0$ for $x > 0$ if $\mu = 1/2$ and $-1/2 < \nu < 1/2$, or if $\mu > 1/2$ and $-\mu \leqq \nu \leqq \mu$.

Equivalent forms of (3.2) and (3.3) were derived in Bailey [11] by using the formula

$$J_\nu^2(x) = \{\Gamma(\nu+1)\}^{-2}(x/2)^{2\nu}\ {}_1F_2[\nu+1/2; 2\nu+1, \nu+1; -x^2] \tag{3.7}$$

to sum the series on the right-hand sides of (3.2) and (3.3). This approach also yields (see Rice [32, (4.1)])

$$\begin{aligned} &{}_3F_2[a, a+1/2; 2a, \nu+1, \nu+1; -x^2] \\ &\quad = \Gamma^2(\nu+1)\left(\frac{2}{x}\right)^{2\nu}\sum_{n=0}^{\infty}\frac{(2\nu+1-2a)_n}{n!}J^2_{n+\nu}(x) \\ &\quad > 0, \qquad \nu+\tfrac{1}{2} > a. \end{aligned} \tag{3.8}$$

Also see Bailey [12] for some expansions in Neumann series and Kapteyn series of the second kind.

Note that by comparing (3.1) and (3.8) we obtain the summation formula

$${}_4F_3\left[\begin{matrix}-n, n+2\nu, a, a+1/2;\\ \nu+1/2, \nu+1, 2a\end{matrix}\right] = \frac{(\nu)_n(2\nu+1-2a)_n}{(\nu+1)_n(2\nu)_n}. \tag{3.9}$$

This formula can also be derived by using known transformation formulas.

To further extend our positivity results we first observe that from (3.1) we have

$$\begin{aligned} &{}_3F_4\left[\begin{matrix}\alpha, \beta, \nu+1/2; -x^2\\ \gamma, \lambda, \mu, \nu+1\end{matrix}\right] \\ &\quad = \Gamma^2(\nu+1)\left(\frac{2}{x}\right)^{2\nu}\sum_{n=0}^{\infty}\frac{(2\nu+1)_n}{n!}\frac{2n+2\nu}{n+2\nu}\,{}_4F_3\left[\begin{matrix}-n, n+2\nu, \alpha, \beta;\\ \gamma, \lambda, \mu\end{matrix}\right]J^2_{n+\nu}(x). \end{aligned} \tag{3.10}$$

The above ${}_4F_3$ series is a Saalschützian when $\mu = \alpha + \beta + 2\nu + 1 - \gamma - \lambda$; and so it follows from Whipple's formula [10, 4.3 (4)] that

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} -n, n + 2\nu, \alpha, \beta; \\ \gamma, \lambda, \alpha + \beta + 2\nu + 1 - \gamma - \lambda \end{matrix}\right] \\
(3.11)\qquad &= \frac{(\gamma + \lambda - \alpha - \beta)_n (1 + \beta + 2\nu - \gamma - \lambda)_n}{(\gamma + \lambda - \beta)_n (1 + \alpha + \beta + 2\nu - \gamma - \lambda)_n} \\
&\cdot {}_7F_6\left[\begin{matrix} a, 1 + a/2, \lambda - \beta, \gamma - \beta, \alpha, n + 2\nu, -n; \\ a/2, \gamma, \lambda, \gamma + \lambda - \alpha - \beta, \gamma + \lambda - \beta - 2\nu - n, \gamma + \lambda - \beta + n \end{matrix}\right],
\end{aligned}
$$

where $a = \gamma + \lambda - \beta - 1$. From (3.10) and (3.11) we find that

$$
(3.12)\qquad {}_3F_4\left[\begin{matrix} \alpha, \beta, \nu + 1/2; -x^2 \\ \gamma, \lambda, \alpha + \beta + 2\nu + 1 - \gamma - \lambda, \nu + 1 \end{matrix}\right] > 0,
$$

when $2\nu + 1 > \gamma + \lambda - \beta > \alpha \geqq 0, \lambda \geqq \beta, \gamma \geqq \beta, \gamma > 0, \lambda > 0$.

Among the other summation formulas which can be used to sum the coefficients in (3.1), we shall here only consider the formula

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} -n, n + 2\nu, b, c + k; \\ 2\nu + 1, b + 1, c \end{matrix}\right] \\
(3.13)\qquad &= \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{n!(2\nu + 1 - b)_{n-1}(c - b)_k}{(b + 1)_n (2\nu + 1)_{n-1}(c)_k} & \text{if } n > 0 \text{ and } k = 0 \text{ or } 1. \end{cases}
\end{aligned}
$$

This formula is a special case of formula (12) in Minton [31] (see Luke [28, 3.13.3 (39)] for the case $k = 0$ of (3.13) and also see Karlsson [23, (10)]). Use of (3.1) and (3.13) gives

$$
\begin{aligned}
{}_3F_4\left[\begin{matrix} \nu + 1/2, b, c + k; -x^2 \\ \nu + 1, 2\nu + 1, b + 1, c \end{matrix}\right] &= \Gamma^2(\nu + 1)\left(\frac{2}{x}\right)^{2\nu} J_\nu^2(x) \\
(3.14)\qquad &+ 2\Gamma^2(\nu + 1)\left(\frac{2}{x}\right)^{2\nu} \frac{(c - b)_k}{(c)_k} \\
&\cdot \sum_{n=1}^{\infty} \frac{(2\nu + 1 - b)_{n-1}}{(b + 1)_n}(n + \nu) J_{n+\nu}^2(x),
\end{aligned}
$$

which is obviously positive if $k = 1, 2\nu + 1 > b > -1, c > b, c > 0$, or if $k = 0$, $2\nu + 1 > b > -1$.

Besides using summation formulas and transformation formulas (see, e.g., [10], [17], [28], [34]) to determine cases in which the coefficients in (3.1) are positive, one can also use expansion formulas [28, Chap. IX] and recurrence relations (see [6] and [21]). In addition, the above positivity results for ${}_pF_{p+1}$

functions can be extended by using the fractional integral [18, 13.1 (95)]

$$\int_0^x (x-t)^{\mu-1}t^{\nu-1}\,{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p;at\\ b_1,\cdots,b_q\end{matrix}\right]dt = \frac{\Gamma(\mu)\Gamma(\nu)x^{\mu+\nu-1}}{\Gamma(\mu+\nu)}\,{}_{p+1}F_{q+1}\left[\begin{matrix}a_1,\cdots,a_p,\nu;ax\\ b_1,\cdots,b_q,\mu+\nu\end{matrix}\right], \tag{3.15}$$

where $\mu > 0$, $\nu > 0$ and either $p \leqq q$ or $p = q + 1$ and $|ax| < 1$.

From each of the above results of the form (for all x)

$${}_pF_{p+1}\left[\begin{matrix}a_1,\cdots,a_r,a_{r+1},\cdots,a_p;-x^2\\ b_1,\cdots,b_s,b_{s+1},\cdots,b_{p+1}\end{matrix}\right] > 0,$$

with $a_1,\cdots,a_r,b_1,\cdots,b_s > 0$, it follows by repeated applications of (3.15) that we also have

$${}_{p+q}F_{p+q+1}\left[\begin{matrix}a_1-\varepsilon_1,\cdots,a_r-\varepsilon_r,a_{r+1},\cdots,a_p,\nu_1,\cdots,\nu_q;-x^2\\ b_1+\delta_1,\cdots,b_s+\delta_s,b_{s+1},\cdots,b_{p+1},\mu_1+\nu_1,\cdots,\mu_q+\nu_q\end{matrix}\right] > 0,$$

provided that each ε_i, δ_i, ν_i, μ_i is positive and $a_1-\varepsilon_1,\cdots,a_r-\varepsilon_r$ are positive. In particular, due to (2.2), from each result in § 2 of the form

$$\int_0^x (x-t)^\lambda t^\mu J_\alpha(t)\,dt > 0, \qquad x > 0, \tag{3.16}$$

with $\alpha > -1$, it follows that if $\varepsilon, \delta, \gamma \geqq 0$ and $\alpha + \mu - \varepsilon > -1$, then we also have

$$\int_0^x (x-t)^{\lambda+\gamma+\varepsilon}t^{\mu-\varepsilon-\delta}J_{\alpha+\delta}(t)\,at > 0, \qquad x > 0. \tag{3.17}$$

Similarly, due to (2.20), from each result of the form

$$\int_0^x (x^2-t^2)^\lambda t^\mu J_\alpha(t)\,dt > 0, \qquad x > 0,$$

with $\alpha > -1$, we also have

$$\int_0^x (x^2-t^2)^{\lambda+\gamma+\varepsilon}t^{\mu-2\varepsilon-\delta}J_{\alpha+\delta}(t)\,dt > 0, \qquad x > 0,$$

when $\varepsilon, \delta, \gamma \geqq 0$ and $\alpha + \mu - 2\varepsilon > -1$. These observations enable us to compare various inequalities; since, for example, from the fact that (3.16) implies (3.17) we see that the case $\alpha > 1/2$ of (1.3) follows from the case $\lambda = \alpha - 1/2 > 0$ of (1.5), so that it is then clear that (1.5) is a stronger inequality than (1.3) when $\alpha > 1/2$.

Note that application of (3.15) to (3.7) gives

$${}_1F_2[\nu+1/2;2\nu+1,\nu+\mu+1;-x^2] > 0, \qquad \nu > -1, \quad \mu > 0, \tag{3.18}$$

which implies the positivity of the ${}_1F_2$ functions in (3.3). This also provides an example of one of the advantages of representing integrals (and other functions) in terms of generalized hypergeometric functions. For from (2.2) it is seen that the integral in (1.3) is a multiple of the ${}_1F_2$ function in (3.18) with $\nu = \alpha$ and

$\mu = \alpha + 1/2$; so that using (3.15) and (3.7) as above we are led directly to formula (2.14).

Since a function $f(x)$, $x > 0$, is completely monotonic if it is the Laplace transform of a nonnegative measure [40, p. 161], one can use the above positivity results and the Laplace transformation formulas in [18, Chap. IV] to obtain some new complete monotonicity results for ${}_pF_q$ functions. For example, after correction of some misprints in [18, 4.23 (18)], we have

$$\begin{aligned}(3.19)\quad &\int_0^\infty e^{-xt}t^{2\sigma-1}\,{}_pF_{p+1}\left[\begin{matrix}a_1,\cdots,a_p;\ -t^2/4\\ b_1,\cdots,b_{p+1}\end{matrix}\right]dt\\ &= \Gamma(2\sigma)x^{-2\sigma}\,{}_{p+2}F_{p+1}\left[\begin{matrix}a_1,\cdots,a_p,\sigma,\sigma+1/2;\ -x^{-2}\\ b_1,\cdots,b_{p+1}\end{matrix}\right],\qquad \sigma > 0,\end{aligned}$$

which can be applied to the above positivity results to obtain extensions of the complete monotonicity results in [20]. In particular, from (3.19), (2.2) and the case $0 \leqq \mu \leqq 1$, $\alpha + \mu = 1/2$ of (2.16) it follows that the functions

$$x^{-3}\,{}_2F_1[(\alpha + 3/2)/2, (\alpha + 5/2)/2; \alpha + 1; -x^{-2}], \qquad -1/2 \leqq \alpha \leqq 1/2,$$

are completely monotonic. Note that both of the end point cases $\alpha = \pm 1/2$ are equivalent to the fact that $x^{-1}(x^2 + 1)^{-1}$ is completely monotonic.

The methods of this paper have been mainly directed at those positivity results which seem to be the most useful (in view of the mentioned applications) and to which they are most applicable. They can also be used to handle some special cases of the general problem of when

$$(3.20)\qquad \int_0^x (x^\nu - t^\nu)^\lambda t^\mu J_\alpha(t)\,dt \geqq 0, \qquad x > 0,$$

which is suggested by (1.12); but one could not expect to use expansions of the form (1.8) to completely determine, as is done in Askey and Steinig [9] and Makai [30], the values of (α, μ) for which (3.20) holds when $\lambda = 0$ and $-1 < \alpha < 1/2$.

In a subsequent paper we shall use analogues of the methods of this paper to give simpler proofs of the positivity of many of the kernels considered in [1], [5], [7], [8], [19] and to prove the positivity of some other important kernels involving orthogonal polynomials.

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Note added in proof. In a paper now in preparation it will be shown that the ${}_5F_4$ series in (2.17) are positive when $0 < \mu < 1$, $\alpha + \mu < 1/2$. For formula (3.9), also see Carlitz, Boll. Un. Mat. Ital., 18 (1963), pp. 90–93.

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DUALITY THEORY FOR *n*th ORDER DIFFERENTIAL OPERATORS UNDER STIELTJES BOUNDARY CONDITIONS*

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Abstract. The adjoint of an nth order vector-valued linear differential system with boundary conditions represented by singular matrix-valued measures is constructed when the system is viewed as an operator with domain and range in a space of L^p integrable functions. Both the operator and its adjoint are shown to be normally solvable. The theory is then applied to the multipoint boundary value problem of Wilder, and some examples are discussed.

1. Introduction. Interest in vector differential systems of the form

$$
\begin{aligned}
& y' + Ay = f, \\
& U(y) = \int_a^b dvy = k
\end{aligned}
\tag{1.1}
$$

goes back to W. M. Whyburn [28], [29], and Mansfield [15] who first systematically studied them.

In the last ten years Cole [5], Bryan [3], Tucker [26], Halanay and Moro [9], Krall [11], Green and Krall [8], Brown [1], Vejvoda and Tvrdy [27] and other writers have made new contributions to the subject.

In many of these articles Green's functions are constructed and adjoint systems are derived or defined for a variety of special cases of (1.1).

In [1], for example, we have studied (1.1) with $k = 0$ in the context of operator theory. If $k = 0$, the system was shown to generate an unbounded closed operator L in the space of n-dimensional L^p integrable functions on a domain prescribed by the functional U. Necessary and sufficient conditions were found determining the existence of the adjoint in the dual space $\mathscr{L}_n^q$ $(1/q + 1/p = 1)$. When these conditions were satisfied, the adjoint operator was constructed; its nullspace was found to correspond exactly to the adjoint system defined, for example, in [9]. Furthermore, both the operator and its adjoint were shown to be normally solvable Fredholm operators with mutually orthogonal ranges and nullspaces, and the index was calculated. Finally, the spectrum of both operators was found to consist only of eigenvalues, and an estimate on their distribution was given.

A natural generalization of (1.1) would be to the higher order "generalized" boundary value problem

$$
\begin{aligned}
& l(y) = \sum_{i=0}^{n} A_i y^{(n-i)}, \\
& U_j(y, \cdots, y^{(n-1)}) = \sum_{i=0}^{n} \int_a^b dw_{ij} y^{(n-i)} = 0, \qquad j = 1, \cdots, q,
\end{aligned}
\tag{1.2}
$$

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where the A_i are $m \times m$ matrices satisfying certain regularity conditions and the w_{ij} are $p \times m$ matrix valued (m.v.) measures of bounded variation.

Even though the general problem was considered (in scalar form) long before (1.1), it has received on the whole less attention. Between 1908 and 1940, Picone [19], Wilder [30], Cioránescu [4], Toyada [25] and Smogorshewsky [23] constructed Green's functions for particular examples. Only Wilder and Toyada seem to have considered the question of adjoint systems. Wilder studied a scalar version of the problem under the multipoint boundary conditions,

$$U_i(y, \cdots, y^{(n-1)}) = \sum_{k=0}^{r} \sum_{j=1}^{n} c_{ij}^{(k)} y^{(n-j)}(t_k) = 0, \qquad i = 1, \cdots, n. \tag{1.3}$$

He pointed out that an adjoint in the conventional sense would not exist because of the discontinuities introduced by the interior points, and that therefore classical self-adjointness was impossible. These facts were also independently arrived at by Toyada who seems to have been unfamiliar with Wilder's work.

More recently the lack of self-adjointness for higher order problems with boundary conditions at interior points has been reconfirmed by Neuberger [17], Loud [12], and Zettl [31], [32].

All of these writers based their work on a careful analysis of the Green's function, and hence did not consider compatible systems. Also, explicit use was made of the finiteness of the interior points, and although Loud in particular constructed adjoint boundary conditions for certain examples of the Wilder type, a precise description of the adjoint system in the general case (1.2) was not achieved.

In this paper we shall study the adjoint theory of (1.2) by extending the point of view of [1] to the higher order case. Section 3 contains the most significant results of the paper. There, an adjoint operator for (1.2) will be constructed in the setting of L_p space and its properties discussed. As in [1] the ranges and nullspaces of both the operator and its adjoint will be shown to be mutually orthogonal. We then (§ 4) apply our theory to Wilder's conditions (1.3) and illustrate it with some simple examples. Section 5 outlines certain mathematical and physical applications of generalized boundary value problems and suggests some new directions for future research.

The reader may wonder perhaps why we regard (1.2) in a vector setting. The theory, however, is no more difficult (aside from some notational complication) to develop from this standpoint than from the scalar one, and we find it useful for certain applications. The real difficulties lie rather in the transition from the matrix system (1.1) to (1.2)—in either scalar or vector form. While it is true that considered as equations, (1.2) can be converted to one similar to (1.1), the operator adjoints of the two systems are essentially different in structure. Our treatment therefore will require the introduction of several new techniques; it will also be necessary to sacrifice some of the generality of [1] by making certain restrictions on the measures w_{ij}.

2. Preliminaries and notation. Before proceeding further we observe that the system (1.2) is equivalent to the system

$$l(y) = \sum_{i=0}^{n} A_i y^{(n-i)},$$

(2.1)

$$U(y, \cdots, y^{(n-1)}) = \sum_{i=1}^{n} \int_a^b dv_i y^{(n-i)} = 0,$$

where v_i is the $l \times m$ m.v. measure

(2.2)
$$\begin{bmatrix} w_{i1} \\ \vdots \\ w_{iq} \end{bmatrix}$$

where $l = pq$. Henceforth therefore, we will deal exclusively with the system (2.1) containing the single functional U.

The possibility that the measures w_i may be nonsingular requires the introduction of artificial smoothness conditions on their absolutely continuous parts, and recourse to a more complex duality theory than is readily at hand. For these reasons we shall assume throughout the paper that the measures w_i are singular; i.e., their support consists only of sets of zero Lebesgue measure.

For more details of a theory of integration of vector-valued functions with respect to m.v. measures we refer the reader to [1].

Suppose $v_1, \cdots, v_n$ are $l \times m$ measures, or matrices. Then the symbol $\tilde{v}$ will denote the $ln \times m$ measure or matrix

(2.3)
$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

and the symbol $\bar{v}$ will denote the $l \times mn$ measure or matrix $(v_n, \cdots, v_1)$. If P is a point, $\mu_{(P)}$ will signify "point mass" measure; that is, $\mu_{(P)}[E] = 1$ if $P \in E$, and $U_{(P)}[E] = 0$ otherwise. It is evident that with this notation we can write an "atomic" m.v. measure v supported at the points $P_1, \cdots, P_r$ as

(2.4)
$$v = \sum_{i=1}^{r} v[P_i]\mu_{(P_i)}.$$

Where necessary (§ 3) we will assume familiarity with the basic theory of linear unbounded operators, particularly with the notions of the *adjoint* and closure of an operator.

If T is a linear operator, $D(T)$, $R(T)$, $N(T)$ will stand for its domain, range and nullspace respectively. T^* will denote the conjugate transpose, dual, or adjoint of a matrix, space or operator depending on the context. We will represent the identity operator on the space X by the symbol I_X and the $n \times n$ identity matrix by I_n. $\mathbb{C}^n$ will denote n-dimensional space over the complex field under the standard Euclidean norm. Finally the notation $\lambda[E](t)$ will stand for the characteristic function acting on the set E.

The setting of this paper is the Banach space $\mathscr{L}_n^p$ consisting of all n-dimensional vector-valued functions with support in $[a, b]$ under the norm

$$\|x\|_p = \left[\int_a^b \left[\sum_{i=1}^n |x_i(t)|^2\right]^{p/2} dt\right]^{1/p} = \left(\int_a^b (x^*x)^{p/2}\right)^{1/p}. \tag{2.5}$$

$\mathscr{L}_n^p$ can be shown to have the usual properties of other complex L^p spaces.

We denote by $\mathscr{A}_m^n$ the class of m-dimensional vector-valued functions f such that f^{n-1} exists and is absolutely continuous. We define

$$\mathscr{D}_m^{pn} = \{f : f \in \mathscr{A}_m^n ; f^{(n)} \in \mathscr{L}_m^p\}, \tag{2.6}$$

and

$$\mathscr{D}_m^{pn[U]} = \{f : f \in \mathscr{D}_m^{pn} \cap N(U)\}. \tag{2.7}$$

When there is no possibility of confusion we will often write $\mathscr{A}_m^n$, $\mathscr{L}_n^p$, $\mathscr{D}_m^{pn}$, or $\mathscr{D}_m^{pn[U]}$ as $\mathscr{A}$, $\mathscr{L}$, $\mathscr{D}$ or $\mathscr{D}^{[U]}$. Also when the indices n, p, or m are unity, they will be omitted.

We define the injection $\hat{}$ of an n-fold differentiable function y in $\mathscr{L}_m^p$ into $\mathscr{L}_{mn}^p$ by $\hat{y} = (y, y^{(1)}, \cdots, y^{(n-1)})^t$. It is easy to see that $\hat{}$ is an injection of $\mathscr{D}_m^{pn[U]}$ into $\mathscr{D}_{mn}^{p[\bar{U}]}$, where $\bar{U}$ is the functional $\int_a^b d\bar{v}\, z$.

We define the operator L' on $\mathscr{L}_m^p$ generated by the system (2.1) to be

$$l(y) = \sum_{i=0}^n A_i y^{n-i} \tag{2.8}$$

on the domain $\mathscr{D}_m^{pn}$, and the operator L as the restriction of L' to $\mathscr{D}_m^{pn[U]}$.

We will assume that the operators L and L' are *regular*, that is, that the matrices A_i are in class $\mathscr{C}^{(n-i)}$ and in particular that the matrix A_0 is nonsingular on $[a, b]$.

We associate with L two other operators $\tilde{L}$ and $\hat{L} \subset \tilde{L}$ on $\mathscr{L}_{mn}^p$. Let

$$\tilde{l}(y) = Cy' + Dy, \tag{2.9}$$

where C is the $mn \times mn$ matrix

$$\begin{bmatrix} I_m & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{bmatrix} \tag{2.10}$$

and D is the $mn \times mn$ matrix

$$\begin{bmatrix} 0 & -I_m & 0 & \cdots & 0 \\ 0 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & & -I_m \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 \end{bmatrix}. \tag{2.11}$$

Then $\tilde{L}(y)$ is given by $\tilde{l}(y)$ on the domain

$$\tilde{\mathscr{D}} = \{y : y \in \mathscr{D}^{p[\bar{U}]}_{mn}\}, \tag{2.12}$$

and $\hat{L}(y)$ is the restriction of $\tilde{L}$ to the domain

$$\hat{\mathscr{D}} = \{y : y = \hat{z} \text{ and } z \in \mathscr{D}^{pn[U]}_{m}\}. \tag{2.13}$$

We call $\tilde{L}$ the *first order operator associated with* L, and $\hat{L}$ the *restricted first order operator associated with* L. The following diagram illustrates the relation between L, $\tilde{L}$ and $\hat{L}$.

$$\begin{tikzcd}
\mathscr{D}^{[U]} \arrow[r, "l"] \arrow[d, "\wedge"'] & \mathscr{L}^{p}_{m} \arrow[d, "j"] \\
\hat{\mathscr{D}} \arrow[r, "\tilde{l}"] \arrow[d, "i"'] & \mathscr{L}^{p}_{mn} \arrow[d, "i"] \\
\tilde{\mathscr{D}} \arrow[r, "\tilde{l}"] & \mathscr{L}^{p}_{mn}
\end{tikzcd} \tag{2.14}$$

$$j(v) = (0, 0, \cdots, 0, v)^t, \qquad i = \text{inclusion}.$$

We close this section by reviewing several results about L and its adjoint proved in [1] for the first order case (1.1).

THEOREM 2.1. *If the measure v is singular, then the domain of L is dense in $\mathscr{L}^{p}_{m}$, $1 \leqq p < \infty$. Therefore the adjoint of L exists as a well-defined operator L^+ in $\mathscr{L}^{q}_{m}$, $1/p + 1/q = 1$. L^+ is given by*

$$l^+(z) = -z' + A^*z \tag{2.15}$$

on the domain

$$\begin{aligned} \mathscr{D}^+ = \bigcup_{\phi \in \mathbb{C}^l} \{z : z(t) + v^*[0,t]\phi \in \mathscr{A}_m;\ z(a^+) = -v^*[a]\phi;\ z(b^-) \\ = v^*[b]\phi;\ l^+(z) \text{ exists a.e. in } \mathscr{L}^{q}_{m}\}. \end{aligned} \tag{2.16}$$

For $p = \infty$, $L = (L^+)^*$, even though $L^+ \neq L^*$.

THEOREM 2.2. *For $1 \leqq p \leqq \infty$, L and L^+ are normally solvable; that is, they are closed operators with closed ranges. Furthermore for $1 \leqq p < \infty$,*

$$R(L^+) = N(L)^{\perp}; \tag{2.17}$$

if $1 < p \leqq \infty$,

$$R(L) = N(L^+)^{\perp}; \tag{2.18}$$

if $p = \infty$,

$$R(L^+) = {}^{\perp}N(L); \tag{2.19}$$

if $p = 1$,

$$R(L) = {}^{\perp}N(L^+).[1] \tag{2.20}$$

3. The adjoint of L. In this section we define an operator L^+ and prove generalizations of Theorems 2.1 and 2.2.

3.1. The partial adjoints and the matrix $\mathscr{B}$. Corresponding to the regular differential expression $l(y)$ we define the $(n + 1)$ partial adjoint expressions l_j^+, $j = 0, 1, \cdots, n$, as follows:

$$\begin{aligned} l_0^+(z) &= A_0^* z, \\ &\vdots \\ l_j^+(z) &= \sum_{i=0}^{j} (-1)^{j-1}(A_i^* z)^{(j-i)}, \\ &\vdots \\ l_n^+(z) &= \sum_{i=0}^{n} (-1)^{n-i}(A_i^* z)^{n-i}. \end{aligned} \tag{3.1}$$

Observe that l_n^+ is simply the formal adjoint to l which we will henceforth call l^+ and that the recursion relation

$$l_{j+1}^+(z) = -l_j^{+\prime}(z) + A_{j+1}^* z \tag{3.2}$$

holds.

LEMMA 3.1. *For* $j = 0, \cdots, n$,

$$l_j^+(z) = \sum_{r=0}^{j} \alpha_{jr} z^{(r)}, \tag{3.3}$$

where

$$\alpha_{jr} = \sum_{i=0}^{j-r} (-1)^{j-i} B_r^{j-i} A_i^{*(j-i-r)} \tag{3.4}$$

and

$$B_r^{j-i} = \binom{j-i}{r}. \tag{3.5}$$

In particular,

$$\alpha_{jj} = (-1)^j A_0^*. \tag{3.6}$$

---

[1] If F is a subset of a space V and F' is a subset of V^*. By $F^{\perp}$ is meant the subset of all elements in V^* that annihilate F. By ${}^{\perp}F'$ is meant the set of all elements in V that are annihilated by every element in F'.

Proof. Use Leibniz's rule and reorder the sums.

We define $\mathscr{B}$ to be the $mn \times mn$ companion matrix

$$\begin{bmatrix} \alpha_{00} & 0 & \cdots & 0 \\ \alpha_{10} & \alpha_{11} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1\,0} & \alpha_{n-1\,1} & \cdots & \alpha_{n-1\,n-1} \end{bmatrix}, \tag{3.7}$$

where the matrices α_{ij} are as given in Lemma 3.1. We observe that the jth row of $\mathscr{B}$ consists of the coefficients of the jth partial adjoint expression l_j^+. Since $\mathscr{B}$ is lower triangular and $\alpha_{jj} = (-1)^j A_0^*$, $j = 0, \cdots, n-1$, which by hypothesis is invertible, $\mathscr{B}$ is invertible.

3.2. The operator L^+ and its associated operators $\tilde{L}$, $\hat{L}$ and $\tilde{L}_K$. Define

$$\begin{aligned} \mathscr{D}^* = \bigcup_{\phi \in \mathbb{C}^l} \{z : \hat{z} + \mathscr{B}^{-1}\tilde{v}^*[0,t]\phi \in \mathscr{A}_{mn};\ \hat{z}' \in \mathscr{L}^q_{mn} \text{ a.e.};\\ \hat{z}(a^+) = -\mathscr{B}^{-1}\tilde{v}^*[a]\phi;\ \hat{z}(b^-) = \mathscr{B}^{-1}\tilde{v}^*[b]\phi\}. \end{aligned} \tag{3.8}$$

Equivalently $\mathscr{D}^*$ consists of all functions z having the property that $l_j^+(z) + v_{j+1}[0,t]\phi$ is absolutely continuous and satisfying the endpoint conditions

$$l_j^+(z)[a] = -v_{j+1}^*[a]\phi, \qquad l_j^+(z)[b] = v_{j+1}^*[b]\phi, \tag{3.9}$$

where ϕ is an arbitrary vector in $\mathbb{C}^l$.

We define L^+ by the formal adjoint $l^+(z)$ acting on the domain $\mathscr{D}^*$. It is evident that if $n = 1$ and $\mathscr{B} = I_{mn}$, L^+ is the adjoint operator defined in [1]. The main result of this section (Theorem 3.2) is that $L^+ = L^*$ for $1 \leqq p < \infty$ and $n > 1$.

Later we shall show that in many practical cases it is possible to eliminate the parameter ϕ so as to obtain nonparametric boundary conditions of a more conventional type. For the present, however, the following lemma is adequate for our purposes.

LEMMA 3.2. *$\mathscr{D}^*$ is nonempty and dense in $\mathscr{L}^q_m$, $1 \leqq q < \infty$. z in $\mathscr{D}^*$ is in class $C^j[a,b]$ if and only if the measures $v_1, \cdots, v_{j+1}$ are nonatomic. Otherwise z and its derivatives have at most countably many discontinuities.*

Proof. The first statement follows trivially taking $\phi = 0$. The last statement is true because $\tilde{v}^*$ is a measure of bounded variation. The second statement is trivial for $j = 0$ by (3.8). Let us assume it is true for $i = 0, \cdots, j-1$. Then from (3.3) and (3.8) and for $t \in (a,b)$,

$$\begin{aligned} l_j^+(z)[t^+] - l_j^+(z)[t^-] &= (-1)^j A_0^*(z^{(j)}(t^+) - z^{(j)}(t^-)) \\ &\quad + \sum_{r=0}^{j-1} \alpha_{jr}(z^{(r)}(t^+) - z^{(r)}(t^-)) \\ &= v_{j+1}^*[t]. \end{aligned} \tag{3.10}$$

By hypothesis $\sum_{r=0}^{j-1} \alpha_{jr}(z^{(r)}(t^+) - z^{(r)}(t^-))\ \tilde{v}$ vanishes if and only if $v_1, \cdots, v_j$ are nonatomic. Hence because of the nonsingularity of A_0^*, $z^{(j)}(t^+) - z^{(j)}(t^-)$ depends only on $v_{j+1}^*[t]$. This proves the lemma.

Following the precedent for L we define the first order operator $\tilde{L}^+$ associated with L^+ in $\mathscr{L}^q_{mn}$ to be

$$\tilde{l}^+(z) = Mz' + Nz, \tag{3.11}$$

where

$$M = \begin{bmatrix} I_m & 0 & \cdots & 0 \\ 0 & I_m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \alpha_{nn} \end{bmatrix}, \tag{3.12}$$

and

$$N = \begin{bmatrix} 0 & -I_m & 0 & \cdots & 0 \\ 0 & 0 & -I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & & -I_m \\ \alpha_{n_0} & \alpha_{n_1} & \alpha_{n_2} & \cdots & \alpha_{n_{n-1}} \end{bmatrix} \tag{3.13}$$

on the domain

$$\begin{aligned} \tilde{\mathscr{D}}^* = \bigcup_{\phi \in \mathbb{C}^l} \{z : z' \text{ exists a.e. in } \mathscr{L}^q_{mn};\ z + \mathscr{B}^{-1}\tilde{v}^*[0,t]\phi \in \mathscr{A}_{mn}; \\ z(a^+) = -\mathscr{B}^{-1}\tilde{v}^*[a]\phi;\ z(b^-) = \mathscr{B}^{-1}\tilde{v}^*[b]\phi\}. \end{aligned} \tag{3.14}$$

The *restricted first order operator* $\hat{L}^+$ *associated with* L^+ *is given by* $\tilde{L}^+$ *on*

$$\hat{\mathscr{D}}^* = \{\hat{z} : z \in \mathscr{D}^*\}. \tag{3.15}$$

Obviously $\hat{\mathscr{D}}^* \subset \tilde{\mathscr{D}}^*$.

Finally the *transformed operator* $\tilde{L}^+_\beta$ *associated with* L^+ is given by

$$\tilde{l}^+_\beta(z) = M\mathscr{B}^{-1}z' + \{(M\mathscr{B}^{-1})' + N\mathscr{B}^{-1}\}z \tag{3.16}$$

on

$$\tilde{\mathscr{D}}^*_{\mathscr{B}} = \{z : \mathscr{B}^{-1}z \in \tilde{\mathscr{D}}^*\}. \tag{3.17}$$

The following diagram indicates the relation between $L^+, \hat{L}^+, \tilde{L}^+$ and $\tilde{L}^+_{\mathscr{B}}$, where the maps $\hat{}$, j and i have the same meaning as in diagram (2.13) and T is the 1–1 onto transformation $\tilde{\mathscr{D}}^* \to \tilde{\mathscr{D}}^*_{\mathscr{B}}$ given by

$$T(z) = \mathscr{B}z. \tag{3.18}$$

We note that the mappings j, i and T are continuous isomorphisms, and that $\tilde{L}^+_{\mathscr{B}}$ and $\tilde{L}^+$ are equivalent under the transformation T; in both diagrams (2.13) and (3.28), however, the mapping $\hat{}$ is *not* necessarily continuous, nor need the domains $\hat{\mathscr{D}}^{[U]}$ or $\hat{\mathscr{D}}^*$ be dense. At this point we also make no assumption about the density of $\mathscr{D}^{[U]}$. Nevertheless it is clear from the diagrams that both L and L^+ depend in some sense on the operators $\tilde{L}$ and $L^+_{\mathscr{B}}$. This is fortunate because both $\tilde{L}$ and $L^+_{\mathscr{B}}$ are operators of the type studied in [1].

$$
\begin{tikzcd}
\mathscr{D}^* \arrow[rr, "l^+"] \arrow[d, "\hat{}"'] & & \mathscr{L}^q_m \arrow[d, "j"] \\
\hat{\mathscr{D}}^* \arrow[rr, "\hat{l}^+"] \arrow[d, "i"'] & & \mathscr{L}^q_{mn} \arrow[d, "i"] \\
\tilde{\mathscr{D}}^* \arrow[rr, "\tilde{l}^+"] \arrow[d, leftrightarrow, "T"'] & & \mathscr{L}^q_{mn} \\
\tilde{\mathscr{D}}^*_{\mathscr{B}} \arrow[urr, "\tilde{l}^+_{\mathscr{B}}"'] & &
\end{tikzcd}
\tag{3.19}
$$

3.3. The closure of L and L^+.

LEMMA 3.3. *The ranges of the operators $\tilde{L}$, $\hat{L}$, L, $\tilde{L}^+_{\mathscr{B}}$, $\tilde{L}^+$, $\hat{L}^+$ and L^+ are closed.*

Proof. The ranges of $\tilde{L}^+_{\mathscr{B}}$ and $\tilde{L}$ are closed by Theorem 2.2. The closure of the other ranges follows from examining the diagrams (2.13) and (3.19).

At this point we introduce the concept of the *minimum modulus* of an operator. The following is proved in Goldberg [7, p. 98, Thms. IV.1.6 and IV.1.7].

LEMMA 3.4. *Let X, Y be Banach spaces and $T:X \to Y$ an operator with closed range and nullspace. Define the minimum modulus $\gamma(T)$ of T by*

$$\gamma(T) = \inf_{x \in D(T)} \frac{\|T(x)\|_Y}{d(x, N(T))}, \tag{3.20}$$

where $0/0 = \infty$. Then T is a closed operator if and only if $\gamma(T) > 0$.

LEMMA 3.5. *Let M be a closed operator. Let $L \subset M$ have closed range. Suppose also that $N(L) = N(M)$. Then L is also a closed operator.*

Proof. Let $z_n \to z$ in $D(L)$ and $L(z_n) \to y$. Since M is closed, $z \in D(M)$ and $M(z) = y$. Since $R(L)$ is closed there exists $\bar{z}$ in $D(L)$ such that $L(\bar{z}) = y$. It follows that $\bar{z} - z \in N(M) = N(L)$. Therefore $z \in D(L)$. Since $M(z) = L(z) = y$, we conclude that L is closed.

LEMMA 3.6. *The operators L and $L^+_{\mathscr{B}}$ are closed.*

Proof. By Theorem 2.2, $\tilde{L}$ and $\tilde{L}^+_{\mathscr{B}}$ are closed operators with closed ranges. Since $\tilde{L}^+_{\mathscr{B}}$ is equivalent to $\tilde{L}^+$ under the transformation $\mathscr{B}$ the same is true for $\tilde{L}^+$. By Lemma 3.3, $\hat{L}$ and $\hat{L}^+$ have closed ranges; moreover it is evident from their definition that $N(\hat{L}) = N(\tilde{L})$ and that $N(\hat{L}^+) = N(\tilde{L}^+)$. Therefore $\hat{L}$ and $\hat{L}^+$ are closed by the previous lemma. It follows from Lemma 3.4 that $\gamma(\hat{L}^+) > 0$ and $\gamma(\hat{L}) > 0$. It is now sufficient to show that $\gamma(L) > 0$ and $\gamma(L^+) > 0$ to conclude by Lemmas 3.3 and 3.4 that L and L^+ are closed. Since the calculation is essentially the same for both L and L^+, we supply the details only for L. Now

$$\gamma(\hat{L}) = \inf_{y \in \mathscr{D}^{[U]}} \frac{\|\hat{l}(\hat{y})\|_{mn}^p}{d(\hat{y}, N(\hat{L}))} > 0. \tag{3.21}$$

Since

$$\begin{aligned}\|\hat{y} - \hat{n}\|_{mn}^p &= \left(\int_a^b \left[\sum_{i=0}^{n-1} |y^{(i)} - n^{(i)}|^2\right]^{p/2} dt\right)^{1/p} \\ &> \|y - n\|_m^p\end{aligned} \tag{3.22}$$

for y in $\mathscr{D}^{[U]}$ and n in $N(L)$,

$$d(\hat{y}, N(\hat{L})) \geqq d(y, N(L)). \tag{3.23}$$

Hence

$$\frac{\|l(y)\|_m^p}{d(y, N(L))} \geqq \frac{\|l(y)\|_m^p}{d(\hat{y}, N(\hat{L}))} \geqq \gamma(\hat{L}) > 0.^2 \tag{3.24}$$

And so

$$\gamma(L) \geqq \gamma(\hat{L}) > 0, \tag{3.25}$$

which is what we wanted to prove.

3.4. L^+ and L are mutally adjoint.

LEMMA 3.7 (Green's relation). *Let $y \in \mathscr{D}_m^{pn}$ and $z \in \mathscr{D}^*$. Then*

$$(y, L^+ z) = (L' y, z) - U(y, \cdots, y^{(n-1)})^* \phi, \tag{3.26}$$

where ϕ is an arbitrary vector in $\mathbb{C}^l$.

Proof. Recall that the definition of the partial adjoint expression l_j^+ implies the recursion relation

$$l_{j+1}^+(z) = -l_j^{+\prime}(z) + A_j^* z, \qquad j = 0, \cdots, n-1. \tag{3.27}$$

Therefore,

$$\int_a^b y^{(n-j)*} l_j^+(z)\, ds = -\int_a^b y^{(n-j)*}[l_{j-1}^+(z) + v_j^*[0, t]\phi]'\, ds + \int_a^b y^{(n-j)*} A_j^* z\, ds. \tag{3.28}$$

For $j = 1, \cdots, n-1$ we can integrate the first integral on the right by parts to get

$$\begin{aligned}&-[y^{(n-j)*}(l_{j-1}^+(z) + v_j^*[a, t]\phi)]_a^b + \int_a^b y^{(n-j+1)*} l_{j-1}^+(z)\, ds \\ &+ \int_a^b y^{(n-j+1)*}(v_j^*[0, t]\phi)\, ds,\end{aligned} \tag{3.29}$$

[2] By (2.9) or (2.14) it is clear that $\|\tilde{l}(\hat{y})\|_{mn}^p = \|l(y)\|_m^p$ for $y \in \mathscr{D}^{[U]}$.

since $l_{j-1}^{+}(z) + v_j^*[0,t]\phi$ and $y^{(n-j)}$ are absolutely continuous functions. Using the endpoint conditions (3.9), the first term in (3.29) becomes

$$-y^{(n-j)*}(b)v_j^*[a,b]\phi. \tag{3.30}$$

Since $v_j^*[0,t]$ is a function of bounded variation, the third integral in (3.39) may be written (cf. McShane [13, p. 332])

$$y^{(n-j)*}(b)v_j^*[a,b]\phi - \left(\int_a^b dv_j y^{n-j}\right)^* \phi. \tag{3.31}$$

Substituting (3.30) and (3.31) into (3.29) and (3.29) into (3.28) we finally get

$$\begin{aligned}\int_a^b y^{(n-1)*}l_j^{+}(z)\,ds &= \int_a^b (A_j y^{(n-j)})^* z\,ds \\ &\quad + \int_a^b y^{(n-j+1)*}l_{j-1}^{*}(z)\,ds - \left(\int_a^b dv_j^* y^{n-j}\right)^* \phi, \quad j = 1, \cdots, n.\end{aligned} \tag{3.32}$$

By successive applications of (3.32) the identity

$$\begin{aligned}(y, L^{+}z) &= \int_a^b y^* l^{+}(z)\,ds \\ &= (L'y, z) - \sum_{j=1}^{n} \left(\int_a^b dv_j y^{(n-j)}\right)^* \phi \\ &= (L'y, z) - U(y, \cdots, y^{(n-1)})^* \phi\end{aligned} \tag{3.33}$$

follows immediately.

LEMMA 3.8. *For $1 < q < \infty$, let L_0^{+} be the operator L^{+} restricted to those functions z in $\mathscr{D}^*$ for which $z^{(i)}$, $i = 0, \cdots, n-1$, is absolutely continuous and vanishes at a and b (i.e., those functions in $\mathscr{D}^*$ for which ϕ is zero.) Then $(L_0^{+})^* = L'$.*

Proof. See Goldberg [7, Thm. VI.2.3, p. 135] for the scalar case ($m = 1$). Goldberg's proof is easily generalized to fit our situation.

THEOREM 3.1. *For $1 < p \leqq \infty$, $(L^{+})^* = L$.*

Proof. If $1 < p \leqq \infty$, then $1 \leqq q < \infty$. Consequently D_0^* is dense in $\mathscr{L}_m^q$. Since $\mathscr{D}_0^* \subset \mathscr{D}^*$, $(L^{+})^*$ is well-defined. We first show that $L \subset (L^{+})^*$. Let $y \in \mathscr{D}^{[U]}$ and $z \in \mathscr{D}^*$. It suffices to prove Green's relation:

$$(Ly, z) - (y, L^{+}z) = 0. \tag{3.34}$$

But this holds by Lemma 3.7.

To show the reverse inclusion, note that $L_0^{+} \subset L^{+}$ implies that $(L^{+})^* \subset (L_0^{+})^*$. By Lemma 3.8 it follows that $(L^{+})^* \subset L'$. For arbitrary z in $\mathscr{D}^*$ and y in $D(L^{+})^*$, we have

$$((L^{+})^*y, z) - (y, L^{+}z) = 0. \tag{3.35}$$

But from Lemma 3.7,

$$((L^{+})^*y, z) - (y, L^{+}z) = \sum_{j=1}^{n} \left(\int_a^b dv_j y^{n-j}\right)^* \phi. \tag{3.36}$$

Since ϕ is arbitrary in $\mathbb{C}^l$, (3.35) and (3.36) imply that

$$U(y, \cdots, y^{(n-1)}) = \sum_{j=1}^{n} \int_a^b dv_j y^{(n-j)} = 0. \tag{3.37}$$

In other words $y \in \mathscr{D}^{[U]}$. Thus $(L^+)^* \subset L$ and the two are equal.

THEOREM 3.2. *If* $1 \leqq p < \infty$, L^* *exists and is equal to* L^+.

Proof. We will first consider the case $p > 1$. Then $\mathscr{L}_m^p$ is reflexive. Since L^+ is closed on the dual of $\mathscr{L}_m^p$, $(L^+)^*$ is densely defined (Goldberg [7, p. 56, Thm. II.2.14]) and moreover,

$$(L^+)^{**} = L^+. \tag{3.38}$$

By Theorem 3.1, $(L^+)^* = L$ and thus $L^* = L^+$. This completes the proof of the theorem in the case $1 < p < \infty$.

Assume now that $p = 1$. It is convenient to adopt the notation $L_p, \mathscr{D}_q^*, L_q^+$ to refer to L defined in $\mathscr{L}_m^p$ and to $\mathscr{D}^*, L^+$ defined in $\mathscr{L}_m^q$. Since $\mathscr{L}_m^p \subset \mathscr{L}_m$ for $p > 1$, it follows that $L_p \subset L_1$. Because $\mathscr{L}_m^p$ is dense in $\mathscr{L}_m$ and because convergence in $\mathscr{L}_m^p$ implies convergence in $\mathscr{L}_m$, the domains $D(L_p)$ are all dense in $\mathscr{L}_m$ for $p \geqq 1$. Thus $L_p^* \supset L_1^*$. By the first part of the proof, $L_p^* = L_q^+$. Hence $L_1^* = l^+$ on some domain $S \subseteq \mathscr{D}_q^* \cap L_n^\infty[0, 1]$. On the other hand, Green's relation shows that $L_\infty^+ \subset L_1^*$; in other words, $\mathscr{D}_\infty^+ \subset S$. Since $\mathscr{D}_q^* \subset L_n^\infty[0, 1]$ and $l^+(s) \subset L_n^\infty$, $L_1^* \subset L_\infty^+$. So $L_\infty^+ = L_1^*$.

The next theorem completes our generalization of Theorems 2.1 and 2.2 to higher order operators.

THEOREM 3.3. *For* $1 \leqq p \leqq \infty$, L *and* L^+ *are normally solvable operators. Consequently, for* $1 \leqq p < \infty$,

$$R(L^+) = N(L)^\perp; \tag{3.39}$$

if $1 < p \leqq \infty$,

$$R(L) = N(L^+)^\perp; \tag{3.40}$$

if $p = \infty$,

$$R(L^+) = {}^\perp N(L); \tag{3.41}$$

if $p = 1$,

$$R(L) = {}^\perp N(L^+). \tag{3.42}$$

Proof. The normal solvability of L and L^+ is just the content of Lemmas 3.3 and 3.6. The statements (3.39)–(3.41) are a consequence of the closed range theorem for closed operators (see for instance, Goldberg [7, p. 95, Thm. IV.1.2]).

4. The problem of Wilder and some examples. Let us now apply our theory to Wilder's scalar multipoint boundary value problem mentioned at the beginning of this paper. We will consider the nth order regular scalar operator

$$l(y) = \sum_{j=0}^{n-1} a_j y^{n-j} \tag{4.1}$$

with boundary conditions

$$U_i(y, \cdots, y^{(n-1)}) = \sum_{k=0}^{r} \sum_{j=1}^{n} c_{ij}^{(k)} y^{(n-j)}(t_k) = 0, \qquad i = 1, \cdots, l. \tag{4.2}$$

We shall specify that $\{t_k\}$ is a set of $r + 1$ points in $[a, b]$ such that $t_0 = a, t_r = b$, and $t_k < t_{k+1}$, $k = 0, \cdots, r - 1$. We will also assume that the forms (4.2) are linearly independent, and that $l < (r + 1)n$. It follows at once from (3.8) and Lemma 3.2 that a function in $\mathscr{D}^*$ is in $\mathscr{D}^{qn}$ in each of the intervals $[t_k, t_{k+1}]$ and also obeys the conditions

$$\begin{aligned} \mathscr{B}(a)\hat{z}(a^+) &= -\tilde{v}^*[a]\phi \\ &\vdots \\ \mathscr{B}(t_k)(\hat{z}(t_k^+) - \hat{z}(t_k^-)) &= -\tilde{v}^*[t_k]\phi \\ &\vdots \\ \mathscr{B}(b)\hat{z}(b^-) &= \tilde{v}^*[b]\phi. \end{aligned} \tag{4.3}$$

We will now show how the parameter ϕ can always be eliminated. Although the following is intended as a formal rather than a practical method, in many instances as will be seen at the end of this section it is almost automatic.

Let $\mathscr{Z}$ be the vector in $\mathbb{C}^{n(r-1)}$,

$$(\hat{z}(a^+), \cdots, \hat{z}(t_k^+) - \hat{z}(t_k^-), \cdots, \hat{z}(b^-))^t.$$

Let $\mathscr{L}$ be the $[n(r + 1)] \times [n(+1)]$ matrix

$$\begin{bmatrix} \mathscr{B}(a) & 0 & \cdots & 0 \\ 0 & \mathscr{B}(t_1) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \mathscr{B}(b) \end{bmatrix}, \tag{4.4}$$

and V the $n(r + 1) \times l$ matrix

$$\begin{bmatrix} -\tilde{v}^*[a] \\ \vdots \\ -\tilde{v}^*[t_k] \\ \vdots \\ -\tilde{v}^*[b] \end{bmatrix}. \tag{4.5}$$

With this notation (4.3) may be written

$$\mathscr{L}\mathscr{Z} = V\phi. \tag{4.6}$$

Let V^+ be the Moore–Penrose generalized inverse of V. If $z \in \mathscr{D}^*$, (4.6) holds for some ϕ in $\mathbb{C}^l$. Then

$$VV^+\mathscr{L}\mathscr{Z} = VV^+V\phi. \tag{4.7}$$

Since

$$VV^+V\phi = V\phi = \mathscr{L}\mathscr{Z}, \tag{4.8}$$

it follows from (3.7) and (3.8) that z must satisfy

$$\mathscr{L}\mathscr{Z} = VV^+\mathscr{L}\mathscr{Z}. \tag{4.9}$$

Conversely if z is a function in $\mathscr{D}^{pn}$ on the intervals $[t_k, t_{k+1}]$ satisfying (4.9) it is trivial (taking $\phi = V^+\mathscr{L}\mathscr{Z}$) that (4.6) is satisfied for some ϕ in $\mathbb{C}^l$. We call (4.9) the nonparametric boundary conditions for $\mathscr{D}^*$. It is interesting to note that if V is invertible, $\mathscr{D}^*$ contains functions satisfying arbitrarily prescribed jumps at the points t_i and limits at the endpoints. This happens (because of the linear independence of the U_i) if and only if $l = (r + 1)n$. In this case the adjoint can be regarded as a kind of "direct sum" of "maximal" differential operators defined on the intervals $[t_k, t_{k+1}]$.

Of course, the extension of the above remarks to the vector version of Wilder's problem is completely routine.

The following are a few simple illustrations of the theory.

1. In his study of self-adjoint multipoint boundary value problems, Loud [12] constructed adjoint boundary conditions for the three-point boundary value problem

$$l(y) = y'', \qquad y(-1) = Ay(0), \qquad y(1) = By(0) \tag{4.10}$$

through an analysis of the discontinuities of the Green's function. Here v_1 is the zero measure and

$$v_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} u_{(-1)} + \begin{pmatrix} A \\ B \end{pmatrix} u_{(0)} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_{(1)}. \tag{4.11}$$

Furthermore,

$$\mathscr{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.12}$$

Then by (4.3) the adjoint parametric boundary conditions become

$$\begin{gathered}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z(-1^+) \\ z'(-1^+) \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \phi, \\
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z(0^+) - z(0^-) \\ z'(0^+) - z'(0^-) \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ -A & B \end{pmatrix} \phi, \\
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z(1^-) \\ z'(1^-) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \phi,
\end{gathered} \tag{4.13}$$

where $\phi = (\phi_1, \phi_2)^t$ is a vector in $\mathbb{C}^2$. From (4.13) it is easily seen that

$$\begin{aligned}
z(-1^+) &= 0, & -z'(-1^+) &= \phi_1, \\
z(0^+) - z(0^-) &= 0, & z'(0^-) - z'(0^+) &= A\phi_1 - B\phi_2, \\
z(1^-) &= 0, & -z'(1^-) &= \phi_2,
\end{aligned} \tag{4.14}$$

or

$$(4.15)\qquad \begin{aligned} &z(-1^+) = 0, && z(1^-) = 0, \\ &z(0^+) - z(0^-) = 0, && z'(0^-) - z'(0^+) = -Az'(-1^+) + Bz'(1^-), \end{aligned}$$

with no constraint on $z'(-1^+)$ or $z'(1^-)$. These results agree with Loud's, but we did not need to construct the Green's function.

2. Consider the degenerate boundary value problem

$$(4.16)\qquad \begin{aligned} &l(y) = y'', \\ &y(-1) = 0; \quad y(0) = 0; \quad y(1) = 0, \\ &y'(-1) = 0; \quad y'(0) = 0; \quad y'(1) = 0. \end{aligned}$$

Here

$$(4.17)\qquad \begin{aligned} v_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{(-1)} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_{(1)}, \\ v_2 &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_{(-1)} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u_{(0)} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_{(1)}. \end{aligned}$$

As before,

$$(4.18)\qquad \mathscr{B} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.$$

Hence the adjoint boundary conditions become

$$(4.19)\qquad \begin{aligned} z(-1^+) &= -\phi_1, & z'(-1^+) &= \phi_4, \\ z(0^+) - z'(0^-) &= -\phi_2, & z'(0^+) - z'(0^-) &= \phi_5, \\ z(1^-) &= -\phi_3, & z'(1^-) &= \phi_6. \end{aligned}$$

Thus the adjoint operator is completely unconstrained and can be viewed as the "direct sum" of maximal operators on the intervals $[-1, 0]$ and $[0, 1]$.

3. The theory is particularly simple in the two-point case. Consider the system

$$l(y) = y' + Py, \qquad Ay(0) + By(1) = 0. \tag{4.20}$$

P is an $m \times m$ continuous matrix, and A, B are $l \times m$ constant matrices. Here there is only one measure ν_1 given by

$$\nu_1 = Au_{(0)} + B_{u(1)}. \tag{4.21}$$

Moreover,

$$\mathscr{B} = I_m.$$

Hence the adjoint parametric boundary conditions are

$$z(0^+) = -A^*\phi, \qquad z(1^-) = B^*\phi. \tag{4.22}$$

If both A and B are invertible ϕ can be eliminated, and (4.22) becomes

$$B^{*-1}z(1^-) + A^{*-1}z(0^+) = 0. \tag{4.23}$$

5. Conclusion. We end the paper with a brief discussion of physical and mathematical applications of boundary value problems similar to (or even more general than) those considered here. Since we hope to study some of the topics outlined below elsewhere in more detail, our treatment will not be exhaustive. Instead we hope to convince the reader that the mathematical theory developed in the previous sections both has practical justification and ought to be regarded as a preliminary effort pointing the way to further research.

The most obvious applications certainly should lie in the theory of elastic, or dynamical systems constrained at interior points or observed at different places or times. Meyer [16, Chap. IV], for instance, has investigated the deflection of an elastic rail as well as the solution of the potential equation in an annulus. Although his point of view is different than ours he does show that both problems lead to *vector* second order systems with conditions prescribed at multiple points. Whyburn [28] suggested that an n-dimensional second order vector-valued system with $2n$-conditions prescribed at n points in time could be used to model the pencil of trajectories of an inaccurately aimed machine gun. In general a dynamical system with many degrees of freedom observed at different times leads to a multi-point vector differential system.

There are also interesting interconnections between generalized boundary value problems, splines, and optimal control.

Consider $l(y)$ under the *nonhomogeneous* side condition

$$U(y, \cdots, y^{(n-1)}) = r, \qquad r \in \mathbb{C}^l. \tag{5.1}$$

Following Jerome and Schumaker [10], define an "Lg-spline" to be the function f satisfying the Stieltjes side conditions (5.1) and minimizing $l(f)$ in $\mathscr{L}_n^2$. Choosing the differential expression $l(y)$ and the measures ν_i appropriately, this definition may be seen to include all the classical splines (e.g., cubic, polynomial, etc.) on finite intervals. If $L(r): \mathscr{L}_n^2 \to \mathscr{L}_n^2$ denotes the nonlinear "r-translate" of L, i.e., the operator generated by $l(y)$ and the side condition (5.1), then it is not difficult to show that the equation

$$L^{+}L(r)[f] = 0 \tag{5.2}$$

(given that $L^{+} = L^{*}$) completely characterizes the spline f. The structure of $\mathscr{D}^{*}$ now makes it possible to analyze the local and global smoothness properties of the spline in terms of the measures ν_i and the coefficients of $l(y)$. The analysis gives as special cases many known results as well as some stronger ones which would be hard to derive by existing methods. For details and proofs see Brown [2].

As Mangasarian and Schumaker [14] have noted, the spline problem itself is a kind of control problem; i.e., we seek to minimize

$$J = \int_a^b u^{*}u\,dt \tag{5.3}$$

subject to

$$u = l(y), \qquad U(y, \cdots, y^{(n-1)}) = r. \tag{5.4}$$

This observation suggests a class of quadratic control problems which would generalize our approach to splines. Suppose

$$\begin{aligned} l_1(y) &= \sum_{i=0}^{n} A_{1i}y^{(n-i)}, \\ l_2(y) &= \sum_{i=0}^{m} A_{2i}y^{(m-i)} \end{aligned} \tag{5.5}$$

are regular k-dimensional vector-valued differential expressions.

Consider the problem of minimizing the functional

$$J = \int_a^b (l_1^{*}(y)l_2(y) + u^{*}u)\,dt \tag{5.6}$$

subject to

$$u = l_2(y), \qquad U(y, \cdots, y^{(n-1)}) = r. \tag{5.7}$$

Define $\mathscr{L} : \mathscr{D}^{[U]} \to \mathscr{L}^2_{2k}$ by

$$\mathscr{L}(y) = \begin{pmatrix} l_1(y) \\ l_2(y) \end{pmatrix}. \tag{5.8}$$

It is easy to see that $\mathscr{L}$ can be regarded as a vector-valued system of order max (m, n) whose coefficients are $2k \times k$ *rectangular* matrices. Furthermore since (5.6) is the square of the $\mathscr{L}^2$-norm of $\mathscr{L}$, a solution u to the control problem exists if and only if $R(\mathscr{L})$ is closed in $\mathscr{L}^2_{2k}$. Moreover if f is a solution of

$$\mathscr{L}^{*}\mathscr{L}(r)[f] = 0, \tag{5.9}$$

then $u = l_2(f)$ is the desired control. Notice that $\mathscr{L}$ is not a regular operator in the sense considered in the paper. It seems reasonable, however, that $\mathscr{L}$ can be analysed by the techniques developed here.

In view of the essentially variational examples above it is perhaps no accident that early activity in multipoint problems arose out of the calculus of variations. For example, Denbow [6] and Reid [20] found that they arose out of the Jacobi necessary conditions connected with problems of the Bolza type where the functional to be minimized involved intermediate as well as endpoints. This discovery motivated the early study of Mansfield [15] which was one of the first in the field.

According to Vejvoda and Tvrdy [27] there are applications of integro-differential systems under interior point boundary conditions to hydrodynamics. Indeed long ago Von Mises investigated in this connection a second order system subject to two side conditions represented by continuous Stieltjes measures (see [28] for details). Unfortunately no fully satisfactory treatment of such side conditions is yet available.

Suppose $D(L)$ is enlarged to include n-fold differentiable functions of bounded variation and $U(y, \cdots, y^{(n-1)})$ involves left or right limits of functions (or derivatives of functions) in $D(L)$. We call $U(y, \cdots, y^{(n-1)})$ an *interface* side condition, and the associated b.v. problem an *interface* problem. For example a typical interface side condition for the nth expression $l(y)$ might be written

$$U(y, \cdots, y^{(n-1)}) = \sum_{k=1}^{m} \sum_{j=1}^{n} A_{kj} y^{(n-j)}(t_s^+) + B_{kj} y^{(n-j)}(t_k^+) \tag{5.10}$$

for appropriate scalars or matrices A_{kj}, B_{kj}.

Interface problems have been studied by Sangren [21], Stallard [24], Zettl [32], Krall [11], and others. The general form of the adjoint problem, Green's function, etc., however, remains open. Such problems, moreover, have wide ranging applications. In [21] for example a single nth order equation with a finite number of interface conditions is studied and applied to problems in heat conduction, potential and vibration theory, and nuclear reactors. Stallard [24] encountered them in the theory of n stage diffusion problems. Applications have also been found to the study of orbits in a cyclotron, e.g., [18], [22]. Additional examples may be found in [16, Chap. III].

In general it is clear that whenever "interface" phenomena are present in the physical problem, they occur in the differential equation modeling it.

A problem having both interface and multipoint features which the writer is currently investigating is that of vibrating membrane constrained at interior points—e.g., by "rivets"—as well as on the boundary. The usual separation of variables technique applied to the two-dimensional wave equation yields two second order self-adjoint equations subject to conditions of the form

$$y(t_i) = 0, \qquad y'(t_i^+) - y'(t_i^-) = \phi_i, \qquad i = 1, \cdots, k. \tag{5.11}$$

Under the conditions (5.10) both systems are self-adjoint and their solutions are splines.

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NOTE ON A BINOMIAL IDENTITY*

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Abstract. Put

$$H(m,n|r,s) = \sum_{i=0}^{\min(m,r)} \sum_{j=0}^{\min(n,s)} \binom{i+j}{i}\binom{m-i+j}{m-i}\binom{i+n-j}{n-j}\binom{r-i+s-j}{r-i}.$$

In a previous paper, the author showed that

$$(*) \qquad H(m,n|r,s) - H(m-1,n|r-1,s) - H(m,n-1|r,s-1) = \binom{m+n}{m}\binom{r+s}{s}$$

provided $m \leqq r$ or $n \leqq s$. In the present paper, the left member of (*) is evaluated for $m > r$, $n > s$. Also the identity

$$H(2r+1,2s+1|r,s) - H(2r,2s+1|r-1,s) - H(2r+1,2s|r,s-1) = \binom{r+s}{r}\binom{2r+2s+1}{r+s}$$

is proved.

1. Introduction. Put

$$H(m,n) = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{i+j}{i}\binom{m-i+j}{m-i}\binom{i+n-j}{n+j}\binom{m-i+n-j}{m-i}.$$

Paul Brock proposed the identity

$$(1) \qquad H(m,n) - H(m-1,n) - H(m,n-1) = \binom{m+n}{m}^2.$$

The published solution [1] by David Slepian proved the identity by means of contour integration. Baer and Brock proved the formula in a later paper [2] by combinatorial methods. The writer [3] gave a proof of (1) as well as certain generalizations by making use of generating functions. In another paper [4], the writer defined

$$(2) \qquad H(m,n|r,s) = \sum_{i=0}^{\min(m,r)} \sum_{j=0}^{\min(n,s)} \binom{i+j}{i}\binom{m-i+j}{m-i}\binom{i+n-j}{n-j}\binom{r-i+s-j}{r-i}$$

and showed that

$$(3) \quad H(m,n|r,s) - H(m-1,n|r-1,s) - H(m,n-1|r,s-1) = \binom{m+n}{m}\binom{r+s}{r}$$

provided $m \leqq r$ or $n \leqq s$. For $m = r$, $n = s$, (3) clearly reduces to (1).

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The purpose of the present note is to see what can be said when $m > r$, $n > s$.

2. Generalization of (3). It is proved in [4] that

$$\begin{aligned}\sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} H(m,n|r,s)x^m y^n u^r v^s \\ = (1-x-y)^{-1}(1-u-v)^{-1}(1-ux-vy)^{-1} \\ - xy(1-x-y)^{-1}(1-ux-vy)^{-1}T,\end{aligned} \tag{4}$$

where

$$T = \{(1-x-y)(1-ux-vy) + xy(1-u-v)\}^{-1}. \tag{5}$$

Since

$$(1-x-y)(1-ux-vy) + xy(1-u-v) = (1-x)(1-y) - ux(1-x) - vy(1-y),$$

we have

$$\begin{aligned} T &= \{(1-x)(1-y) - ux(1-x) - vy(1-y)\}^{-1} \\ &= (1-x)^{-1}(1-y)^{-1}\left\{1 - \frac{ux}{1-y} - \frac{vy}{1-x}\right\}^{-1} \\ &= \sum_{r,s=0}^{\infty} \binom{r+s}{r} \frac{(ux)^r (vy)^s}{(1-x)^{s+1}(1-y)^{r+1}} \\ &= \sum_{r,s=0}^{\infty} \binom{r+s}{r} (ux)^r (vy)^s \sum_{i=0}^{\infty} \binom{s+i}{i} x^i \sum_{j=0}^{\infty} \binom{r+j}{j} y^j. \end{aligned}$$

If we put $m = r + i$, $n = s + j$, this becomes

$$T = \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{r+s}{r} \binom{m-r+s}{s} \binom{n+r-s}{r} u^r v^s. \tag{6}$$

Substituting from (6) in (4), we get

$$\begin{aligned}(1-ux-vy) \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} H(m,n|r,s)x^m y^n u^r v^s \\ = \sum_{m,n=0}^{\infty} \sum_{r,s=0}^{\infty} \binom{m+n}{m} \binom{r+s}{r} x^m y^n u^r v^s \\ -xy(1-x-y)^{-1} \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{r+s}{r} \binom{m-r+s}{s} \binom{n+r-s}{r} u^r v^s.\end{aligned} \tag{7}$$

If $m \leqq r$ or $n \leqq s$, we again get (1). Now

$$
\begin{aligned}
&(1-x-y)^{-1} \sum_{m,n=0}^{\infty} x^m y^n \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{r+s}{r}\binom{m-r+s}{s}\binom{n+r-s}{r} u^r v^s \\
(8)\qquad & = \sum_{m,n=0}^{\infty} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{r+s}{r} G(m,n|r,s) x^m y^n u^r v^s,
\end{aligned}
$$

where

$$
(9)\qquad G(m,n|r,s) = \sum_{i=0}^{m-r} \sum_{j=0}^{n-s} \binom{i+j}{i}\binom{m-i-r+s}{s}\binom{n-j+r-s}{r}.
$$

We now assume that

$$
(10)\qquad m > r, \qquad n > s.
$$

It follows immediately from (7) and (8) that

$$
\begin{aligned}
&H(m,n|r,s) - H(m-1,n|r-1,s) - H(m,n-1|r,s-1) \\
(11)\qquad & = \binom{m+n}{m}\binom{r+s}{r} - \binom{r+s}{r} G(m-1,n-1|r,s).
\end{aligned}
$$

3. Continuation. The sum $G(m, n, r, s)$ can be simplified. We recall the identity

$$
(12)\qquad \sum_{j=0}^{n} \binom{j+m}{j}\binom{n-j+k}{k} = \binom{m+n+k+1}{n}.
$$

Thus

$$
\sum_{j=0}^{n-s} \binom{i+j}{j}\binom{n-j+r-s}{r} = \binom{i+n+k-s+1}{n-s},
$$

so that

$$
G(m,n|r,s) = \sum_{i=0}^{m-r} \binom{m-i-r+s}{s}\binom{i+n+r-s+1}{n-s}.
$$

If i is replaced by $m-r-i$, this becomes

$$
(13)\qquad G(m,n|r,s) = \sum_{i=0}^{m-r} \binom{i+s}{i}\binom{m+n-i-s+1}{n-s}.
$$

Since the upper limit is less than $m+1$, (12) does not apply. However,

$$
\begin{aligned}
\sum_{i=0}^{m-r} \binom{s+i}{i}\binom{m+n-i-s+1}{n-s} &= \sum_{i=0}^{m+1} \binom{i+s}{i}\binom{m+n-i-s+1}{n-s} \\
&\quad - \sum_{i=m-r+1}^{m+1} \binom{i+s}{i}\binom{m+n-i-s+1}{n-s} \\
&= \binom{m+n+2}{m+1} - \sum_{i=0}^{r} \binom{i+n-s}{i} \\
&\quad \cdot \binom{m+s-i+1}{s}.
\end{aligned}
$$

Thus

$$G(m, n|r, s) = \binom{m+n+2}{m+1} - \sum_{i=0}^{r} \binom{i+n-s}{i}\binom{m+s-i+1}{s}. \tag{14}$$

Comparing (14) with (13), we get

$$G(m, n|r, s) = \binom{m+n+2}{m+1} - G(m, n|m-r, n-s), \qquad m \geqq r, \quad n \geqq s. \tag{15}$$

Substitution of (14) in (11) gives

$$\begin{aligned} H(m, n|r, s) - H(m-1, n|r-1, s) - H(m, n-1|r, s-1) \\ = \binom{r+s}{r}\Bigg) \sum_{i=0}^{r} \binom{i+n-s-1}{i}\binom{m+s-i}{s}. \end{aligned} \tag{16}$$

Since

$$H(m, n|r, s) = H(n, m|s, r),$$

we have also

$$\begin{aligned} H(m, n|r, s) - H(m-1, n|r-1, s) - H(m, n-1|r, s-1) \\ = \binom{r+s}{r} \sum_{i=0}^{s} \binom{i+m-r-1}{i}\binom{n+r-i}{r}. \end{aligned} \tag{17}$$

We may now state the following theorem.

THEOREM. *For* $m \leqq r$ *or* $n \leqq s$, $H(m, n|r, s)$ *satisfies* (1). *For* $m > r$, $n > s$, $H(m, n|r, s)$ *satisfies* (16) *and* (17). *Thus*

$$H(m, n|r, s) - H(m-1, n|r-1, s) - H(m, n-1|r, s-1) > 0 \tag{18}$$

for all nonnegative m, n, r, s.

It follows from (16) and (17) that

$$\sum_{i=0}^{r} \binom{i+n-s-1}{i}\binom{m+s-i}{s} = \sum_{i=0}^{s} \binom{i+m-r-1}{i}\binom{n+r-i}{r}. \tag{19}$$

It is not difficult to give a direct proof of (19).

For $m = 2r$, $n = 2s$, (15) reduces to

$$G(2r, 2s|r, s) = \frac{1}{2}\binom{2r+2s+2}{r+s+1} = \binom{2r+2s+1}{r+s}. \tag{20}$$

Therefore, taking $m = 2r + 1$, $n = 2s + 1$ in (11), we get

$$\begin{aligned} H(2r+1, 2s+1|r, s) - H(2r, 2s+1|r-1, s) - H(2r+1, 2s|r, s-1) \\ = \binom{r+s}{r}\binom{2r+2s+1}{r+s}, \end{aligned} \tag{21}$$

an identity that may be compared with (1).

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AN INTEGRAL REPRESENTATION FOR THE SOLUTION W_{km} OF WHITTAKER'S DIFFERENTIAL EQUATION*

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Abstract. Recent results concerning the existence of solutions of ordinary differential equations with Laplace–Stieltjes transforms for coefficients are applied to Whittaker's equation. As a result, integral representations involving, in certain cases, the Legendre functions are obtained for the solutions W_{km} to Whittaker's equation and their products.

1. Introduction. Recently, Hartman [3], D'Archangelo [1], and D'Archangelo and Hartman [2] obtained results concerning the existence and, in certain cases, the uniqueness of solutions of a particular form involving Laplace–Stieltjes transforms for the Nth order linear differential equation

$$D^N u + \sum_{j=0}^{N-1} [a_j + g_j(t)]D^j u = 0, \tag{1.1}$$

where $Du = du/dt = u'$, and for the first order system of dimension N,

$$y' = [A + g(t)]y, \tag{1.2}$$

where a_j is a (complex) constant in (1.1) and A is a (complex) constant $N \times N$ matrix in (1.2). In (1.1) or (1.2), it was assumed that $g_j(t)$ or $g(t)$ is representable as a Laplace–Stieltjes transform

$$g_j(t) = \int_0^\infty e^{-st}\, dG_j(s), \qquad G_j(+0) = G_j(0) = 0, \tag{1.3}$$

or

$$g(t) = \int_0^\infty e^{-st}\, dG(s), \qquad G(+0) = G(0) = 0, \tag{1.4}$$

absolutely convergent for Re $t > 0$ and satisfying

$$\int_{+0} s^{-\gamma} |dG_j(s)| < \infty, \tag{1.5}$$

or

$$\int_{+0} s^{-\gamma} |dG(s)| < \infty, \tag{1.6}$$

for a suitable constant $\gamma > 0$. We considered the problem of the representation of certain solutions in terms of Laplace–Stieltjes transforms.

The following theorem and corollary for the nth order equation (1.1) were proved by Hartman [3, Thm. 2.1 and Cor. 2.1]. They are basic examples of the

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results obtained for the above problem, and they will suffice for the construction of integral representations for the Whittaker functions, W_{km}, and their products.

THEOREM 1.1 *Let* $P_N(\lambda)$ *be the polynomial*

$$P_N(\lambda) \equiv \lambda^N + \sum_{j=0}^{N-1} a_j \lambda^j. \tag{1.7}$$

For a fixed real number τ, *let*

$$\{\lambda(1), \cdots, \lambda(K)\} = \textit{set of distinct zeros of } P_N(\lambda) : \operatorname{Im} \lambda = \tau, \tag{1.8}$$

$$\operatorname{Re} \lambda(1) < \cdots < \operatorname{Re} \lambda(K), \tag{1.9}$$

$$k = \textit{multiplicity of zero } \lambda = \lambda(1) \textit{ of } P_N(\lambda). \tag{1.10}$$

Let the coefficient functions $g_j(t)$ *be representable as Laplace–Stieltjes transforms* (1.3) *absolutely convergent for* $\operatorname{Re} t > 0$, *satisfying* (1.5) *for* $\gamma \geqq k$. *Then* (1.1) *has a unique solution* $u = u(t)$ *representable in the form*

$$u = e^{\lambda(1)t}\left\{1 + \int_0^\infty e^{-st}\, d\omega(s)\right\}, \qquad \omega(0) = \omega(+0) = 0, \tag{1.11}$$

absolutely convergent for $\operatorname{Re} t > 0$, *so that* $u \sim e^{\lambda(1)t}$.

COROLLARY 1.1. *Assume that* $\lambda(1) = 0$ in (1.8), *and that therefore* (1.1) *can be written in the form*

$$D^k P(-D)u + \sum_{j=0}^{k+m-1} (-1)^{k+j} g_j(t) D^j u = 0, \tag{1.12}$$

where $m \geqq 0$, $k \geqq 0$, $m + k = N$ *and*

$$P(t) = a_m t^m + \cdots + a_1 t + a_0 \neq 0 \quad \textit{for } t \geqq 0,\ a_m \neq 0. \tag{1.13}$$

If for $\gamma \geqq k$, *we assume* (1.5) *for* $j \leqq k - 1$, *and we assume that* $P > 0$ *on* $(0, \infty)$, *and* $dG_j \leqq 0$ *in* (1.3), *then* $d\omega \geqq 0$ *in* (1.11).

2. Whittaker's equation. The change of variables $u = We^{-t/2}$ transforms the differential equation

$$u'' + u' + (k/t - (m^2 - 1/4)/t^2)u = 0 \tag{2.1}$$

into its normal form, called Whittaker's equation:

$$W'' + \{-1/4 + k/t - \beta/t^2\}W = 0, \quad \text{where } \beta = m^2 - 1/4. \tag{2.2}$$

For $\operatorname{Re} t > 0$, a particular solution of (2.2) which Whittaker denoted by W_{km} in [4, p. 340], is given by

$$W_{km}(t) = ct^k e^{-t/2} \int_0^\infty s^{-k+m-1/2}(1 + s/t)^{k+m-1/2} e^{-s}\, ds, \tag{2.3}$$

where $c = 1/\Gamma(-k + m + 1/2)$, provided these expressions make sense. Since $W_{km}(t) = t^k e^{-t/2}\{1 + O(t^{-1})\}$ (cf. [4, p. 343]) the following is a consequence of Corollary 1.1.

PROPOSITION 2.1. (a) *Whittaker's function, W_{km}, has an integral representation as*

$$W_{km}(t) = e^{-t/2}t^k\left\{1 + \int_0^\infty e^{-st}\, d\omega_{km}(s)\right\}, \tag{2.4}$$

where the integral is absolutely convergent for Re $t > 0$.

(b) *If $k \geqq 0$ and $m^2 \geqq (k-1/2)^2$, then $d\omega_{km} \geqq 0$.*

Proof. Let

$$W = Ve^{-t/2}t^k \tag{2.5}$$

in (2.2). Then V satisfies

$$-V'' + V' + (-2k/t)V' - [(k^2 - k - \beta)/t^2]V = 0. \tag{2.6}$$

Equation (2.6) is in the form (1.12) where $\lambda(1) = 0$ with multiplicity 1, $P(t) = 1 + t > 0$, $G_1(s) = -2ks$, and $G_0(s) = (k^2 - k - \beta)s^2/2$. Therefore (2.6) satisfies the integrability conditions of Corollary 1.1, and also if $k \geqq 0$ and $(k^2 - k - \beta) \leqq 0$, the monotonicity conditions of Corollary 1.1. Hence, assertions (a) and (b) are proved.

In [5], Aurel Wintner showed that for $k \leqq 0$ and $m \geqq 1/2$ but $(k, m) \neq (0, 1/2)$, $W_{km}(t) = \int_0^\infty e^{-st}\, d\mu(s)$ for $t > 0$, where $\mu = \mu_{km}(t)$ is a nondecreasing function of t.

Originally, the author considered Proposition 2.1 only for $k = 0$, in which case (2.2) already satisfies the conditions of Theorem 1.1. Philip Hartman pointed out that Proposition 2.1 also holds for $k \neq 0$ by making the change of variables (2.5) and applying Corollary 1.1 to (2.6).

However, the case $k = 0$ is interesting for the following reason.

PROPOSITION 2.2. *Whittaker's function W_{0m} has an integral representation as*

$$W_{0m}(t) = e^{-t/2}\left\{1 + \int_0^\infty e^{-st}\, d\omega_{0m}(s)\right\}, \tag{2.7}$$

where the integral is absolutely convergent for Re $t > 0$, *where*

$$\omega_{0m}(s) = P(1 + 2s) \quad \text{for } s \geqq 0, \tag{2.8}$$

and where $P = P_\nu(\sigma)$, $\nu = m - 1/2$ is the unique solution of the Legendre differential equation

$$(d/d\sigma)[(1 - \sigma^2)\, dP/d\sigma] + \nu(\nu + 1)P = 0, \tag{2.9}$$

regular at $\sigma = 1$ and normalized by $P(1) = 1$.

Unlike (1.11) in Theorem 1.1, $\omega(0) = \omega(+0) = 1$, instead of the usual normalization $\omega(0) = \omega(+0) = 0$. Formulas (2.7) and (2.8) are analogous to those obtained for the Hankel functions in Proposition 8.1 in [2], as are the arguments used in their proof.

Proof. Since $W = W_{0m}(t)$ is a solution of (2.2) with $k = 0$, it follows that if v is the Laplace–Stieltjes transform in (2.7), then

$$v'' - v' - \beta/t^2 - \beta v/t^2 = 0.$$

By Corollary 2.1 of [3], if the G_j's of (1.3) are $C^\infty([0,\infty))$, then ω of (1.11) is $C^\infty((0,\infty))$. Therefore, $\omega=\omega_{0m}(s)$ is C^∞ for $s>0$, so that

$$\frac{1}{t^2}=\int_0^\infty e^{-st}s\,ds$$

implies that

$$(s^2+s)\omega'-\beta s-\beta\int_0^s (s-r)\omega'(r)\,dr=0.$$

Differentiating gives

$$[(s^2+s)\omega']'-\beta-\beta\int_0^s \omega'(r)\,dr=0. \tag{2.10}$$

Hence, if ω is normalized by $\omega(0)=1$, equation (2.10) becomes

$$[(s^2+s)\omega']'-\beta\omega=0. \tag{2.11}$$

The point $s=0$ is a regular singular point for (2.11) with indicial exponents 0 and 0, so that there exists a unique solution $\omega(s)$ (up to constant factor) such that $\omega'(s)$ is of class L^1 on $0<s\leqq\varepsilon$. This solution is regular at $s=0$ and is uniquely determined by the normalization $\omega(0)=1$. The change of variables, $\sigma=2s+1$, transforms (2.11) into (2.9) with $P_\nu=\omega$, $\nu=m-1/2$. This completes the proof.

As an application of the above results, consider the generalized Laguerre differential equation

$$tL''+(1+\alpha-t)L'+\nu L=0. \tag{2.12}$$

After the change of dependent variable, $L=t^\nu V$, V satisfies the equation

$$-V''+V'-\{(2\nu+1+\alpha)/t\}V'-\{\nu(\nu+\alpha)/t^2\}V=0, \tag{2.13}$$

which is just (2.6) with $k=\nu+(1+\alpha)/2$ and $\beta=(\alpha^2-1)/4$. Therefore by Propositions 2.1 and 2.2, we get the following.

PROPOSITION 2.3. (a) *The generalized Laguerre equation* (2.12) *has a unique solution of the form*

$$L(t)=t^\nu\left\{1+\int_0^\infty e^{-st}\,d\omega_{km}(s)\right\}, \tag{2.14}$$

where $k=\nu+(1+\alpha)/2$, $m=\alpha/2$, ω_{km} *is as in* (2.4), *and the integral is absolutely convergent for* $\operatorname{Re} t>0$.

(b) *If* $\nu+(1+\alpha)/2\geqq 0$ *and* $\nu(\nu+\alpha)\leqq 0$, *then* $d\omega\geqq 0$ *in* (2.14).

(c) *If* $\alpha=0$, (2.12) *reduces to what is usually called Laguerre's equation, and if* $\nu=-1/2$, *then*

$$L(t)=t^{-1/2}\left\{1+\int_0^\infty e^{-st}\,d(P_{-1/2}(1+2s))\right\}, \tag{2.15}$$

where $P_{-1/2}$ *is the Legendre function of Proposition* 2.2.

3. Products of Whittaker functions. Recall that if $v=v(t)$ satsifies the equation $v''+Iv=0$ and $w=w(t)$ satisfies $w''+Jw=0$, then $y\equiv vw$ satisfies

$$y'''+2(I+J)y'+(I'+J')y=(I-J)(v'w-w'v).$$

In particular, if $I=J$, we get that $y'''+4Iy'+2I'y=0$. Using $v=w=W_{0m}$ and $I=-(1/4+\beta/t^2)$ as in (2.2) with $k=0$, we see that $(W_{0m})^2\equiv y$ satisfies the differential equation

$$y'''-y'-(4\beta/t^2)y'+(4\beta/t^3)y=0, \tag{3.1}$$

which satisfies the conditions of Theorem 1.1 for the zero $\lambda=-1$. That is, Theorem 1.1 implies that there exists a unique solution to (3.1) of the form

$$y=e^{-t}\left\{1+\int_0^\infty e^{-st}\,dU(s)\right\}, \qquad U(0)=U(+0)=1, \tag{3.2}$$

absolutely convergent for $\operatorname{Re} t>0$. Arguing as in the proof of Proposition 2.2, if u denotes the Laplace–Stieltjes transform in the last formula, then

$$u'''-3u''+2u'-(4\beta/t^2)u'+(4\beta/t^2)(1+u)+(4\beta/t^3)(1+u)=0.$$

Thus we get that

$$-s(s^2+3s+2)U'(s)+2\beta\int_0^s[(s^2-r^2)+2(s-r)]U'(r)\,dr+4\beta s+2\beta s^2=0.$$

Differentiating this equation gives

$$-[s(s+1)(s+2)U']'+4\beta(s+1)U=0 \quad \textit{if } U(0)=1.$$

Now if we let $\sigma=2(s+1)^2-1$ and $U=P$, we get Legendre's equation (2.9) again. Hence $U(s)=P(-1+2(s+1)^2)$, and we get the following proposition.

PROPOSITION 3.1. *Whittaker's functions, W_{km}, satisfy the equations*

$$[W_{km}(t)][W_{\bar{k}\bar{m}}(t)]=e^{-t}t^{k+\bar{k}}\left\{1+\int_0^\infty e^{-st}\,d\omega(s)\right\},$$

where the integral is absolutely convergent for $\operatorname{Re} t>0$. *In particular, if $k=0$,*

$$[W_{0m}(t)]^2=e^{-t}\left\{1+\int_0^\infty e^{-st}\,dP_\nu(-1+2(s+1)^2)\right\},$$

$\nu=m-1/2$, and P_ν is the Legendre function of Proposition 2.2.

By Proposition 2.2, we also have that

$$[W_{0m}(t)]^2=\left[e^{-t/2}\left\{1+\int_0^\infty e^{-st}\,dP_\nu(1+2s)\right\}\right]^2,$$

which equals $e^{-t}\{1+\int_0^\infty e^{-st}\,dQ(s)\}$, where if Q is normalized by $Q(0)=Q(+0)=1$,

$$Q(s)=P_\nu(1+2s)+2\int_0^s P_\nu(1+2(s-r))P_\nu'(1+2r)\,dr. \tag{3.3}$$

If we compare this with the result of Proposition 3.1, we arrive at the following functional relationship for $P = P_\nu$.

PROPOSITION 3.2. *For arbitrary ν and real $s \geqq 0$, the Legendre function $P = P_\nu(\sigma)$ satisfies*

$$P(-1+2(s+1)^2) = P(1+2s) + 2\int_0^s P(1+2(s-r))P'(1+2r)\,dr,$$

where $P'(1+2r) = [dP(\sigma)/d\sigma]_{\sigma=1+2r}$.

As a corollary to Propositions 3.1 and 3.2 we get the following.

PROPOSITION 3.3. *The solution to Laguerre's equation, $L = L(t)$, defined in* (2.15), *satisfies the equation*

$$\begin{aligned}(L(t))^2 &= t^{-1}\left\{1 + \int_0^\infty e^{-st}\,dP_{-1/2}(-1+2(s+1)^2)\right\}\\ &= t^{-1}\left\{1 + \int_0^\infty e^{-st}\,dQ(s)\right\},\end{aligned}$$

where $Q(s)$ is as in (3.3), *with $\nu = -1/2$.*

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NONLINEAR AXISYMMETRIC BUCKLED STATES OF SHALLOW SPHERICAL CAPS*

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Abstract. Nonlinear axisymmetric buckled states of a thin, elastic spherical cap satisfying edge conditions for which the spherical shape remains one possible state of equilibrium when the cap is subjected to a uniform radial load are studied by constructive methods. The approach used is based upon the Lyapunov–Schmidt method and shows that buckled states exist near every eigenvalue of the linearized problem.

1. Introduction. In this paper we study the nonlinear axisymmetric buckling of a thin, elastic spherical cap satisfying edge conditions for which the spherical shape remains one possible state of equilibrium (the unbuckled state) when the cap is subjected to a uniform radial load. Our approach is a constructive one that in some cases yields information on the number of buckled states branching at an eigenvalue of the linearized problem and also shows that these states depend *continuously* upon the load parameter. The method is suited to deal with branching problems in which an eigenvalue of the linearized problem has multiplicity exceeding one. For the problem at hand only multiplicities one and two are possible but both of these multiplicities can occur; even the first eigenvalue (the critical buckling load) may be of multiplicity two in some cases. We show rigorously that buckled states branch from the spherical state at every eigenvalue of the linearized problem.

In the axisymmetric case considered here, functions corresponding to stress and deflection are determined by a pair of coupled nonlinear second order ordinary differential equations together with suitable boundary conditions. We are able to reformulate the problem in an appropriate *real* Hilbert space $\mathscr{H}$ so that the basic problem of determining the buckled states of the cap is equivalent to one of finding nontrivial solutions of a single operator equation of the form

$$f - \lambda Af + \alpha^2 A^2 f + \alpha Q(f) + C(f) = 0, \tag{*}$$

where $A:\mathscr{H} \to \mathscr{H}$ is a linear, self-adjoint, positive, compact operator, $Q:\mathscr{H} \to \mathscr{H}$ and $C:\mathscr{H} \to \mathscr{H}$ are continuous, homogeneous, polynomial operators of degree two and three, respectively, $Q(f)$ is the gradient of $\frac{1}{3}(Q(f), f)$ and $C(f)$ is the gradient of $\frac{1}{4}(C(f), f)$. The parameter λ is a measure of the radial load and the constant α is a measure of the maximum depth of the shallow cap.

In order to set the terminology we say that λ is an eigenvalue of the linear operator L_λ, defined by $L_\lambda f \equiv f - \lambda Af + \alpha^2 A^2 f$, if $L_\lambda f = 0$ for some nonzero f in $\mathscr{H}$. The eigenvalues of L_λ are related to the characteristic values μ of A (i.e., of $f - \mu Af = 0$) by $\lambda = \mu + \alpha^2\mu^{-1}$. The operator A has only simple eigenvalues so that the eigenvalues of L_λ are simple unless $\alpha^2 = \mu_m\mu_n$ for distinct eigenvalues

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μ_m, μ_n of A, in which case $\lambda = \mu_m + \mu_n$ has multiplicity two. Let λ_0 be an eigenvalue of L_λ and let $\mathcal{N}$ denote the nullspace of L_{λ_0}. When $\mathcal{N}$ is one-dimensional, the methods used here can be applied (e.g., see [12]) to show that there is exactly one nontrivial buckled state associated with λ_0. We therefore restrict our attention to the more difficult case in which $\mathcal{N}$ has dimension two.

Since $\mathcal{N}$ is two-dimensional and L_λ is self-adjoint, the method of Lyapunov–Schmidt (e.g., see [12], [13], [15]) reduces the problem of solving (*) in $\mathcal{H}$ to that of finding small solutions $\xi = (\xi_1, \xi_2)$, in the Euclidean plane $\mathbb{R}^2$, of a system

$$(**) \quad -\eta\xi_i + (Q(V + \xi_1 v_1 + \xi_2 v_2) + \alpha^{-1}C(V + \xi_1 v_1 + \xi_2 v_2), v_i) = 0 \quad (i = 1, 2).$$

Here $\{v_1, v_2\}$ is a basis for $\mathcal{N}$, orthonormal with respect to the positive operator A (i.e., $(Av_i, v_j) = \delta_{ij}$),

$$\eta = \alpha^{-1}(\lambda - \lambda_0) \tag{1.1}$$

is a real parameter, and $V = V(\xi, \eta)$ is an element of $\mathcal{N}^\perp$ (the orthogonal complement of $\mathcal{N}$ in $\mathcal{H}$) that is analytic in ξ and η for $|\xi| < \rho_0$, $|\eta| < \eta_0$ (e.g., see [15, p. 19]) and satisfies

$$\|V(\xi, \eta)\| \leqq K|\xi|^2 \quad \text{for } |\xi| < \rho_0, \quad |\eta| < \eta_0, \tag{1.2}$$

with constant K depending only on ρ_0 and η_0. Thus the problem of finding buckled states is reduced to the problem of solving an appropriate system of two analytic equations in $\mathbb{R}^2$ involving the load parameter λ.

Of course, finding real solutions of a system of analytic equations in $\mathbb{R}^2$ may be a difficult matter. As an added complication we also wish to show that the solutions vary continuously with the load parameter λ in an interval containing λ_0. We shall show that the appropriate system (**) has, for $|\lambda - \lambda_0|$ sufficiently small, at least one real nontrivial solution $\xi(\lambda)$ continuous in λ with $\xi(\lambda_0) = 0$. Consequently, equation (*) has at least one buckled state associated with each eigenvalue of L_λ, i.e., bifurcation from the spherical state takes place at every eigenvalue of the linearized problem.

In an earlier work on spherical caps, Reiss [11] proves by Poincaré's method that bifurcation takes place at the simple eigenvalues of L_λ and shows that bifurcation from several of the double eigenvalues is indicated. He also conjectures that there are solutions near each double eigenvalue of L_λ. Our principal result shows that this is indeed the case.

The method used to solve the system (**) is one developed in our earlier papers [7], [12], [13]. The relevant equations are first solved in the case $\eta = 0$, then implicit function theorem arguments are given to extend the solution obtained to some interval $|\eta| < \eta_0$. When the Jacobian involved in this argument is not zero, the solution obtained is analytic in η. In the degenerate case recourse is had to a more refined implicit function theorem due to MacMillan [9] and Bliss [4] which is based on the Weierstrass preparation theorem; in this case the solution is Hölder continuous for $|\eta| < \eta_0$ but not necessarily analytic at $\eta = 0$. Nevertheless, it should be pointed out that even continuity in the entire interval $|\eta| < \eta_0$ is a stronger result than those obtained by methods relying only upon topological degree or category arguments (e.g., see the results of Berger [3] for thin shallow elastic shells).

2. Formulation of the problem. The mathematical model adopted here for the problem of the shallow spherical cap is the one used by Reiss [11]. If spherical coordinates (r, θ, ψ) are chosen with polar angle θ measured from the axis of symmetry of the cap and the middle surface of the undeformed cap given by $r = R, 0 \leqq \theta \leqq \theta_0, 0 \leqq \psi \leqq 2\pi$, then the small axisymmetric displacements are assumed to be governed by the nondimensional equations

$$Gf + \lambda f = fg + \alpha g, \tag{2.1a}$$

$$Gg = -\tfrac{1}{2}f^2 - \alpha f, \tag{2.1b}$$

where $Gh(x) \equiv x^{-3}(d/dx)[x^3(dh/dx)]$, $x = \theta\theta_0^{-1}$, $0 < x < 1$, λ is proportional to the constant inward radial pressure applied to the cap and α is proportional to the depth, $R - R\cos\theta_0$, of the cap. Here $f(x) \equiv (\alpha/R\theta)(d/d\theta)w(\theta)$, where w is the radial displacement of the surface, and $g(x)$ is a corresponding "excess" stress function. In addition, symmetry and smoothness at $x = 0$ and a clamped-sliding requirement at the edge $x = 1$ (see [11, p. 69]) yield the boundary conditions

$$f'(0) = g'(0) = f(1) = g(1) = 0. \tag{2.2}$$

DEFINITION 1. A *classical solution of Problem* I is a pair of functions f, g that belong to $C^2(0, 1) \cap C^1[0, 1]$ and satisfy (2.1) and (2.2) pointwise.

We next proceed to a Hilbert space formulation of a generalized solution of Problem I. Let $\mathscr{A}$ denote the linear space of real-valued functions u that belong to $C^1[0, 1]$ and satisfy $u(1) = 0$. An inner product $(\cdot, \cdot)$ and associated norm $\|\cdot\|$ are defined on $\mathscr{A}$ by

$$(u, v) = \int_0^1 u'v'x^3\,dx, \tag{2.3}$$

where a prime denotes differentiation with respect to x (that (2.3) actually defines a norm follows from part (iii) of Lemma 1 below). The Hilbert space $\mathscr{H}$ is then defined as the closure of $\mathscr{A}$ in the norm defined by (2.3).

LEMMA 1. *If f is in $\mathscr{H}$, then*

(i) *f may be identified with a function which is continuous on* $(0, 1]$,

(ii) $f(1) = 0$,

(iii) $[\int_0^1 |f(x)|^3 x^3\,dx]^{1/3} \leqq 2^{-1/2}\|f\|$.

Proof. For f in $\mathscr{A}$ we have

$$f(x) = -\int_x^1 f'(t)\,dt = -\int_x^1 [f'(t)t^{3/2}]t^{-3/2}\,dt, \tag{2.4}$$

so that by Schwarz's inequality,

$$|f(x)| \leqq \|f\|\left[\int_x^1 t^{-3}\,dt\right]^{1/2} \leqq 2^{-1/2}x^{-1}\|f\|. \tag{2.5}$$

For each fixed x in $(0, 1]$ it follows from (2.5) by a standard limiting argument that $f(x)$ is a bounded linear functional on $\mathscr{H}$ and that (2.4) and (2.5) hold for all f in $\mathscr{H}$. Properties (i) and (ii) then follow from (2.4) and property (iii) is obtained directly from (2.5).

Remark 1. It also follows from (2.5) that on $\mathscr{A}$ the norm $\|\cdot\|$ is equivalent to the "weighted" Sobolev norm ${}_\mu\|u\|_{1,2}^2 \equiv \int_0^1 u^2\,d\mu + \int_0^1 (u')^2\,d\mu$, where $d\mu = x^3\,dx$. Thus $\mathscr{H}$ is equivalent to the "weighted" Sobolev space $W_{1,2}^\mu(0,1)$ defined as the completion of $\mathscr{A}$ in the norm ${}_\mu\|\cdot\|_{1,2}$.

Now suppose that $\{f, g\}$ is a classical solution of Problem I and that φ is in $\mathscr{A}$. If we multiply (2.1a) and (2.1b) by φ and integrate over $0 < x < 1$, we obtain

$$(f, \varphi) = \int_0^1 [\lambda f - fg - \alpha g]\varphi x^3\,dx, \tag{2.6a}$$

$$(g, \varphi) = \int_0^1 [\tfrac{1}{2}f^2 + \alpha f]\varphi x^3\,dx, \tag{2.6b}$$

where we have used the relation $(h, \varphi) = -\int_0^1 (Gh)\varphi x^3\,dx$ valid for h in $\mathscr{A} \cap C^2(0,1)$ and φ in $\mathscr{A}$. Equations (2.6) suggest the following definition.

DEFINITION 2. A *generalized solution of Problem* I is a pair of functions f, g that belong to $\mathscr{H}$ and satisfy (2.6) for all φ in $\mathscr{H}$.

LEMMA 2. *A pair of functions f, g is a generalized solution of Problem* I *if and only if it is a classical solution of Problem* I.

This lemma is essentially due to Friedrichs and Stoker [5]; a proof is sketched in Appendix A. Note that the boundary conditions $f'(0) = g'(0) = 0$ are "lost" in Definition 2 and must be regained in the proof of Lemma 2.

From (iii) of Lemma 1, it follows for fixed f, g in $\mathscr{H}$ that each term on the right in (2.6) is a bounded linear functional of φ in $\mathscr{H}$, e.g.,

$$\begin{aligned}\left|\int_0^1 fg\varphi x^3\,dx\right| &\leqq \left(\int_0^1 |f|^3x^3\,dx\right)^{1/3}\left(\int_0^1 |g|^3x^3\,dx\right)^{1/3}\left(\int_0^1 |\varphi|^3x^3\,dx\right)^{1/3} \\ &\leqq 2^{-3/2}\|f\|\,\|g\|\,\|\varphi\|.\end{aligned} \tag{2.7}$$

The Riesz representation theorem then enables us to write (2.6) as a system of operator equations:

$$f = \lambda Af - B(f, g) - \alpha Ag, \tag{2.8a}$$

$$g = \tfrac{1}{2}B(f, f) + \alpha Af, \tag{2.8b}$$

where the linear operator $A:\mathscr{H} \to \mathscr{H}$ and bilinear operator $B:\mathscr{H} \to \mathscr{H}$ are defined by

$$(Af, \varphi) = \int_0^1 f\varphi x^3\,dx \quad \text{and} \quad (B(f, g), \varphi) = \int_0^1 fg\varphi x^3\,dx, \qquad \varphi \in \mathscr{H}. \tag{2.9}$$

Remark 2. We see from (2.9) that for f, g, φ in $\mathscr{H}$, the forms (Af, φ) and $(B(f, g), \varphi)$ are symmetric in all entries. It follows from (2.7) that the operators A and B are bounded, i.e.,

$$|(Af, \varphi)| \leqq 2^{-5/3}\|f\|\,\|\varphi\| \quad \text{and} \quad |(B(f, g), \varphi)| \leqq 2^{-3/2}\|f\|\,\|g\|\,\|\varphi\|.$$

If we now replace g in (2.8a) by the right-hand side of (2.8b) we obtain equation (*), namely,

$$f - \lambda Af + \alpha^2A^2f + \alpha Q(f) + C(f) = 0, \tag{*}$$

where

$$Q(f) \equiv \tfrac{1}{2}AB(f, f) + B(f, Af) \tag{2.10}$$

and

$$C(f) \equiv \tfrac{1}{2}B(f, B(f, f)). \tag{2.11}$$

The system (2.8) is thereby uncoupled; if we solve equation (*) for f in $\mathscr{H}$, then g is determined by means of (2.8b). Such an approach is analogous to that employed previously by Berger and Fife [2] and Berger [3] to treat certain buckling problems in nonlinear elasticity (see also [7]); in particular, an equation analogous to (*) is derived in [3, p. 594].

In view of the above discussion, to determine a nontrivial classical solution of Problem I it suffices to determine a nontrivial solution f in $\mathscr{H}$ of the single operator equation (*).

3. The branching results for Problem I. It is standard to show (using properties of A, Q, C stated below) that nontrivial solutions of (*) can branch from $f = 0$ only at values of λ that are eigenvalues of the linear part of (*). The next two lemmas show that the eigenvalues of L_λ are closely related to the characteristic values of A, i.e., values of μ for which there is an f in $\mathscr{H}$ such that $f \neq 0$ and $f - \mu Af = 0$.

LEMMA 3. *A is a bounded, linear, compact, self-adjoint, positive operator. Its characteristic values are simple; in fact, $\mu_n = \omega_n^2$ $(n = 1, 2, 3 \cdots)$, where ω_n is the n-th positive zero of the Bessel function $J_1(x)$. The eigenfunction corresponding to ω_n^2 is $x^{-1}J_1(\omega_n x)$.*

Proof. From the definition of A in (2.9), it is easy to see that A is bounded, linear, self-adjoint and positive on $\mathscr{H}$. For the sake of completeness we show that A is also compact. It suffices to show that if $\{u_n\}$ in $\mathscr{H}$ is a weakly convergent sequence, then $\{Au_n\}$ has a strongly convergent subsequence. We note first of all that if $\{u_n\}$ in $\mathscr{H}$ is weakly convergent and if $\varphi \in \mathscr{H}$, then Hölder's inequality and part (iii) of Lemma 1 imply

$$\begin{aligned} |(Au_m - Au_n, \varphi)| &= \int_0^1 (u_m - u_n)\varphi x^3\,dx \\ &\leqq 2^{-1/2}\|\varphi\|\left(\int_0^1 |u_m - u_n|^{3/2}x^3\,dx\right)^{2/3}. \end{aligned} \tag{3.1}$$

Thus, if the associated sequence $\{x^2u_n(x)\}$ has a strongly convergent subsequence $\{x^2u_{n_k}(x)\}$ in the Lebesque space $L_{3/2}(0, 1)$ and if we set $\varphi = Au_{n_k} - Au_{n_j}$ in (3.1), then $\{Au_{n_k}\}$ is a Cauchy sequence in $\mathscr{H}$. In order to show that $\{x^2u_n(x)\}$ is sequentially compact in $L_{3/2}(0, 1)$, it suffices to show that $\{x^2u_n(x)\}$ is uniformly bounded in $L_{3/2}(0, 1)$ and the limit

$$\lim_{t\to 0^+} J_n(t) \equiv \lim_{t\to 0^+} \int_0^1 |(x + t)^2u_n(x + t) - x^2u_n(x)|^{3/2}\,dx = 0 \tag{3.2}$$

exists uniformly with respect to n (e.g., see [14, p. 30]). Since $\{u_n\}$ is bounded in $\mathscr{H}$, inequality (2.5) implies the uniform boundedness of $\{x^2u_n(x)\}$ in $L_{3/2}(0, 1)$. In addition, for $x > 0$ and $t > 0$, the representation (2.4) yields

$$
\begin{aligned}
(3.3)\qquad |u_n(x+t) - u_n(x)| &\leqq \|u_n\| \left(\int_x^{x+t} s^{-3}\, ds \right)^{1/2} \\
&\leqq 2^{-1/2} \|u_n\| x^{-2} [t(2x+t)]^{1/2},
\end{aligned}
$$

which together with the inequality

$$
\begin{aligned}
J_n^{2/3}(t) &\leqq \left(\int_0^1 |(x+t)^2 - x^2|^{3/2} u_n(x+t)|^{3/2}\, dx \right)^{2/3} \\
&\quad + \left(\int_0^1 |u_n(x+t) - u_n(x)|^{3/2} x^3\, dx \right)^{2/3} \\
&\leqq 2t \int_0^1 |u_n(x+t)|^{3/2} (x+t)^{3/2}\, dx + 2^{-1/2} \|u_n\| t^{1/2} \left(\int_0^1 (2x+t)^{3/4}\, dx \right)^{2/3} \\
&\leqq (\text{const.})\, t^{1/2}
\end{aligned}
$$

implies the desired uniform limit in (3.2). Thus, A is compact. Finally, it can be shown as in the proof of Lemma 2 that the characteristic value problem for A is equivalent to the classical problem

$$
xf'' + 3f' + \mu x f = 0, \qquad f'(0) = f(1) = 0,
$$

whose eigenvalues and eigenfunctions are those specified in the statement of Lemma 3.

The properties of L_λ which we require are contained in the following.

LEMMA 4. (i) *L_λ is self-adjoint.*

(ii) *λ_0 is an eigenvalue of L_λ if and only if at least one of the two numbers $\mu_+ \equiv \frac{1}{2}(\lambda_0 + \sqrt{\lambda_0^2 - 4\alpha^2})$, $\mu_- \equiv \frac{1}{2}(\lambda_0 - \sqrt{\lambda_0^2 - 4\alpha^2})$ is a characteristic value of A, i.e., if and only if $\lambda_0 = \mu_0 + \alpha^2 \mu_0^{-1}$ for some characteristic value μ_0 of A.*

(iii) *If λ_0 is an eigenvalue of L_λ, then the corresponding eigenfunctions of A (i.e., at μ_+, μ_- or both) are also eigenfunctions of L_{λ_0} and span the nullspace of L_{λ_0}. In particular, λ_0 is a simple eigenvalue of L_λ unless μ_+ and μ_- are distinct characteristic values of A, in which case $\mathcal{N}(L_{\lambda_0})$ is two-dimensional.*

The proof of the lemma follows from the factorizations

$$
L_\lambda = (I - \mu_+ A)(I - \mu_- A) = (I - \mu_- A)(I - \mu_+ A)
$$

and the self-adjoint property of A.

Remark 3. The relationship $\lambda = \mu + \alpha^2 \mu^{-1}$ shows that $\lambda \geqq 2\alpha$ and that a double eigenvalue of L_λ occurs if and only if α^2 is the product of distinct characteristic values of A. The eigenvalue of L_λ of greatest interest is usually the smallest one (related to the load at which the cap first buckles) which has multiplicity two if and only if α^2 is the product of *successive* characteristic values of A.

We collect some properties of the nonlinear operators Q and C in the following lemma; the lemma is easily proved using definitions (2.9) through (2.11) together with Remark 2.

LEMMA 5. (i) *Q is a continuous, homogeneous, polynomial operator of degree two and the gradient of the real-valued functional*

$$\tau(f) \equiv \tfrac{1}{3}(Q(f), f) \qquad f \in \mathscr{H}. \tag{3.4}$$

For each f in $\mathscr{H}$ the operator Q has a differential D_f, which satisfies

$$D_f g = B(f, Ag) + B(g, Af) + AB(f, g)$$

for all g in $\mathscr{H}$, and is Lipschitz continuous in f.

(ii) *C is a continuous, homogeneous, polynomial operator of degree three and the gradient of the functional*

$$\sigma(f) \equiv \tfrac{1}{4}(C(f), f), \qquad f \in \mathscr{H}. \tag{3.5}$$

For each f in $\mathscr{H}$, the operator C has a differential δ_f which satisfies

$$\delta_f g = \tfrac{1}{2}B(g, B(f, f)) + B(f, B(g, f))$$

for all g in $\mathscr{H}$, and is Lipschitz continuous in f.

An additional important property of the functional $\tau(u)$ is expressed in the next lemma, whose proof is outlined in Appendix B.

LEMMA 6. *Let $u_m(x) = x^{-1}J_1(\omega_m x)$ be an eigenfunction of the operator A. Then*

$$0 < \tau(u_m) = (2\omega_m^2)^{-1} \int_0^1 J_1^3(\omega_m t)\, dt. \tag{3.6}$$

In particular, $\tau(u)$ does not vanish identically on any of the nullspaces of the operator L_λ.

We shall also make use of the following lemma due essentially to MacMillan [9] and Bliss [4] (see also [7] and [13]); the indicated Hölder continuity follows from a result of Lojasiewicz [8, p. 123]. It is convenient to state the lemma in connection with the problem of the existence of solutions $x = x(\sigma)$ of systems of the form

$$\Phi^i(x, \sigma) = 0, \qquad x \in \mathbb{R}^n, \quad \sigma \in \mathbb{R}^1 \quad (i = 1, 2, \cdots, n), \tag{3.7}$$

where the Φ^i are real and analytic in a ball $|x|^2 + \sigma^2 < r^2$ in $\mathbb{R}^{n+1}$.

LEMMA 7. *Suppose that $\Phi^i(x, 0) = \varphi^i_{k_i}(x) + \rho^i(x)$ $(i = 1, 2, \cdots, n)$, where the $\varphi^i_{k_i}$ are homogeneous polynomials of degree k_i and the ρ^i satisfy $|\rho^i(x)|/(|x|^{k_i}) \to 0$ as $|x| \to 0$. Suppose that $\prod_{i=1}^n k_i$ is odd and the resultant of the homogeneous polynomials $\varphi^i_{k_i}$ does not vanish. Then there exist positive constants x_0 and σ_0 such that for $|x| < x_0$ and $|\sigma| < \sigma_0$, the system* (3.7) *has at least one Hölder continuous real solution $x = x(\sigma)$ with $x(0) = 0$ and Hölder exponent $h = (\prod_{i=1}^n k_i)^{-1}$.*

Our principal result is contained in the following theorem; the operator S denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{N} \equiv \mathscr{N}(L_{\lambda_0})$.

THEOREM 1. *Let λ_0 be an eigenvalue of L_λ of multiplicity two. Then there exists a positive constant δ such that, for $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$, equation (*) has at least one nontrivial solution $f(\lambda)$ of the form*

$$f(\lambda) = (\alpha a)^{-1}(\lambda - \lambda_0)u^* + U^*. \tag{3.8}$$

Here u^* in the nullspace $\mathcal{N}$ of L_{λ_0} maximizes the functional $\tau(u)$ restricted to the "ellipse" $\mathcal{E} \equiv \{u \in \mathcal{N} : (Au, u) = 1\}$, $a = 3\tau(u^*)$, and U^* in $\mathcal{N}^\perp$, the orthogonal complement of $\mathcal{N}$, depends continuously on λ and satisfies $\lim_{\lambda \to \lambda_0} (\lambda - \lambda_0)^{-1} U^* = 0$. If, in addition, a is not an eigenvalue of the operator SD_{u^*}, then U^* is analytic in λ.

Because of Lemma 2, the solution determined in Theorem 1 yields a classical solution of Problem I so that, taken together with a result of Reiss [11] for the simple eigenvalue case, Theorem 1 has the following result as a corollary.

THEOREM 2. *At every eigenvalue λ_0 of L_λ, there branches from the spherical state of the cap a buckled state that exists for λ in an open interval containing λ_0, i.e., branching is both "upward" and "downward" from every eigenvalue of the linearized problem.*

Remark 4. While it is true that exactly one buckled state branches from the unbuckled solution at a simple eigenvalue, it might happen that multiple branches appear at a double eigenvalue (the computations of Reiss [11] indicate three branches at some of the double eigenvalues). Explicit and complete information on the number of branches can be obtained by the procedure given below in the proof of Theorem 1 (see Remark 5, below) if sufficient information is known regarding the coefficients in the bifurcation equations (**). For the present problem, such information depends upon a knowledge of certain integrals involving the Bessel function $J_1(x)$.

Proof of Theorem 1. The proof is a variation of a general approach used previously by the authors [7], [12], [13] and is based upon the Lyapunov–Schmidt method. The elements f of $\mathcal{H}$ are resolved into components $f = v + V$ with $v \equiv Sf \in \mathcal{N}$ and $V \in \mathcal{N}^\perp$. Equation (*) is similarly decomposed by projection onto $\mathcal{N}$ and $\mathcal{N}^\perp$. The resulting equation on $\mathcal{N}^\perp$ is solved for analytic V satisfying (1.2), by the method of contraction, whenever v is a sufficiently small member of $\mathcal{N}$ and λ is sufficiently close to λ_0. Since λ_0 has multiplicity two, $\mathcal{N}$ is spanned by $f_1 \equiv c_1 x^{-1} J_1(\omega_m x)$ and $f_2 \equiv c_2 x^{-1} J_1(\omega_n x)$, when $\lambda_0 = \omega_m^2 + \omega_n^2$ with $m \neq n$ (see Lemmas 3 and 4); here c_1 and c_2 are arbitrary constants. Let v_1, v_2 be any basis for $\mathcal{N}$ such that $(Av_i, v_j) = \delta_{ij}$ (e.g., $v_1 = f_1$ and $v_2 = f_2$ for suitable c_1, c_2). Then $v \in \mathcal{N}$ may be represented as $v = \xi_1 v_1 + \xi_2 v_2$ and the projection of (*) onto $\mathcal{N}$ by means of S takes the form (**); in particular, we use here that A maps $\mathcal{N}$ into $\mathcal{N}$ and $AS = SA$.

If we now set

$$z = (z_1, z_2) \equiv \eta^{-1}(\xi_1, \xi_2), \qquad \eta = \alpha^{-1}(\lambda - \lambda_0) \neq 0, \tag{3.9}$$

in (**) and cancel a factor η^2, there results the system

$$0 = -z_i + (Q(z_1 v_1 + z_2 v_2 + \eta^{-1} V) + \eta\alpha^{-1} C(z_1 v_1 + z_2 v_2 + \eta^{-1} V), v_i) \qquad (i = 1, 2). \tag{3.10}$$

Formally setting $\eta = 0$ in (3.10) and using (1.2), we obtain the system in $\mathbb{R}^2$.

$$0 = -\beta_i + (Q(\beta_1 v_1 + \beta_2 v_2), v_i) \equiv F_i(\beta) \qquad (i = 1, 2). \tag{3.11}$$

It is clear that every continuous solution $z(\eta)$ of (3.10) yields a continuous solution of (**) by means of the substitution (3.9). On the other hand, if $\beta^* = (\beta_1^*, \beta_2^*)$ is a nontrivial solution of (3.11) at which the Jacobian $\partial(F_1, F_2)/\partial(\beta_1, \beta_2)$ is not zero, then an argument involving the implicit function theorem (e.g., as in Theorems 3 and 4 of [12]) shows that (3.10) has a nontrivial solution $z(\eta)$, analytic in η for $|\eta|$ sufficiently small and satisfying $z(0) = \beta^*$.

In order to simplify the study of the system (3.11) we make a special choice of basis for $\mathcal{N}$. If u^* is defined as in the statement of Theorem 1, we set $v_1 = u^*$ and choose v_2 *on* $\mathscr{E}$ such that $(Av_1, v_2) = 0$. Since $SQ(u)$ is the gradient of $\tau(u)$ on $\mathcal{N}$, at the maximizer v_1, $SQ(v_1) = \text{const.}\, Av_1$ and $(Q(v_1), v_2) = \text{const.}\,(Av_1, v_2) = 0$. It follows that (3.11) reduces to

$$
\begin{aligned}
0 &= -\beta_1 + a\beta_1^2 + c\beta_2^2, \\
0 &= -\beta_2 + 2c\beta_1\beta_2 + e\beta_2^2,
\end{aligned}
\tag{3.12}
$$

where $a = (Q(v_1), v_1), c = (Q(v_2), v_1) = \frac{1}{2}(D_{v_1}(v_2), v_2)$ and $e = (Q(v_2), v_2)$. The real solutions of (3.12) are then given by the intersections of the lines $\beta_2 = 0$ and $-1 + 2c\beta_1 + e\beta_2 = 0$ with the conic $a\beta_1^2 + c\beta_2^2 - \beta_1 = 0$; since $a > 0$ because of Lemma 6, the system (3.12) always has the solution $\beta = (a^{-1}, 0)$ at which $\partial(F_1, F_2)/\partial(\beta_1, \beta_2) = -1 + 2ca^{-1}$. Since SD_{u^*} restricted to $\mathcal{N}$ is diagonalized by the basis $\{v_1, v_2\}$, it is easily seen that the eigenvalues of that transformation are $2a$ and $2c$. From the last two statements we see that when $a \neq 2c$, equation (*) has a nontrivial solution of the form (3.8) which is analytic in λ.

It remains to verify that even in the degenerate case, $a = 2c$, there exists a nontrivial solution of the form (3.8). First of all, for the same special basis v_1, v_2 as above, we note the identity (s and t are real numbers)

$$Q(sv_1 + tv_2) = s^2Q(v_1) + stD_{v_1}(v_2) + t^2Q(v_2).$$

Using this identity, we see that if $a = 2c$ and if $u = (v_1 + tv_2)/(1 + t^2)^{1/2}$, then $u \in \mathscr{E}$ and

$$3\tau(u) = (Q(u), u) = a + et^3 + (15a - 36c)t^4/8 + O(t^5),$$

so that we must have $e = 0$ in order to maximize $\tau(u)$ at $t = 0$. Thus the system (3.10) may be written as

$$
\begin{aligned}
0 &= -z_1 + az_1^2 + \frac{a}{2}z_2^2 + s^{(1)}(z, \eta), \\
0 &= -z_2 + az_1z_2 + s^{(2)}(z, \eta),
\end{aligned}
\tag{3.13}
$$

where $s^{(i)}(z, \eta) \equiv (Q(z_1v_1 + z_2v_2 + \eta^{-1}V) - Q(z_1v_1 + z_2v_2) + \eta\alpha^{-1}C(z_1v_1 + z_2v_2 + \eta^{-1}V), v_i)$ for $i = 1, 2$. In order to apply Lemma 7 we first make the change of variables $x_1 = z_1 - a^{-1}, x_2 = z_2$ to obtain from (3.13),

$$
\begin{aligned}
0 &= x_1 + ax_1^2 + \frac{a}{2}x_2^2 + t^{(1)}(x, \eta) \equiv \Psi^1(x, \eta), \\
0 &= ax_1x_2 + t^{(2)}(x, \eta) \equiv \Psi^2(x, \eta),
\end{aligned}
\tag{3.14}
$$

where $t^{(i)}(x, \eta) \equiv s^{(i)}(z, \eta)$. Next we define $\Phi^1(x, \eta) \equiv \Psi^1(x, \eta)$ and

$$\Phi^2(x, \eta) \equiv \Psi^2(x, \eta) - ax_2\Psi^1(x, \eta) = -ax_1^2x_2 - \frac{a}{2}x_2^3 + q(x, \eta),$$

where $q(x,\eta) \equiv t^{(2)}(x,\eta) - ax_2 t^{(1)}(x,\eta)$. Then, in the notation of Lemma 7, $\varphi_1^1(x) = x_1, \varphi_3^2(x) = -ax_1^2x_2 - (a/2)x_2^3, k_1 \cdot k_2 = 3$ is odd and φ_1^1, φ_3^2 have nonzero resultant since $(x_1, x_2) = (0,0)$ is an isolated zero (in fact the only zero) of the vector field $\{\varphi_1^1(x), \varphi_3^2(x)\}$. Lemma 7 then asserts that the system $\Phi^1(x,\eta) = 0$, $\Phi^2(x,\eta) = 0$ has at least one real Hölder continuous solution $x(\eta)$ for $|\eta| < \eta_0$ with $x(0) = 0$. Then $z(\eta) \equiv (a^{-1} + x_1(\eta), x_2(\eta))$ solves (3.10) for $|\eta| < \eta_0$ and generates the required solution of (*).

Remark 5. By considering the cases $c = 0$ and $c \neq 0$, it is easy to check that, regardless of the particular values of a and e, the system (3.12) has at most a finite number of real solutions. Since $a > 0$, it follows from Theorem 4.3 in [13] (see also [6]), that if $ae^2 + 4c^3 \neq 0$, then *every* solution $\xi(\eta)$ of (**) continuous near $\eta = 0$ with $\xi(0) = 0$ is obtained by means of the substitution (3.9) from a solution $z(\eta)$ of (3.10) continuous near $\eta = 0$ with $\beta \equiv z(0) \neq (0,0)$. Thus, the problem of determining *all* possible continuous solutions of (**) with $\xi(0) = 0$ is reduced to one of verifying certain inequalities among the coefficients a, c and e.

Appendix A. We give here a sketch of the proof of Lemma 2. Since the components f, g of a classical solution of Problem I are easily seen to lie in $\mathscr{H}$ and satisfy (2.6), a classical solution is clearly a generalized solution.

For the converse proposition, we see first of all from Lemma 1 that the components f and g of a generalized solution of Problem I are at least continuous in $(0, 1]$ and satisfy the classical boundary condition (2.2) at $x = 1$. Let us suppose for the moment that we have proved the following lemma.

LEMMA A.1. *The components of a generalized solution of Problem* I *lie in* $C^2(0,1)$.

Then for an arbitrary smooth function φ of compact support in $(0, 1)$, the left sides of (2.6) may be integrated by parts to give

$$\text{(A.1)} \qquad \begin{aligned} &\int_0^1 [Gf + \lambda f - fg - \alpha g]\varphi x^3\,dx = 0, \\ &\int_0^1 [Gg + \tfrac{1}{2}f^2 + \alpha f]\varphi x^3\,dx = 0. \end{aligned}$$

Since the bracketed terms are continuous by Lemma A.1 and φ is sufficiently arbitrary, we easily see that f, g satisfy (2.1) pointwise for $0 < x < 1$; we must also establish that f and g are continuously differentiable at $x = 0$ and $x = 1$ and satisfy $f'(0) = g'(0) = 0$. Each of the equations (2.1) may be written in the form

$$\text{(A.2)} \qquad (x^3h'(x))' = x^3\psi(x),$$

where h is either f or g and ψ is one of the quantities $-\lambda f + fg + \alpha g$ or $-\frac{1}{2}f^2 - \alpha f$. Then, using $h'(x) = h'(\frac{1}{2}) - \int_x^{1/2} h''(t)\,dt$, $h''(t) = t^{-3}[(t^3h')' - 3t^2h']$ and (A.2), we see that

$$\text{(A.3)} \qquad h'(x) = h'(\tfrac{1}{2}) - \int_x^{1/2} \psi(t)\,dt + 3\int_x^{1/2} t^{-1}h'(t)\,dt, \qquad 0 < x < 1.$$

Since $h \in \mathscr{H}$, the right side of (A.3) is continuous up to $x = 1$, so that f and g are at least in $C^1(0, 1]$. We now examine the growth of f and g near $x = 0$. From (2.5) we know that $|\psi(x)| \leqq cx^{-2}$ for $0 < x \leqq 1$; here, and in the sequel, c denotes a positive constant. Hence, for $0 < x < \frac{1}{2}$,

$$|h'(x)| \leqq |h'(\tfrac{1}{2})| + c\left|\int_x^{1/2} t^{-2}\,dt\right| + 3\|h\|\left|\int_x^{1/2} t^{-5}\,dt\right|^{1/2} \leqq cx^{-2}.$$

In particular, $x^3h'(x)$ is continuous on $[0, 1]$ with value zero at $x = 0$. Consequently,

$$x^3h'(x) = \int_0^x (t^3h'(t))'\,dt = \int_0^x t^3\psi(t)\,dt. \tag{A.4}$$

Since $|\psi(x)| \leqq cx^{-2}$, we see from (A.4) that $|h'(x)| \leqq cx^{-1}$ and therefore that $|h(x)| \leqq c(1 + |\log x|)$ and $|\psi(x)| \leqq c(1 + |\log x|)^2$. Use of this latter estimate with (A.4) shows *successively* that $|h'(x)| \leqq c_\varepsilon x^{1-\varepsilon}$ for any $\varepsilon > 0$, $|h(x)| \leqq c$, $|\psi(x)| \leqq c$. These bounds for x near $x = 0$ reveal that f and g are, in fact, continuously differentiable up to $x = 0$ and satisfy $f'(0) = g'(0) = 0$ (an additional iteration in (A.4) even shows that $|h'(x)| \leqq cx$).

It remains to verify Lemma A.1. The proof is basically an application as in [5] of the du Bois–Reymond lemma of the calculus of variations. Equations (2.6) take the form

$$\int_0^1 (x^3h')\varphi'\,dx = -\int_0^1 (x^3\psi)\varphi\,dx \tag{A.5}$$

for all φ in $\mathscr{H}$, where h and ψ are defined as in (A.2); in particular, ψ is continuous on $(0, 1)$. Since φ is sufficiently arbitrary, (A.5) states that the continuous function $x^3\psi$ is the distribution derivative of x^3h', so that x^3h', $(x^3h')'$ and, therefore, h' and h'', may be identified with continuous functions on $(0, 1)$. An alternative approach to establishing Lemma A.1 is to make use of some well-known interior regularity results such as those in Agmon [1] together with some standard Sobolev embedding theorems (e.g., see the discussion in Appendix B of [7] for an analogous but somewhat more involved application of the results in [1]).

Appendix B. Lemma 6 is a special case of a result of Makai [10]. For the convenience of the reader we shall give the ideas here. Since $J_1(x)$ changes sign at ω_n $(n = 1, 2, \cdots)$, to establish $\int_0^{\omega_m} J_1^3(x)\,dx > 0$ it suffices to show that

$$H_n \equiv \int_{\omega_{n-1}}^{\omega_{n+1}} J_1^3(x)\,dx > 0 \tag{B.1}$$

for n odd (we take $\omega_0 \equiv 0$). If we set $z(x) \equiv \sqrt{x}J_1(x)$, then z satisfies $z'' + h(x)z = 0$, $-k \equiv z'(\omega_n) = \sqrt{\omega_n}J_1'(\omega_n) < 0$, where $h(x) \equiv 1 - (3/(4x^2))$. Let $z_1(t) \equiv z(\omega_n - t)$, $z_2(t) \equiv -z(\omega_n + t)$. Then $z_i(t)$ satisfies $z_i''(t) + h(\omega_n + (-1)^i t)z_i(t) = 0$, $z_i(0) = 0$, $z_i'(0) = k > 0$ and $z_i(t) > 0$ for $0 < t < s \equiv \min(\omega_{n+1} - \omega_n, \omega_n - \omega_{n-1})$. Now

$$(z_1'z_2 - z_1z_2')' = [h(\omega_n + t) - h(\omega_n - t)]z_1z_2 > 0, \qquad 0 < t < s,$$

and $z_1'z_2 - z_1z_2' = 0$ at $t = 0$ so that $z_1'z_2 - z_1z_2' > 0$ for $0 < t < s$. Consequently, $z_1(t)/z_2(t)$ is strictly increasing on $0 < t < s$. Since $\lim_{t\to 0^+}(z_1(t)/z_2(t)) = 1$, we have

$$z_1(t) > z_2(t) \quad \text{for } 0 < t \leqq s. \tag{B.2}$$

From (B.2) we see that $s = \omega_{n+1} - \omega_n < \omega_n - \omega_{n-1}$ and also

$$\frac{z_1^3(t)}{(\omega_n - t)^{3/2}} > \frac{z_2^3(t)}{(\omega_n + t)^{3/2}}, \qquad 0 < t \leqq s. \tag{B.3}$$

The inequality (B.1) now follows from (B.3) because

$$H_n = \int_{\omega_{n-1}}^{\omega_n - s} J_1^3(x)\,dx + \int_0^s \left[\frac{z_1^3(t)}{(\omega_n - t)^{3/2}} - \frac{z_2^3(t)}{(\omega_n + t)^{3/2}}\right] dt.$$

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STABILITY OF SYMMETRIC HYPERBOLIC SYSTEMS WITH NONLINEAR FEEDBACK*

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Abstract. This paper studies the stability of distributed parameter systems represented by the symmetric hyperbolic system $\Phi_t + \sum_{i=1}^m A_i\Phi_{x_i} + B\Phi = \mathscr{F}\Phi$, with the nonlinear feedback operator $\mathscr{F}$. The trajectory solution is derived, with the system states as elements in the space of square integrable functions. Stability and instability criteria are obtained for the trajectory solutions. Also, global and local asymptotic and L_2-stability criteria, and analogues of Lyapunov's first method are derived.

1. Introduction. This paper is concerned with the stability and instability of distributed parameter systems represented by systems of symmetric hyperbolic partial differential equations with nonlinear feedback terms (which will always include the special case of linear feedback terms). The importance of symmetric hyperbolic systems is due to the fact that many equations of mathematical physics occur in this form. Specifically, every second order hyperbolic equation can be reduced to a symmetric hyperbolic system (see Courant–Hilbert [2]).

In the recent literature several authors have investigated the stability of distributed parameter systems. Stability criteria, applicable mainly to equations of the parabolic type are due, among others, to Kastenberg [5], [6], [7], Crawford and Kastenberg [1], Wang [10], [11], [12] and Zubov [13]. The work of Wang [9] is also applicable to some types of linear symmetric hyperbolic systems. The objective of this paper is to derive stability and instability criteria for symmetric hyperbolic systems with nonlinear terms.

In general the methods used in the literature follow corresponding procedures for ordinary differential equations. The existence of the classical pointwise solution is assumed and stability criteria are then obtained for this solution. Whereas the assumption of the existence of the classical pointwise solution is generally justified for systems represented by ordinary differential equations, this may not always be true for partial differential equations and in particular for our case of quasi-linear symmetric hyperbolic systems. We shall derive instead the generalized "strong" trajectory solution, such that the system states are elements in the space of square integrable functions. The stability criteria are then developed for this generalized solution.

This paper is restricted to the initial value problem, for which the strong trajectory solution and the corresponding stability criteria are derived. However, it is readily seen (even though this will not be done here) that if the existence of the generalized solution of the initial-boundary value problem with appropriate homogeneous data is assumed, then the stability criteria can be extended to that case. It should also be noted that from the physical point of view the solution of the initial value problem, with the system states in the space of square integrable functions, may be considered as an approximation of the dynamic behavior of the given system, with the boundary conditions requiring the solution to vanish at points sufficiently far out in space.

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In § 2 we introduce continuous trajectories in the space of square integrable functions, and in § 3 the initial value problem is shown to be well-posed in the space of these trajectories. Our main tool is the theory of strong solutions of linear symmetric hyperbolic systems due to Friedrichs [4] (see also Lax [8]). In § 4 three types of stability are considered, namely, Lyapunov stability, asymptotic stability and L_2-stability. Section 5 considers the existence of local solutions and criteria for local asymptotic and local L_2-stability.

2. Preliminaries. We shall consider in this section some preliminary concepts which will be required in the analysis of the system of equations, to be introduced in the next section. Throughout the paper we use the following notations: S is the m-dimensional Euclidean space with the points $x \equiv (x_1, \cdots, x_m)$, $-\infty < x_i < \infty$. H denotes the $(m+1)$-dimensional half-space $(x, t) \equiv (x_1, \cdots, x_m, t)$, $0 \leqq t < \infty$. H_T is the slab $0 \leqq t \leqq T$, $-\infty < x_i < \infty$, and S_T is the hyperplane $t = T$. $U \equiv (u_1, \cdots, u_n)$ will denote n-dimensional vector functions of the variables x or (x, t). A function is said to be *smooth* if its components are continuous and square integrable in the domain under consideration. A function is *very smooth* if its components are in C^∞ and vanish for sufficiently large values of $|x|$, where $|\cdot|$ denotes the Euclidean norm.

$\mathcal{S}$ will denote the Hilbert space of square integrable functions on S with inner product and norm

$$\begin{aligned} &\{U, V\} = \int_S U \cdot V \, dx, \qquad U \cdot V \equiv u_1 v_1 + \cdots + u_n v_n, \quad dx \equiv dx_1 \cdots dx_m, \\ &|||U|||^2 = \{U, U\}. \end{aligned} \tag{1}$$

$\mathcal{H}_T$ is the Hilbert space of square integrable functions in the slab H_T with inner product and norm:

$$[W, R]_T = \iint_{H_T} W \cdot R \, dx \, dt, \qquad \|W\|_T^2 = [W, W]_T. \tag{2}$$

DEFINITION 1. A *trajectory*, denoted by $\Phi(x; t)$, is defined as a mapping from $0 \leqq t < \infty$ into the state space $\mathcal{S}$, $t \to \Phi(x; t)$. Further, the trajectory is *continuous* at t_0 if

$$\lim_{t \to t_0} |||\Phi(\cdot\,; t) - \Phi(\cdot\,; t_0)||| = 0. \tag{3}$$

A *continuous trajectory* is a trajectory which is continuous at all $t \geqq 0$.

LEMMA 1. *If $\Phi(x; t)$ is a continuous trajectory, then for every $T > 0$, there exists a unique function $U \in \mathcal{H}_T$, such that U can be identified with the trajectory in the interval $0 \leqq t \leqq T$, in the following sense. There exists a sequence of functions $U^{(k)}$, smooth in H_T, such that*

$$\|U^{(k)} - U\|_T \to 0 \tag{4}$$

and

$$|||U^{(k)}(\cdot\,, t) - \Phi(\cdot\,; t)||| \to 0, \tag{5}$$

uniformly for all $t \in [0, T]$.

Proof. We apply the *integral mollifiers*, due to Friedrichs [3], to the x-variables of $\Phi(x; t)$. Let $j(x)$ be an infinitely continuously differentiable nonnegative function of x with support in the sphere $|x|<1$, and such that $\int j(x)\,dx = 1$. We set

$$U^{(k)}(x, t) = k^m \int_{-\infty}^{\infty} j(k(x-\bar{x}))\Phi(\bar{x}; t)\,d\bar{x}. \tag{6}$$

For fixed t, $U^{(k)}(x, t)$ is infinitely continuously differentiable in the x-variables. Furthermore,

$$\begin{aligned} &|||U^{(k)}(\cdot\,, t)||| \leqq |||\Phi(\cdot\,; t)|||, \\ &|||U^{(k)}(\cdot\,, t) - \Phi(\cdot\,; t)||| \to 0 \quad \text{as } k \to \infty, \\ &|U^{(k)}(x, t)|^2 \leqq k^m \sup j(y) |||\Phi(\cdot\,; t)|||^2. \end{aligned} \tag{7}$$

From these properties of the mollifiers and the assumed continuity of the given trajectory, it follows that the functions $U^{(k)}(x, t)$ are smooth (pointwise continuous and square integrable in H_T) and, as functions in $\mathcal{S}$, form an equicontinuous sequence for $t \in [0, T]$. The Arzela–Ascoli theorem now implies that the convergence in $\mathcal{S}$ of $U^{(k)}(x, t)$ to Φ is uniform for $t \in [0, T]$. It follows that $U^{(k)}$ converges in $\mathcal{H}_T$ to a unique function U, which is the required function in the statement of the lemma. This completes the proof.

From the uniform convergence of $U^{(k)}$ we have the following lemma.

LEMMA 2. *If the continuous trajectory Φ corresponds in* $[0, T]$ *to the function* $U \in \mathcal{H}_T$, *in the sense of Lemma 1, then*

$$\|U\|_T^2 = \int_0^T |||\Phi|||^2\,dt \tag{8}$$

and

$$[U_1, U_2]_T = \int_0^T \{\Phi_1, \Phi_2\}\,dt. \tag{9}$$

The implication of Lemmas 1 and 2 is that for a given continuous trajectory the restriction to the interval $[0, T]$ is a function in $\mathcal{H}_T$ with the norm and inner product given by (8) and (9).

We shall utilize in this paper the results regarding *strong solutions* of linear symmetric hyperbolic systems, due to Friedrichs [4] (see also Lax [8]). We now summarize these results in the form in which they will be used here. Given in the slab H_T, $T>0$, the linear system

$$\mathscr{E}U \equiv U_t + \sum_{i=1}^{m} C_i U_{x_i} + DU = G, \qquad G \in \mathcal{H}_T, \tag{10}$$

where

$$U_t = \frac{\partial U}{\partial t}, \qquad U_{x_i} = \frac{\partial U}{\partial x_i}.$$

$C_i(x, t)$ are $n \times n$ symmetric matrices with elements that are bounded and have bounded continuous derivatives. D is an $n \times n$ matrix with bounded continuous elements. The initial data, given on $t=0$, is $U = U_o \in \mathcal{S}$. The principal result states that there exists a unique strong solution of this system, satisfying the initial data in the strong sense. Specifically, there exists a unique function $U \in \mathcal{H}_T$ and a sequence $W^{(k)}$ of very smooth functions in H_T, such that

$$\|\mathscr{E}W^{(k)} - G\|_T + \|W^{(k)} - U\|_T + |||W^{(k)}(\cdot, 0) - U_0||| \to 0.$$

We shall also need the following, well-known, *energy identity* which we derive for the sake of completeness. If V is very smooth in H_T, then for $0 < t \leqq T$,

$$[V, \mathscr{E}V]_t = \tfrac{1}{2}|||V(\cdot, t)|||^2 - \tfrac{1}{2}|||V(\cdot, 0)|||^2 - \tfrac{1}{2}[V, \sum (C_i)_{x_i}V - 2DV]_t. \tag{11}$$

Since the matrices C_i are symmetric and V vanishes for large $|x|$, it follows from integration by parts that

$$[V, C_iV_{x_i}]_t = -[V_{x_i}, C_iV]_t - [V, (C_i)_{x_i}V]_t = -[V, C_iV_{x_i}]_t - [V, (C_i)_{x_i}V]_t.$$

Hence,

$$[V, C_iV_{x_i}]_t = -\tfrac{1}{2}[V, (C_i)_{x_i}V]_t.$$

Also

$$[V, V_t]_t = \tfrac{1}{2}|||V(\cdot, t)|||^2 - \tfrac{1}{2}|||V(\cdot, 0)|||^2.$$

These identities imply (11).

LEMMA 3. *The strong solution $U(x, t)$ of* (10) *with strong initial data U_0 belongs to the space $\mathscr{S}$ for every t, $0 \leqq t \leqq T$, and is a continuous trajectory in this interval. Furthermore, U satisfies the identity:*

$$|||U(\cdot, t)|||^2 = |||U_0|||^2 + [U, \sum (C_i)_{x_i}U - 2DU + 2G]_t. \tag{12}$$

Proof. We apply the energy identity (11) to the approximating functions $W^{(k)}$. It follows that the sequence converges uniformly for $t \in [0, T]$. The limit is therefore a continuous trajectory in this interval. It follows from Lemma 1 that U is this continuous trajectory. Finally, (12) follows from the energy identity for $W^{(k)}$ as $k \to \infty$.

3. Strong trajectory solution. We introduce the system of equations representing the distributed parameter system with the *nonlinear feedback operator* $\mathscr{F}$:

$$\Phi_t + \sum_{i=1}^{m} A_i\Phi_{x_i} + B\Phi = \mathscr{F}\Phi. \tag{13}$$

Adopting systematically the summation convention for double indices, (13) will be written as

$$\Phi_t + A_i\Phi_{x_i} + B\Phi = \mathscr{F}\Phi.$$

$A_i = A_i(x, t)$, $i = 1, \cdots, m$, are symmetric $n \times n$ matrices with continuously differentiable elements, which together with their first order derivatives are bounded in H_T for every $T > 0$. $B = B(x, t)$ is an $n \times n$ matrix with continuous and bounded elements in H_T. The feedback operator $\mathscr{F} = \mathscr{F}(t)$ is assumed to have the following properties:

(a) For fixed $t \geqq 0$, $\mathscr{F}$ maps the space $\mathscr{S}$ into itself; that is, if $\psi \in \mathscr{S}$, then $\mathscr{F}\psi \in \mathscr{S}$.

(b) For every $t \geqq 0$, $\mathscr{F}$ satisfies a Lipschitz condition,

$$|||\mathscr{F}(t)\psi_1 - \mathscr{F}(t)\psi_2||| \leqq K(t)|||\psi_1 - \psi_2|||, \tag{14}$$

where $K(t)$ is a positive continuous nondecreasing function, which will be referred to as a Lipschitz function.

(c) $\mathscr{F}$ depends continuously on t, uniformly in $\mathscr{S}$. Specifically, given $\varepsilon > 0$ and $t_0 \geqq 0$, there exists $\delta > 0$ such that for $|t - t_0| < \delta$ and all $\psi \in \mathscr{S}$,

$$|||\mathscr{F}(t)\psi - \mathscr{F}(t_0)\psi||| < \varepsilon\, |||\psi|||. \tag{15}$$

LEMMA 4. *$\mathscr{F}$ maps continuous trajectories into continuous trajectories.*

Proof. If Φ is a continuous trajectory, then by property (a), $\mathscr{F}\Phi$ is a trajectory and therefore only the continuity has to be shown. Indeed,

$$\begin{aligned} &|||\mathscr{F}(t)\Phi(\,\cdot\,;t) - \mathscr{F}(t_0)\Phi(\,\cdot\,;t_0)|||\\ &\quad \leqq |||\mathscr{F}(t)\Phi(\,\cdot\,;t) - \mathscr{F}(t)\Phi(\,\cdot\,;t_0)||| + |||\mathscr{F}(t)\Phi(\,\cdot\,;t_0) - \mathscr{F}(t_0)\Phi(\,\cdot\,;t_0)|||\\ &\quad \leqq K(t)|||\Phi(\,\cdot\,;t) - \Phi(\,\cdot\,;t_0)||| + |||\mathscr{F}(t)\Phi(\,\cdot\,;t_0) - \mathscr{F}(t_0)\Phi(\,\cdot\,;t_0)|||. \end{aligned}$$

As $t \to t_0$, the right-hand side approaches zero by the continuity of the trajectory and by (15). This proves the lemma.

Let the operators $\mathscr{L}_0$ and $\mathscr{L}_{\mathscr{F}}$ be defined as follows:

$$\mathscr{L}_0 \equiv \frac{\partial}{\partial t} + A_i \frac{\partial}{\partial x_i} + B, \qquad \mathscr{L}_{\mathscr{F}} \equiv \mathscr{L}_0 - \mathscr{F}.$$

The system (13) can now be written in the form

$$\mathscr{L}_{\mathscr{F}}\Phi \equiv (\mathscr{L}_0 - \mathscr{F})\Phi = 0.$$

We proceed to define the concept of a strong trajectory solution of the initial value problem for the system (13) in the space of continuous trajectories.

DEFINITION 2. The trajectory $\Phi(x;t)$ is a *strong trajectory solution* (or simply, a *solution*) of the system $\mathscr{L}_{\mathscr{F}}\Phi = 0$ with the *initial state* $\Phi_0 \in \mathscr{S}$, if $\Phi(x;0) = \Phi_0$ and if, for every $T > 0$, there exists a sequence $U^{(k)}(x,t)$, of very smooth functions in H_T, such that as $k \to \infty$,

$$|||\Phi(\,\cdot\,;t) - U^{(k)}(\,\cdot\,,t)||| \to 0 \quad \text{uniformly for } t \in [0,T]. \tag{16}$$

and

$$\|U_t^{(k)} + A_i U_{x_i}^{(k)} + BU^{(k)} - \mathscr{F}\Phi\|_T \to 0. \tag{17}$$

It follows from the uniform convergence condition (16) that a strong trajectory solution is a continuous trajectory.

THEOREM 1. *There exists a unique strong trajectory solution of the system*

$$\mathscr{L}_{\mathscr{F}}\Phi \equiv \Phi_t + A_i\Phi_{x_i} + B\Phi - \mathscr{F}\Phi = 0 \tag{18}$$

with the initial state

$$\Phi(x;0) = \Phi_0 \in \mathscr{S}. \tag{19}$$

Proof. We denote by $\mathscr{P}_T$, $T > 0$, the space of continuous trajectories in $[0,T]$. For $\zeta \in \mathscr{P}_T$ let ω denote the strong solution in $\mathscr{H}_T$ of the system $\mathscr{L}_0\omega = \mathscr{F}\zeta$ with initial data Φ_0. It follows from Lemma 3 that $\omega \in \mathcal{P}_T$. We now introduce the mapping $\mathcal{T}_T$ of $\mathcal{P}_T$ into itself: $\omega = \mathcal{T}_T\zeta$. It will be shown that $\mathcal{T}_T$ has a fixed point in $\mathcal{P}_T$. Let $\omega_1 = \mathcal{T}_T\zeta_1$ and $\omega_2 = \mathcal{T}_T\zeta_2$; then it follows from (12) that

$$|||\omega_1(\,\cdot\,;t)-\omega_2(\,\cdot\,;t)|||^2 \leqq \gamma(t)(\|\zeta_1-\zeta_2\|_t^2+\|\omega_1-\omega_2\|_t^2), \tag{20}$$

where $\gamma(t)$ is a positive nondecreasing function dependent on the Lipschitz function $K(t)$ of the operator $\mathcal{F}$, and on the coefficients of the matrices A_i and B in H_T.

We introduce the norm $\|\cdot\|_{T,\lambda}$ in $\mathcal{H}_T$, defined by $\|V\|_{T,\lambda} = \|\exp(-\lambda t/2)V\|_T$, with $\lambda > 0$ a constant at our disposal. Clearly the norms $\|\cdot\|_T$ and $\|\cdot\|_{T,\lambda}$ are equivalent in $\mathcal{H}_T$. We multiply (20) by $\exp(-\lambda t)$ and integrate from 0 to T, using integration by parts on the right. It follows that

$$\|\omega_1-\omega_2\|_{T,\lambda}^2 \leqq \frac{\gamma(t)}{\lambda}(\|\zeta_1-\zeta_2\|_{T,\lambda}^2+\|\omega_1-\omega_2\|_{T,\lambda}^2).$$

We now choose λ such that $\gamma(T)/\lambda < 1/5$. Then

$$\|\omega_1-\omega_2\|_{T,\lambda} \leqq \tfrac{1}{2}\|\zeta_1-\zeta_2\|_{T,\lambda}. \tag{21}$$

Since $\mathcal{P}_T$ is dense in $\mathcal{H}_T$, the mapping $\mathcal{T}_t$ can be extended to a mapping of $\mathcal{H}_T$ into itself with (21) continuing to hold. This contraction mapping has a fixed point $\sigma \in \mathcal{H}_T$, such that $\sigma = \mathcal{T}_T\sigma$. Next we show that σ is a continuous trajectory in the interval $[0, T]$, and that it is a strong trajectory solution of (18) with the initial state (19). There exists a sequence $\sigma^{(k)} \in \mathcal{P}_T$ such that $\sigma^{(k)} = \mathcal{T}_T\sigma^{(k-1)}$ and $\|\sigma-\sigma^{(k)}\|_T \to 0$. For each k there exists a very smooth function in H_T, which we denote by $V^{(k)}(x, t)$, such that

$$\begin{gathered} |||\sigma^{(k)}(\,\cdot\,;t)-V^{(k)}(\,\cdot\,;t)||| \leqq \frac{1}{k} \quad \text{for all } t\in[0, T], \\ \|\mathcal{L}_0V^{(k)}-\mathcal{F}\sigma^{(k-1)}\|_T \leqq \frac{1}{k}, \qquad |||V^{(k)}(x,0)-\Phi_0||| \leqq \frac{1}{k}. \end{gathered} \tag{22}$$

This implies that as $k \to \infty$,

$$\|\sigma - V^{(k)}\|_T \to 0 \quad \text{and} \quad \|\mathcal{L}_0V^{(k)}-\mathcal{F}V^{(k)}\|_T \to 0. \tag{23}$$

We apply the energy identity (11) to $V^{(k)}$. It then follows from (23) that for fixed t, $V^{(k)}(x, t)$ converges in $\mathcal{S}$, and the convergence is uniform for all $t\in[0, T]$. σ is therefore a continuous trajectory, and $\|\mathcal{L}_0V^{(k)}-\mathcal{F}\sigma\|_T \to 0$. Hence σ is a strong trajectory solution, in the interval $[0, T]$ of (18) with the initial state (19). It now follows from (11) that σ satisfies the identity

$$|||\sigma(\,\cdot\,;t)|||^2 = |||\Phi_0|||^2 + [\sigma, ((A_i)_{x_i}-2B)\sigma]_t + 2[\sigma, \mathcal{F}\sigma]_t. \tag{24}$$

Finally, we show the uniqueness of the trajectory solution and its independence of the particular value of T. If σ_1 and σ_2 are solutions, in H_{T_1} and H_{T_2}, respectively, with $T_1 \leqq T_2$, of the system (18) with the initial state (19), then it follows from (24) that for $t \leqq T_1$,

$$|||\sigma_1(\cdot\,;t)-\sigma_2(\cdot\,;t)|||^2=[\sigma_1-\sigma_2,((A_i)_{x_i}-2B)(\sigma_1-\sigma_2)]_t$$
$$+2[\sigma_1-\sigma_2,\mathscr{F}\sigma_1-\mathscr{F}\sigma_2]_t.$$

Hence,

$$|||\sigma_1(\cdot\,;t)-\sigma_2(\cdot\,;t)|||^2\leqq\gamma(t)\|\sigma_1-\sigma_2\|_t^2$$
$$=\gamma(t)\int_0^t|||\sigma_1(\cdot\,;\tau)-\sigma_2(\cdot\,;\tau)|||^2\,d\tau.$$

It follows that $|||\sigma_1(\cdot\,;t)-\sigma_2(\cdot\,;t)|||=0$. This shows the uniqueness of the strong trajectory solution σ and its independence of the particular value of T. We now let $T\to\infty$, which yields the strong trajectory solution of (18) with initial state (19) for $0\leqq t<\infty$. This completes the proof of Theorem 1.

Now, σ satisfies (24) which by (9) can be written as

$$|||\sigma(\cdot\,;t)|||^2=|||\Phi_0|||^2+\int_0^t\{\sigma,((A_i)_{x_i}-2B)\sigma+2\mathscr{F}\sigma\}\,d\tau.$$

The integrand is a continuous function of τ, since σ and $\mathscr{F}\sigma$ are continuous trajectories. This implies the following.

COROLLARY 1.1. *The norm* $|||\sigma(\cdot\,;t)|||$ *of the strong trajectory solution of system* (18) *satisfies the ordinary differential equation*

$$\frac{d}{dt}|||\sigma(\cdot\,;t)|||^2=\{\sigma,((A_i)_{x_i}-2B)\sigma+2\mathscr{F}\sigma\}. \tag{25}$$

4. Stability. For the study of stability it will now be assumed that the given system has been formulated about an equilibrium state. Specifically, it is assumed that $\mathscr{F}0=0$, and therefore $\Phi=0$ is an equilibrium state. We introduce the following types of stability:

DEFINITION 3. (a) The trivial solution of the system $\mathscr{L}_{\mathscr{F}}\Phi=0$ is (Lyapunov) *stable* if for given $\varepsilon>0$, there exists $\eta>0$ such that for all initial states Φ_0, with $|||\Phi_0|||<\eta$, the solution Φ satisfies $|||\Phi|||<\varepsilon$ for all $0\leqq t<\infty$.

(b) The trivial solution is *globally asymptotically stable*, if it is stable, and if, for all arbitrary initial states, $|||\Phi|||\to0$ as $t\to\infty$.

(c) The trivial solution is *locally asymptotically stable*, if it is stable, and if there exists $\eta>0$ such that for all initial states with $|||\Phi_0|||<\eta$ the solution satisfies $|||\Phi|||\to0$ as $t\to\infty$.

(d) The trivial solution is *globally L_2-stable* if, for any arbitrary initial state the solution satisfies $\int_0^\infty|||\Phi|||^2\,dt<\infty$.

(e) The trivial solution is *locally L_2-stable*, if there exists $\eta>0$ such that $|||\Phi_0|||<\eta$ implies that $\int_0^\infty|||\Phi|||^2\,dt<\infty$.

Instability will be considered as the negation of stability. In particular, the trivial solution is defined to be *unstable*, if it is not stable.

We now develop stability criteria.

DEFINITION 4. The *upper response* of the operator $\mathcal{L}_{\mathcal{F}}$, denoted by $\alpha_{\mathcal{F}}(t)$, at time $t \geqq 0$, is defined by

$$\alpha_{\mathcal{F}}(t) = \sup_{\psi \in \mathcal{S}} \{\psi, ((A_i)_{x_i} - 2B)\psi + 2\mathcal{F}\psi\}/|||\psi|||^2. \tag{26}$$

LEMMA 5. *$\alpha_{\mathcal{F}}(t)$ is a continuous function of t.*

Proof. We denote $\mathcal{R}(t) \equiv (A_i)_{x_i} - 2B + 2\mathcal{F}$. For all $\psi \in \mathcal{S}$ and sufficiently small $|t - t_0|$,

$$\{\psi, \mathcal{R}(t)\psi - \mathcal{R}(t_0)\psi\}/|||\psi|||^2 \leqq |||\mathcal{R}(t)\psi - \mathcal{R}(t_0)\psi|||/|||\psi||| < \varepsilon.$$

This follows from (15) and the properties of A_i and B. Hence $\{\psi, \mathcal{R}(t)\psi\}/|||\psi|||^2 - \alpha_{\mathcal{F}}(t_0) \leqq \varepsilon$. Since this holds for all $\psi \in \mathcal{S}$, it also holds for the supremum. Hence $\alpha_{\mathcal{F}}(t) - \alpha_{\mathcal{F}}(t_0) \leqq \varepsilon$. In the same way we also have $\alpha_{\mathcal{F}}(t_0) - \alpha_{\mathcal{F}}(t) \leqq \varepsilon$. This concludes the proof.

DEFINITION 5. The *lower response* of the operator $\mathcal{L}_{\mathcal{F}}$, denoted by $\beta_{\mathcal{F}}(t)$, at time $t \geqq 0$, is defined by

$$\beta_{\mathcal{F}}(t) = \inf_{\psi \in \mathcal{S}} \{\psi, ((A_i)_{x_i} - 2B)\psi + 2\mathcal{F}\psi\}/|||\psi|||^2. \tag{27}$$

The proof of the next lemma is analogous to the proof of Lemma 5.

LEMMA 6. *$\beta_{\mathcal{F}}(t)$ is a continuous function of t.*

THEOREM 2. (i) *If $\int_0^t \alpha_{\mathcal{F}}(\tau)\, d\tau \leqq M < \infty$ for all $t > 0$, then the trivial solution is stable.*

(ii) *If $\int_0^\infty \alpha_{\mathcal{F}}(\tau)\, d\tau = -\infty$, then the trivial solution is globally asymptotically stable.*

(iii) *If $\int_0^\infty \exp\left(\int_0^t \alpha_{\mathcal{F}}(\tau)\, d\tau\right) dt < \infty$, then the trivial solution is globally L_2-stable.*

Proof. It follows from (25) that

$$\frac{d}{dt}|||\Phi(\cdot\,;t)|||^2 \leqq \alpha_{\mathcal{F}}(t)|||\Phi(\cdot\,;t)|||^2.$$

Hence,

$$|||\Phi(\cdot\,;t)|||^2 \leqq |||\Phi_0|||^2 \exp\left(\int_0^t \alpha_{\mathcal{F}}(\tau)\, d\tau\right). \tag{28}$$

In case (i) it follows from (28) that $|||\Phi(\cdot\,,t)|||^2 \leqq |||\Phi_0|||^2\, e^M$, which implies stability. In case (ii), in addition to stability, (28) implies that $|||\Phi||| \to 0$, for arbitrary Φ_0, which shows that the trivial solution is globally asymptotically stable. In case (iii), (28) clearly implies that $\int_0^\infty |||\Phi|||^2\, dt < \infty$. This completes the proof.

THEOREM 3. (i) *If $\limsup_{t\to\infty} \int_0^t \beta_{\mathcal{F}}(\tau)\, d\tau = \infty$, then the trivial solution is unstable.*

(ii) *If $\int_0^t \beta_{\mathcal{F}}(\tau)\, d\tau \geqq N > -\infty$ for all $t > 0$, then the trivial solution is not globally or locally asymptotically stable.*

(iii) *If $\int_0^\infty \exp\left(\int_0^t \beta_{\mathcal{F}}(\tau)\, d\tau\right) dt = \infty$, then the trivial solution is not globally or locally L_2-stable.*

Proof. It follows from (25) that

$$\frac{d}{dt}|||\Phi(\cdot\,;t)|||^2 \geqq \beta_{\mathscr{F}}(t)|||\Phi(\cdot\,;t)|||^2.$$

Hence,

$$|||\Phi(\cdot\,;t)|||^2 \geqq |||\Phi_0|||^2 \exp\left(\int_0^t \beta_{\mathscr{F}}(\tau)\,d\tau.\right) \tag{29}$$

In case (i), (29) implies that for $\Phi_0 \neq 0$ the solution is unbounded, in the norm, as $t \to \infty$. Hence the trivial solution is unstable. In case (ii), the trivial solution may be stable, but for arbitrary $\Phi_0 \neq 0$ it follows from (29) that $|||\Phi|||^2 \geqq |||\Phi_0|||^2\, e^N > 0$, and hence the trivial solution is neither globally nor locally asymptotically stable. In case (iii), (29) implies that $\int_0^\infty |||\Phi|||^2\,dt = \infty$. This completes the proof.

5. Local stability. Local asymptotic and local L_2-stability are related to the local behavior of the feedback operator $\mathscr{F}$.

For $\rho > 0$, let $\mathscr{S}_\rho$ denote the sphere of radius ρ in $\mathscr{S}$, that is, the set of all $\psi \in \mathscr{S}$ such that $|||\psi||| \leqq \rho$. In addition to the previous stipulation that $\mathscr{F}0 = 0$, it is now required throughout this section (except when specifically stated otherwise) that $\mathscr{F}$ satisfies only local Lipschitz and local continuity conditions. We assume that there exists $\rho > 0$, such that for $\psi_1, \psi_2 \in \mathscr{S}_\rho$,

$$|||\mathscr{F}\psi_1 - \mathscr{F}\psi_2||| \leqq K(t)|||\psi_1 - \psi_2|||. \tag{30}$$

Also, for $t_0 \geqq 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for $|t - t_0| < \delta$ and all $\psi \in \mathscr{S}_\rho$,

$$|||\mathscr{F}(t)\psi - \mathscr{F}(t_0)\psi||| \leqq \varepsilon|||\psi|||. \tag{31}$$

The existence of the solution is connected to the stability in the following way. Even if the initial state lies in the sphere $\mathscr{S}_\rho$, the solution of $\mathscr{L}_{\mathscr{F}}\Phi = 0$, for increasing t, may not remain in $\mathscr{S}_\rho$, and may therefore cease to exist.

We now denote by $\alpha_{\mathscr{F}}(t;\rho)$ the *local upper response* of $\mathscr{F}$ in the sphere $\mathscr{S}_\rho$:

$$\alpha_{\mathscr{F}}(t;\rho) = \sup_{\psi \in \mathscr{S}_\rho} \{\psi, ((A_i)_{x_i} - 2B)\psi + 2\mathscr{F}\psi\}/|||\psi|||^2. \tag{32}$$

We now consider the existence of the local solution, that is, the existence of the strong trajectory solution of $\mathscr{L}_{\mathscr{F}}\Phi = 0$, provided the initial state Φ_0 is such that $|||\Phi_0||| < \eta$, where $\eta > 0$ is sufficiently small.

THEOREM 4. *If $\mathscr{F}$ satisfies the local Lipschitz and continuity conditions* (30) *and* (31) *in the sphere $\mathscr{S}_\rho$, and if the local upper response* (32) *satisfies $\int_0^t \alpha_{\mathscr{F}}(\tau;\rho)\,d\tau \leqq M < \infty$ for all $t \geqq 0$, then there exists $\eta = \eta(\rho) > 0$ such that the system $\mathscr{L}_{\mathscr{F}}\Phi = 0$ has a strong trajectory solution for all initial states Φ_0, with $|||\Phi_0||| < \eta$.*

Proof. We introduce the auxiliary feedback operator $\bar{\mathscr{F}}$, which coincides with $\mathscr{F}$ in $\mathscr{S}_\rho$ and satisfies the Lipschitz and continuity conditions (14), (15) for all $\mathscr{S}$:

$$\bar{\mathscr{F}}\psi = \begin{cases} \mathscr{F}\psi & \text{for } \psi \in \mathscr{S}_\rho, \\ (|||\psi|||/\rho)\mathscr{F}(\rho\psi/|||\psi|||) & \text{for } |||\psi||| > \rho. \end{cases} \tag{33}$$

To continue with the proof we introduce the following lemma.

LEMMA 7. *The feedback operator* $\bar{\bar{\mathscr{F}}}$ *satisfies the Lipschitz conditions* (12) *with the Lipschitz function* $3K(t)$, *and the continuity condition* (13).

Proof. By definition of $\bar{\bar{\mathscr{F}}}$, the Lipschitz condition holds for $\psi_1, \psi_2 \in \mathscr{S}_\rho$. Now, consider first the case $|||\psi_1||| > \rho$ and $|||\psi_2||| > \rho$. Then

$$\begin{aligned}
|||\bar{\bar{\mathscr{F}}}\psi_1 - \bar{\bar{\mathscr{F}}}\psi_2||| &= \left|\left|\left|\frac{1}{\rho}\left(|||\psi_1|||\mathscr{F}\frac{\rho\psi_1}{|||\psi_1|||} - |||\psi_2|||\mathscr{F}\frac{\rho\psi_2}{|||\psi_2|||}\right)\right|\right|\right| \\
&\leqq \left|\left|\left|\frac{1}{\rho}\left(|||\psi_1|||\mathscr{F}\frac{\rho\psi_1}{|||\psi_1|||} - |||\psi_2|||\mathscr{F}\frac{\rho\psi_1}{|||\psi_1|||}\right)\right|\right|\right| \\
&\quad + \left|\left|\left|\frac{1}{\rho}\left(|||\psi_2|||\mathscr{F}\frac{\rho\psi_1}{|||\psi_1|||} - |||\psi_2|||\mathscr{F}\frac{\rho\psi_2}{|||\psi_2|||}\right)\right|\right|\right| \\
&\leqq K(t)\left(|(|||\psi_1||| - |||\psi_2|||)| + |||\psi_2|||\left|\left|\left|\frac{\psi_1}{|||\psi_1|||} - \frac{\psi_2}{|||\psi_2|||}\right|\right|\right|\right) \\
&\leqq K(t)\left(|(|||\psi_1||| - |||\psi_2|||)| + \left|\left|\left|\frac{|||\psi_2|||}{|||\psi_1|||}\psi_1 - \psi_1\right|\right|\right| + |||\psi_1 - \psi_2|||\right) \\
&\leqq 3K(t)|||\psi_1 - \psi_2|||.
\end{aligned}$$

Consider next the case $|||\psi_1||| \leqq \rho$, $|||\psi_2||| > \rho$. Then

$$\begin{aligned}
|||\bar{\bar{\mathscr{F}}}\psi_1 - \bar{\bar{\mathscr{F}}}\psi_2||| &= \left|\left|\left|\mathscr{F}\psi_1 - \frac{|||\psi_2|||}{\rho}\mathscr{F}\frac{\rho\psi_2}{|||\psi_2|||}\right|\right|\right|. \\
&\leqq \left|\left|\left|\mathscr{F}\psi_1 - \frac{|||\psi_2|||}{\rho}\mathscr{F}\psi_1\right|\right|\right| + \frac{|||\psi_2|||}{\rho}\left|\left|\left|\mathscr{F}\psi_1 - \mathscr{F}\frac{\rho\psi_2}{|||\psi_2|||}\right|\right|\right| \\
&\leqq K(t)\left(|||\psi_1|||\left|1 - \frac{|||\psi_2|||}{\rho}\right| + \frac{|||\psi_2|||}{\rho}\left|\left|\left|\psi_1 - \frac{\rho\psi_2}{|||\psi_2|||}\right)\right|\right|\right| \\
&\leqq K(t)\left(\frac{|||\psi_1|||}{\rho}|\rho - |||\psi_2|||| + \frac{1}{\rho}|||(\psi_1|||\psi_2||| - \rho\psi_1)||| + |||\psi_1 - \psi_2|||\right) \\
&\leqq 3K(t)|||\psi_1 - \psi_2|||.
\end{aligned}$$

Following (31) the continuity of $\bar{\bar{\mathscr{F}}}$ requires verification only for $|||\psi||| > \rho$. Indeed, if $|t - t_0| < \delta$, then

$$\begin{aligned}
|||\bar{\bar{\mathscr{F}}}(t)\psi - \bar{\bar{\mathscr{F}}}(t_0)\psi||| &= \frac{|||\psi|||}{\rho}\left|\left|\left|\mathscr{F}(t)\frac{\rho\psi}{|||\psi|||} - \mathscr{F}(t_0)\frac{\rho\psi}{|||\psi|||}\right|\right|\right| \\
&\leqq \varepsilon\frac{|||\psi|||}{\rho}\frac{|||\rho\psi|||}{|||\psi|||} = \varepsilon|||\psi|||.
\end{aligned}$$

This completes the proof of Lemma 7.

Continuing with the proof of Theorem 4, it follows, from Lemma 7 and Theorem 1, that for arbitrary $\psi_0 \in \mathscr{S}$ there exists the strong trajectory solution of

$$(34) \qquad \mathscr{L}_{\bar{\bar{\mathscr{F}}}}\Phi = 0, \qquad \Phi(x; 0) = \Phi_0.$$

Consider $\alpha_{\bar{\bar{\mathscr{F}}}}(t)$, the upper response of $\mathscr{L}_{\bar{\bar{\mathscr{F}}}}$, and $\alpha_{\mathscr{F}}(t; \rho)$, the local upper response of $\mathscr{L}_{\mathscr{F}}$ in the sphere $\mathscr{S}_\rho$. We show next that $\alpha_{\bar{\mathscr{F}}}(t)=\alpha_{\mathscr{F}}(t;\rho)$. Indeed, for $|||\psi|||>\rho$,

$$\begin{aligned}&\{\psi, ((A_i)_{x_i}-2B)\psi+2\bar{\mathscr{F}}\psi\}/|||\psi|||^2\\&=\{\psi, ((A_i)_{x_i}-2B)\psi+2(|||\psi|||/\rho)\mathscr{F}(\rho\psi/|||\psi|||)\}/|||\psi|||^2\\&=\{(\rho\psi/|||\psi|||), ((A_i)_{x_i}-2B)(\rho\psi/|||\psi|||)+2\mathscr{F}(\rho\psi/|||\psi|||)\}/|||(\rho\psi/|||\psi|||)|||^2.\end{aligned}$$

This implies that $\alpha_{\bar{\mathscr{F}}}(t)=\alpha_{\mathscr{F}}(t;\rho)$. It now follows from the assumption $\int_0^t \alpha_{\bar{\mathscr{F}}}(\tau)\,d\tau \leqq M$ and from (28), that

$$|||\Phi(\cdot\,;t)|||^2 \leqq |||\Phi_0|||^2 \exp\left(\int_0^t \alpha_{\mathscr{F}}(\tau;\rho)\,d\tau\right) \leqq |||\Phi_0|||^2\, e^M. \tag{35}$$

Now let $\eta=\rho^{1/2}\,e^{-M/2}$; then for all Φ_0 with $|||\Phi_o|||<\eta$, the solution of (34) is such that $|||\Phi(\cdot\,;t)|||<\rho$ for all $t\geqq 0$ and hence $\bar{\mathscr{F}}\Phi=\mathscr{F}\Phi$. It follows that Φ is also a strong trajectory solution of $\mathscr{L}_{\mathscr{F}}\Phi=0$. This completes the proof of Theorem 4.

From (35) follows the next corollary.

COROLLARY 4.1. (a) *Under the assumptions of Theorem* 5 *the trivial solution is stable.*

(b) *If* $\int_0^\infty \alpha_{\mathscr{F}}(\tau;\rho)d\tau=-\infty$, *then the trivial solution is locally asymptotically stable.*

(c) *If* $\int_0^\infty \left(\exp\int_0^t \alpha_{\mathscr{F}}(\tau;\rho)\,d\tau\right)dt<\infty$, *then the trivial solution is locally* L_2*-stable.*

The next stability and instability criteria possess an analogy to Lyapunov's direct method for systems of ordinary differential equations.

THEOREM 5. *If the eigenvalues of the symmetric matrix* $(A_i)_{x_i}-(B+B^T)$ *are uniformly negative in the half-space* $H(t\geqq 0)$, *and if*

$$\lim_{|||\psi|||\to 0} |||\mathscr{F}\psi|||/|||\psi|||=0 \tag{36}$$

uniformly for all $t\geqq 0$, *then the trivial solution of* $\mathscr{L}_{\mathscr{F}}\Phi=0$ *is locally asymptotically stable and locally* L_2*-stable.*

Proof. There exists $\gamma>0$ such that for all $\psi\in\mathscr{S}$ and all $t\geqq 0$, $\{\psi, ((A_i)_{x_i}-2B)\psi\}\leqq -\gamma|||\psi|||^2$. It follows from (36) that there exists $\rho_0>0$ such that for all $|||\psi|||<\rho_0$ and all $t\geqq 0$, $\{\psi, \mathscr{F}\psi\}/|||\psi|||^2\leqq |||\mathscr{F}\psi|||/|||\psi|||\leqq \gamma/4$. Therefore,

$$\begin{aligned}\alpha_{\mathscr{F}}(t;\rho_0)&=\sup_{\psi\in\mathscr{S}_{\rho_0}}\{\psi, ((A_i)_{x_i}-2B)\psi+2\mathscr{F}\psi\}/|||\psi|||^2\\&\leqq \sup_{\psi\in\mathscr{S}}\{\psi, ((A_i)_{x_i}-2B)\psi\}/|||\psi|||^2+\sup_{\psi\in\mathscr{S}_{\rho_0}}\{\psi,\mathscr{F}\psi\}/|||\psi|||^2\\&\leqq -\frac{\gamma}{2}.\end{aligned}$$

Hence, $\int_0^\infty \alpha_{\mathscr{F}}(\tau;\rho_o)\,d\tau=-\infty$ and $\int_0^\infty\left(\exp\int_0^t \alpha_{\mathscr{F}}(\tau;\rho_0)\,d\tau\right)dt\leqq 2/\gamma<\infty$. It now follows from Corollary 4.1 that the trivial solution is locally asymptotically stable and locally L_2-stable.

THEOREM 6. *If the eigenvalues of* $(A_i)_{x_i}-(B+B^T)$ *are uniformly positive in the half-space H, if* $\mathscr{F}$ *satisfies the global Lipschitz and continuity conditions* (14), (15) *and the local condition* (36), *then the trivial solution of* $\mathscr{L}_{\mathscr{F}}\Phi=0$ *is unstable and further it is not globally and not locally* L_2*-stable.*

Proof. Since $\mathscr{F}$ satisfies the global Lipschitz and continuity conditions it follows from Theorem 1 that the strong trajectory solution of $\mathscr{L}_{\mathscr{F}}\Phi = 0$ exists for arbitrary initial states $\Phi_0 \in \mathscr{S}$. The uniform positivity of the eigenvalues implies that there exists $\gamma > 0$ such that for all $\psi \in \mathscr{S}$ and all $t \geqq 0$, $\{\psi, ((A_i)_{x_i} - 2B)\psi\} \geqq \gamma |||\psi|||^2$. It follows from (36) that there exists $\rho_0 > 0$ such that for all $|||\psi||| < \rho_0$ and all $t \geqq 0$, $\{\psi, \mathscr{F}\psi\}/|||\psi|||^2 \geqq -|||\mathscr{F}\psi|||/|||\psi||| > -\gamma/4$, and therefore

$$\{\psi, ((A_i)_{x_i} - 2B)\psi + 2\mathscr{F}\psi\} \geqq \frac{\gamma}{2}|||\psi|||^2. \tag{37}$$

We now show that for arbitrary nonzero initial state Φ_0 there exists $t_0 \geqq 0$ such that the solution of $\mathscr{L}_{\mathscr{F}}\Phi = 0$ satisfies $|||\Phi(\cdot\,;t_0)||| \geqq \rho_0$. Assume to the contrary that for all $t \geqq 0$, $|||\Phi(\cdot\,;t)||| < \rho_0$. Then it follows from (25) and (37) that

$$\frac{d}{dt}|||\Phi|||^2 \geqq \frac{\gamma}{2}|||\Phi|||^2.$$

This implies that $|||\Phi(\cdot\,;t)|||^2 \geqq |||\Phi_0|||^2\, e^{\gamma t/2}$, which clearly contradicts the assumption. We show next that for all $t \geqq t_0$, $|||\Phi(\cdot\,;t)||| \geqq \rho_0$. Indeed, if we assume that for some $t_1 > t_0$, $|||\Phi(\cdot\,;t_1)||| < \rho_0$, then there must exist t_2, $t_0 < t_2 < t_1$, such that $|||\Phi(\cdot\,;t_2)||| < \rho_0$ and also $(d/dt)|||\Phi(\cdot\,;t_2)|||^2 < 0$. This contradicts the fact that (25) and (37) imply that this derivative must be nonnegative. We conclude from the above that the trivial solution is unstable. It also follows that $\int_0^\infty |||\Phi(\cdot\,;t)|||^2\,dt = \infty$, and therefore the trivial solution is not globally and not locally L_2-table.

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COMPLETE FAMILIES OF SOLUTIONS FOR PARABOLIC EQUATIONS WITH ANALYTIC COEFFICIENTS*

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Abstract. A complete family of solutions is constructed for the general linear second order parabolic equation in one space variable with entire coefficients defined in a domain with moving boundary and for a class of second order parabolic equations in two space variables with entire coefficients defined in a cylindrical domain. The construction is based on the use of integral operators and results on the analytic continuation of solutions to partial differential equations with analytic coefficients. A numerical example is given which uses a complete family of solutions to approximate the solution to the first initial-boundary value problem for a parabolic equation in one space variable defined in a cylindrical domain.

1. Introduction. One of the more important applications of integral operators for elliptic equations is their use in constructing a complete family of solutions for the equation under investigation and thus providing a method for approximating the solutions to a wide variety of boundary value problems associated with equations of elliptic type (cf. [1], [7], [9], [11], [13], [15]). In recent papers ([3], [5]) the author has constructed an integral operator for the parabolic equation

$$u_{xx} + a(x, t)u_x + b(x, t)u = u_t \tag{1.1}$$

and showed how this operator could be used to construct a complete family of solutions to (1.1) in a rectangle. These last two papers lay the foundation for using integral operator methods to solve initial-boundary value problems for parabolic equations in a manner analogous to their use in the solution of elliptic boundary value problems. It is the purpose of this paper to extend the results of [5] in three different directions:

(a) Instead of (1.1) we will consider the general linear second order parabolic equation

$$u_{xx} + a(x, t)u_x + b(x, t)u = c(x, t)u_t. \tag{1.2}$$

(b) We will construct a set of solutions to (1.2) which are complete with respect to the maximum norm over the closure of domains with moving boundaries instead of only in a rectangle.

(c) We will show how these results can be extended to the case of parabolic equations in two space variables defined in cylindrical domains.

Numerical experiments on using the methods described in this paper to solve initial-boundary value problems for parabolic equations are presently being carried out by Y. F. Chang of the Data Systems and Services Department at Indiana University, and we hope to report on this in detail in the near future. A preliminary numerical example taken from this work is given in § 4 of this paper.

2. Complete families of solutions for parabolic equations in one space variable. We consider (1.2) and for the sake of simplicity assume that the coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are entire functions of their independent (complex) variables.

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We will further assume that $c(x, t) > 0$ for $-\infty < x < \infty, 0 \leqq t \leqq t_0$ and, again for the sake of simplicity, that

$$\int_{-\infty}^{0} \sqrt{c(s, t)}\, ds = \int_{0}^{\infty} \sqrt{c(s, t)}\, ds = \infty. \tag{2.1}$$

We note at this point that due to the analyticity of the coefficients, every classical solution of (1.2) (i.e., a solution of (1.2) that is twice continuously differentiable with respect to x and continuously differentiable with respect to t) in a domain D is in fact analytic with respect to x and infinitely differentiable (but not necessarily analytic) with respect to t. Our aim is to construct a complete family of solutions with respect to the maximum norm for (1.2) defined in a region D bounded by the characteristics $t = 0$ and $t = t_0$ as well as the analytic curves $x = s_1(t)$ and $x = s_2(t)$, where $s_1(t) < s_2(t)$ for $0 \leqq t \leqq t_0$. The one-to-one analytic transformation

$$\xi = \int_0^x \sqrt{c(s, t)}\, ds, \qquad \tau = t, \tag{2.2}$$

reduces (1.2) to an equation of the same form but with $c(x, t) = 1$. The domain D is transformed into a domain in the (ξ, τ)-plane of the same form as that described above. Hence we can assume $c(x, t) = 1$ in (1.2) to begin with. If we now set

$$u(x, t) = v(x, t) \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\}, \tag{2.3}$$

we arrive at an equation for $v(x, t)$ of the same form as (1.2) but with $a(x, t) = 0$. Hence, without loss of generality, we can restrict ourselves to equations of the canonical form

$$u_{xx} + q(x, t)u = u_t, \tag{2.4}$$

where (due to the assumption (2.1)) $q(x, t)$ is analytic for $-\infty < x < \infty, 0 \leqq t \leqq t_0$, and consider classical solutions of (2.4) which continuously assume the initial-boundary data

$$\begin{gathered} u(s_1(t), t) = f(t), \quad u(s_2(t), t) = g(t), \qquad 0 \leqq t \leqq t_0, \\ u(x, 0) = h(x), \qquad s_1(0) \leqq x \leqq s_2(0), \end{gathered} \tag{2.5}$$

where $x = s_1(t)$ and $x = s_2(t)$ are analytic arcs satisfying

$$s_1(t) < s_2(t) \quad \text{for } 0 \leqq t \leqq t_0, \quad f(0) = h(s_1(0)), \quad g(0) = h(s_2(0)),$$

and $f(t)$, $g(t)$ and $h(x)$ are continuous functions of their independent variables.

Now suppose that for a given $\varepsilon > 0$ we are able to construct a solution $w(x, t)$ of (2.4) defined in a rectangle $R = \{(x, t)\colon -x_0 \leqq x \leqq x_0, 0 \leqq t \leqq t_0\}$ such that $D \subset R$ and

$$\max_{(x,t)\in \bar{D}} |u(x, t) - w(x, t)| < \varepsilon/2, \tag{2.6}$$

where $\bar{D}$ denotes the closure of D. Let $h_n(x, t)$ be defined by

$$h_n(x, t) = \sum_{k=0}^{[n/2]} \frac{x^{n-2k} t^k}{(n - 2k)!k!}, \tag{2.7}$$

and let $u_n(x, t)$ be the solution of (2.4) defined by

$$u_n(x, t) = h_n(x, t) + \int_{-x}^{x} P(s, x, t) h_n(s, t)\, ds, \tag{2.8}$$

where $P(s, x, t)$ is the (unique) solution of the initial value problem

$$P_{xx} - P_{ss} + q(x, t)P = P_t, \tag{2.9a}$$

$$P(x, x, t) = -\frac{1}{2}\int_0^x q(s, t)\, ds, \tag{2.9b}$$

$$P(-x, x, t) = 0. \tag{2.9c}$$

The existence of the function $P(s, x, t)$ and the fact that $u_n(x, t)$ is a solution of (2.4) follows from the results of [3] and [5]. In particular, $\tilde{P}(\xi, \eta, t) = P(\xi - \eta, \xi + \eta, t)$ can be constructed by the iterative scheme

$$\begin{aligned} \tilde{P}(\xi, \eta, t) &= \lim_{n\to\infty} \tilde{P}_n(\xi, \eta, t), \\ \tilde{P}_1(\xi, \eta, t) &= -\frac{1}{2}\int_0^{\xi} q(s, t)\, ds, \\ \tilde{P}_{n+1}(\xi, \eta, t) &= -\frac{1}{2}\int_0^{\xi} q(s, t)\, ds \\ &\quad + \int_0^{\eta}\int_0^{\xi} \left(\frac{\partial}{\partial t}\tilde{P}_n(\xi, \eta, t) - q(\xi + \eta, t)\tilde{P}_n(\xi, \eta, t)\right) d\xi\, d\eta, \\ &\qquad n \geqq 1. \end{aligned} \tag{2.10}$$

The convergence of the sequence $\{\tilde{P}_n\}$ is quite rapid and good approximations can be found by terminating the recursion process after several iterations. From the results of [5] we can now conclude that there exists an integer N and constants $a_1, \cdots, a_N$ such that

$$\max_{(x,t)\in R} \left| w(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \varepsilon/2 \tag{2.11}$$

and hence from (2.6),

$$\max_{(x,t)\in \bar{D}} \left| u(x, t) - \sum_{n=0}^{N} a_n u_n(x, t) \right| < \varepsilon; \tag{2.12}$$

i.e., the set $\{u_n(x, t)\}$ is a complete family of solutions for (2.4) defined in D. If we first orthonormalize the set $\{u_n(x, t)\}$ over the base and sides of D, it is seen that on compact subsets of D we can approximate the solution to the first initial-boundary value problem for (2.4) in D by the sum $\sum_{n=0}^{N} a_n \varphi_n(x, t)$, where

$$\begin{aligned} a_n &= \int_0^{t_0} f(t)\varphi_n(s_1(t), t)\, dt + \int_{s_1(0)}^{s_2(0)} h(x)\varphi_n(x, 0)\, dx \\ &\quad + \int_0^{t_0} g(t)\varphi_n(s_2(t), t)\, dt \end{aligned} \tag{2.13}$$

and the set $\{\varphi_n(x, t)\}$ is obtained by applying the Gram–Schmidt process to the set $\{u_n(x, t)\}$. Since each $\varphi_n(x, t)$ is a solution of (2.4), error estimates can be obtained by either applying the maximum principle for parabolic equations or the pointwise bounds for solutions established by Sigillito in [14]. Thus the problem we are considering will be solved if we can construct a function $w(x, t)$ defined in R and satisfying (2.6), and we now turn our attention to this problem.

From the existence theorem for the first initial-boundary value problem for parabolic equations, the maximum principle for parabolic equations and the Weierstrass approximation theorem, it is seen that there exists a solution $w(x, t)$ of (2.4) in D satisfying analytic boundary data on $x = s_1(t)$, $x = s_2(t)$ and $t = 0$ such that (2.6) is valid. From the reflection principle for parabolic equations ([3], [4]) (and the regularity theorems for solutions to initial-boundary value problems for parabolic equations—c.f. [8]) we can conclude that $w(x, t)$ can be uniquely continued as a solution of (2.4) across the arc $s_1(t)$ into the region bounded by the characteristics $t = t_0$, $t = 0$, and the analytic curves $x = 2s_1(t) - s_2(t)$, $x = s_2(t)$. Applying the reflection principle a second time, but this time continuing $w(x, t)$ across the arc $s_2(t)$, shows that $w(x, t)$ can be continued into the region bounded by $t = t_0$, $t = 0$, $x = 2s_1(t) - s_2(t)$ and $x = 3s_2(t) - 2s_1(t)$. Due to the fact that $s_1(t) < s_2(t)$ for $0 \leqq t \leqq t_0$, it is seen that by repeating the above procedure we can continue $w(x, t)$ into the entire infinite strip $-\infty < x < \infty$, $0 \leqq t \leqq t_0$. In particular, there exists a rectangle $R \supset D$ into which $w(x, t)$ can be continued and we have thus established the existence of the desired function $w(x, t)$.

We now make use of the above results to construct a complete family of solutions to (1.2) without first reducing it to the canonical form (2.4). This is desirable from a computational point of view in order to eliminate the problem of inverting the transformation (2.2). From the above analysis and the fact that $P(s, x, t)$ is analytic for $-\infty < s < \infty$, $-\infty < x < \infty$, $0 \leqq t \leqq t_0$ (cf. [3]) it is seen from equations (2.2)–(2.3) and (2.7)–(2.8) that every classical solution of (1.2) in D can be approximated arbitrarily closely in the maximum norm over $\bar{D}$ by a solution of (1.2) which is an analytic function of x and t in the strip $-\infty < x < \infty$, $0 \leqq t \leqq t_0$. Hence from the results of [2] we have that a complete family of solutions to (1.2) with respect to the maximum norm over $\bar{D}$ is given by

$$
\begin{aligned}
u_{2n}(x, t) &= \frac{1}{2\pi i} \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\} \oint_{|t-\tau|=\delta} E^{(1)}(x, t, \tau)\tau^n \, d\tau, \\
u_{2n+1}(x, t) &= \frac{1}{2\pi i} \exp\left\{-\frac{1}{2}\int_0^x a(s, t)\, ds\right\} \oint_{|t-\tau|=\delta} E^{(2)}(x, t, \tau)\tau^n \, d\tau, \\
&\qquad n = 0, 1, 2, \cdots,
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
E^{(1)}(x, t, \tau) &= \frac{1}{t - \tau} + \sum_{n=2}^{\infty} x^n p^{(1,n)}(x, t, \tau), \\
E^{(2)}(x, t, \tau) &= \frac{x}{t - \tau} + \sum_{n=3}^{\infty} x^n p^{(2,n)}(x, t, \tau)
\end{aligned}
\tag{2.15}
$$

with

$$p^{(1,1)} = 0,$$

$$p^{(1,2)} = -\frac{c(x,t)}{2(t-\tau)^2} - \frac{q(x,t)}{2(t-\tau)},$$

$$\text{(2.16a)}\quad p^{(1,k+2)} = -\frac{2}{k+2}\,p_x^{(1,k+1)} - \frac{1}{(k+2)(k+1)}[p_{xx}^{(1,k)} + q(x,t)p^{(1,k)} - c(x,t)p_t^{(1,k)}], \qquad k \geqq 1,$$

$$p^{(2,2)} = 0,$$

$$p^{(2,3)} = -\frac{c(x,t)}{6(t-\tau)^2} - \frac{q(x,t)}{6(t-\tau)},$$

$$\text{(2.16b)}\quad p^{(2,k+2)} = -\frac{2}{k+2}p_x^{(2,k+1)} - \frac{1}{(k+2)(k+1)}[p_{xx}^{(2,k)} + q(x,t)p^{(2,k)} - c(x,t)p_t^{(2,k)}], \qquad k \geqq 2,$$

and

$$\text{(2.17)}\qquad q(x,t) = b(x,t) - \tfrac{1}{2}\left[a_x(x,t) + a^2(x,t) - c(x,t)\int_0^x a_t(s,t)\,ds\right].$$

The convergence of the series (2.15) for $t \neq \tau$ and estimates on the rate of this convergence can be found in [2]. An approximation of the solution $u_n(x, t)$ can be obtained by truncating the series (2.15) and computing the residue in (2.14).

3. Complete families of solutions for parabolic equations in two space variables. In this section we will show how the methods developed in [5] and the previous section of this paper can be extended to include the case of the parabolic equation in two space variables

$$\text{(3.1)}\qquad u_{xx} + u_{yy} + c(x,y)u = d(x,y)u_t$$

defined in a cylindrical domain $\Omega \times T$, where $T = [0, t_0]$ and Ω is a bounded simply connected domain whose boundary $\partial\Omega$ is three times continuously differentiable. We will assume for the sake of simplicity that $c(x, y)$ and $d(x, y)$ are entire functions of their independent (complex) variables and that furthermore $c(x, y) \leqq 0$, $d(x, y) > 0$, for $(x, y) \in \bar{\Omega} = \Omega \cup \partial\Omega$.

Let $u(x, y, t)$ be a (classical) solution of (3.1) which continuously assumes prescribed initial-boundary data on $\partial\Omega \times T$ and $\Omega_0 = \{(x, y, t):(x, y) \in \bar{\Omega}, t = 0\}$. From the maximum principle for parabolic equations and the Weierstrass approximation theorem, we can assume, without loss of generality, that the boundary data assumed by $u(x, y, t)$ on $\partial\Omega \times T$ is a polynomial in t, i.e.,

$$u(x, y, t) = \sum_{n=0}^{N} f_n(x, y)t^n, \qquad (x, y, t) \in \partial\Omega \times T, \tag{3.2}$$

where the $f_n(x, y)$ are Hölder continuous functions defined on $\partial\Omega$. We now look for a solution of (3.1) in the form

$$w(x, y, t) = \sum_{n=0}^{N} w_n(x, y)t^n \tag{3.3}$$

such that $w(x, y, t) = u(x, y, t)$ for $(x, y, t) \in \partial\Omega \times T$. From (3.1) and (3.2) it is seen that the functions $w_n(x, y)$ must satisfy the recursive scheme

$$\begin{aligned}
&\frac{\partial^2 w_N}{\partial x^2} + \frac{\partial^2 w_N}{\partial y^2} + c(x, y)w_N = 0, && (x, y) \in \Omega,\\
&w_N(x, y) = f_N(x, y), && (x, y) \in \partial\Omega,\\
&\frac{\partial^2 w_n}{\partial x^2} + \frac{\partial^2 w_n}{\partial y^2} + c(x, y)w_n = (n + 1)\, d(x, y)w_{n+1}, && (x, y) \in \Omega,\\
&w_n(x, y) = f_n(x, y), && (x, y) \in \partial\Omega,
\end{aligned} \tag{3.4}$$

for $n = 0, 1, \cdots, N - 1$. The existence of the $w_n(x, y)$ for $n = 0, 1, \cdots, N$ follows from the smoothness of $\partial\Omega$ and the fact that $c(x, y) \leqq 0$ in $\bar{\Omega}$. From the results of Vekua ([15, p. 156, p. 19]) and the fact that $w_n(x, y)$ depends continuously on the nonhomogeneous term $(n + 1)\, d(x, y)w_{n+1}(x, y)$, we can conclude that for $\varepsilon > 0$ there exists a solution $w_1(x, y, t)$ of (3.1) which is an entire function of its independent (complex) variables such that

$$\max_{\bar{\Omega} \times T} |w_1(x, y, t) - w(x, y, t)| < \varepsilon/2. \tag{3.5}$$

Now let $v(x, y, t) = u(x, y, t) - w(x, y, t)$ and let λ_n and $\varphi_n(x, y)$ be the eigenvalues and eigenfunctions, respectively, that correspond to the eigenvalue problem

$$\begin{aligned}
&u_{xx} + u_{yy} + c(x, y)u + \lambda\, d(x, y)u = 0, && (x, y) \in \Omega,\\
&u(x, y) = 0, && (x, y) \in \partial\Omega.
\end{aligned} \tag{3.6}$$

From (3.2)–(3.4) and the expansion theorem for the eigenvalue problem (3.6) (c.f. [10, p. 229]) we can conclude that

$$\begin{aligned}
v(x, y, t) &= \sum_{n=0}^{\infty} a_n\varphi_n(x, y) \exp(-\lambda_n t),\\
a_n &= \iint_{\Omega} v(x, y, 0)\varphi_n(x, y)\, d(x, y)\, dx\, dy,
\end{aligned} \tag{3.7}$$

where the series in (3.7) converges absolutely and uniformly in $\bar{\Omega} \times T$. By truncating the series in (3.7) and again appealing to the results of Vekua, we can conclude that there exists a solution $w_2(x, y, t)$ of (3.1) which is an entire function of its independent (complex) variables such that

$$\max_{\bar{\Omega} \times T} |w_2(x, y, t) - v(x, y, t)| < \varepsilon/2. \tag{3.8}$$

The inequalities (3.5) and (3.8) now imply that there exists a solution $\tilde{u}(x, y, t)$ of (3.1) which is an entire function of its independent complex variables such that

$$\max_{\bar{\Omega} \times T} |\tilde{u}(x, y, t) - u(x, y, t)| < \varepsilon. \tag{3.9}$$

The above analysis shows that in order to approximate classical solutions of (3.1) with respect to the maximum norm over $\bar{\Omega} \times T$, it suffices to construct a family of solutions which are complete in the maximum norm over $\bar{\Omega} \times T$ with respect to the class of solutions to (3.1) which are entire functions of their independent complex variables. From the results of [6] it is seen that such a complete family of solutions is given by

$$\begin{aligned} u_{2n,m}(x, y, t) &= \mathrm{Re}\left[-\frac{z^n}{2\pi i} \oint_{|t-\tau|=\delta} \int_{-1}^{1} E(z, \bar{z}, t-\tau, s)\tau^m (1 - s^2)^{n-(1/2)}\, ds\, d\tau \right], \\ u_{2n+1,m}(x, y, t) &= \mathrm{Im}\left[-\frac{z^n}{2\pi i} \oint_{|t-\tau|=\delta} \int_{-1}^{1} E(z, \bar{z}, t-\tau, s)\tau^m (1 - s^2)^{n-(1/2)}\, ds\, d\tau \right], \\ & \qquad n, m = 0, 1, 2, \cdots, \end{aligned} \tag{3.10}$$

where "Re" denotes "take the real part", "Im" denotes "take the imaginary part", $z = x + iy$, $\bar{z} = x - iy$, and

$$E(z, z^*, t, s) = \frac{1}{t} + \sum_{n=1}^{\infty} s^{2n} z^n \int_0^{z^*} P^{(2n)}(z, \zeta^*, t)\, d\zeta^* \tag{3.11}$$

with

$$\begin{aligned} P^{(2)} &= -\frac{2C(z, z^*)}{t} - \frac{2D(z, z^*)}{t^2}, \\ (2n+1)P^{(2n+2)} &= -2\left[P_z^{(2n)} + C(z, z^*) \int_0^{z^*} P^{(2n)}\, d\zeta^* - D(z, z^*) \int_0^{z^*} P_t^{(2n)}\, d\zeta^* \right], \\ & \qquad n \geqq 1, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} C(z, z^*) &= \frac{1}{4} c\left(\frac{z + z^*}{2}; \frac{z - z^*}{2i} \right), \\ D(z, z^*) &= \frac{1}{4} d\left(\frac{z + z^*}{2}; \frac{z - z^*}{2i} \right). \end{aligned} \tag{3.13}$$

Estimates on the rate of convergence of the series (3.11) can be found in [6], and approximations of the solution $u_{n,m}(x, y, t)$ can be obtained by truncating the series (3.11) and computing the residue in (3.10). In particular, for the special case of the heat equation ($c = 0$, $d = 1$), we have

$$E(z, \bar{z}, t - \tau, s) = \frac{1}{t - \tau} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\frac{1}{2})}{\Gamma(k + \frac{1}{2})} \left(\frac{r^2 s^2}{t - \tau} \right)^k, \tag{3.14}$$

where $r^2 = z\bar{z} = x^2 + y^2$, and using the result

$$\int_{-1}^{1} (1 - s^2)^{n-(1/2)} s^{2k}\, ds = \frac{\Gamma(n + \frac{1}{2})\Gamma(k + \frac{1}{2})}{\Gamma(n + k + 1)}, \tag{3.15}$$

we have

$$\begin{aligned} u_{2n,m}(x, y, t) &= \cos n\theta \sum_{k=0}^{m} \frac{\Gamma(\frac{1}{2})\Gamma(m + 1)\Gamma(n + \frac{1}{2})}{\Gamma(k + 1)\Gamma(m - k + 1)\Gamma(n + k + 1)} r^{2k+n} t^{m-k}, \\ u_{2n+1,m}(x, y, t) &= \sin n\theta \sum_{k=0}^{m} \frac{\Gamma(\frac{1}{2})\Gamma(m + 1)\Gamma(n + \frac{1}{2})}{\Gamma(k + 1)\Gamma(m - k + 1)\Gamma(n + k + 1)} r^{2k+n} t^{m-k}, \end{aligned} \tag{3.16}$$

where $x = r\cos\theta$, $y = r\sin\theta$. Noting that since in this special case $u_n(x, y, t)$ is a polynomial in x, y and t, it follows from the results of § 2 and the uniqueness theorem for Cauchy's problem for the heat equation that another complete family of solutions for the heat equation defined in $\Omega \times T$ is given by

$$v_{n,m}(x, y, t) = h_n(x, t)h_m(y, t) \tag{3.17}$$

for $n, m = 0, 1, 2, \cdots$, where $h_n(x, t)$ is defined in (2.7).

4. A numerical example. In this section we given an example of the use of the methods discussed in [5] and this paper to approximate the solution of the initial-boundary value problem

$$u_{xx} - x^2 u = u_t, \qquad -1 < x < 1, \quad 0 < t < 1, \tag{4.1}$$

$$\begin{aligned} &u(-1, t) = \exp(-\tfrac{1}{2} - t), \qquad u(1, t) = \exp(-\tfrac{1}{2} - t), \qquad 0 \leqq t \leqq 1, \\ &u(x, 0) = \exp(-\tfrac{1}{2}x^2), \qquad -1 \leqq x \leqq 1. \end{aligned} \tag{4.2}$$

Initial-boundary value problems for (4.1) defined in a domain with moving boundary can of course be treated in an identical manner. A complete family of solutions for (4.1) was constructed by using the operator (2.8). Since the coefficients of (4.1) are independent of t, so is $P(s, x, t)$, i.e., $P(s, x, t) = P(s, x)$. As an approximation to the kernel $P(s, x)$ we used $P_{10}(s, x)$ as defined by (2.10). A short calculation using (2.10) shows that

$$\max_{\substack{-1 \leqq x \leqq 1 \\ 1 \leqq s \leqq 1}} |P(s, x) - P_{10}(s, x)| \leqq 1.6 \times 10^{-20}. \tag{4.3}$$

The set $\{u_n(x, t)\}$ obtained from (2.8) was then orthonormalized over the base and vertical sides of the rectangle $-1 \leqq x \leqq 1$, $0 \leqq t \leqq 1$, to obtain the set $\{\varphi_n(x, t)\}$ and the solution to the initial-boundary value problem (4.1), (4.2) was approximated by the sum

$$u^*(x, t) = \sum_{n=0}^{14} a_n \varphi_n(x, t) \tag{4.4}$$

with the coefficients a_n, $n = 0, 1, \cdots, 14$, given by (2.13). Note that since the solution of the initial-boundary value problem (4.1), (4.2) is an even function of x, the odd coefficients $a_1, a_3, \cdots, a_{13}$ in (4.4) all turn out to be identically zero.

The exact solution of the initial-boundary value problem (4.1), (4.2) is

$$u(x, t) = \exp\left(-\tfrac{1}{2}x^2 - t\right). \tag{4.5}$$

In Table 1 we give the values of $u^*(x, t)$ at selected grid points and also the relative error defined by

$$\text{relative error} = \frac{u^*(x, t) - u(x, t)}{u(x, t)}. \tag{4.6}$$

TABLE 1

x	t	Approximate solution	Relative error
0	0	1.00000	-6.9580×10^{-9}
0.2	0	0.98020	-3.3762×10^{-9}
0.4	0	0.92312	4.2696×10^{-9}
0.6	0	0.83527	8.0803×10^{-9}
0.8	0	0.72615	-4.3613×10^{-10}
1.0	0	0.60653	-2.2466×10^{-8}
0	0.2	0.81873	4.0571×10^{-10}
0.2	0.2	0.80252	9.3730×10^{-10}
0.4	0.2	0.75578	2.7202×10^{-9}
0.6	0.2	0.68386	6.1830×10^{-9}
0.8	0.2	0.59452	1.1356×10^{-8}
1.0	0.2	0.49659	1.5536×10^{-8}
0	0.4	0.67032	2.2209×10^{-9}
0.2	0.4	0.65705	1.7415×10^{-9}
0.4	0.4	0.61878	3.2910×10^{-11}
0.6	0.4	0.55990	-3.6697×10^{-9}
0.8	0.4	0.48675	-1.0332×10^{-8}
1.0	0.4	0.40657	-2.0325×10^{-8}
0	0.6	0.54881	-1.1541×10^{-9}
0.2	0.6	0.53794	-8.6797×10^{-10}
0.4	0.6	0.50662	3.4421×10^{-10}
0.6	0.6	0.45841	3.6095×10^{-9}
0.8	0.6	0.39852	1.0898×10^{-8}
1.0	0.6	0.33287	2.4115×10^{-8}
0	0.8	0.44933	2.7676×10^{-9}
0.2	0.8	0.44043	2.5721×10^{-9}
0.4	0.8	0.41478	1.5339×10^{-9}
0.6	0.8	0.37531	-1.8415×10^{-9}
0.8	0.8	0.32628	-1.0323×10^{-8}
1.0	0.8	0.27253	-2.7649×10^{-8}
0	1.0	0.36788	-7.3333×10^{-11}
0.2	1.0	0.36059	4.9411×10^{-10}
0.4	1.0	0.33960	2.3005×10^{-9}
0.6	1.0	0.30728	4.1011×10^{-9}
0.8	1.0	0.26714	-5.4443×10^{-9}
1.0	1.0	0.22313	-8.4473×10^{-8}

Since $u(x, t)$ and $u^*(x, t)$ are even functions of x, values of the approximate solution and relative error are only given for $0 \leqq x \leqq 1$, $0 \leqq t \leqq 1$. Note that since each $\varphi_n(x, t)$ is a solution of (4.1), the maximum error (in absolute value) occurs on the base or vertical sides of the rectangle $-1 \leqq x \leqq 1$, $0 \leqq t \leqq 1$; in this case at the points $(x, t) = (\pm 1, 1)$, where the relative error is 8.4473×10^{-8} in absolute value.

The computation time to construct $u^*(x, t)$ (i.e., to find the coefficients a_n, the Taylor coefficients of $\varphi_n(x, t)$, and to evaluate $u^*(x, t)$ at selected grid points) using the CDC 6600 computer was approximately six seconds.

5. Concluding remarks. The main problem in constructing a complete family of solutions through the use of integral operators as discussed in this paper, is to show that every classical solution in a given domain can be approximated with respect to the maximum norm over the closure of the domain by a solution of the parabolic equation that is an entire function of its independent complex variables. In the case of both one and two space variables, this was established through the use of results on the (global) analytic continuation of solutions to partial differential equations, in particular, the reflection principle for parabolic equations in one space variable ([3], [4]) and the results of Vekua which are based on knowledge of the domain of regularity in the complex domain of solutions to elliptic equations in two independent variables (cf. [15, p. 32]). What has been established is the analogue of Runge's theorem in analytic function theory for classical solutions to parabolic equations in one and two space variables. In order to extend our results to parabolic equations in two space variables defined in domains with moving boundaries and to parabolic equations in more than two space variables, it is necessary to obtain sharper results on the analytic continuation (with respect to the space variables) of classical solutions to parabolic equations with analytic coefficients in several independent variables. This is a difficult problem and only partial results have been obtained so far. One notable result in this direction is the reflection principle obtained by C. D. Hill for analytic solutions of parabolic equations in two space variables ([12]). It is to be hoped that more refined results in this direction will be forthcoming in the not too distant future.

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A SET OF RATIONAL FUNCTIONS RELATED TO THE EULER–FROBENIUS POLYNOMIALS*

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Abstract. The asymptotic behavior of the fundamental cardinal spline (Schoenberg's terminology) of odd degree n is governed by that particular root of the Euler–Frobenius polynomial Π_n which lies inside of and nearest to the unit circle. Rigorous bounds for this *dominant* root and an asymptotic formula for the remaining roots are given. These results are based on the partial fraction expansion of $q_n(-e^{\pi w})$, where $q_n(z) = z\Pi_n(z)(1 - z)^{-n-1}$. Various other properties of the functions q_n are discussed, as is their use in the representation of the fundamental cardinal splines of odd degree.

1. Introduction. Consider the Euler–Frobenius polynomials Π_n (Schoenberg [4, p. 22]), defined by their generating function

$$\frac{z-1}{z-e^t} = \sum_{m=0}^{\infty} \frac{\Pi_m(z)}{(z-1)^m} \frac{t^m}{m!}. \tag{1}$$

The following facts about these polynomials are well known. For $n \geqq 1$, Π_n is a monic polynomial of degree $n - 1$, whose roots (for $n \geqq 2$) are simple and negative; if λ is a root, so is $1/\lambda$. For odd $n \geqq 3$, i.e., for $n = 2m - 1$, $m \geqq 2$, the roots, $\lambda_k^{(2m-1)}$, may be ordered

$$\lambda_{2m-2}^{(2m-1)} < \cdots < \lambda_m^{(2m-1)} < -1 < \lambda_{m-1}^{(2m-1)} < \cdots < \lambda_1^{(2m-1)} < 0, \tag{2}$$

where $\lambda_{m+l-1}^{(2m-1)} = 1/\lambda_{m-l}^{(2m-1)}$.

We call $\lambda_{m-1}^{(2m-1)}$ the *dominant* root of Π_{2m-1} for the following reason. For each integer $m \geqq 2$, let $L_{2m-1}:\mathbb{R} \to \mathbb{R}$ be the fundamental cardinal spline (Schoenberg [4, p. 35]) of degree $2m - 1$. (We recall here that L_{2m-1} is uniquely determined by the requirements (i) $L_{2m-1} \in C^{2m-2}(\mathbb{R})$; (ii) $\forall k \in \mathbb{Z}$, the restriction of L_{2m-1} to $[k, k + 1]$ is a polynomial of degree at most $2m - 1$; (iii) $\forall k \in \mathbb{Z}$, $L_{2m-1}(k) = \delta_{k0}$; (iv) L_{2m-1} is bounded.) If we define $\alpha_{2m-1} > 0$ by

$$\lambda_{m-1}^{(2m-1)} = -e^{-\alpha_{2m-1}}, \tag{3}$$

then there exists a constant $A_{2m-1} > 0$ such that

$$|L_{2m-1}(x)| < A_{2m-1}\, e^{-\alpha_{2m-1}|x|}, \qquad x \in \mathbb{R};$$

and no such constant exists if α_{2m-1} is replaced by any larger positive number. The preceding statement follows, for example, from Nilson's explicit representation [3, p. 448] of L_{2m-1}, there denoted by $A(\theta)$; see also equations (42), (43) and (44) of this paper.

We shall establish a fairly precise result concerning the dominant root.

THEOREM 1. *Let m be an integer $\geqq 2$. Then there exists a positive number η_m,*

$$1 < \eta_m < 1 + 3^{1-2m}, \tag{4}$$

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such that the dominant root of Π_{2m-1} *is given by*

$$\lambda_{m-1}^{(2m-1)} = -e^{-\pi w_m}, \qquad w_m = \tan\frac{\pi\eta_m}{4m}. \tag{5}$$

We shall prove this theorem in § 2. The proof will depend on properties of certain rational functions q_n that are closely related to the Euler–Frobenius polynomials. In § 3 we shall consider some additional properties of these functions and use them in constructing a representation of L_{2m-1} equivalent to Nilson's.

Before turning to the proof, we note that the theorem implies

$$\alpha_{2m-1} \sim \pi \tan \pi/4m, \qquad m \to \infty,$$

where α_{2m-1} is defined in (3). This relation is, in fact, the special case $l = 1$ of the following asymptotic formula: for fixed l, $-m + 2 \leqq l \leqq m - 1$,

$$-\log\left(-\lambda_{m-l}^{(2m-1)}\right) \equiv \alpha_{2m-1,l} \sim \pi \tan\frac{\pi(2l-1)}{4m}, \qquad m \to \infty, \tag{6}$$

This formula is suggested by our proof of Theorem 1; heuristically it results from keeping only the $k = 0$ term in the series expansion (20). The method of proof of Theorem 1 can easily be extended to yield the rather weak assertion made by (6); however, I have not yet succeeded in establishing error bounds comparable to those contained in Theorem 1 for $l = 1$.

A comparison with the exact values of the roots gives an indication of the remarkable accuracy of formula (6). Table 1 shows the error made in using (6) as an approximation to $\lambda_{m-l}^{(2m-1)}$; Table 2 gives the relative error in calculating $\alpha_{2m-1,l}$. (Exact values of the roots were taken from Schoenberg and Silliman [5].

TABLE 1

$[\lambda_{m-l}^{(2m-1)}(\text{approximate}) - \lambda_{m-l}^{(2m-1)}(\text{exact})] \times 10^8$

m	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
2	−423,119					
3	−38,618	−11,763				
4	−3,365	−565	6,909			
5	−322	−40	869	2,111		
6	−31	−3	77	61	228	
7	−3	0	6	−5	−116	−104

TABLE 2

$$\frac{\pi}{\alpha_{2m-1,l}} \tan\frac{\pi(2l-1)}{4m} - 1$$

m	$l = 1$	$l = 2$	$l = 3$	$l = 4$	$l = 5$	$l = 6$
2	−1.2E − 2					
3	−1.1E − 3	−8.8E − 4				
4	−1.0E − 4	−2.5E − 5	1.6E − 3			
5	−1.1E − 5	−1.2E − 6	6.4E − 5	1.6E − 3		
6	−1.1E − 6	−8.5E − 8	3.6E − 6	8.9E − 6	5.9E − 5	
7	−1.2E − 7	O(E − 9)	2.2E − 7	−3.7E − 7	−3.4E − 5	−9.2E − 4

Because of the symmetry of the roots, the range of l in the tables is $1 \leqq l \leqq m-1$; E $-$ integer $\equiv 10^{-\text{integer}}$ in Table 2.)

2. Proof of Theorem 1. Define the sequence $\{q_n\}$ of complex rational functions by

$$q_n(z) = \left(z\frac{d}{dz}\right)^n \frac{1}{1-z}, \qquad n = 0, 1, 2, \cdots. \tag{7}$$

Hille [2, p. 47] has shown that

$$q_n(z) = \frac{P_n(z)}{(1-z)^{n+1}}, \tag{8}$$

where P_n is a monic polynomial of degree n, with $P_n(0) = 0$ for $n \geqq 1$. Except for an additional factor z, the Hille polynomials, as they have come to be known in spline theory [1, p. 134], [3], are identical with the Euler–Frobenius polynomials Π_n. (This follows from a comparison of (8) with (11), below.) Our proof of Theorem 1 depends on proving a corresponding result for q_{2m-1}. We shall proceed by a series of lemmas, the first of which exhibits a generating function for the q's.

LEMMA 1. *For any complex $z \neq 1$, the functions q_n defined by* (7) *satisfy*

$$\frac{1}{1-ze^t} = \sum_{m=0}^{\infty} q_m(z)\frac{t^m}{m!}, \tag{9}$$

the series having a positive radius of convergence.

Proof. Let $A \subset \mathbb{C}$ be open and let $F: A \to \mathbb{C}$ be regular. Then for any fixed τ, $z \in \mathbb{C}$ such that $ze^\tau \in A$, $F(ze^t)$ is clearly an analytic function of t in some disk $|t-\tau| < \eta$, $\eta > 0$. Moreover, an easy induction on n shows

$$\left(z\frac{d}{dz}\right)^n F(ze^\tau) = \frac{\partial^n}{\partial t^n}F(ze^t)|_{t=\tau}, \qquad n = 0, 1, 2, \cdots,$$

so that

$$F(ze^t) = \sum_{m=0}^{\infty}\left(z\frac{d}{dz}\right)^m F(ze^\tau)\frac{(t-\tau)^m}{m!}, \qquad |t-\tau| < \eta.$$

With F defined by $F(\zeta) \equiv (1-\zeta)^{-1}$, and $\tau = 0$, $z \neq 1$, we obtain (9).

With the aid of (9), the relation of the q's to the Euler–Frobenius polynomials becomes readily apparent. Writing (1) in the form

$$\frac{z-1}{z-e^t} = 1 + \sum_{m=1}^{\infty}\frac{\Pi_m(z)}{(z-1)^m}\frac{t^m}{m!},$$

multiplying this equation by $z(z-1)^{-1}$, and replacing t by $-t$, yields, after a slight rearrangement,

$$\frac{1}{1-ze^t} = \frac{1}{1-z} + \sum_{m=1}^{\infty}\frac{z\Pi_m(z)}{(1-z)^{m+1}}\frac{t^m}{m!}. \tag{10}$$

Comparing (10) with (9), we find the required relation,

$$q_n(z) = \frac{z\Pi_n(z)}{(1-z)^{n+1}}, \qquad n = 1, 2, 3, \cdots. \tag{11}$$

This result is hardly new, although I have been unable to find a specific literature reference. Equation (11) could be obtained very simply by comparing the known power series representations [4, p. 22]

$$\frac{\Pi_n(z)}{(1-z)^{n+1}} = \sum_{j=0}^{\infty} (j+1)^n z^j, \qquad |z| < 1,$$

and [2, p. 49]

$$q_n(z) = \sum_{j=0}^{\infty} j^n z^j, \qquad |z| < 1;$$

however, the generating function appearing in (9) and (10) is of intrinsic interest and will be used extensively in § 3. (Electrical engineers, by the way, will recognize $q_n(1/z)$ as the Z-transform of the function x^n.)

Our next result follows directly from the definition (7); we omit the obvious proof.

LEMMA 2. *Let $A \subset \mathbb{C}$ be open, let $w \in A$, and let $f: A \to \mathbb{C}$ be regular, with $f(w) \neq 1$, $f'(w) \neq 0$. Then*

$$q_n(f(w)) = \begin{cases} \left(\dfrac{f(w)}{f'(w)} \dfrac{d}{dw}\right)^n \dfrac{1}{1 - f(w)}, & n \geqq 0, \\[2ex] \left(\dfrac{f(w)}{f'(w)} \dfrac{d}{dw}\right)^{n-1} \dfrac{f(w)}{(1 - f(w))^2}, & n \geqq 1. \end{cases} \tag{12}$$

We now use (12) and the well-known expansion

$$\frac{1}{\sin^2 \pi z} = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(k+z)^2}, \qquad z \notin \mathbb{Z}, \tag{13}$$

to obtain an analogous expansion for $q_n(-e^{\pi w})$.

LEMMA 3. *Let $A \subset \mathbb{C}$ be compact and not contain any point of the form $(2k+1)i$, $k \in \mathbb{Z}$. Then for each $n \geqq 1$,*

$$q_n(-e^{\pi w}) = \left(\frac{i}{\pi}\right)^{n+1} n! \sum_{k=-\infty}^{\infty} \frac{1}{(1 + 2k - iw)^{n+1}} \tag{14}$$

or, equivalently,

$$q_n(-e^{\pi w}) = \left(\frac{i}{\pi}\right)^{n+1} n! \sum_{k=0}^{\infty} \left(\frac{1}{(1 + 2k - iw)^{n+1}} + \frac{(-1)^{n+1}}{(1 + 2k + iw)^{n+1}}\right), \tag{15}$$

the series converging uniformly for $w \in A$.

Proof. Set $f(w) \equiv -e^{\pi w}$. Then $(f(w)/f'(w))\, d/dw = \pi^{-1}\, d/dw$ and $f(w) \cdot (1 - f(w))^{-2} = -e^{\pi w}(1 + e^{\pi w})^{-2} = -(4\cosh^2(\pi w/2))^{-1}$. Now $\cosh^2(\pi w/2) = \cos^2(i\pi w/2) = \sin^2 \pi(\frac{1}{2} - iw/2)$. Hence (13) gives us

$$\frac{f(w)}{(1 - f(w))^2} = -\frac{1}{4\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(\frac{1}{2} + k - (iw/2))^2}$$
$$= -\frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(1 + 2k - iw)^2},$$

and (12) takes the form

$$q_n(-e^{\pi w}) = -\frac{1}{\pi^{n+1}} \frac{d^{n-1}}{dw^{n-1}} \sum_{k=-\infty}^{\infty} \frac{1}{(1 + 2k - iw)^2}.$$

Clearly the series and its term-by-term derivatives converge uniformly for $w \in A$; term-by-term differentiation is valid and establishes (14); and the equivalence of (14) and (15) is trivial.

The last lemma of this section contains the essence of Theorem 1.

LEMMA 4. *For each integer $m \geqq 2$ there exists a positive number w_m,*

$$w_m = \tan \pi\eta_m/4m, \qquad 1 < \eta_m < 1 + 3^{1-2m}, \tag{16}$$

such that $q_{2m-1}(-e^{\pm\pi w_m}) = 0$ and $q_{2m-1}(-e^{\pi w}) \neq 0$ for $w \in (-w_m, w_m)$.

For the proof of this lemma, we shall require three simple inequalities:

$$\theta \geqq \sin\theta \geqq 2\theta/\pi, \qquad 0 \leqq \theta \leqq \pi/2, \tag{17}$$

$$\sum_{k=1}^{\infty} (1 + 2k)^{-n} < 3^{1-n}/2, \qquad n \geqq 4, \tag{18}$$

$$(1 - n^{-2})^n > \tfrac{1}{2}, \qquad n \geqq 2. \tag{19}$$

The first of these is standard; the second follows readily from

$$\sum_{k=2}^{\infty} \frac{1}{(1 + 2k)^n} < \int_1^{\infty} \frac{dx}{(1 + 2x)^n} = \frac{3^{1-n}}{2(n - 1)}.$$

To prove (19) we note that, for $-1 < x < 1$,

$$(1 - x^2)^{1/x} = \exp(x^{-1} \log(1 - x^2)) = \exp\left(-\sum_{k=1}^{\infty} k^{-1} x^{2k-1}\right).$$

Hence, for $0 < x \leqq \frac{1}{2}$,

$$(1 - x^2)^{1/x} \geqq (1 - t^2)^{1/t}|_{t=1/2} = \tfrac{9}{16} > \tfrac{1}{2},$$

and setting $x = n^{-1}$ establishes (19).

Proof of Lemma 4. From Lemma 3 it follows that $q_{2m-1}(-e^{\pi w})$, regarded as a function of w, is real and analytic on $\mathbb{R}$. Let $w \in \mathbb{R}$, and let m be an integer $\geqq 2$.

From (15) we find

$$F_m(w) \equiv \frac{(-\pi^2)^m}{2(2m-1)!} q_{2m-1}(-e^{\pi w})$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(1+2k+iw)^{2m} + (1+2k-iw)^{2m}}{((1+2k)^2+w^2)^{2m}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(1+2k)^{2m}} \cdot \frac{1}{(1+w^2/(1+2k)^2)^m}$$

$$\cdot \left\{ \frac{1}{2}\left(\frac{1+2k+iw}{\sqrt{(1+2k)^2+w^2}} \right)^{2m} + \frac{1}{2}\left(\frac{1+2k-iw}{\sqrt{(1+2k)^2+w^2}} \right)^{2m} \right\}.$$

We define, for $k = 0, 1, 2, \cdots$, functions $\theta_k : \mathbb{R} \to (-\pi/2, \pi/2)$ by

$$\theta_k(w) = \tan^{-1} \frac{w}{1+2k},$$

the principal value of $\tan^{-1}$ being understood. In terms of these,

$$F_m(w) = \sum_{k=0}^{\infty} \frac{1}{(1+2k)^{2m}} \cos^{2m} \theta_k \cos 2m\theta_k, \tag{20}$$

where we have written θ_k for $\theta_k(w)$. Since $\theta_k(0) = 0$, all k, and $|\theta_k(w)| < |\theta_0(w)|$ for $k = 1, 2, 3, \cdots$ and $w \neq 0$, it is easy to see that $F_m(w)$ is strictly positive for $|2m\theta_0| \leqq \pi/2$, i.e., for $|w| \leqq \tan(\pi/(4m)) \equiv a_m$. We show now that $F_m(w)$ is strictly negative for $2m\theta_0 = (1 + 3^{1-2m})\pi/2$, i.e., for $w = \tan((1 + 3^{1-2m})\pi/(4m)) \equiv b_m$. Using inequalities (17) and (19), we have

$$\begin{aligned} \cos^{2m} \theta_0 &= (1 - \sin^2 \theta_0)^m \geqq (1 - \theta_0^2)^m \\ &= \left\{ 1 - \frac{1}{m^2}\left(\frac{\pi(1+3^{1-2m})}{4} \right)^2 \right\}^m \\ &> (1 - m^{-2})^m > \tfrac{1}{2} \end{aligned}$$

and

$$\cos 2m\theta_0 = -\sin(3^{1-2m}\pi/2) \leqq -3^{1-2m}.$$

Hence the $k = 0$ term $< -3^{1-2m}/2$; together with (18), this implies that $F_m(b_m)$ is negative. The function F_m, being real and analytic on $[a_m, b_m]$, must, therefore, have at least one and at most a finite number of zeros in (a_m, b_m). Since $F_m(w) = F_m(-w)$, the smallest of these zeros, w_m, has the properties asserted by the lemma.

The transformation $z = -e^{\pi w}$ maps the interval $[-w_m, w_m]$ one-to-one onto the interval $[-e^{\pi w_m}, -e^{-\pi w_m}]$. Hence it follows from Lemma 4 that $q_{2m-1}(z)$ has no zeros in the interior of this interval. Since by (11), Π_n and q_n have the same zeros in $\mathbb{C}\setminus\{0, 1\}$, we clearly must have—see (2)—that $\lambda_m^{(2m-1)} = -e^{\pi w_m}$, $\lambda_{m-1}^{(2m-1)} = -e^{-\pi w_m}$, and Theorem 1 is proved.

3. Some further remarks about the functions q_n. In the notation of this paper, the exponential Euler polynomial $A_n(\cdot\,; z)$ of Schoenberg [4, p. 21] may be represented in the form

$$A_n(x; z) = x^n + (1 - z^{-1}) \sum_{m=1}^{n} (-1)^{m+1} \binom{n}{m} q_m(z) x^{n-m};$$

with the aid of (21) below, this can be written more succinctly as

$$A_n(x; z^{-1}) = (1 - z) \sum_{m=0}^{n} \binom{n}{m} q_m(z) x^{n-m}.$$

The exponential Euler polynomials are the basic building blocks of the exponential Euler splines and eigensplines; they are also, although somewhat disguised, the ingredients used by Nilson [3] in his construction of the fundamental cardinal spline. Consequently, the functions q_n are of interest in their own right, and it is this interest that provides the motivation for the present section. We note, incidentally, that (7) leads to an analogous operational definition of the exponential Euler polynomials,

$$A_n(x; z^{-1}) = (1 - z)\left(x + z\frac{\partial}{\partial z}\right)^n \frac{1}{1 - z}.$$

The statements about the roots of the Euler–Frobenius polynomials Π_n given at the beginning of § 1 are usually derived from the recursion relation [4, p. 22]

$$\Pi_{n+1}(z) = (1 + nz)\Pi_n(z) + z(z - 1)\Pi_n'(z), \qquad \Pi_0(z) = 1,$$

and the symmetry of the coefficients of the Π's. Corresponding statements about the functions q_n may be obtained directly from Lemmas 1 and 2, without reference to properties of the Euler–Frobenius or Hille polynomials.

First we apply the expansion (9) to both sides of the identity

$$\frac{1}{1 - ze^t} = -\frac{1}{1 - z^{-1}e^{-t}} + 1, \qquad ze^t \neq 0, 1,$$

to find

$$\sum_{m=0}^{\infty} q_m(z)\frac{t^m}{m!} = 1 - \sum_{m=0}^{\infty} (-1)^m q_m\left(\frac{1}{z}\right)\frac{t^m}{m!}, \qquad z \neq 0, 1.$$

Hence

$$\begin{aligned} q_0(1/z) &= 1 - q_0(z), \\ q_n(1/z) &= (-1)^{n+1} q_n(z), \qquad n \geqq 1; \end{aligned} \tag{21}$$

regarded as identities between rational functions, these relations are valid without the restriction $z \neq 0, 1$.

Next, letting $f(w) \equiv w(w + 1)^{-1}$ in (12) yields

$$q_n\left(\frac{w}{w + 1}\right) = \left(w(w + 1)\frac{d}{dw}\right)^n (w + 1). \tag{22}$$

Therefore, if we define a sequence of polynomials $\{G_n\}$ by

$$G_{n+1}(w) = w(w+1)G_n'(w), \qquad G_0(w) = 1 + w, \tag{23}$$

then $q_n(w(w+1)^{-1}) = G_n(w)$. Since the transformation $z = w(w+1)^{-1}$, with inverse $w = z(1-z)^{-1}$, maps the extended complex plane onto itself one-to-one, we have, equivalently,

$$q_n(z) = G_n\left(\frac{z}{1-z}\right); \tag{24}$$

an alternative form of this equation is obtained by substituting $1/z$ for z and using (21):

$$q_n(z) = (-1)^{n+1}G_n\left(\frac{1}{z-1}\right), \qquad n \geqq 1. \tag{25}$$

We shall need the following properties of the polynomials G_n.

LEMMA 5. (i) *The polynomials G_n defined by* (23) *are of the form*

$$G_n(w) = w + \cdots + n!w^{n+1}, \qquad n \geqq 1. \tag{26}$$

(ii) *For $n \geqq 1$, G_n has simple roots at -1 and* 0, *and $n-1$ simple roots in* $(-1, 0)$.

Proof. The inductive proof of (i) is trivial. Suppose that (ii) is true for a particular n. By Rolle's theorem, the polynomial G_n' has at least n distinct roots in $(-1, 0)$. By (23), G_{n+1} has the same n roots, plus those at -1 and 0. Since G_{n+1} is of degree $n+2$, all these roots are simple, and there are no others; (ii) is thus also true for $n+1$. The inductive proof is completed by noting that (ii) is true for $n = 1$.

(We note, in passing, that the polynomials G_n are interesting in a variety of ways. For example, let us recall that the "differences of zero" $\Delta^j 0^m$ of the finite difference calculus [1, pp. 124–125] satisfy $\Delta^1 0^1 = 1$ and

$$\Delta^1 0^{m+1} = \Delta^1 0^m; \qquad \Delta^{m+1}0^{m+1} = (m+1)\Delta^m 0^m;$$
$$\Delta^j 0^{m+1} = j(\Delta^j 0^m + \Delta^{j-1}0^m), \qquad j = 2, 3, \cdots, m,$$

for all positive integers m. From this recursion relation, we easily verify that (23) implies

$$G_m(w) = (w+1)\sum_{j=1}^{m} \Delta^j 0^m w^{m-j}, \qquad m \geqq 1;$$

using this expression in conjunction with (11) and (25) yields the expansion of $\Pi_m(z)$ about $z = 1$:

$$\Pi_m(z) = \sum_{j=1}^{m} \Delta^j 0^m (z-1)^{m-j}, \qquad m \geqq 1.)$$

We can now prove the following.

THEOREM 2. *The only zeros of q_n, $n \geqq 1$, in the extended complex plane are simple zeros at* 0 *and* ∞, *and $n-1$ simple zeros in $(-\infty, 0)$. If λ is a zero, so is $1/\lambda$.*

Proof. As noted earlier, the transformation $w = z(1 - z)^{-1}$ maps the extended complex plane onto itself one-to-one; in particular, $[-\infty, 0]$ is mapped onto $[-1, 0]$. From Lemma 5 and (24) it follows then that the only zeros of q_n are at 0, at ∞, and at $n - 1$ distinct points in $(-\infty, 0)$. Since $dw/dz = (1 - z)^{-2}$, the mapping $z \mapsto w$ is conformal except at $z = 1$ and $z = \infty$. Hence simplicity of the roots of G_n implies simplicity of the finite zeros of q_n; the simplicity of the zero at ∞ follows from (21), as does the last statement of the theorem.

Various identities satisfied by the q's follow from corresponding identities satisfied by their generating function. We have already seen one application of this method in the derivation of (21); we sketch here some further examples.

Denote the generating function by Φ:

$$\Phi(z, t) = \frac{1}{1 - ze^t}. \tag{27}$$

From the identity

$$\Phi(z, t) = ze^t\Phi(z, t) + 1$$

we find, using (9) and comparing coefficients of the representation of the two sides in powers of t,

$$q_n(z) = z \sum_{m=0}^{n} \binom{n}{m} q_m(z), \qquad n \geqq 1; \tag{28}$$

solving (28) for $q_n(z)$ gives the recursion relation

$$q_n(z) = \frac{z}{1 - z} \sum_{m=0}^{n-1} \binom{n}{m} q_m(z), \qquad n \geqq 1. \tag{29}$$

(We note here that (28) is equivalent to a recursion relation derived from (1) by Schoenberg [4, p. 23] in the same way.)

The identity

$$z\Phi(z, -t) = e^t\Phi(z, -t) - e^t$$

leads similarly to

$$q_n(z) = (-1)^{n+1} \frac{1}{1 - z} \left\{ \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} q_m(z) - 1 \right\}, \qquad n \geqq 1, \tag{30}$$

which could also be obtained by substituting $1/z$ for z in (29) and using (21). Forming appropriate linear combinations of (29) and (30), we have

$$q_{2m}(z) = \frac{z}{(1 - z)^2} \left\{ 2 \sum_{j=0}^{m-1} \binom{2m}{2j} q_{2j}(z) - 1 \right\}, \qquad m \geqq 1, \tag{31}$$

and

$$q_{2m+1}(z) = \frac{z}{(1 - z)^2} \left\{ 2 \sum_{j=0}^{m-1} \binom{2m+1}{2j+1} q_{2j+1}(z) + 1 \right\}, \qquad m \geqq 1, \tag{32}$$

which may be combined in a single formula,

$$q_n(z) = \frac{z}{(1-z)^2}\left\{\sum_{j=1}^{[n/2]} 2\binom{n}{n-2j} q_{n-2j} + (-1)^{n-1}\right\}, \qquad n \geqq 2. \tag{33}$$

Finally, the identity

$$x\Phi(x, t) - y\Phi(y, t) = (x - y)\Phi(x, t)\Phi(y, t)$$

gives us

$$xq_m(x) - yq_m(y) = (x - y)\sum_{j=0}^{m}\binom{m}{j} q_j(x)q_{m-j}(y), \qquad m \geqq 0; \tag{34}$$

dividing (34) by $x - y$, letting $y \to x$, and recalling that $zq_m'(z) = q_{m+1}(z)$, we further find

$$q_{m+1}(z) + q_m(z) = \sum_{j=0}^{m}\binom{m}{j} q_j(z)q_{m-j}(z), \qquad m \geqq 0. \tag{35}$$

We next prove a result of a rather different nature.

THEOREM 3. *Let m, n be positive integers, with $n > m$. Then*

$$\sum_{j=1}^{n-1} \frac{q_m(\lambda_j^{(n)})}{q_{n+1}(\lambda_j^{(n)})} + 1 = (-1)^{n-m}, \tag{36}$$

the sum being taken over the $n - 1$ zeros of q_n (and hence of Π_n) in $(-\infty, 0)$.

Proof. Consider the complex line integral

$$I = \frac{1}{2\pi i}\int_C \frac{q_m(z)}{zq_n(z)}\,dz \equiv \frac{1}{2\pi i}\int_C g(z)\,dz,$$

the contour C being a circle about the origin containing in its interior all poles of $g(z)$ in the finite z-plane. Using (25) and (26), we find

$$g(z) = (-1)^{n-m}\frac{1}{z}\,\frac{1 + \cdots + m!(z-1)^{-m}}{1 + \cdots + n!(z-1)^{-n}}; \tag{37}$$

similarly, (24) and (26) lead to

$$g(z) = \frac{1}{z}\,\frac{1 + \cdots + m!z^m(1-z)^{-m}}{1 + \cdots + n!z^n(1-z)^{-n}}. \tag{38}$$

Equation (37) shows that $g(z) = (-1)^{n-m}z^{-1} + O(z^{-2})$ as $z \to \infty$, and hence $I = (-1)^{n-m}$. Now I equals the sum of the residues of $g(z)$ at its poles. These can only occur at $z = 1$, at $z = 0$, and at the zeros of $q_n(z)$ in $(-\infty, 0)$. Since $n > m$, (37) shows that $g(z)$ is regular at $z = 1$. From (38) we see that $g(z)$ has residue 1 at $z = 0$. Finally, at $z = \lambda_j^{(n)}$, $j = 1, 2, \cdots, n - 1$, $q_n(z)$ has simple zeros; the corresponding residues, therefore, are $q_m(\lambda_j^{(n)})/(\lambda_j^{(n)}q_n'(\lambda_j^{(n)})) = q_m(\lambda_j^{(n)})/q_{n+1}(\lambda_j^{(n)})$, and the theorem is proved.

The significance of the theorem stems from a corollary that follows readily if one combines terms corresponding to mutually reciprocal zeros and uses (21).

COROLLARY. *Let m be an integer* $\geqq 2$. *Then*

$$\sum_{j=1}^{m-1} \frac{q_{2k}(\lambda_j^{(2m-1)})}{q_{2m}(\lambda_j^{(2m-1)})} = -1, \qquad k = 1, 2, \cdots, m-1, \tag{39}$$

the sum in each case being taken over the zeros of q_{2m-1} *in* $(-1, 0)$.

Equation (39) can be used, in conjunction with Schoenberg's results on eigensplines [4, p. 26], to establish a computationally attractive representation of the fundamental cardinal spline L_n of odd degree $n \geqq 3$. We merely state the result; the rather simple proof, and an algorithm for evaluating expressions of the form $\sum_{m=M}^{N} c_m L_n(x - m)$, will be given in a later paper.

Let n be an odd integer $\geqq 3$. For all $z \in \mathbb{C}$ such that $q_{n+1}(z) \neq 0$, we define particular multiples of the exponential Euler polynomials $A_n(x; z)$ and $A_n(x; z^{-1})$,

$$\begin{aligned} p_n^+(x; z) &= -z \frac{q_0(z)}{q_{n+1}(z)} A_n(x; z) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l \frac{q_{n-l}(z)}{q_{n+1}(z)} x^l - z \frac{q_0(z)}{q_{n+1}(z)} x^n, \end{aligned} \tag{40}$$

$$\begin{aligned} p_n^-(x; z) &= \frac{q_0(z)}{q_{n+1}(z)} A_n(x; z^{-1}) \\ &= \sum_{l=0}^{n-1} \binom{n}{l} \frac{q_{n-l}(z)}{q_{n+1}(z)} x^l + \frac{q_0(z)}{q_{n+1}(z)} x^n; \end{aligned} \tag{41}$$

in terms of these, we further define the continuous function $\Phi_n(\cdot\,; z): \mathbb{R} \to \mathbb{C}$ by setting, with $k = [x]$,

$$\Phi_n(x; z) = \begin{cases} z^k p_n^+(x - k; z), & 0 \leqq k \leqq x \leqq k + 1, \\ z^{-k} p_n^-(x - k; z), & k \leqq x \leqq k + 1 \leqq 0. \end{cases} \tag{42}$$

In particular, we note the limiting form of $\Phi_n(x; z)$ as $z \to 0$:

$$\Phi_n(x; 0) = \begin{cases} \sum_{l=0}^{n-1} \binom{n}{l} (-1)^l x^l - x^n = (1 - x)^n, & 0 \leqq x \leqq 1, \\ (x + 1)^n, & -1 \leqq x \leqq 0, \\ 0, & |x| > 1. \end{cases} \tag{43}$$

With these definitions we have

$$L_n(x) = \Phi_n(x; 0) + \sum_{j=1}^{(n-1)/2} \Phi_n(x; \lambda_j^{(n)}), \tag{44}$$

the sum being taken over the zeros of q_n in $(-1, 0)$. This representation of L_n is equivalent to Nilson's [3, p. 448] but considerably simpler in form. (Note that in evaluating (44), we may omit the $l = 0$ terms in (40) and (41).)

By setting $\lambda_0^{(n)} = 0$ we can write (44) alternatively as

$$L_n(x) = \sum_{j=0}^{(n-1)/2} \Phi_n(x; \lambda_j^{(n)}), \tag{45}$$

the sum now being taken over all the zeros of q_n inside the unit circle. We may, therefore, think of the zero at the origin as contributing the inhomogeneous component of L_n, i.e., the only component which does not vanish at $x = 0$.

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APPELL'S FUNCTION F_4 AS A DOUBLE AVERAGE*

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Abstract. A quadratic transformation of a double hypergeometric series of order two (Appell's F_4 with equal denominator parameters) into a series of order three revives longstanding doubts about the accepted classification by order. Both series can be represented as double Dirichlet averages of x^t. Appell's F_4 with unrestricted parameters can be represented as such an average with two rows and three columns. There are six cases in which a restriction on the parameters reduces the number of columns to two.

1. Introduction. After Appell introduced his four hypergeometric series in two variables, Horn approached the subject systematically in 1889, defining a double hypergeometric series to be a series $\sum\sum A(m, n)x^m y^n$ such that $A(m + 1, n)/A(m, n)$ and $A(m, n + 1)/A(m, n)$ are rational functions of m and n. The larger degree of the two rational functions is the order of the series. Series of order one are reducible to series in a single variable. The four Appell functions belong to a class of 14 series which, together with their 20 confluent limits, exhaust the possible series of order two. (See [10, § 5.7] for more precise statements.) The analytic continuation of Appell functions entails other series of order two, and hence Appell functions, which have substantial importance in applied mathematics, must be studied as part of a larger class. It would be gratifying if series of order two formed a class sufficient unto itself, but in 1948, Erdélyi [9, p. 380] found an equality (not explicitly recorded in his paper) connecting two series of second order with a series of third order. He remarked, "This appears to be an indication of the inadequacy of the classification of hypergeometric series of two variables accepted at present". A quarter century later the same classification is still accepted for lack of a better, and the author has found no reference to Erdélyi's remark in the subsequent literature. The matter is raised again by § 2 of the present paper, where Appell's F_4 with equal denominator parameters is transformed quadratically into a double series of order three.

Thus it is not clear that the class of double hypergeometric series of order two is the right object to study. A different approach to Appell functions was suggested by the author [7], who discussed Appell's F_1, F_2, and F_3 as double Dirichlet averages of x^t, i.e., as special cases of the function

$$\mathscr{R}_t(b, Z, \beta) = \int (u \cdot Z \cdot v)^t \, d\mu_b(u) \, d\mu_\beta(v), \tag{1.1}$$

where u is a k-tuple of positive weights, v a κ-tuple of positive weights, and $u \cdot Z \cdot v = \sum\sum u_i Z_{ij} v_j$. The Dirichlet measure μ_b has the form of a beta distribution in $k - 1$ variables, but in the present context the parameters $b_1, \cdots, b_k$ may be complex with positive real parts. One component of b is associated with each row (and one component of β with each column) of the $k \times \kappa$ matrix Z with entries in

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the right-half complex plane. Permutation of rows or columns, along with their corresponding parameters, does not change the value of $\mathscr{R}_t$. The function is plainly homogeneous of degree t in the elements of Z.

Appell's F_2 is a double Dirichlet average with $k = \kappa = 2$ and $Z_{11} + Z_{22} = Z_{12} + Z_{21}$, which means that the entries of Z are the vertices of a parallelogram in the complex plane. Appell's F_1 has a representation with $k = \kappa = 2$, but also it has more symmetrical representations with $k = 1$, $\kappa = 3$ (a single Dirichlet average since $k = 1$ signifies no integration) and $k = 2$, $\kappa = 3$. The permutation symmetry of $\mathscr{R}$ replaces the linear transformations of F_1 and F_2. For F_3 there are representations with $k = 2$, $\kappa = 3$ and $k = \kappa = 3$. The double series of order three encountered in § 2 has a representation with $k = \kappa = 2$ and $Z_{21} = Z_{22}$, which means that the entries of Z are the vertices of a triangle.

When [7] was written it seemed unlikely that $\mathscr{R}$ could represent the general case of Appell's F_4,

$$F_4(\alpha, \beta; \gamma, \delta; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\delta)_n m!\,n!} x^m y^n, \qquad |x|^{1/2} + |y|^{1/2} < 1. \tag{1.2}$$

In § 3, however, we shall give such a representation with $k = 2$, $\kappa = 3$. In § 4 we shall list six cases in which a restriction on the parameters of F_4 permits a representation with $k = \kappa = 2$. These latter formulas will be used in a subsequent paper [12] on quadratic transformations of $\mathscr{R}$.

2. Restricted F_4 as a series of third order. The single Dirichlet average of x^n is the R-polynomial [4, (3.6)],

$$R_n(\rho, \sigma; x, y) = \frac{n!}{(\rho + \sigma)_n} \sum_{m=0}^{n} \frac{(\rho)_m(\sigma)_{n-m}}{m!(n-m)!} x^m y^{n-m}. \tag{2.1}$$

Elementary manipulations show that (1.2) can be written as

$$F_4(\alpha, \beta; \gamma, \delta; x, y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(\gamma + \delta - 1 + n)_n}{(\gamma)_n(\delta)_n n!} R_n(1 - \delta - n, 1 - \gamma - n; x, y). \tag{2.2}$$

The R-polynomial on the right side is a Jacobi polynomial which, if $\gamma = \delta$, admits a quadratic transformation [6, (2.8)] for Gegenbauer polynomials,

$$(2\nu)_n R_n(\nu, \nu; \xi^2, \eta^2) = (\nu)_n R_n[\tfrac{1}{2} - \nu - n, \tfrac{1}{2} - \nu - n; (\xi + \eta)^2, (\xi - \eta)^2]. \tag{2.3}$$

Replacing (x, y) by (x^2, y^2) in (2.2), we find

$$F_4(\alpha, \beta; \gamma, \gamma; x^2, y^2) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} R_n[\gamma - \tfrac{1}{2}, \gamma - \tfrac{1}{2}; (x + y)^2, (x - y)^2]. \tag{2.4}$$

Substituting (2.1) and replacing n by $m + n$,

$$F_4(\alpha, \beta; \gamma, \gamma; x^2, y^2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}(\gamma - \frac{1}{2})_m(\gamma - \frac{1}{2})_n}{(\gamma)_{m+n}(2\gamma - 1)_{m+n} m!\,n!} (x + y)^{2m}(x - y)^{2n}. \tag{2.5}$$

In Horn's classification, the left side is a series of second order, while the right side is of third order. This suggests that the order of a double series is not a very fundamental property. Some caution is advisable, for the order of a hypergeometric series in one variable appears significant even though certain series of different order are connected by nonlinear transformations [5, (10), (11)]. These, however, result from separating a series into even and odd terms, and they consequently involve three series, not two as in (2.5). Erdélyi's equality of 1948 involves three series, but the variables are related by linear fractional transformations rather than nonlinear. See the note added in proof.

The series of order three on the right side of (2.5) can be written as an $\mathscr{R}$-function. From [8, (3.4)] and (2.1) we find

$$(2.6)\qquad (\mu+\mu')_n\mathscr{R}_n(\mu,\mu';W;\nu,\nu') = (\mu)_n R_n(\nu,\nu';\xi,\eta),\qquad W=\begin{bmatrix}\xi & \eta\\ 0 & 0\end{bmatrix}.$$

Thus the right side of (2.4) is

$$(2.7)\qquad \sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\mathscr{R}_n(\beta,\gamma-\beta;W';\gamma-\tfrac{1}{2},\gamma-\tfrac{1}{2}),\qquad W'=\begin{bmatrix}(x+y)^2 & (x-y)^2\\ 0 & 0\end{bmatrix}.$$

By [7, (6.3)] the sum of the series is

$$(2.8)\qquad \mathscr{R}_{-\alpha}(\beta,\gamma-\beta;W'';\gamma-\tfrac{1}{2},\gamma-\tfrac{1}{2}),\qquad W''=\begin{bmatrix}1-(x+y)^2 & 1-(x-y)^2\\ 1 & 1\end{bmatrix}.$$

A transformation[1] of F_4 with $\beta=\alpha+\frac{1}{2}$ into F_2, due to Bailey [3], can be proved in a different way using (2.2). Replacement of (x, y) by (x^2, y^2) and substitution of [8, (5.9)], with use of [7, (2.9)(iii)], gives

$$(2.9)\qquad \begin{aligned} F_4(\alpha,\beta;\gamma,\delta;x^2,y^2) &= \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\frac{1}{2})_n n!}\mathscr{R}_{2n}(\gamma-\tfrac{1}{2},\gamma-\tfrac{1}{2};X;\delta-\tfrac{1}{2},\delta-\tfrac{1}{2}),\\ X &= \begin{bmatrix}x+y & x-y\\ -x+y & -x-y\end{bmatrix}.\end{aligned}$$

If $\beta=\alpha+\frac{1}{2}$, the coefficient of this series becomes $(2\alpha)_{2n}/(2n)!$. Since the corresponding $\mathscr{R}$-polynomials of odd degree vanish [8, (5.8)], the series can be summed by [7, (6.3)] to yield

$$(2.10)\qquad \mathscr{R}_{-2\alpha}(\gamma-\tfrac{1}{2},\gamma-\tfrac{1}{2};X';\delta-\tfrac{1}{2},\delta-\tfrac{1}{2}),\qquad X'=\begin{bmatrix}1-x-y & 1-x+y\\ 1+x-y & 1+x+y\end{bmatrix}.$$

This is an F_2 since the entries of X' are the vertices of a parallelogram.

3. Unrestricted F_4 in terms of $\mathscr{R}$. Burchnall and Chaundy's representation of F_4 [10, Eq. 5.8.1(4)] is

$$(3.1)\qquad \begin{aligned} &F_4[\alpha,\beta;\gamma,\delta;x(1-y),y(1-x)]\\ &\quad= F_4[\beta,\alpha;\gamma,\delta;x(1-y),y(1-x)]\\ &\quad= \int_0^1\int_0^1 (1-ux)^{1+\beta-\gamma-\delta}(1-vy)^{1+\alpha-\gamma-\delta}(1-ux-vy)^{\gamma+\delta-\alpha-\beta-1}\\ &\qquad\qquad \cdot d\mu_{(\alpha,\delta-\alpha)}(v)\,d\mu_{(\beta,\gamma-\beta)}(u),\end{aligned}$$

[1] The first listing on p. 242 of [10, vol. 1] should read $\beta=\alpha+\frac{1}{2}$ instead of $\gamma+\gamma'=\alpha+1$.

where $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, $\operatorname{Re}\delta > \operatorname{Re}\alpha > 0$, and

$$d\mu_{(\rho,\sigma)}(u) = \frac{\Gamma(\rho+\sigma)}{\Gamma(\rho)\Gamma(\sigma)} u^{\rho-1}(1-u)^{\sigma-1}\,du. \tag{3.2}$$

The double integral provides the analytic continuation of the F_4-series to the region defined by $\operatorname{Re} x < 1$, $\operatorname{Re} y < 1$, and $\operatorname{Re}(x+y) < 1$. By [11, (T.1)] the integral with respect to v is

$$\begin{aligned}\int_0^1 &(1-vy)^{1+\alpha-\gamma-\delta}(1-ux-vy)^{\gamma+\delta-\alpha-\beta-1}\,d\mu_{(\alpha,\delta-\alpha)}(v)\\ &= (1-ux)^{+\delta-\alpha-\beta-1}\\ &\quad\cdot R_{-\alpha}\left(\gamma+\delta-\alpha-1, 1+\alpha+\beta-\gamma-\delta, \delta-\beta; 1-y, \frac{1-ux-y}{1-ux}, 1\right).\end{aligned} \tag{3.3}$$

The power of $1-ux$ in the integrand of (3.1) combines with the power on the right side of (3.3) to leave $(1-ux)^{-\alpha}$, and even this can be removed by using the homogeneity of $R_{-\alpha}$. Thus the right side of (3.1) equals

$$\begin{aligned}\int_0^1 &R_{-\alpha}[\gamma+\delta-\alpha-1, 1+\alpha+\beta-\gamma-\delta, \delta-\beta;\\ &\qquad (1-y)(1-ux), 1-ux-y, 1-ux]\,d\mu_{(\beta,\gamma-\beta)}(u).\end{aligned} \tag{3.4}$$

By [7, (2.8)] we find the desired result,

$$\begin{aligned}&F_4[\alpha,\beta;\gamma,\delta;x(1-y),y(1-x)]\\ &\qquad = \mathscr{R}_{-\alpha}(\beta,\gamma-\beta;Z;\gamma+\delta-\alpha-1,1+\alpha+\beta-\gamma-\delta,\delta-\beta),\\ &Z = \begin{bmatrix}(1-x)(1-y) & 1-x-y & 1-x\\ 1-y & 1-y & 1\end{bmatrix}.\end{aligned} \tag{3.5}$$

Since a vanishing column parameter can be omitted [7, p. 422] along with the corresponding column of Z, inspection of (3.5) yields three cases in which one restriction on the parameters ($\gamma+\delta = \alpha+1$, $\gamma+\delta = \alpha+\beta+1$, or $\delta = \beta$) allows F_4 to be written in terms of $\mathscr{R}$ with $k = \kappa = 2$. These three are respectively equivalent to known transformations of F_4 into F_2, a product of ${}_2F_1$-series, or F_1 [2, pp. 81, 102]. The case $\gamma = \beta$ has a vanishing row parameter but leads only to a different representation of F_1. It is not essentially distinct from $\delta = \beta$ because (1.2) is unchanged by interchanging γ with δ and x with y.

Because of the peculiar arguments on the left sides of (3.1) and (3.5) it is sometimes convenient to put

$$\begin{aligned}x &= \sin^2\frac{\theta+\varphi}{2}, & y &= \sin^2\frac{\theta-\varphi}{2},\\ x(1-y) &= \left(\frac{\sin\theta+\sin\varphi}{2}\right)^2, & y(1-x) &= \left(\frac{\sin\theta-\sin\varphi}{2}\right)^2,\\ (1-x)(1-y) &= \left(\frac{\cos\theta+\cos\varphi}{2}\right)^2, & 1-x-y &= \cos\theta\cos\varphi.\end{aligned} \tag{3.6}$$

4. Restricted F_4 in terms of $\mathscr{R}$. For convenient reference we list the six cases in which F_4 with a single restriction on the parameters is an $\mathscr{R}$-function with $k = \kappa = 2$. The six restrictions are $\gamma + \delta = \alpha + \beta + 1$, $\delta = \beta$, $\gamma + \delta = \alpha + 1$, $\beta = \alpha + \frac{1}{2}$, $\delta = \alpha - \beta + 1$, $\gamma = \delta$, and the respective $\mathscr{R}$-functions are expressible as a product of ${}_2F_1$-series, F_1, F_2, F_2, F_2, and a series of third order. The first three come from (3.5), the fourth from (2.9) and (2.10), the fifth from [1, p. 27], and the sixth from (2.7) and (2.8).

$$\begin{aligned} &F_4[\alpha, \beta; \gamma, 1 + \alpha + \beta - \gamma; x(1 - y), y(1 - x)] \\ &\quad = \mathscr{R}_{-\alpha}(\beta, \gamma - \beta; X_1; \beta, 1 + \alpha - \gamma) \\ &\quad = R_{-\alpha}(\beta, \gamma - \beta; 1 - x, 1)R_{-\alpha}(\beta, 1 + \alpha - \gamma; 1 - y, 1), \\ &\qquad X_1 = \begin{bmatrix} (1 - x)(1 - y) & 1 - x \\ 1 - y & 1 \end{bmatrix}; \end{aligned} \tag{4.1}$$

$$\begin{aligned} &F_4[\alpha, \beta; \gamma, \beta; x(1 - y), y(1 - x)] \\ &\quad = \mathscr{R}_{-\alpha}(\beta, \gamma - \beta; X_2; 1 + \alpha - \gamma, \beta - 1 - \alpha + \gamma), \\ &\qquad X_2 = \begin{bmatrix} 1 - x - y & (1 - x)(1 - y) \\ 1 - y & 1 - y \end{bmatrix}; \end{aligned} \tag{4.2}$$

$$\begin{aligned} &F_4[\alpha, \beta; \gamma, 1 + \alpha - \gamma; x(1 - y), y(1 - x)] \\ &\quad = \mathscr{R}_{-\alpha}(\beta, \gamma - \beta; X_3; \beta, 1 + \alpha - \gamma - \beta), \\ &\qquad X_3 = \begin{bmatrix} 1 - x - y & 1 - x \\ 1 - y & 1 \end{bmatrix}; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &F_4(\alpha, \alpha + \tfrac{1}{2}; \gamma, \delta; x^2, y^2) = \mathscr{R}_{-2\alpha}(\gamma - \tfrac{1}{2}, \gamma - \tfrac{1}{2}; X_4; \delta - \tfrac{1}{2}, \delta - \tfrac{1}{2}), \\ &\qquad X_4 = \begin{bmatrix} 1 - x - y & 1 - x + y \\ 1 + x - y & 1 + x + y \end{bmatrix}; \end{aligned} \tag{4.4}$$

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, \alpha - \beta + 1; x^2, y^2) = \mathscr{R}_{-\alpha}(\beta, \gamma - \beta; X_5; \alpha - \beta + \tfrac{1}{2}, \alpha - \beta + \tfrac{1}{2}), \\ &\qquad X_5 = \begin{bmatrix} (1 + y)^2 - x^2 & (1 - y)^2 - x^2 \\ (1 + y)^2 & (1 - y)^2 \end{bmatrix}; \end{aligned} \tag{4.5}$$

$$\begin{aligned} &F_4(\alpha, \beta; \gamma, \gamma; x^2, y^2) = \mathscr{R}_{-\alpha}(\beta, \gamma - \beta; X_6; \gamma - \tfrac{1}{2}, \gamma - \tfrac{1}{2}), \\ &\qquad X_6 = \begin{bmatrix} 1 - (x + y)^2 & 1 - (x - y)^2 \\ 1 & 1 \end{bmatrix}. \end{aligned} \tag{4.6}$$

The factorization in (4.1) follows from [7, (6.5)]. These formulas will be used in a subsequent paper [12] on quadratic transformations of $\mathscr{R}$.

Note added in proof. Reference [13] contains a linear transformation connecting one double series of order two with one series of order three.

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THE EVALUATION OF CERTAIN CLASSES OF NONABSOLUTELY CONVERGENT DOUBLE SERIES*

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Abstract. Certain classes of nonabsolutely convergent double series are evaluated in closed form. Their values are expressible in terms of various types of Dedekind sums. In many cases, it is shown that the inversion in order of summation yields a different value. In certain other cases, it is shown that inversion yields the same value.

1. Introduction. In [3] we employed the Poisson summation formula to evaluate in closed form the nonabsolutely convergent double series

$$S(d, c) = \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ md \neq nc}}^{\infty} \frac{1}{(md)^2 - (nc)^2},$$

where, here and throughout the sequel, c and d denote coprime, positive integers. The value of $S(d, c)$ was shown to be expressible in terms of the classical Dedekind sum $s(d, c)$, defined below in § 2. Moreover, $S(d, c)$ is always a rational multiple of π^2. As a numerical example, we have

$$S(11, 29) = \frac{39\pi^2}{74008} = \frac{39\pi^2}{2^3 \cdot 11 \cdot 29^2}.$$

Observe that if we invert the order of summation in $S(d, c)$, we obtain the sum $-S(c, d)$. We showed in [3] that inverting the order of summation yields a different value. In fact, by using the reciprocity theorem for Dedekind sums, given in § 2, we showed that

$$S(d, c) + S(c, d) = \frac{\pi^2}{4cd}. \tag{1.1}$$

The objective of this paper is to demonstrate that the method which we employed in [3] can be used to evaluate in closed form a rather wide variety of nonabsolutely convergent double series. By no means have we attempted to be exhaustive in our presentation. We have chosen to present only some of the most elegant illustrations of the method. For most of the double series summed here, the values obtained involve various types of Dedekind sums.

Dedekind sums are very important in number theory. In particular, the ordinary Dedekind sums appear in the transformation formulas of the Dedekind eta-function, and as a consequence, play an important role in the theory of the partition function [2, chap. 3]. The most important property of Dedekind sums is the reciprocity theorem. In particular, the reciprocity theorem makes feasible the numerical calculation of Dedekind sums. A short table of values for the ordinary Dedekind sums has been published by D. Zagier [9]. A more extensive table has

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recently been computed by L. Pinzur [6]. For an excellent introduction to Dedekind sums, see the monograph of Rademacher and Grosswald [8].

For the first large class of series evaluated here, we will give in full all of the details. For the remaining classes, we will be content to give just brief sketches of the proofs, as only the details, not the methods, change from example to example.

2. Double series with periodic coefficients. For this class, we shall need the periodic Poisson summation formula developed by L. Schoenfeld and the author [5].

Let $A = \{a_n\}$, $-\infty < n < \infty$, be a sequence of complex numbers with period k, i.e., $a_n = a_{n+k}$ for every integer n. Define the complementary sequence $B = \{b_n\}$, $-\infty < n < \infty$, by

$$b_n = \frac{1}{k}\sum_{j=0}^{k-1} a_j e^{-2\pi i jn/k}. \tag{2.1}$$

In fact, (2.1) is valid if and only if

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi i jn/k}, \qquad -\infty < n < \infty.$$

We may now state the periodic Poisson summation formula. If f is of bounded variation on $[\alpha, \beta]$,

$$\begin{aligned}\frac{1}{2}\sum_{n=\alpha}^{\beta}{}' a_n\{f(n+0)+f(n-0)\} &= b_0\int_\alpha^\beta f(x)\,dx \\ &\quad + \sum_{n=1}^{\infty}\int_\alpha^\beta (b_n e^{2\pi i nx/k} + b_{-n}e^{-2\pi i nx/k})f(x)\,dx,\end{aligned} \tag{2.2}$$

where the prime $'$ on the summation sign at the left indicates that if $n = \alpha$ or $n = \beta$, only $a_\alpha f(\alpha+0)$ or $a_\beta f(\beta - 0)$, respectively, is counted. If $A = I = \{1\}$, (2.2) reduces to the ordinary Poisson summation formula

$$\begin{aligned}\frac{1}{2}\sum_{n=\alpha}^{\beta}{}' \{f(n+0)+f(n-0)\} &= \int_\alpha^\beta f(x)\,dx \\ &\quad + 2\sum_{n=1}^{\infty}\int_\alpha^\beta f(x)\cos(2\pi nx)\,dx.\end{aligned} \tag{2.3}$$

The periodic Bernoulli numbers $B_n(A)$, $0 \leqq n < \infty$, and periodic Bernoulli functions $\mathscr{B}_n(x, A)$, $0 \leqq n < \infty$, are defined recursively as follows. Let

$$\mathscr{B}_0(x, A) = B_0(A) = \frac{1}{k}\sum_{j=0}^{k-1} a_j = b_0,$$

$$B_1(A) = \frac{1}{k}\sum_{j=0}^{k-1}(j - \tfrac{1}{2}k)a_j,$$

and for $x \geqq 0$,

$$\mathscr{B}_1(x, A) = B_0(A)x - B_1(A) - \sum_{0 \leqq j \leqq x}{}' a_j,$$

where the prime on the summation sign indicates that if $j = x$, only $\frac{1}{2}a_x$ is counted. For $n \geqq 2$ and $x \geqq 0$, let

$$\mathscr{B}_n(x, A) = n \int_0^x \mathscr{B}_{n-1}(u, A)\, du + (-1)^n B_n(A),$$

where

$$B_n(A) = \frac{(-1)^{n+1} n}{k} \int_0^k (k - u)\mathscr{B}_{n-1}(u, A)\, du.$$

It can be shown that $\mathscr{B}_n(x, A)$ has period k [5]. The definition of $\mathscr{B}_n(x, A)$ is then extended to $-\infty < x < \infty$ by periodicity. If $A = I$, $B_n(I) = B_n$ and $\mathscr{B}_n(x, I) = \mathscr{B}_n(x)$, where B_n and $\mathscr{B}_n(x)$ denote the ordinary Bernoulli numbers and functions, respectively. The periodic Bernoulli functions $\mathscr{B}_n(x, A)$, $-\infty < x < \infty$, have the following Fourier series expansions [5]

$$\mathscr{B}_n(x, A) = -n! \sum_{m=1}^{\infty} (k/2\pi i m)^n \{b_m e^{2\pi i m x/k} + (-1)^n b_{-m} e^{-2\pi i m x/k}\}. \tag{2.4}$$

From (2.4) it can be deduced that [5]

$$\mathscr{B}_n(x, A) = k^{n-1} \sum_{j=0}^{k-1} a_j \mathscr{B}_n\left(\frac{x - j}{k}\right). \tag{2.5}$$

Let c and d be positive, coprime integers. Then the Dedekind sum $s(d, c; A)$ associated with the sequence A is defined by

$$s(d, c; A) = \sum_{j (\mathrm{mod}\, ck)} a_j \mathscr{B}_1(dj/c, B)\mathscr{B}_1(j/ck),$$

where B is the complementary sequence to A. If $A = I$, then $s(d, c; I) = s(d, c)$, the ordinary Dedekind sum.

In (2.2), put $\alpha = 0$, $\beta = ck$, and $f(x) = \mathscr{B}_1(dx/c, B)\mathscr{B}_1(x/ck)$. As in [4, (7.3)], we find that

$$\frac{1}{4}a_0 b_0 + s(d, c; A) = \frac{c}{2d} B_0(A) B_2(B) + R(c, d, B), \tag{2.6}$$

where

$$R(c, d, B) = \sum_{n=1}^{\infty} \int_0^{ck} (b_n e^{2\pi i n x/k} + b_{-n} e^{-2\pi i n x/k})\mathscr{B}_1(dx/c, B)\mathscr{B}_1(x/ck)\, dx. \tag{2.7}$$

Since $\mathscr{B}_1(x)$ is boundedly convergent on any interval, we deduce from (2.5) that $\mathscr{B}_1(x, B)$ is boundedly convergent as well. Hence, after using (2.4), we may invert the order of summation and integration below to obtain

$$\int_0^{ck} e^{2\pi inx/k}\mathscr{B}_1(dx/c, B)\mathscr{B}_1(x/ck)\,dx$$

$$\begin{aligned}
&= ck\int_0^1 e^{2\pi incy}\mathscr{B}_1(dky, B)\mathscr{B}_1(y)\,dy \\
&= -\frac{ck}{2\pi i}\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 (y-\tfrac{1}{2})(a_m e^{2\pi i(nc+md)y} - a_{-m}e^{2\pi i(nc-md)y})\,dy \\
&= \frac{ck}{4\pi^2}\sum_{\substack{m=1\\ md\neq nc}}^{\infty}\frac{1}{m}\left\{\frac{a_m}{md+nc}+\frac{a_{-m}}{md-nc}\right\} \\
&\quad + \frac{ck}{4\pi^2}\sum_{\substack{m=1\\ md=nc}}^{\infty}\frac{a_m}{m(md+nc)}.
\end{aligned}$$

Thus, from (2.7),

$$\begin{aligned}
R(c,d,B) = \frac{ck}{4\pi^2}\sum_{n=1}^{\infty}\Bigg[& b_n\sum_{\substack{m=1\\ md\neq nc}}^{\infty}\frac{1}{m}\left\{\frac{a_m}{md+nc}+\frac{a_{-m}}{md-nc}\right\} \\
& + b_n\sum_{\substack{m=1\\ md=nc}}^{\infty}\frac{a_m}{m(md+nc)} + b_{-n}\sum_{\substack{m=1\\ md\neq nc}}^{\infty}\frac{1}{m}\left\{\frac{a_m}{md-nc}+\frac{a_{-m}}{md+nc}\right\} \\
& + b_{-n}\sum_{\substack{m=1\\ md=nc}}^{\infty}\frac{a_{-m}}{m(md+nc)}\Bigg].
\end{aligned} \tag{2.8}$$

Suppose now that A is even, i.e., $a_n = a_{-n}$ for every integer n. With the use of (2.1), it is not difficult to show that B is also even. Hence, (2.8) reduces to

$$R(c,d,B) = \frac{cdk}{\pi^2}S(d,c;A,B) + \frac{ck}{2\pi^2}\sum_{n=1}^{\infty}\sum_{\substack{m=1\\ md=nc}}^{\infty}\frac{a_m b_n}{m(md+nc)}, \tag{2.9}$$

where

$$S(d,c;A,B) = \sum_{n=1}^{\infty}\sum_{\substack{m=1\\ md\neq nc}}^{\infty}\frac{a_m b_n}{(md)^2-(nc)^2}.$$

Since $(c,d)=1$, $md = nc$ if and only if $n = rd$, $1 \leqq r < \infty$. Thus, $m = rc$, and so

$$\begin{aligned}
\frac{ck}{2\pi^2}\sum_{n=1}^{\infty}\sum_{\substack{m=1\\ md=nc}}^{\infty}\frac{a_m b_n}{m(md+nc)} &= \frac{k}{4cd\pi^2}\sum_{r=1}^{\infty}\frac{a_{rc}b_{rd}}{r^2} \\
&= \frac{k}{4cd\pi^2}\zeta(c,d,A,B;2),
\end{aligned}$$

say. ($\zeta(c, d, A, B; 2)$ can be evaluated in closed form by several methods. For example, see [5, § 6]). Putting the above into (2.9) and then substituting (2.9) into (2.6), we find that

$$\frac{1}{4}kB_0(A)B_0(B) + s(d, c; A)$$
$$= \frac{c}{2d}B_0(A)B_2(B) + \frac{cdk}{\pi^2}S(d, c; A, B) + \frac{k}{4cd\pi^2}\zeta(c, d, A, B; 2),$$

since $a_0 = kB_0(B)$ and $b_0 = B_0(A)$. Rearranging the above, we conclude that

$$\begin{aligned} S(d, c; A, B) = {} & \frac{\pi^2}{4cd}B_0(A)B_0(B) - \frac{\pi^2}{2d^2k}B_0(A)B_2(B) \\ & - \frac{1}{4c^2d^2}\zeta(c, d, A, B; 2) + \frac{\pi^2}{cdk}s(d, c; A). \end{aligned} \tag{2.10}$$

By reversing the roles of both A and B and c and d, we also have

$$\begin{aligned} S(c, d; B, A) = {} & \frac{\pi^2}{4cd}B_0(A)B_0(B) - \frac{\pi^2}{2c^2k}B_0(B)B_2(A) \\ & - \frac{1}{4c^2d^2}\zeta(c, d, A, B; 2) + \frac{\pi^2}{cdk}s(c, d; B). \end{aligned} \tag{2.11}$$

For an arbitrary periodic sequence A, the periodic Dedekind sum $s(d, c; A)$ satisfies the reciprocity law [4, Thm. 7.3]

$$\begin{aligned} s(d, c; A) + s(c, d; B) = {} & -\frac{k}{4}B_0(A)B_0(B) + \frac{c}{2d}B_0(A)B_2(B) \\ & + \frac{d}{2c}B_0(B)B_2(A) + \frac{1}{2cd}C(A, B) - \mathscr{B}_1(0, A)\mathscr{B}_1(0, B), \end{aligned} \tag{2.12}$$

where

$$C(A, B) = \sum_{r=1}^{k}\sum_{j=1}^{k} b_r a_{-j}\mathscr{B}_2\left(\frac{cr + dj}{k}\right).$$

If A is even, it is clear from (2.4) that $\mathscr{B}_1(0, A) = 0$. Adding (2.10) and (2.11) and then employing the reciprocity law (2.12), we find that

$$\begin{aligned} S(d, c; A, B) + S(c, d; B, A) = {} & \frac{\pi^2}{4cd}B_0(A)B_0(B) \\ & + \frac{\pi^2}{2c^2d^2k}C(A, B) - \frac{1}{2c^2d^2}\zeta(c, d, A, B; 2). \end{aligned} \tag{2.13}$$

Equation (2.13) is more than a reciprocity theorem for double series. By interchanging m and n in $S(c, d; B, A)$, we see that $-S(c, d; B, A)$ is the double sum obtained from $S(d, c; A, B)$ by inverting the order of summation in the latter. Thus, the right side of (2.13) gives the "error" made by inverting the order of summation in $S(d, c; A, B)$.

We now consider some special cases for even A.

First, let $A = I$. Then $B_0(A) = B_0(B) = 1$, $C(A, B) = 1/6$, and $\zeta(c, d, A, B; 2) = \pi^2/6$. We thus find that (2.13) reduces to (1.1).

Next, suppose that $c \equiv 0 \pmod{k}$. Define d' by $dd' \equiv 1 \pmod{k}$, and let $A' = \{a_{nd'}\}$. Then, $C(A, B) = B_0(B)B_2(A')$ [4, Proposition 7.4]. Hence, from (2.13),

$$S(d, c; A, B) + S(c, d; B, A) = \frac{\pi^2}{4cd} B_0(A)B_0(B) + \frac{\pi^2}{2c^2d^2k} B_0(B)B_2(A') - \frac{kB_0(B)}{2c^2d^2} \sum_{r=1}^{\infty} b_{rd} r^{-2}. \tag{2.14}$$

A similar formula holds if $d \equiv 0 \pmod{k}$

Let χ be a primitive character with modulus k. Let

$$G(n, \chi) = \sum_{j \pmod{k}} \chi(j)\, e^{2\pi i j n/k}$$

be the Gaussian sum. For primitive characters [2, p. 312],

$$G(n, \chi) = \bar{\chi}(n) G(\chi), \tag{2.15}$$

where $G(\chi) = G(1, \chi)$. Thus, if $A = \{\chi(n)\}$, where χ is primitive, we deduce from (2.1) and (2.15) that

$$b_n = G(-n, \chi)/k = \bar{\chi}(-n) G(\chi)/k.$$

Suppose that χ is primitive and even. Since $B_0(\chi) = 0$, from (2.10) we deduce that

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ md \neq nc}}^{\infty} \frac{\chi(m)\bar{\chi}(n)}{(md)^2 - (nc)^2} = \frac{\pi^2}{cdG(\chi)} s(d, c; \chi) - \frac{\chi(c)\bar{\chi}(d)\pi^2}{24c^2d^2}.$$

In particular, if c or d is $\equiv 0 \pmod{k}$,

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ md \neq nc}}^{\infty} \frac{\chi(m)\bar{\chi}(n)}{(md)^2 - (nc)^2} = \frac{\pi^2}{cdG(\chi)} s(d, c; \chi)$$

and

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ md \neq nc}}^{\infty} \frac{\chi(m)\bar{\chi}(n)}{(md)^2 - (nc)^2} = \sum_{m=1}^{\infty} \sum_{\substack{n=1 \\ md \neq nc}}^{\infty} \frac{\chi(m)\bar{\chi}(n)}{(md)^2 - (nc)^2}$$

from (2.14), where we interchanged m and n in the second sum on the left side of (2.14). Thus, in this case, inverting the order of summation does not alter the value of the double sum.

Now assume that A is odd, i.e., $a_n = -a_{-n}$ for every integer n. Then it is easy to show from (2.1) that B is odd. From (2.8),

$$R(c, d, B) = -\frac{c^2k}{\pi^2} T(d, c; A, B) + \frac{k}{4cd\pi^2} \zeta(c, d, A, B; 2),$$

by the same argument as before, where

$$T(d, c; A, B) = \sum_{\substack{n=1 \\ md \neq nc}}^{\infty} \sum_{m=1}^{\infty} \frac{n a_m b_n}{m\{(md)^2 - (nc)^2\}}.$$

Since A is odd, $a_0 = b_0 = 0$. Thus, (2.6) and the calculation above yield

$$T(d, c; A, B) = \frac{1}{4c^3 d} \zeta(c, d, A, B; 2) - \frac{\pi^2}{c^2 k} s(d, c; A). \tag{2.16}$$

By reversing the roles of both A and B and c and d, we get

$$T(c, d; B, A) = \frac{1}{4cd^3} \zeta(c, d, A, B; 2) - \frac{\pi^2}{d^2 k} s(c, d; B). \tag{2.17}$$

Again, recall that $B_0(A) = B_0(B) = 0$. Thus, by adding (2.16) and (2.17) and then using (2.12), we find that

$$\begin{aligned} &c^2 T(d, c; A, B) + d^2 T(c, d; B, A) \\ &\quad = \frac{1}{2cd} \zeta(c, d, A, B; 2) - \frac{\pi^2}{k} \left\{ \frac{1}{2cd} C(A, B) - \mathscr{B}_1(0, A) \mathscr{B}_1(0, B) \right\}. \end{aligned} \tag{2.18}$$

Equation (2.18) is not quite as interesting as its analogue (2.13) because $-d^2 T(c, d; B, A)$ is not the sum one gets by inverting the order of summation in $c^2 T(d, c; A, B)$.

We consider a couple of special cases for odd A.

If $c \equiv 0 \pmod{k}$, we find that $C(A, B) = 0$, since A is odd. From the definition of $\mathscr{B}_1(x, A)$, $\mathscr{B}_1(0, A) = -B_1(A)$ since $a_0 = 0$. Also,

$$\zeta(c, d, A, B; 2) = k B_0(B) \sum_{r=1}^{\infty} b_{rd} r^{-2} = 0.$$

Hence, (2.16) and (2.18) reduce respectively to

$$T(d, c; A, B) = -\frac{\pi^2}{c^2 k} s(d, c; A) \tag{2.19}$$

and

$$c^2 T(d, c; A, B) + d^2 T(c, d; B, A) = \frac{\pi^2}{k} B_1(A) B_1(B).$$

We obtain the same results if $d \equiv 0 \pmod{k}$.

If $a_n = \chi(n)$ is an odd, primitive character of modulus k, then $b_n = -\bar{\chi}(n) G(\chi)/k$. Hence, (2.19) gives

$$\sum_{\substack{n=1 \\ md \neq nc}}^{\infty} \sum_{m=1}^{\infty} \frac{n \chi(m) \bar{\chi}(n)}{m\{(md)^2 - (nc)^2\}} = \frac{\pi^2}{c^2 G(\chi)} s(d, c; \chi).$$

3. A double series with terms of alternating sign. In this section, we indicate briefly how to evaluate

$$U(d, c) = \sum_{\substack{n=1 \\ md \neq nc}}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+n}}{(md)^2 - (nc)^2}.$$

$U(d,c)$ was not summed in the previous section; for if we put $a_n = (-1)^n$, then $b_n = 0$ if n is even and $b_n = 1$ if n is odd. The value of $U(d,c)$ will be given in terms of $s(d,c;\frac{1}{2},\frac{1}{2})$, where for arbitrary real numbers x and y, $s(d,c;x,y)$ denotes the Dedekind–Rademacher sum [7], [4]

$$s(d,c;x,y) = \sum_{j(\mathrm{mod}\, c)} \left(\left(d\frac{j+y}{c} + x\right)\right)\left(\left(\frac{j+y}{c}\right)\right).$$

Here, we have used the customary notation $\mathscr{B}_1(x) = ((x))$. Note that $s(d,c;0,0) = s(d,c)$.

To sum $U(d,c)$, put $\alpha = -\frac{1}{2}$, $B = c - \frac{1}{2}$, and $f(x) = ((d(x+\frac{1}{2})/c + \frac{1}{2}))(((x + \frac{1}{2})/c))$ in the Poisson summation formula (2.3). As in [4, § 4], we get

$$\text{(3.1)}\qquad s(d,c;\tfrac{1}{2},\tfrac{1}{2}) = \frac{c}{2d}\mathscr{B}_2(\tfrac{1}{2}) + 2c\sum_{n=1}^{\infty}(-1)^n\int_0^1 ((dt+\tfrac{1}{2}))((t))\cos(2\pi nct)\,dt.$$

The integrals on the right side of (3.1) may be evaluated by substituting in the Fourier series for $((dt+\frac{1}{2}))$ and then inverting the order of summation and integration. Accordingly, we deduce that

$$\text{(3.2)}\qquad s(d,c;\tfrac{1}{2},\tfrac{1}{2}) = \frac{c}{2d}\mathscr{B}_2(\tfrac{1}{2}) + \frac{cd}{\pi^2}U(d,c) + \frac{c}{2\pi^2}\sum_{\substack{n=1 \\ md=nc}}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{m(md+nc)}.$$

Since $(c,d) = 1$, $md = nc$ if and only if $n = rd$, $1 \leqq r < \infty$. Thus,

$$\frac{c}{2\pi^2}\sum_{\substack{n=1 \\ md=nc}}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{m(md+nc)} = \frac{1}{4\pi^2 cd}\sum_{r=1}^{\infty}\frac{(-1)^{r(c+d)}}{r^2} = \frac{1}{4cd}\mathscr{B}_2(\tfrac{1}{2}(c+d)).$$

Since also $\mathscr{B}_2(\frac{1}{2}) = -1/12$, we find that (3.2) yields

$$\text{(3.3)}\qquad U(d,c) = \frac{\pi^2}{cd}s(d,c;\tfrac{1}{2},\tfrac{1}{2}) + \frac{\pi^2}{24d^2} - \frac{\pi^2}{4c^2d^2}\mathscr{B}_2(\tfrac{1}{2}(c+d)).$$

By interchanging c and d we get

$$\text{(3.4)}\qquad U(c,d) = \frac{\pi^2}{cd}s(c,d;\tfrac{1}{2},\tfrac{1}{2}) + \frac{\pi^2}{24c^2} - \frac{\pi^2}{4c^2d^2}\mathscr{B}_2(\tfrac{1}{2}(c+d)).$$

In the special instance at hand, the reciprocity theorem for Dedekind–Rademacher sums is given by [7], [4, § 4]

$$\text{(3.5)}\qquad s(d,c;\tfrac{1}{2},\tfrac{1}{2}) + s(c,d;\tfrac{1}{2},\tfrac{1}{2}) = -\frac{d}{24c} - \frac{c}{24d} + \frac{1}{2cd}\mathscr{B}_2(\tfrac{1}{2}(c+d)).$$

Hence, upon adding (3.3) and (3.4) and then employing (3.5), we obtain

$$\text{(3.6)}\qquad U(d,c) + U(c,d) = 0.$$

This is quite interesting, because (3.6) shows that reversing the order of summation in $U(d, c)$ does not change the value of the double sum, which contrasts with the result (1.1).

4. A generalization of $S(d, c)$. For a real, we shall evaluate

$$V_a(d, c) = \sum_{\substack{n=-\infty \\ md \neq \pm(nc+a)}}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(md)^2 - (nc + a)^2},$$

where here and in the sequel we write $\sum_{n=-\infty}^{\infty}$ for $\lim_{N\to\infty} \sum_{n=-N}^{N}$.

In (2.3) put $\alpha = 0$, $\beta = c$ and $f(x) = \exp(2\pi i x a/c)((dx/c))((x/c))$ to obtain

$$1/4 + v_a(d, c) = \sum_{n=-\infty}^{\infty} \int_0^c e^{2\pi i(n+a/c)x}((dx/c))((x/c))\, dx, \tag{4.1}$$

where

$$v_a(d, c) = \sum_{j=1}^{c-1} e^{2\pi i ja/c}((dj/c))((j/c)).$$

After the change of variable $x = cy$, the integrals in the series on the right are easily evaluated upon the substitution of the Fourier series for $((dy))$ and inverting the order of summation and integration. After this calculation, (4.1) becomes

$$\begin{aligned} 1/4 + v_a(d, c) &= \frac{cd}{2\pi^2} V_a(d, c) \\ &+ \frac{c}{4\pi^2}\left\{ \sum_{\substack{n=-\infty \\ md=-(nc+a)}}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(md - nc - a)} + \sum_{\substack{n=-\infty \\ md=nc+a}}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m(md + nc + a)} \right\}. \end{aligned} \tag{4.2}$$

Since $(c, d) = 1$, the congruence $md \equiv -a \pmod c$ has a solution if a is an integer. Let m_0 denote the least positive solution. Then all positive solutions are given by $m_0 + rc$, $0 \leqq r < \infty$. Similarly, let m_0' denote the least positive solution of $md \equiv a \pmod c$ if a is an integer, and so all positive solutions are given by $m_0' + rc$, $0 \leqq r < \infty$. If $\lambda(a)$ denotes the characteristic function of the integers, we see that we can write the expression in curly brackets on the right side of (4.2) as

$$\begin{aligned} \frac{\lambda(a)}{2d} \sum_{r=0}^{\infty} \frac{1}{(m_0 + rc)^2} + \frac{\lambda(a)}{2d} \sum_{r=0}^{\infty} \frac{1}{(m_0' + rc)^2} \\ = \frac{\lambda(a)}{2c^2 d}\{\zeta(2, m_0/c) + \zeta(2, m_0'/c)\}, \end{aligned}$$

where $\zeta(s, b)$ denotes the Hurwitz zeta function. Substituting the above into (4.2) and rearranging, we find that

$$\begin{aligned} V_a(d, c) = \frac{\pi^2}{2cd} + \frac{2\pi^2}{cd} v_a(d, c) \\ - \frac{\lambda(a)}{4c^2 d^2}\{\zeta(2, m_0/c) + \zeta(2, m_0'/c)\}. \end{aligned}$$

If $a = 0$,

$$\begin{aligned} V_a(d, c) &= 2S(d, c) + \frac{\pi^2}{6d^2} \\ &= \frac{\pi^2}{2cd} + \frac{2\pi^2}{cd} s(d, c) - \frac{\pi^2}{12c^2d^2}, \end{aligned}$$

which is in agreement with the result obtained in [3].

5. Double series and higher order Dedekind sums. We briefly indicate how to sum

$$S_p(d, c) = \sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ md \neq nc}}^{\infty} \frac{1}{m^{p-1}\{(md)^2 - (nc)^2\}},$$

where $p > 1$ is odd.

In the Poisson summation formula (2.3), put $\alpha = 0$, $\beta = c$ and $f(x) = \mathscr{B}_p(dx/c)\mathscr{B}_1(x/c)$, where $p > 1$ is odd. We then obtain

$$\begin{aligned} s_p(d, c) &\equiv \sum_{n=1}^{c-1} \mathscr{B}_p(dn/c)\mathscr{B}_1(n/c) \\ &= \int_0^c \mathscr{B}_p(dx/c)\mathscr{B}_1(x/c)\, dx + 2 \sum_{n=1}^{\infty} \int_0^c \mathscr{B}_p(dx/c)\mathscr{B}_1(x/c) \cos(2\pi nx)\, dx \\ &= \frac{c}{d(p+1)} B_{p+1} + \frac{4cd(-1)^{(p-1)/2} p!}{(2\pi)^{p+1}} S_p(d, c) \\ &\quad + \frac{1}{2c^p d(p+1)} B_{p+1}. \end{aligned}$$

The omitted calculations are in the same spirit as those in the previous sections. The higher order Dedekind sums $s_p(d, c)$ were first introduced by T. M. Apostol [1]. One can obtain a reciprocity theorem for $S_p(d, c)$ by employing the reciprocity theorem for $s_p(d, c)$ [1].

6. The last example. All of the previous examples involved the use of the Fourier series of Bernoulli functions. Our last example uses the Fourier series of $R(x)$, where

$$R(x) = \begin{cases} 1 & \text{if } 0 < x - [x] < 1/2, \\ -1 & \text{if } 1/2 < x - [x] < 1, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to show that

$$R(x) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2\pi(2m-1)x)}{2m-1}, \qquad -\infty < x < \infty. \tag{6.1}$$

For simplicity, we make the added assumption here that both c and d are odd. In (2.3) put $\alpha = 0$, $\beta = c$, and $f(x) = R(dx/c)((x/c))$ to get

$$\begin{aligned}
-1/2 + w(d,c) &\equiv -1/2 + \sum_{n=1}^{c-1} R(dn/c)((n/c)) \\
&= \int_0^c R(dx/c)((x/c))\,dx \\
&\quad + 2\sum_{n=1}^{\infty}\int_0^c R(dx/c)((x/c))\cos(2\pi nx)\,dx.
\end{aligned} \tag{6.2}$$

To calculate the first integral on the right side of (6.2), write

$$I \equiv \int_0^c R(dx/c)((x/c))\,dx = c\sum_{j=0}^{d-1}\int_{j/d}^{(j+1)/d} R(dx)((x))\,dx$$

and then let $x = (j + y)/d$ to get

$$\begin{aligned}
I &= \frac{c}{d}\int_0^1 R(y)\sum_{j=0}^{d-1}\left(\left(\frac{j+y}{d}\right)\right)dy \\
&= \frac{c}{d}\int_0^1 R(y)((y))\,dy = -\frac{c}{4d},
\end{aligned} \tag{6.3}$$

where we have used the familiar multiplication theorem for the first Bernoulli functions [8, p. 4].

To calculate the latter integrals on the right side of (6.2), substitute into the integrals the Fourier series from (6.1) and invert the order of summation and integration. Since $R(x) = 2((2x)) - 4((x))$, the Fourier series of $R(x)$ is boundedly convergent, and the inversion is justified. Hence,

$$\begin{aligned}
&\int_0^c R(dx/c)((x/c))\cos(2\pi nx)\,dx \\
&\quad = \frac{4c}{\pi}\sum_{m=1}^{\infty}\frac{1}{2m-1}\int_0^1 x\sin(2\pi(2m-1)\,dx)\cos(2\pi ncx)\,dx \\
&\quad = -\frac{2cd}{\pi^2}\sum_{\substack{m=1\\(2m-1)d\neq nc}}^{\infty}\frac{1}{((2m-1)d)^2-(nc)^2} \\
&\qquad -\frac{c}{\pi^2}\sum_{\substack{m=1\\(2m-1)d=nc}}^{\infty}\frac{1}{(2m-1)\{(2m-1)d+nc\}}.
\end{aligned} \tag{6.4}$$

Since $(c, d) = 1$ and c and d are both odd, $(2m - 1)d = nc$ if and only if $n = (2r - 1)d$, $1 \leqq r < \infty$. Thus,

$$\begin{aligned}
\sum_{\substack{n=1\\(2m-1)d=nc}}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(2m-1)\{(2m-1)d+nc\}} &= \frac{1}{2c^2d}\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2} \\
&= \frac{\pi^2}{16c^2d}.
\end{aligned} \tag{6.5}$$

Hence, substituting (6.3) and (6.4) into (6.2) and using (6.5), we deduce that

$$\sum_{n=1}^{\infty} \sum_{\substack{m=1 \\ (2m-1)d \neq nc}}^{\infty} \frac{1}{((2m-1)d)^2 - (nc)^2} = \frac{\pi^2}{8cd} - \frac{\pi^2}{16d^2} - \frac{\pi^2}{32c^2d^2} - \frac{\pi^2}{4cd} w(d, c),$$

where $w(d, c)$ is defined by (6.2).

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ON THE BEHAVIOR OF PRINCIPAL FUNDAMENTAL SOLUTIONS OF ELLIPTIC EQUATIONS*

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Abstract. Fundamental solutions which decay exponentially at ∞ together with their first derivatives are called principal fundamental solutions. Such solutions of elliptic equations in R^m, $m \geqq 2$, play an important role in the study of pseudo-parabolic equations in $R^m \times R$. We establish the behavior as $|x - y| \to \infty$ and as $|x - y| \to 0$ of a function H defined for $x, y \in R^m$, $x \neq y$, which is basic for the construction of the principal fundamental solution G. It is known that $G - H = O(H)$. Since H is closely related to Bessel functions, complex analysis is used to determine its behavior.

1. Introduction. We devote this section to describe (i) notations and the terminology, (ii) a problem and a method of construction of its solution that has led to this study, and (iii) a class of functions closely related to the function whose behavior is the object of this study.

Let R^m, $m \geqq 2$, denote the m-dimensional Euclidean space and any x in R^m be written $x = (x_1, \cdots, x_m)$. For $1 < p < \infty$, let $W^{2,p}(R^m)$ denote the Sobolev space of functions which together with their first and second order derivatives belong to $L^p(R^m)$. This is a Banach space with norm given by

$$\|v\|_{2,p} = \sum_1^m \left\| \frac{\partial^2 v}{\partial x_i \partial x_j} \right\|_p + \sum_1^m \left\| \frac{\partial v}{\partial x_i} \right\|_p + \|v\|_p,$$

for functions $v \in W^{2,p}(R^m)$.

Let A be a second order elliptic differential operator defined by

$$Au(x) = \sum_1^m a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum a_i(x) \frac{\partial u}{\partial x_i} + a_0(x)u,$$

where $a_0(x)$ is strictly negative in R^m and for some positive constants λ and Λ, and $\xi \in R^m - \{0\}$,

$$\lambda|\xi| \leqq \rho(x;\xi) \equiv \left(\sum a_{ij}(x)\xi_i\xi_j\right)^{1/2} \leqq \Lambda|\xi|. \tag{1.1}$$

A principal fundamental solution (p.f.s.) of $Au = 0$ is a fundamental solution $G(x, y)$ of $Au = 0$, defined for all $x \neq y$, which decays exponentially together with its first derivatives. In [3], Giraud has shown that a unique p.f.s. exists for $Au = 0$, provided that the coefficients of A satisfy a uniform Hölder condition in addition to the other assumptions already indicated. For a brief review of his construction see [4]. If $(A_{ij}(x))$ denotes the inverse of $(a_{ij}(x))$ and $S(x)$ denotes the square root of the determinant of $a_{ij}(x)$, then

$$G(x, y) = H_k(x, y) + \int_{R^m} H_k(x, \xi)\varphi(\xi, y)\, d\xi,$$

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where for $0 < k < \infty$,

$$(1.2) \qquad H_k(x, y) = \frac{e^{-k\rho}(2\pi\rho)^{(1-m)/2}k^{(m-3)/2}}{S(y)2\Gamma((m-1)/2)} \int_0^\infty t^{(m-3)/2}\left(1 + \frac{t}{2k\rho}\right)^{(m-3)/2} e^{-t}\,dt$$

is the p.f.s. of the equation $\sum a_{ij}(y)(\partial^2 u/\partial x_i\,\partial x_j) - k^2 u = 0$. This article concerns the function H_k. We define below another function denoted by $L(x, y)$ which will later be shown to be the principal part of $H_k(x, y)$.

$$(1.3) \qquad L(x, y) = \begin{cases} \dfrac{\Gamma((m-2)/2)}{S(y)4\pi^{m/2}}\rho^{2-m}, & m > 2, \\[2ex] \dfrac{-1}{S(y)2\pi}\log \rho, & m = 2. \end{cases}$$

Observe that L does not involve the positive constant k.

Now we describe a problem and a method of construction of its solution which prompted this study. Let B be another second order elliptic differential operator with bounded continuous coefficients defined in R^m and A be as before. Let $u_0 \in W^{2,p}(R^m)$ and consider the problem of finding a function $u: R \to W^{2,p}(R^m)$ satisfying $A(\partial u/dt) + Bu = 0$ in $W^{2,p}(R^m)$ and $u(x, 0) = u_0(x)$. This problem has been solved in [5] by setting up the equivalent integro-differential equation

$$(1.4) \qquad u = u_0 - \int_0^t \int_{R^m} G(\cdot, y)Bu(y)\,dy\,dt$$

and solving it by successive approximations. Indeed, the solution is given by

$$u = u_0 + \sum_1^\infty (-1)^n K^n u_0,$$

where $K^1 u_0$ is the double integral in (1.4) with u replaced by u_0 and $K^n = K^1(K^{n-1})$. In [5], it has been shown that this series converges in $W^{2,p}(R^m)$ provided that

$$\left\| \int_{R^m} G(\cdot, y)f(y)\,dy \right\|_{2,p} \leqq \text{const.}\ \|f\|_p,$$

for $f \in L^p(R^m)$ which in turn is valid if

$$(1.5) \qquad \left\| \int_{R^m} H(\cdot, y)f(y)\,dy \right\|_{2,p} \leqq \text{const.}\ \|f\|_p.$$

To prove the validity of this inequality it is enough to show that precisely the same inequality is satisfied by each of the three functions defined by

$$\int_{B(x,R_0)} L(x, y)f(y)\,dy, \qquad \int_{B(x,R_0)} (H - L)(x, y)f(y)\,dy$$

and

$$\int_{R^m - B(x,R_0)} H(x, y)f(y)\,dy,$$

where $B(x, R_0)$ is a ball of radius R_0. The first of these three integrals may be handled in the framework of the L^p-theory of singular integrals. *However, in the case of the second and third integrals we need to know the behavior of* $H - L$ *and its first and second order derivatives as* $|x - y| \to 0$ *and that of* H *and its derivatives up to order two as* $|x - y| \to \infty$.

We now come to the third part of this section as indicated earlier. A one-parameter family of functions known as Bessell kernels (see [1]) is given by

$$(1.6)\qquad \begin{aligned} G_\alpha(x) &= C_\alpha e^{-|x|} \int_0^\infty e^{-|x|}\left(t + \frac{t^2}{2}\right)^{(m-\alpha-1)/2} dt, \\ C_\alpha^{-1} &= (2\pi)^{(m-1)/2} 2^{\alpha/2} \Gamma\left(\frac{m - \alpha + 1}{2}\right) \end{aligned}$$

for $0 < \alpha < m + 1$. See also [9].

When A is the m-dimensional Laplacian and $k = 1$, the function $H_k(x, y)$ coincides with $G_2(x - y)$. For integral α and f in $L^p(R^m)$, it is known that

$$\|G_\alpha * f\|_{\alpha,p} \leqq \text{const. } \|f\|_p.$$

For results of this nature and additional references, see [9]. If one observes that for $0 < \alpha < m + 1$,

$$(I - A)^{-\alpha/2} f = G_\alpha * f,$$

one quickly notes that the best way of dealing with $G_\alpha * f$ is through Fourier analysis and related techniques. This is due to the fact that the operator involved in the case of G_α is of constant coefficients. Since the same is not true of H_k, an alternate approach is needed to deal with it.

Throughout this article we shall work with H_1 which will be referred to as H. *It should be understood that in all references to* (1.2), $k = 1$. For the case $k \neq 1$ see the remark at the end of § 3.

2. The behavior of $H - L$. In this section we study the behavior of $(H - L)(x, y)$ as $x \to y$. In [2] Giraud has shown, by complex variable methods, that $H = (1 + O(1))L$. Since this is not sufficient for our purposes, we shall first obtain an exact expression for $H - L$ and use it to derive an estimate for the same in $B(x, R_0)$.

THEOREM 2.1. *For the functions H and L defined by* (1.2) *and* (1.3), *the following relations are valid.*

(i) *If* $m \geqq 3$ *and is an odd integer, then*

$$(2.1)\qquad \begin{aligned} H(x, y) - L(x, y) = \frac{C}{S(y)} \Bigg\{ & e^{-\rho} \rho^{(1-m)/2} \sum_{l=n}^{2n-1} \binom{n}{l - n} \Gamma(l + 1)(2\rho)^{n-l} \\ & + \Gamma(m - 2)(e^{-\rho} - 1)\rho^{(1-m)/2}(2\rho)^{(3-m)/2} \Bigg\}, \end{aligned}$$

where $C = (\frac{1}{2})(2\pi)^{(1-m)/2}\Gamma((m - 1)/2)$ *and* $n = (m - 3)/2$.

(ii) *If $m \geqq 4$ and is an even integer, then*

$$H(x, y) - L(x, y)$$

$$= \frac{C_1}{S(y)} e^{-\rho} \rho^{(1-m)/2} \Big\{ \text{const. } \rho^{(5-m)/2} + \cdots + \text{const. } \rho^{-1/2} + \text{const. } \rho^{1/2}$$

$$+ \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{\Gamma(s)\Gamma(-s-(m-3)/2)\Gamma(-s+(m-1)/2)}{\Gamma((m-1)/2)\Gamma((3-m)/2)} (2\rho)^s \, ds \Big\} \tag{2.2}$$

$$+ \frac{1}{S(y)} \frac{\Gamma((m-2)/2)}{4\pi^{m/2}} \frac{e^{-\rho} - 1}{\rho^{m-2}},$$

where $C_1 = \frac{1}{2}(2\pi)^{(1-m)/2}$.

(iii) *If $m = 2$ and C_1 is as above, then*

$$\begin{aligned} H(x, y) - L(x, y) = \frac{1}{S(y)} \Big\{ & \frac{e^{-\rho} - 1}{2\pi} \log \left(\frac{1}{\rho} \right) - \frac{e^{-\rho}}{2\pi} \log 2 \\ & + \frac{C_1}{\pi} \frac{e^{-\rho} \rho^{-1/2}}{2\pi i} \int_{1/2+\alpha-i\infty}^{1/2+\alpha+i\infty} \Gamma(s)\Gamma^2(\tfrac{1}{2} - s)(2\rho)^s \, ds \Big\}, \end{aligned} \tag{2.3}$$

where α is any positive number such that $0 < \alpha < 1$.

Proof of (i). If $n \equiv (m-3)/2$, then, from (1.2) we have after expanding $(1 + t/(2\rho))^n$ in powers of t and integrating,

$$\begin{aligned} H(x, y) &= \frac{C}{S(y)} e^{-\rho} \rho^{(1-m)/2} \sum_{l=n}^{2n} \binom{n}{l-n} \Gamma(l+1)(2\rho)^{n-l} \\ &= \frac{C}{S(y)} \Big\{ e^{-\rho} \rho^{(1-m)/2} \sum_{l=n}^{2n-1} \binom{n}{l-n} \Gamma(l+1)(2\rho)^{n-l} \\ &\quad + (e^{-\rho} - 1)\Gamma(m-2)\rho^{(1-m)/2}(2\rho)^{(3-m)/2} + \Gamma(m-2)2^{(3-m)/2}\rho^{2-m} \Big\}. \end{aligned} \tag{2.4}$$

The last term of this equation may be simplified, with the help of the multiplication theorem of Gauss and Legendre for gamma functions applied to $\Gamma((m-2)/2)$ and $\Gamma((m-1)/2)$, and reduced to $L(x, y)$. This shows that $H(x, y) - L(x, y)$ is in the required form when $m \geqq 3$ and is odd.

Proof of (ii). When m is an even positive integer not equal to 2, it can be shown as in Whittaker and Watson [10], that

$$H(x, y)$$

$$= \frac{C_1}{S(y)} \frac{e^{-\rho}}{2\pi i} \rho^{(1-m)/2} \int_{-i\infty}^{+i\infty} \frac{\Gamma(s)\Gamma(-s+(m-3)/2)\Gamma(-s+(m-1)/2)}{\Gamma((m-1)/2)\Gamma((3-m)/2)} (2\rho)^s \, \mathrm{d}s, \tag{2.5}$$

where C_1 is the constant appearing in (2.2). The path of integration begins and ends on the half-lines of the imaginary axis and is curved in such a way that the origin and the negative integers are to the left of the path while the poles of $\Gamma(-s(m-3)/2) \cdot \Gamma(-s+(m-1)/2)$, that is,

$$s = n - (m-3)/2, \qquad n = 0, 1, 2, \cdots,$$

are all to the right of the path. If I denotes the integral appearing in (2.5), then to calculate I we first integrate around the contour which is essentially a rectangle with corners $+i\xi$, $-i\xi$, $1 + i\xi$ and $1 - i\xi$ with a provision to include the points

$$n - (m - 3)/2, \qquad\qquad n = 0, 1, \cdots, (m - 2)/2,$$

where $m \geqq 4$ is an even integer. If we now let $\xi \to \infty$, we obtain

$$I = 2\pi i \left\{ - \sum_{l=(4-m)/2}^{1} R_{l-1/2} + \int_{1-i\infty}^{1+i\infty} \right\},$$

where R_n is the residue of the integrand at n, and the integral in the above equation is taken along the line $x = 1$. It is not difficult to verify that

$$-R_{(3-m)/2} = \frac{\Gamma(m-2)}{\Gamma((m-1)/2)}(2\rho)^{(3-m)/2}.$$

After calculating the other residues and using them in (2.5), we have

$$\begin{aligned} H(x, y) = \frac{C_1}{S(y)} e^{-\rho} \rho^{(1-m)/2} \Bigg\{ &\frac{\Gamma(m-2)}{\Gamma((m-1)/2)}(2\rho)^{(3-m)/2} \\ &+ \text{const.}\, \rho^{(5-m)/2} + \cdots + \text{const.}\, \rho^{-1/2} + \text{const.}\, \rho^{1/2} + \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Bigg\}. \end{aligned} \tag{2.6}$$

The integral appearing in the above equation may be shown to be bounded by a constant multiple of ρ, the constant being independent of ρ. Moreover, the constants appearing in (2.6) are all numerical. To complete the proof of (ii) we have only to show that

$$\frac{C_1}{S(y)} \frac{1}{\rho^{(m-1)/2}} \frac{\Gamma(m-2)}{\Gamma((m-1)/2)}(2\rho)^{(3-m)/2} = L(x, y), \qquad m \geqq 4.$$

This follows by employing reasoning similar to the reasoning used in the proof of (i) and the fact that $C_1 = C \cdot (m - 1)/2$.

Proof of (iii). As mentioned in the proof of (ii), we have for $m = 2$,

$$H(x, y) = \frac{C_1}{S(y)} e^{-\rho} \rho^{-1/2} \frac{1}{2\pi i} I, \tag{2.7}$$

where

$$I \equiv \frac{1}{\pi} \int_{-i\infty}^{i\infty} \Gamma(s)\Gamma^2(\tfrac{1}{2} - s)(2\rho)^s \, ds.$$

The integrand has a double pole at $s = 1/2$. To calculate the residue at $s = 1/2$, we first note that

$$(2\rho)^s = (2\rho)^{1/2}(1 + (s - \tfrac{1}{2}) \log 2\rho + \cdots),$$

$$\Gamma(\tfrac{1}{2} - s) = \frac{-1}{s - \frac{1}{2}} - \gamma + \cdots,$$

and

$$\Gamma(s) = \sqrt{\pi}(1 - (\gamma + 2\log 2)(s - \tfrac{1}{2}) + \cdots),$$

where, in obtaining the above Taylor expansions around $s = \frac{1}{2}$, we used the Gauss–Legendre product formula for Γ-functions and γ is Euler's constant. The residue at $s = \frac{1}{2}$ may now be shown to equal $\sqrt{2\pi\rho}\log 2\rho$, by using the above mentioned Taylor expansions. To obtain the value of I, we first integrate around the contour which is a rectangle with corners $+i\xi$, $-i\xi$, $\frac{1}{2} + \alpha + i\xi$ and $\frac{1}{2} + \alpha - i\xi$, $0 < \alpha < 1$, and indented at the origin. Now we let $\xi \to \infty$ and observe that the integrals along the two horizontal parts of the contour vanish. Thus, we have

$$I = 2\pi i\left\{-\left(\frac{2\rho}{\pi}\right)^{1/2}\log 2\rho + \frac{1}{2\pi^2 i}\int_{1/2+\alpha-i\infty}^{1/2+\alpha+i\infty}\right\}.$$

By substituting the above expression for I in (2.7), we obtain

$$\begin{aligned} H(x, y) = \frac{1}{S(y)}\Bigg\{&\frac{e^{-\rho}}{2\pi}\log\left(\frac{1}{\rho}\right) - \frac{e^{-\rho}}{2\pi}\log 2 \\ &+ \frac{e^{-\rho}\rho^{-1/2}}{2\pi^2 i}C_1\cdot\int_{1/2+\alpha-i\infty}^{1/2+\alpha+i\infty}\Bigg\}. \end{aligned} \tag{2.8}$$

The integral appearing in the above equation may be shown to be bounded by a constant multiple of $\rho^{\alpha+1/2}$, the constant being independent of ρ. The proof of part (iii) is now clear from (2.8).

In the rest of this article we will on several occasions use the estimates

$$\left|\frac{\partial\rho}{\partial x_i}\right| \leqq \text{const.}, \qquad \left|\frac{\partial^2\rho}{\partial x_i\,\partial x_j}\right| \leqq \text{const.}\, r^{-1}, \tag{2.9}$$

where the constants are independent of $x, y \in R^m$. The next theorem concerns the behavior of $\partial^2(H - L)/\partial x_i\,\partial x_j$ as $r \to 0$.

THEOREM 2.2. *Let H and L be as in equations* (1.2) *and* (1.3) *with $k = 1$. Then for some constant C independent of $x \in R^m$ and all y such that $0 < r \leqq R_0$, we have*

$$\left|\frac{\partial^2(H - L)}{\partial x_i\,\partial x_j}\right| \leqq \begin{cases} Cr^{1-m} & \text{if } m \geqq 3, \\ Cr^{\alpha-2} & \text{if } m = 2, \end{cases} \tag{2.10}$$

where R_0 is any positive constant and $0 < \alpha < 1$.

Proof. We base this on the three different expressions for $H - L$ obtained in Theorem 2.1.

Case (i). *$m \geqq 3$ and is an odd integer.* From (2.1) we have

$$\begin{aligned}(H - L)(x, y) &= \frac{1}{S(y)}\left\{\sum_{l=n}^{m-4} d_l e^{-\rho}\rho^{-(l+1)} + d_{m-3}(e^{-\rho} - 1)\rho^{2-m}\right\} \\ &\equiv \frac{1}{S(y)}\{I_1(\rho) + I_2(\rho)\},\end{aligned}$$

where the constants d_k, $k = n, n+1, \cdots, m-3$, are all independent of ρ, and $n = (m - 3)/2$. Since $S(y)$ is bounded below by a positive constant, it is enough to obtain the desired conclusion for $\partial^2 I_1(\rho)/\partial x_i \partial x_j$ and $\partial^2 I_2(\rho)/\partial x_i \partial x_j$. Since $I_1(\rho)$ is a linear combination of the functions

$$e^{-\rho}\rho^{3-m}, e^{-\rho}\rho^{4-m}, \cdots, e^{-\rho}\rho^{(1-m)/2},$$

each of which has second derivatives obeying inequality (2.10), $\partial^2 I_1(\rho)/\partial x_i\, \partial x_j$ also obeys inequality (2.10). On the other hand,

$$\begin{aligned}\frac{\partial^2 I_2(\rho)}{\partial x_i\, \partial x_j} = &-\frac{\partial^2 \rho}{\partial x_i\, \partial x_j}\{(m-2)(e^{-\rho}-1)\rho^{1-m} + e^{-\rho}\rho^{2-m}\}\\ &+\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j}\{e^{-\rho}\rho^{2-m} + (m-2)\, e^{-\rho}\rho^{1-m}\\ &\qquad + (m-2)(m-1)(e^{-\rho}-1)\rho^{-m}\}.\end{aligned} \tag{2.11}$$

Now let R_0 be any positive constant. Then $0 < r \leqq R_0$ implies that $\rho \leqq \rho_0 \equiv \Lambda R_0$. By using the fact that $(e^{-\rho} - 1)/\rho$ is bounded for all $\rho > 0$ together with the inequalities (2.9), we conclude, from (2.11), that for all $x \in R^m$ and for all y such that $0 < r \leqq R_0$, the function $\partial^2 I_2(\rho)/\partial x_i\, \partial x_j$ also satisfies inequality (2.10).

Case (ii). $m \geqq 4$ *and is even.* We have, from (2.2),

$$\begin{aligned}H(x, y) - L(x, y) &= \frac{1}{S(y)}\left\{I_3(\rho) + d_1 \frac{e^{-\rho}-1}{\rho^{m-2}} + d_2\, e^{-\rho}\rho^{(3-m)/2}\int_{-\infty}^{\infty} f(\rho, t)\, dt\right\}\\ &\equiv \frac{1}{S(y)}\{I_3(\rho) + I_4(\rho) + I_5(\rho)\},\end{aligned}$$

where d_1 and d_2 are constants independent of x and y,

$$f(\rho, t) = \rho^{it}\Gamma(1 + it)\Gamma\left(-it - \frac{m-1}{2}\right)\Gamma\left(-it + \frac{m-3}{2}\right),$$

and where $I_3(\rho)$ is of the type $I_1(\rho)$ introduced in the proof of Case (i). Also, $I_4(\rho)$ and $I_2(\rho)$ are identical apart from a constant factor. Thus, $I_3(\rho)$ and $I_4(\rho)$ may be handled just as we handled $I_1(\rho)$ and $I_2(\rho)$. Hence, we obtain an estimate for $\partial^2 I_5(\rho)/\partial x_i\, \partial x_j$. To this extent we have, if $g(\rho) \equiv \int_{-\infty}^{\infty} f(\rho, t)\, dt$, then

$$\begin{aligned}\frac{\partial^2 I_5(\rho)}{\partial x_i\, \partial x_j} = \frac{d_2}{S(y)}\Bigg\{&\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j}\left[2\frac{\partial}{\partial \rho} e^{-\rho}\rho^{(3-m)/2}\frac{\partial g}{\partial \rho}\right.\\ &\left. + \frac{\partial^2}{\partial \rho^2}(e^{-\rho}\rho^{(3-m)/2})g(\rho) + e^{-\rho}\rho^{(3-m)/2}\frac{\partial^2 g}{\partial \rho^2}\right]\\ &+\frac{\partial^2 \rho}{\partial x_i\, \partial x_j}\left[\frac{\partial}{\partial \rho}(e^{-\rho}\rho^{(3-m)/2})g(\rho)\right.\\ &\left. + e^{-\rho}\rho^{(3-m)/2}\frac{\partial g}{\partial \rho}\right]\Bigg\}.\end{aligned} \tag{2.12}$$

By using the asymptotic expansion of the logarithm of the gamma function, it is not difficult to verify that $\int f(\rho, t)\, dt$ and its first and second derivatives with respect to ρ are of orders $O(1)$, $O(r^{-1})$, $O(r^{-2})$, respectively, uniformly with respect to x and $y \in R^m$. Because of this, the dominant term in each of the terms on the right-hand side of (2.12) is of order $O(r^{-(m+1)/2})$, and hence

$$\left|\frac{\partial^2 I_5(\rho)}{\partial x_i\, \partial x_j}\right| \leqq \text{const.}\, r^{1-m},$$

the constant being independent of $x \in R^m$, and y such that $0 < r < R_0$.

Case (iii). $m = 2$. If, in (2.3), s is replaced by $\frac{1}{2} + \alpha + it$, we get

$$\begin{aligned} H(x, y) - L(x, y) &= \frac{1}{S(y)}\left\{\frac{e^{-\rho} - 1}{2\pi} \log\left(\frac{1}{\rho}\right) - \frac{e^{-\rho}}{2\pi}\log 2 \right. \\ &\qquad \left. + d_3\, e^{-\rho}\rho^{\alpha} \int_{-\infty}^{\infty} \Gamma(\tfrac{1}{2} + \alpha + it)\Gamma^2(-\alpha - it)\rho^{it}\, \mathrm{d}t\right\} \\ &= \frac{1}{S(y)}\{I_6(\rho) + I_7(\rho) + I_8(\rho)\}, \end{aligned}$$

where d_3 is a constant independent of x and y. To obtain the required estimate for $\partial^2(H - L)/\partial x_i\, \partial x_j$, it is enough to obtain the same estimate for the corresponding second derivatives of $I_6(\rho)$, $I_7(\rho)$ and $I_8(\rho)$. It is easy to obtain this estimate for the second derivative of $I_7(\rho)$. Accordingly we concentrate on $I_6(\rho)$ and $I_8(\rho)$. We have

$$\begin{aligned} \frac{\partial^2 I_6(\rho)}{\partial x_i\, \partial x_j} &= \left\{-\frac{e^{-\rho}}{2\pi}\log\left(\frac{1}{\rho}\right) - \frac{e^{-\rho} - 1}{2\pi\rho}\right\}\frac{\partial^2 \rho}{\partial x_i\, \partial x_j} \\ &\quad + \left\{\frac{e^{-\rho}}{\pi\rho} + \frac{e^{-\rho}}{2\pi}\log\left(\frac{1}{\rho}\right) + \frac{e^{-\rho} - 1}{2\pi\rho^2}\right\}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j}. \end{aligned}$$

The last four terms in the right-hand side of this equation may be shown to be of order $O(r^{-1})$ and hence, if $0 < r \leqq R_0$, of orders $O(r^{\alpha-2})$, $0 < \alpha < 1$, uniformly, in $x \in R^m$. On the other hand, the first term is of order $O(r^{-1}|\log r^{-1}|)$, and hence of order $O(r^{\alpha-2})$, $0 < \alpha < 1$, uniformly in $x \in R^m$, provided that $0 < r \leqq R_0$. Thus $\partial^2 I_6(\rho)/\partial x_i\, \partial x_j$ obeys the inequality (2.10). For $I_8(\rho)$, we observe that the integral appearing in the definition of $I_8(\rho)$ and its first and second derivatives are of orders $O(1)$, $O(r^{-1})$ and $O(r^{-2})$, uniformly with respect to x and y. Thus, it follows that $\partial^2 I_8(\rho)/\partial x_i\, \partial x_j$ also obeys the inequality (2.10) provided that $0 < r \leqq R_0$. This completes the proof of Theorem 2.2.

3. Global estimates for H and its derivatives. The estimates in this section describe the behavior of H (and its derivatives) as $|x - y| \to 0$ and as $|x - y| \to \infty$. A single expression describes both these aspects. Since L (and therefore H) behaves like $\log r$ as $x \to y$ in the case $m = 2$ and like r^{m-2} in the case $m > 2$, these two cases are examined separately. The first result of this section concerns the case $m > 2$ and the second one concerns the case $m = 2$.

THEOREM 3.1. *Let $H(x, y)$ be defined as in* (1.2), *and $k = 1$. Let $\alpha_1, \cdots, \alpha_m$ be such that the α_i are all nonnegative integers with $0 \leqq |\alpha| \leqq 2$, where $|\alpha| = \sum \alpha_i$. Then there exists a positive number a, $0 < a < \lambda$, such that for any integer $m \geqq 3$,*

$$|D_x^{\alpha} H(x, y)| \leqq C\, e^{-ar} r^{2-m-|\alpha|}, \tag{3.1}$$

where the constant C is independent of x, $y \in R^m$.

Proof. This is based on the expressions for $H(x, y)$ obtained in the course of the proof of Theorem 2.1.

Case (i). *m is an odd integer exceeding* 1. Equation (2.4) may be written as

$$H(x, y) = \frac{e^{-\rho}}{S(y)}\{\text{const.}\,\rho^{(1-m)/2} + \text{const.}\,\rho^{-((1+m)/2)} + \cdots + \text{const.}\,\rho^{2-m}\}, \tag{3.2}$$

where the constants are all independent of ρ. Notice that there are $1 + (m - 3)/2$ terms on the right-hand side of the above equation, arranged in decreasing powers of ρ. Thus each term in (3.2) is of the form

$$\text{const.}\,\frac{1}{S(y)}e^{-\rho}\rho^{((1-m)/2)-l}, \qquad 0 \leqq l \leqq \frac{m-3}{2},$$

and hence is of order $O(e^{-ar}r^{2-m})$ uniformly for $x, y \in R^m$. The constant a may be chosen between 0 and λ. Thus the conclusion of the theorem is valid when $|\alpha| = 0$.

By differentiating both sides of (3.2) with respect to the coordinates of x, it is easy to see that

$$\frac{\partial H}{\partial x_i} = \frac{\partial \rho}{\partial x_i}\frac{e^{-\rho}}{S(y)}\{\text{const.}\,\rho^{(1-m)/2} + \text{const.}\,\rho^{-((1+m)/2)} + \cdots + \text{const.}\,\rho^{1-m}\},$$

where the constants once again are independent of ρ. Since $\partial\rho/\partial x_i$ is bounded in $R^m \times R^m - \delta$, *where δ is the diagonal of* $R^m \times R^m$, an analysis similar to the one used in obtaining an estimate for $H(x, y)$ yields

$$\left|\frac{\partial H}{\partial x_i}\right| \leqq \text{const.}\,e^{-ar}r^{1-m}, \qquad 1 \leqq i \leqq m,$$

the constant being independent of $x, y \in R^m$. The desired estimate for second derivatives may be obtained in a similar fashion.

Case (ii). *$m \geqq 4$ and is even.* From (2.6), we have

$$\begin{aligned} H(x, y) = \frac{1}{S(y)}e^{-\rho}\Big\{&\text{const.}\,\rho^{2-m} + \text{const.}\,\rho^{3-m} + \cdots \\ &+ \text{const.}\,\rho^{-m/2} + \text{const.}\,\rho^{(2-m)/2} \\ &+ \int_{-\infty}^{\infty} g(t)\rho^{((3-m)/2)+it}\,dt\Big\}, \end{aligned} \tag{3.3}$$

where $g(t)$ is an absolutely integrable function and the constants are independent of ρ. By using an argument similar to that used in Case (i), we can show that all the terms on the right-hand side of (3.2) except that involving the integral satisfy the estimate (3.1) uniformly in $R^m \times R^m - \delta$. The integral in (3.3) may be differentiated twice under the integral sign with respect to ρ. Consequently, this integral and its first and second derivatives are of orders $O(r^{(3-m)/2})$, $O(r^{(1-m)/2})$ and $O(r^{-(1+m)/2})$, respectively. Thus, if we define $F(x, y)$ by

$$F(x, y) \equiv \frac{1}{S(y)}e^{-\rho}\int_{-\infty}^{\infty} g(t)\rho^{((3-m)/2)+it}\,dt,$$

then we have

$$|F(x, y)| \leqq \text{const.}\, e^{-\lambda r} r^{(3-m)/2}$$
$$\leqq \text{const.}\, e^{-ar} r^{2-m}, \qquad 0 < a < \lambda,$$

uniformly in $R^m \times R^m - \delta$. Similarly, since

$$\frac{\partial F}{\partial x_i} = \frac{e^{-\rho}}{S(y)} \frac{\partial \rho}{\partial x_i} \left\{ - \int_{-\infty}^{\infty} + \frac{\partial}{\partial \rho} \int_{-\infty}^{\infty} \right\},$$

we also have

$$\left| \frac{\partial F}{\partial x_i} \right| \leqq \text{const.}\, e^{-\lambda r} \{ r^{(1-m)/2} + r^{(3-m)/2} \}$$
$$\leqq \text{const.}\, e^{-ar} r^{1-m}, \qquad 0 < a < \lambda,$$

the constant being independent of x, $y \in R^m$. In a similar manner, we can obtain the desired estimate for the second derivatives.

THEOREM 3.2. *Let $H(x, y)$ be as in* (1.2) *and $k = 1$. Let $\beta = (\beta_1, \beta_2)$ be such that β_1 and β_2 are nonnegative integers with $1 \leqq |\beta| \leqq 2$. If $m = 2$, then for some constant a, $0 < a < \lambda$, we have*

$$|H(x, y)| \leqq C\, e^{-ar} |\log r|, \tag{3.4}$$

and

$$|D_x^\beta H(x, y)| \leqq C\, e^{-ar} r^{-|\beta|}, \qquad 1 \leqq |\beta| \leqq 2, \tag{3.5}$$

the constant C being independent of x, $y \in R^m$.

Proof. From (2.8), we have

$$H(x, y) = \frac{1}{S(y)} \left\{ \frac{e^{-\rho}}{2\pi} \log \left(\frac{1}{2\rho} \right) + \text{const.}\, e^{-\rho} \int_{-\infty}^{\infty} h(t) \rho^{\alpha + it}\, dt \right\}, \tag{3.6}$$

where the constant is independent of ρ, $h(t)$ is absolutely integrable and $0 < \alpha < 1$. The first term on the right-hand side of (3.6) is of order $O(e^{-ar}|\log r|)$, uniformly in $x, y \in R^m$. Since the second term is of order $O(e^{-\lambda r} r^\alpha)$, it is also of order $O(e^{-ar}|\log r|)$ uniformly in x, $y \in R^m$. Before we proceed to estimate the derivatives of H, we remark that the first and second derivatives with respect to ρ of the integral appearing in (3.6) are of orders $O(r^{\alpha-1})$, $O(r^{\alpha-2})$, respectively, uniformly in $x, y \in R^m$. Therefore

$$\left| \frac{\partial H}{\partial x_i} \right| \leqq \text{const.}\, e^{-\rho} \left\{ \frac{1}{\rho} + \left| \log \frac{1}{2\rho} \right| + r^\alpha + r^{\alpha - 1} \right\}$$
$$\leqq \text{const.}\, e^{-ar} r^{-1}$$

uniformly in x, $y \in R^m$, since $0 < \alpha < 1$. The estimate for the second derivatives may be obtained in a similar way.

Remark. The conclusions of Theorems 2.2, 3.1 and 3.2 are valid even if k is not equal to 1, but the constant C in each case will then depend on k. The same comment applies to the constant a appearing in the exponential function involved in the conclusion of Theorems 2.2, 3.1, and 3.2.

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DUAL TRIGONOMETRIC SERIES—A RELATED PROBLEM*

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Abstract. For any n, let $\varphi(e^{inx})$ be defined as $\cos nx$ on $[0, \pi/2)$ and $\sin nx$ on $[\pi/2, \pi]$. In this paper we consider both $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ and $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ and study whether or not they are complete in $L^2[0, \pi]$, and whether or not they are bases in $L^2[0, \pi]$.

1. Introduction. Problems in dual trigonometric series usually take either of two general forms: Given $f \in L^2[0, \pi]$, $c \in (0, \pi)$ and a sequence $\{k_n\}$, does there exist a unique sequence $\{a_n\}$ (and if so, find it) such that either

$$\text{(a)} \qquad \sum_{n=1}^{\infty} a_n k_n \sin nx = f(x), \qquad 0 \leqq x < c,$$

$$\sum_{n=1}^{\infty} a_n \sin nx = f(x), \qquad c \leqq x \leqq \pi,$$

or

$$\text{(b)} \qquad \sum_{n=0}^{\infty} a_n k_n \cos nx = f(x), \qquad 0 \leqq x < c,$$

$$\sum_{n=0}^{\infty} a_n \cos nx = f(x), \qquad c \leqq x \leqq \pi,$$

where convergence is in the $L^2[0, \pi]$ sense? (Variations involve $\sin (n-1/2)x$ and $\cos (n-1/2)x$ instead of $\sin nx$ and $\cos nx$ respectively. See [3] for a long discussion of such problems.) It was the study of such problems that led us to the study of the following problem: For $f \in L^2[0, \pi]$, does there exist a unique $\{a_n\}$ such that

$$\sum a_n \cos nx = f(x), \qquad 0 \leqq x < \pi/2,$$
$$\sum a_n \sin nx = f(x), \qquad \pi/2 \leqq x \leqq \pi,$$

(where again convergence is in the $L^2[0, \pi]$ sense)?

In this paper we shall deal with the case $c = \pi/2$, but it seems fairly clear that the methods could be adapted to be used with any $c \in (0, \pi)$.

Our notation is made simpler by the following operator: For any $g(x)$ defined on $[0, \pi]$, let

$$\varphi(g(x)) = \begin{cases} \operatorname{Re} g(x), & 0 \leqq x < \dfrac{\pi}{2}, \\[2ex] \operatorname{Im} g(x), & \dfrac{\pi}{2} \leqq x \leqq \pi. \end{cases}$$

Thus our previously stated problem becomes: Given $f \in L^2 [0, \pi]$, does there exist a unique $\{a_n\}$ such that $\sum a_n \varphi(e^{inx}) = f(x)$? We are thus studying the question of

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whether or not $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ and $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ are bases in $L^2[0,\pi]$. We shall also study the completeness of those two sets of functions and that is done first.

2. Completeness questions.

THEOREM 1. *$\{\varphi(e^{inx})\}_{n=0}^{\infty}$ is complete in $L^p[0,\pi]$ for $p\geqq 1$ but is incomplete in $C[0,\pi]$.*

Proof. Assume there is an $f(x)\in L^q[0,\pi]$ (where $1/p+1/q=1$ if $p>1$, and $q=\infty$ if $p=1$) such that

$$0=\int_0^{\pi}\varphi(e^{inx})f(x)\,dx=\int_0^{\pi/2}\cos(nx)f(x)\,dx+\int_{\pi/2}^{\pi}\sin(nx)f(x)\,dx$$

for $n=0,1,2,\cdots$. Then

$$\int_0^{\pi/2}\cos(nx)f(x)\,dx+(-1)^n\int_{-\pi/2}^{0}\sin(nx)f(x+\pi)\,dx=0,\qquad n=0,1,2,\cdots.$$

Then certainly

$$\left(\int_0^{\pi/2}\cos(nx)f(x)\,dx\right)^2-\left(\int_{-\pi/2}^{0}\sin(nx)f(x+\pi)\,dx\right)^2=0,$$
$$n=0,\pm1,\pm2,\pm3,\cdots.$$

Let $F(z)=(\int_0^{\pi/2}\cos(zx)f(x)\,dx)^2-(\int_{-\pi/2}^{0}\sin(zx)f(x+\pi)\,dx)^2$. Then $F(z)$ is entire, vanishes at all the integers (positive, negative, and zero) and $|F(z)|\leqq Me^{\pi|z|}$.

By a well-known theorem on the growth of entire functions (see [1])

$$F(z)\equiv C\sin\pi z\qquad\text{for some constant } C.$$

However, $F(z)$ is an even function of z and $C\sin\pi z$ is odd. Therefore, $C=0$ and $F(z)\equiv 0$. Then

$$\int_0^{\pi/2}\cos(zx)f(x)\,dx=\pm\int_{-\pi/2}^{0}\sin(zx)f(x+\pi)\,dx,$$

where we either have $+$ for all z or $-$ for all z. In either case we have an even function equal to an odd function which is impossible unless both are identically zero. Thus $\int_0^{\pi/2}\cos(nx)f(x)\,dx=0$, $n=0,1,2,\cdots$, which implies $f(x)=0$ a.e. on $[0,\pi/2]$.

Similarly, $\int_{-\pi/2}^{0}\sin(nx)f(x+\pi)\,dx=\int_{\pi/2}^{\pi}\sin(nx)f(x)\,dx=0$, for $n=1,2,\cdots$, which implies $f(x)=0$ a.e. on $[\pi/2,\pi]$.

Thus, $f(x)=0$ a.e. on $[0,\pi]$, and $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ is complete in $L^p[0,\pi]$.

To prove $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ is incomplete in $C[0,\pi]$, we have only to notice that for each n, $\varphi(e^{inx})$ is zero at π. Therefore $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ could not approximate any function which is not 0 at π.

In our next theorem, we prove that for $p>2$ we need $\varphi(1)$ for completeness.

THEOREM 2. *$\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is incomplete in $L^p[0,\pi]$ if $p>2$.*

Proof. We consider the function $(1+z)(1+z^2)^{-1/2}$ which is in H^q for any $q\in[1,2)$. Thus

$$\int_{-\pi}^{\pi}\frac{1+e^{i\theta}}{(1+e^{2i\theta})^{1/2}}e^{in\theta}\,d\theta=0,\qquad n=1,2,3,\cdots.$$

By simplifying the integrand, we obtain

$$\int_{-\pi/2}^{\pi/2} \frac{2\cos(\theta/2)\,e^{in\theta}}{(2\cos\theta)^{1/2}}\,d\theta + i\int_{\pi/2}^{\pi} \frac{2\cos(\theta/2)\,e^{in\theta}}{(2|\cos\theta|)^{1/2}}\,d\theta - i\int_{-\pi}^{-\pi/2} \frac{2\cos(\theta/2)\,e^{in\theta}}{(2|\cos\theta|)^{1/2}}\,d\theta = 0,$$

$$n = 0, 1, 2, 3, \cdots.$$

Simplifying this expression (by taking into account evenness and oddness), we get

$$\int_0^{\pi/2} \frac{\cos(\theta/2)\cos n\theta}{(\cos\theta)^{1/2}}\,d\theta - \int_{\pi/2}^{\pi} \frac{\cos(\theta/2)\sin n\theta}{|\cos\theta|^{1/2}}\,d\theta = 0,$$

$$n = 1, 2, 3, \cdots.$$

Thus the function

$$f(\theta) = \begin{cases} \dfrac{\cos(\theta/2)}{(\cos\theta)^{1/2}}, & 0 < \theta < \dfrac{\pi}{2}, \\[2ex] -\dfrac{\cos(\theta/2)}{|\cos\theta|^{1/2}}, & \dfrac{\pi}{2} < \theta < \pi, \end{cases}$$

is in $L^q[0, \pi]$, $1 \leqq q < 2$, and is orthogonal to $\{\varphi(e^{inx})\}_{n=1}^{\infty}$. Thus $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is incomplete in $L^p[0, \pi]$, $p > 2$.

This function, $f(\theta)$, can also be used to prove the completeness of $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ in $L^p[0, \pi]$, $1 \leqq p \leqq 2$. The argument goes as follows. Since the addition of $\varphi(1)$ to the collection $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ makes a complete set in $L^p[0, \pi]$, $p \geqq 1$, we know there is at most one function orthogonal to $\{\varphi(e^{inx})\}_{n=1}^{\infty}$. If, for some $q \geqq 2$, there were a $g \in L^q[0, \pi]$, orthogonal to $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ then, since that g would also be in all $L^q[0, \pi]$ for $1 \leqq q \leqq 2$, that g would have to be equal to $Cf(\theta)$ for some constant C. But since $Cf(\theta)$ is not in $L^2[0, \pi]$ unless $C = 0$, we must conclude that $g(\theta) = 0$ a.e.

Thus, we have proved the following theorem.

THEOREM 3. *$\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is complete in $L^p[0, \pi]$ where $1 \leqq p \leqq 2$.*

3. Basis questions. We now restrict our attention to $L^2[0, \pi]$ and consider the question of whether $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ (or $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ is a basis.

One fact about a basis that we shall need is (see [2]): if $\{f_n\}_{n=1}^{\infty}$ is a basis, there is an associated sequence of continuous functionals (abbreviated a.s.c.f.) $\{L_n\}_{n=1}^{\infty}$ such that $L_n(f_m) = \delta_{mn}$.

We essentially have already proved the following theorem.

THEOREM 4. *$\{\varphi(e^{inx})\}_{n=0}^{\infty}$ is not a basis in $L^2[0, \pi]$.*

Proof. We have proved that $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is complete in $L^2[0, \pi]$. If $\{\varphi(e^{inx})\}_{n=0}^{\infty}$ were a basis, then L_0, from the a.s.c.f., would have the property that $L_0(\varphi(e^{inx})) = 0$, $n = 1, 2, \cdots$, while $L_0(\varphi(1)) = 1$, thereby contradicting the completeness of $\{\varphi(e^{inx})\}_{n=1}^{\infty}$.

We notice that the proof is based on the fact that $\varphi(1)$ is an "extra" function. We shall now prove that $\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is also not a basis. Ironically enough this proof will be based on the fact that in some sense $\varphi(1)$ is "missing."

In the rest of this paper we shall refer to the function $(1-z)(1+z^2)^{-1/2}$ as $h(z)$. We note that:

(i) For $0 \leqq x \leqq \pi/2$, $h(e^{ix})$ is pure imaginary, while, for $\pi/2 < x \leqq \pi$, $h(e^{ix})$ is real. Thus $\varphi(h(e^{ix})) = 0$ a.e.

(ii) If $h(z)$ is expressed as $\sum_{n=0}^{\infty} h_n z^n$, then $h_0 = 1$ and all the h_n are real.

THEOREM 5. *Let* $S_N(x) = \sum_{n=0}^{N} h_n e^{inx}$ *be the N-th partial sum of the Fourier series of* $h(e^{ix})$. *Then* $\lim_{N\to\infty} \int_0^\pi \varphi(S_N(x)) \cos(kx)\, dx = 0$ *for* $k = 0, 1, 2, \cdots$.

Proof. Since $\varphi(h(e^{ix})) = 0$ a.e. it will suffice to prove that $\lim_{N\to\infty} \int_0^\pi \varphi(S_N(x)) \cos(kx) dx = \int_0^\pi \varphi(h(e^{ix})) \cos(kx)\, dx$, or that $\lim_{N\to\infty} \int_0^\pi \varphi(S_N(x) - h(e^{ix})) \cos(kx)\, dx = 0$, $k = 0, 1, 2, \cdots$.

By Hölder's inequality we have that $|\int_0^\pi \varphi(S_N(x) - h(e^{ix})) \cos(kx)\, dx| \leqq \|\varphi(S_N(x) - h(e^{ix}))\|_{3/2} \|\cos kx\|_3 \leqq \varphi(S_N(x) - h(e^{ix}))\|_{3/2}$, where $\|\cdot\|_p$ is the $L^p[0, \pi]$ norm. We notice that, for any x, $|\varphi(S_N(x) - h(e^{ix}))| \leqq |S_N(x) - h(e^{ix})|$, and then $\|\varphi(S_N(x) - h(e^{ix}))\|_{3/2} \leqq \|S_N(x) - h(e^{ix})\|_{3/2}$. Moreover, since $h(e^{ix}) \in L^{3/2}[0, \pi]$, its Fourier series converges to it in the $L^{3/2}[0, \pi]$ norm, and, consequently,

$$\|S_N(x) - h(e^{ix})\|_{3/2} \to 0 \quad \text{as } N \to \infty.$$

Combining this with our previously derived inequality, we get our desired result.

THEOREM 6. *There exist positive m and* M_1 *such that*

$$0 < m \leqq \|\varphi(S_N(x))\|_2 \leqq M_1 \quad \textit{for all } N.$$

Proof. See p. 993.

We now combine the previous two theorems to give us the following theorem.

THEOREM 7. $\varphi(S_N(x))$ *converges to* 0 *weakly in* $L^2[0, \pi]$.

Proof. Take any bounded linear functional on $L^2[0, \pi]$ or, equivalently, take any $g \in L^2[0, \pi]$. What we must show is that $\int_0^\pi \varphi(S_N(x)) g(x)\, dx$ converges to 0. Given $\varepsilon > 0$, we find $\sum_{n=0}^{M} b_n \cos nx$ such that $\|\sum_{n=0}^{M} b_n \cos nx - g(x)\|_2 < \varepsilon$. Then, by Theorem 5, we choose N so large that $|\int_0^\pi \varphi(S_N(x))(\sum_{n=0}^{M} b_n \cos nx)\, dx| < \varepsilon$. Then,

$$\begin{aligned} &\left|\int_0^\pi \varphi(S_N(x)) g(x)\, dx\right| \\ &\quad \leqq \left|\int_0^\pi \varphi(S_N(x))\Big(g(x) - \sum_{n=0}^{M} b_n \cos nx\Big)\, dx\right| \\ &\qquad\qquad + \left|\int_0^\pi \varphi(S_N(x))\left(\sum_{n=0}^{M} b_n \cos nx\right) dx\right| \\ &\quad \leqq \left\|\varphi(S_N(x))\|_2 \|g(x) - \sum_{n=0}^{M} b_n \cos nx\right\|_2 + \varepsilon \\ &\quad \leqq M_1 \varepsilon + \varepsilon. \end{aligned}$$

As ε was arbitrary we are done.

Since $S_N(x) = \sum_{n=0}^{N} h_n e^{inx}$, this theorem can also be stated as: $\varphi(\sum_{n=0}^{N} h_n e^{inx})$ converges to 0 weakly, or $\sum_{n=1}^{N} h_n \varphi(e^{inx})$ converges to $-\varphi(1)$ weakly (we recall that $h_0 = 1$ and all the h_n are real). Moreover, since by Theorem 6, $\|\varphi(S_N(x))\|_2 \geqq m > 0$, $\varphi(S_N(x))$ does not converge to 0 strongly in $L^2[0, \pi]$, or $\sum_{n=1}^{N} h_n \varphi(e^{inx})$ does not converge to $-\varphi(1)$ strongly in $L^2[0, \pi]$. Using this we can prove our previously mentioned theorem.

THEOREM 8. *$\{\varphi(e^{inx})\}_{n=1}^{\infty}$ is not a basis in $L^2[0, \pi]$.*

Proof. If it were a basis, there would exist a unique sequence $\{b_k\}$ such that $\lim_{N\to\infty} \sum_{n=1}^{N} b_n\varphi(e^{inx}) = -\varphi(1)$ strongly (in $L^2[0, \pi]$). Let $\{L_k\}_{k=1}^{\infty}$ be the a.s.c.f. Then, for any k, $-L_k(\varphi(1)) = b_k$. However, since $\lim_{N\to\infty} \sum_{n=1}^{N} h_n\varphi(e^{inx}) = -\varphi(1)$ weakly, we would also have that $-L_k(\varphi(1)) = h_k$, and thus have $h_k = b_k$ for all k. However, we proved that $\sum_{n=1}^{N} h_n\varphi(e^{inx})$ does not converge strongly to $-\varphi(1)$, which contradicts the assumption that $\sum_{n=1}^{N} b_n\varphi(e^{inx})$ converged strongly to $-\varphi(1)$.

Proof of Theorem 6. Before we can prove Theorem 6 we shall need the following lemma.

LEMMA 1. *Let $F(x)$ be positive and nonincreasing in $(0, A)$. Then, for any B,*

$$\left|\int_0^A F(x)\sin(x+B)\,dx\right| \leqq \int_0^{2\pi} F(x)|\sin(x+B)|\,dx.$$

Proof. Let $x_1, \cdots, x_k$ be the zeros of $\sin(x+B)$ in $(0, A)$ (where $0 < x_1 < x_2 < \cdots < x_k < A$), let $x_0 = 0$, $x_{k+1} = A$, and let

$$A_i = \int_{x_i}^{x_{i+1}} F(x)|\sin(B+x)|\,dx, \qquad i = 0, 1, \cdots, k.$$

Then, by the monotonicity of $F(x)$ and the periodicity of $\sin(x+B)$, $A_1 \geqq A_2 \geqq \cdots \geqq A_k$. Thus

$$A_0 - A_1 + A_2 - A_3 + \cdots \pm A_k = A_0 - (A_1 - A_2) - (A_3 - A_4) - \cdots - (\;) \leqq A_0$$

and

$$A_0 - A_1 + A_2 - A_3 + \cdots \pm A_k = (A_0 - A_1) + (A_2 - A_3) + \cdots + (\;) \geqq A_0 - A_1.$$

Therefore

$$\left|\int_0^A F(x)\sin(B+x)\,dx\right| = |A_0 - A_1 + A_2 - A_3 + \cdots \pm A_k|$$

$$\leqq A_0 + A_1 = \int_0^{x_2} F(x)|\sin(B+x)|\,dx,$$

and the proof is finished by observing that $x_2 \leqq 2\pi$.

We can now proceed with the proof of Theorem 6. We have

$$S_N(x) = \int_{-\pi}^{\pi} h(e^{i\theta})D_N(x-\theta)\,d\theta,$$

where $D_N(t)$, the Dirichlet kernel, is

$$\frac{1}{2\pi}\,\frac{\sin(2N+1)(t/2)}{\sin(t/2)}$$

and

$$h(e^{i\theta})=\begin{cases} -\dfrac{i\sqrt{2}\sin(\theta/2)}{(\cos\theta)^{1/2}}, & -\pi/2<\theta<\pi/2,\\[2ex] \dfrac{\sqrt{2}\sin(\theta/2)}{|\cos\theta|^{1/2}}, & \pi/2<\theta<\pi,\\[2ex] -\dfrac{\sqrt{2}\sin(\theta/2)}{|\cos\theta|^{1/2}}, & -\pi<\theta<-\pi/2.\end{cases}$$

Therefore

$$\varphi(S_N(x))=\begin{cases} \sqrt{2}\displaystyle\int_{\pi/2}^{\pi}\frac{\sin(\theta/2)}{|\cos\theta|^{1/2}}[D_N(x-\theta)+D_N(x+\theta)]\,d\theta, & 0<x<\dfrac{\pi}{2},\\[2ex] \sqrt{2}\displaystyle\int_{0}^{\pi/2}\frac{\sin(\theta/2)}{(\cos\theta)^{1/2}}[D_N(x-\theta)-D_N(x+\theta)]\,d\theta, & \dfrac{\pi}{2}<x<\pi.\end{cases}$$

Thus

$$\begin{aligned}\|\varphi(S_N(x))\|_2^2&=\int_0^{\pi}|\varphi(S_N(x))|^2\,dx\\ &=2\int_0^{\pi/2}\left[\int_{\pi/2}^{\pi}\frac{\sin(\theta/2)}{|\cos\theta|^{1/2}}(D_N(x-\theta)+D_N(x+\theta))\,d\theta\right]^2dx\\ &\quad+2\int_{\pi/2}^{\pi}\left[\int_0^{\pi/2}\frac{\sin(\theta/2)}{(\cos\theta)^{1/2}}(D_N(x-\theta)-D_N(x+\theta))\,d\theta\right]^2dx\\ &=2I_1(N)+2I_2(N).\end{aligned}$$

If we let

$$I_3(N)=\int_0^{\pi/2}\left[\int_{\pi/2}^{\pi}\frac{\sin(\theta/2)}{|\cos\theta|^{1/2}}D_N(x-\theta)\,d\theta\right]^2dx,$$

$$I_4(N)=\int_0^{\pi/2}\left[\int_{\pi/2}^{\pi}\frac{\sin(\theta/2)}{|\cos\theta|^{1/2}}D_N(x+\theta)\,d\theta\right]^2dx,$$

$$I_5(N)=\int_{\pi/2}^{\pi}\left[\int_0^{\pi/2}\frac{\sin(\theta/2)}{(\cos\theta)^{1/2}}D_N(x-\theta)\,d\theta\right]^2dx,$$

and

$$I_6(N)=\int_{\pi/2}^{\pi}\left[\int_0^{\pi/2}\frac{\sin(\theta/2)}{(\cos\theta)^{1/2}}D_N(x+\theta)\,d\theta\right]^2dx,$$

then we get the following inequalities:

$$(I_3(N))^{1/2}-(I_4(N))^{1/2}\leqq(I_1(N))^{1/2}\leqq(I_3(N))^{1/2}+(I_4(N))^{1/2},$$

and

$$(I_5(N))^{1/2}-(I_6(N))^{1/2} \leqq (I_2(N))^{1/2} \leqq (I_5(N))^{1/2}+(I_6(N))^{1/2}.$$

Then Theorem 6 will be proved once we have proved the following lemma.

LEMMA 2. (a) $\lim_{N\to\infty} I_3(N)=\lim_{N\to\infty} I_5(N)=\alpha>0$, *while*

(b) $\lim_{N\to\infty} I_4(N)=\lim_{N\to\infty} I_6(N)=0$.

Note. We shall be using the following inequalities:

(a) $\sin x \geqq 2x/\pi$ for $0\leqq x\leqq \pi/2$,

(b) $|(\sin mx)/\sin x|\leqq m$.

We shall also write M instead of $2N+1$.

Proof. (a) If we make the changes of variable $U=(M\theta/2)-M\pi/4$ and $V=M\pi/4-Mx/2$, $I_3(N)$ becomes

$$\frac{2}{\pi^2}\int_0^{M\pi/4}\left[\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)}{M\sin((U+V)/M)}\,dU\right]^2 dV.$$

We shall now use the dominated convergence theorem twice. First, for $U\leqq 1$,

$$\left|\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)}{M\sin((U+V)/M)}\right|\leqq\frac{1}{(4U/\pi)^{1/2}},$$

while for $U>1$ it is bounded by

$$\frac{1}{(4U/\pi)^{1/2}}\,\frac{1}{2(U+V)/\pi}\leqq\frac{\sqrt{\pi}\cdot\pi}{4}\,\frac{1}{U^{3/2}}.$$

Therefore,

$$\lim_{M\to\infty}\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)}{M\sin((U+V)/M)}\,dU$$
$$=\int_0^{\infty}\frac{\sin(\pi/4)\sin(U+V)\,dU}{(2U)^{1/2}(U+V)}.$$

We now consider

$$\left(\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)\,dU}{M\sin((U+V)/M)}\right)^2.$$

Since, for $0<U<M/4$ (and any $V>0$),

$$\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{1}{M\sin((U+V)/M)}$$

is positive and nonincreasing (as seen by taking its logarithmic derivative), we apply Lemma 1.

Thus

$$\left(\int_0^{M\pi/4} \frac{\sin(U/M+\pi/4)\sin(U+V)}{(M\sin(2U/M))^{1/2}M\sin((U+V)/M)}\,dU\right)^2$$

$$\leqq\left(\int_0^{2\pi} \frac{\sin(U/M+\pi/4)|\sin(U+V)|\,dU}{(M\sin(2U/M))^{1/2}M\sin((U+V)/M)}\right)^2$$

$$\leqq\begin{cases}\left(\displaystyle\int_0^{2\pi}\frac{1}{(4U/\pi)^{1/2}}\,dU\right)^2=2\pi^2 & \text{for } V\leqq 1,\\ \left(\displaystyle\int_0^{2\pi}\frac{1}{(4U/\pi)^{1/2}}\,\frac{1}{2V/\pi}\,dU\right)^2=\dfrac{\pi^4}{2V^2} & \text{for } V\geqq 1.\end{cases}$$

Therefore

$$\lim_{N\to\infty} I_3(N)=\lim_{M\to\infty}\frac{2}{\pi^2}\int_0^{M\pi/4}\left[\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)}{M\sin((U+V)/M)}\,dU\right]^2 dV$$

$$=\frac{2}{\pi^2}\int_0^{\infty}\lim_{M\to\infty}\left[\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V)\,dU}{M\sin((U+V)/M)}\right]^2 dV$$

$$=\frac{2}{\pi^2}\int_0^{\infty}\left(\int_0^{\infty}\frac{\sin(\pi/4)}{(2U)^{1/2}}\,\frac{\sin(U+V)}{(U+V)}\,dU\right)^2 dV$$

$$=\frac{1}{2\pi^2}\int_0^{\infty}\left(\int_0^{\infty}\frac{\sin(U+V)}{\sqrt{U}(U+V)}\,dU\right)^2 dV.$$

(The proof that this is also the limit of $I_5(N)$ is essentially the same and is omitted.)

To finish the proof of part (a) we need prove only that this limit is nonzero. That, however, is easily shown by the fact that the integrand

$$\left(\int_0^{\infty}(\sin(U+V))/\sqrt{U}(U+V)\,dU\right)^2$$

is, obviously, nonnegative and nonidentically zero by continuity and by considering the integrand at 0 (or at $K\pi$).

(b) If we make the changes of variable, $V=Mx/2$ and $U=M\theta/2-M\pi/4$, $I_4(N)$ becomes

$$\frac{2}{\pi^2}\int_0^{M\pi/4}\left(\int_0^{M\pi/4}\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{\sin(U+V+M\pi/4)}{M\sin(U/M+V/M+\pi/4)}\,dU\right)^2 dV,$$

which, since

$$\frac{\sin(U/M+\pi/4)}{(M\sin(2U/M))^{1/2}}\,\frac{1}{M\sin(U/M+V/M+\pi/4)}$$

is positive and nonincreasing, is, by Lemma 1, less than or equal to

$$\frac{2}{\pi^2}\int_0^{M\pi/4}\left(\int_0^{2\pi}\frac{\sin(U/M+\pi/4)|\sin(U+V+M\pi/4)|}{(M\sin(2U/M))^{1/2}M\sin(U/M+V/M+\pi/4)}\,dU\right)^2 dV$$

$$\leqq\frac{2}{\pi^2}\int_0^{M\pi/4}\left(\int_0^{2\pi}\frac{1}{(4U/\pi)^{1/2}}\cdot\frac{1}{M/\sqrt{2}}\,dU\right)^2 dV=\frac{2\pi}{M}.$$

Thus $\lim_{N\to\infty} I_4(N) = 0$. (The proof that $\lim_{N\to\infty} I_6(N) = 0$ is essentially the same.)

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A NEW INTEGRAL EQUATION FOR CERTAIN PLANE DIRICHLET PROBLEMS*

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Abstract. The solution of the Dirichlet problem for a plane simply connected region, part of whose boundary contains an analytic arc, is simplified by making use of reflection through the analytic arc. An integral equation is derived for the double layer potential. A geometrical lower bound on the smallest eigenvalue is derived when the nonanalytic arc is convex.

1. Introduction. Consider a plane simply connected domain Ω whose boundary is a piecewise smooth curve $\gamma = \gamma_0 \cup \gamma_1$. Moreover, let γ_0 be part of a simple closed analytic curve Γ satisfying certain additional conditions to be specified and let γ_1 be smooth. We shall show that the Dirichlet problem for Ω with Hölder continuous boundary data can be reduced to an integral equation on γ_1 only. The integral equation involves a kernel made up of a double layer potential and a certain geometrical quantity dependent on γ_0. We shall show that this integral equation always has a solution. We shall also consider the special case when γ_1 is convex with continuous curvature and give estimates for the smallest eigenvalue. The estimates will, in turn, give a rate of convergence which is useful for the numerical solution of the integral equation by iteration.

The most interesting theoretical aspect of the paper is the use of geometrical reflection for the construction of the integral equation, which in a certain sense is simpler than the one found in potential theory. The most interesting practical aspect of the paper is the fact that we obtain a new integral equation for the treatment of certain potential problems and when γ_1 is convex we obtain a rate of convergence estimate given in terms of geometrical quantities.

2. Formulation of the problem. We shall restrict ourselves to curves Γ whose interior contains γ_0 and Γ is of the form

$$x(\theta) = \sum_{k=0}^{n} a_k \cos k\theta + b_k \sin k\theta,$$

$$y(\theta) = \sum_{k=0}^{m} \alpha_k \cos k\theta + \beta_k \sin k\theta,$$

$0 \leqq \theta < 2\pi$, $n \geqq m$, with

$$x'^2(\theta) + y'^2(\theta) \neq 0, \qquad (a_n, b_n) \neq (0, 0) \neq (\alpha_m, \beta_m)$$

and if $m = n$, then either $\alpha_n^2 + \beta_n^2 \neq a_n^2 + b_n^2$ or $\alpha_n a_n + \beta_n b_n \neq 0$.

We consider

$$z(\sigma) = \overline{g[e^{i\sigma}]} - \overline{ih[e^{i\sigma}]} + \overline{g[e^{i\bar\sigma}]} - \overline{ih[e^{i\bar\sigma}]}$$

and the reflection of z in parametric form

$$\hat{z}(\sigma) = g[e^{i\sigma}] + ih[e^{i\sigma}] + g[e^{i\bar\sigma}] + ih[e^{i\bar\sigma}],$$

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where $\sigma = \theta + i\xi$ with

$$g(t) = \sum_{k=0}^{n} c_k t^k, \qquad h(t) = \sum_{k=0}^{n} \gamma_k t^k;$$

$$c_k = \tfrac{1}{2}(a_k - ib_k), \qquad \gamma = \tfrac{1}{2}(\alpha_k - i\beta_k).$$

One can easily check, see e.g. [4], that for a circle, the reflection gives inversion in the circle. We also require of Ω that if τ_l is a zero of

$$Q^1_{2n}(\tau) = \sum_{p=0}^{2n} D_p \tau^p \quad \text{or} \quad Q^2_{2n}(\tau) = \sum_{p=0}^{2n} E_p \tau^p,$$

where

$$D_p = (p-n)B_{n-p}, \qquad E_p = (p-n)\bar{A}_{n-p}, \qquad 0 \leqq p \leqq n-1;$$

$$D_p = (p-n)A_{p-n}, \qquad E_p = (p-n)\bar{B}_{p-n}, \qquad n \leqq p \leqq 2n;$$

$$A_k = c_k + i\gamma_k, \qquad B_k = \bar{c}_k + i\bar{\gamma}_k, \qquad \gamma_k = 0 \quad \text{if } k > m;$$

then

$$g(\tau_l) + ih(\tau_l) + \overline{g(1/\bar{\tau}_l)} + \overline{ih(1/\bar{\tau}_l)} \notin \Omega$$

and

$$\overline{g(\tau_l)} + \overline{ih(\tau_l)} + g(1/\bar{\tau}_l) + ih(1/\bar{\tau}_l) \notin \Omega.$$

We are looking for a real function $u(z)$, $z = x + iy$, harmonic in Ω, continuous on $\Omega \cup \gamma$ and such that

$$u(z) \to F(t) \quad \text{as } z \to t, \quad t \text{ on } \gamma_0 \cup \gamma_1$$

for which $F(t)$ is real-valued and Hölder continuous. Let z_1 and z_2 be the two common points of γ_0 and γ_1. Without loss of generality we can assume

$$F(z_1) = F(z_2) = 0$$

since if not, we consider

$$u_2(z) = u(z) - \left\{ F(z_2) \operatorname{Re} \frac{z - z_1}{z_2 - z_1} + F(z_1) \operatorname{Re} \frac{z - z_2}{z_1 - z_2} \right\}.$$

Before we can formulate the integral equation we need to recall certain facts concerning geometrical reflection. It is shown in [3], [4], that if γ_0 is of the form we are considering, then there exists a "reflection function", $G(z)$ for γ_0, a single-valued and analytic function in Ω such that if $z \in \Omega$ and

$$\hat{z} = \overline{G(z)},$$

then for z close enough to γ_0, $\hat{z} \notin \Omega$. We shall assume for all $z \in \Omega$, $\hat{z} \notin \Omega$. We also have

(i) $\overline{G(\Omega)}$ is a region bounded by γ_0;

(ii) $\hat{z} = z$ for $z \in \gamma_0$;

(iii) $G(z)$ can be extended as a single-valued function analytic on $\Omega \cup \gamma_0 \cup \hat{\Omega}$ where

$$\hat{\Omega} = \{\zeta : \zeta = \hat{z} = \overline{G(z)},\ z \in \Omega\} \equiv \overline{G(\Omega)};$$

(iv) $G'(z) \neq 0$ for $z \in \Omega \cup \gamma_0 \cup \hat{\Omega} \cup \gamma_1 \cup \hat{\gamma}_1$, and

(v) $\hat{\hat{z}} = z$ for $z \in \Omega \cup \gamma_0 \cup \hat{\Omega}$.

We shall assume $\gamma_1 \cap [\Omega \cup \gamma_0 \cup \hat{\Omega}] = \varnothing$, (Here we mean γ_1 does not include its endpoints.)

We look for $u(z)$ of the form:

$$\begin{aligned} u(z) = \operatorname{Re} \frac{1}{2\pi i} \oint_{\gamma_0} dt \mu_0(t) \left\{ \frac{1}{t-z} - \frac{1}{t-\hat{z}} \right\} \\ + \operatorname{Re} \frac{1}{2\pi i} \oint_{\gamma_1} dt \mu_1(t) \left\{ \frac{1}{t-z} - \frac{1}{t-\hat{z}} \right\}, \end{aligned} \tag{2.1}$$

where $\mu(t)$ is a Hölder continuous function on $\gamma_0 \cup \gamma_1$ with

$$\mu(t) = \begin{cases} \mu_0(t) & \text{for } t \text{ on } \gamma_0, \\ \mu_1(t) & \text{for } t \text{ on } \gamma_1, \end{cases}$$

$\mu(z_1) = \mu(z_2) = \mu_j(z_k) = 0$, $j = 1, 2$, $k = 1, 2$.

We recall some facts [2] about Cauchy integrals. Let

$$\Phi(z) = \frac{1}{2\pi i} \int_{\gamma_0 \cup \gamma_1} \frac{\varphi(t)}{t-z}\, dt$$

where $\varphi(t)$ is Hölder continuous on $\gamma_0 \cup \gamma_1$. Let t^* be on $\gamma_0 \cup \gamma_1$ and let α $(0 < \alpha \leqq 2\pi)$ be the angle through which the vector t^*t rotates when the point t on the left of $\gamma_0 \cup \gamma_1$ and close to $\gamma_0 \cup \gamma_1$ moves from a point before t^* ($\gamma_0 \cup \gamma_1$ has an orientation) to points beyond t^*. At all ordinary points (not corners, i.e., not z_1 or z_2) $\alpha = \pi$ and at z_1, $\alpha = \alpha_1$, and at z_2, $\alpha = \alpha_2$, see Fig. 1. Then the Plemelj

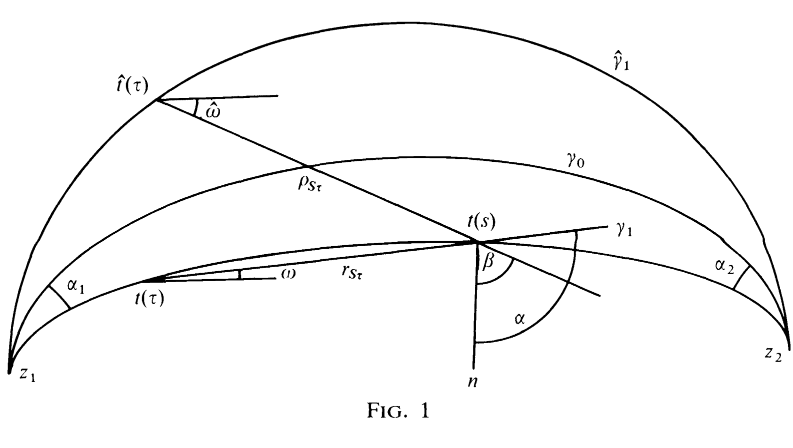

FIG. 1

formulas tell us that (see e.g. [2, p. 428])

$$\Phi(z)\to\Phi^+(t^*)=\left(1-\frac{\alpha}{2\pi}\right)\varphi(t^*)+\frac{1}{2\pi i}\int_{\gamma_0\cup\gamma_1}\frac{\varphi(t)}{t-t^*}\,dt$$

as $z\to t^*\in\gamma_0\cup\gamma_1$ along any path in Ω and

$$\Phi(z)\to\Phi^-(t^*)=-\frac{\alpha}{2\pi}\varphi(t^*)+\frac{1}{2\pi i}\int_{\gamma_0\cup\gamma_1}\frac{\varphi(t)}{t-t^*}\,dt$$

as $z\to t^*\in\gamma_0\cup\gamma_1$ along any path outside Ω.

Utilizing the Plemelj formulas along with (i) and (ii) we get

$$u(z)\xrightarrow[\substack{z\to t_0\in\gamma_0\\ z\in\Omega}]{}\mu_0(t_0),$$

which is valid even for $t_0=z_1$, z_2, and

$$u(z)\xrightarrow[\substack{z\to t_1\in\gamma_1\\ z\in\Omega}]{}\operatorname{Re}\frac{1}{2\pi i}\oint_{\gamma_0}dt\mu_0(t)\left\{\frac{1}{t-t_1}-\frac{1}{t-\hat{t}_1}\right\}$$

$$+\frac{1}{2}\mu_1(t_1)+\operatorname{Re}\frac{1}{2\pi i}\oint_{\gamma_1}dt\mu_1(t)\left\{\frac{1}{t-t_1}-\frac{1}{t-\hat{t}_1}\right\}$$

which is also valid even for $t_1=z_1$, z_2. Thus we see we must have

$$\mu_0(t)=F_0(t),\qquad t\in\gamma_0,\quad \mu_i=F_i \text{ on } \gamma_i$$

and

$$\mu_1(t_1)=-\operatorname{Re}\frac{1}{\pi i}\oint_{\gamma_1}dt\mu_1(t)\left\{\frac{1}{t-t_1}-\frac{1}{t-\hat{t}_1}\right\}+g(t_1),\tag{2.2}$$

where

$$g(t_1)=2F_1(t_1)-\operatorname{Re}\frac{1}{\pi i}\oint_{\gamma_0}dtF_0(t)\left\{\frac{1}{t-t_1}-\frac{1}{t-\hat{t}_1}\right\}.\tag{2.3}$$

It is with the integral equation (2.2) that we shall concern ourselves. An alternative way of writing (2.2) is

$$\nu(\tau)=-\int_0^{s_1}K(\tau,s)\nu(s)\,ds+h(\tau),\tag{2.4}$$

where

$$s=\text{arc length of }\gamma_1,$$
$$s_1=\text{length of }\gamma_1,$$
$$t=t(s),\qquad t_1=t(\tau),$$
$$K(\tau,s)=\frac{1}{\pi}\operatorname{Im}\left\{\frac{t'(s)}{t(s)-)t(\tau)}-\frac{t'(s)}{t(s)-\hat{t}(\tau)}\right\},\tag{2.5}$$
$$h(\tau)=g[t(\tau)],$$
$$\nu(\tau)=\mu_1[t(\tau)].$$

As is easily seen and as is well known, see e.g. [1, p. 121],

$$K(\tau, s) = \frac{1}{\pi}\left\{\frac{\cos \alpha(s, \tau)}{r_{s\tau}} - \frac{\cos \beta(s, \tau)}{\rho_{s\tau}}\right\} = \frac{1}{\pi}\left\{\frac{d\omega}{ds} - \frac{d\hat{\omega}}{ds}\right\}, \tag{2.6}$$

where

$$r_{s\tau} = |t(s) - t(\tau)|, \qquad \rho_{s\tau} = |t(s) - \hat{t}(\tau)|, \tag{2.7}$$

and $\alpha(s, \tau)$ and $\beta(s, \tau)$ are the angles formed by the exterior normal n constructed at the point $t(s)$ and the vectors $t(s) - t(\tau)$, and $t(s) - \hat{t}(\tau)$ respectively; ω and $\hat{\omega}$ are the angles that $t(s) - t(\tau)$ and $t(s) - \hat{t}(\tau)$ form respectively with the positive x-axis (see Fig. 1).

3. Solution of the problem. We shall show that (2.2) always has a solution. Toward this end we consider

$$v(\tau) = \lambda \int_0^{s_1} K(\tau, s) v(s)\, ds + h(\tau), \tag{3.1}$$

$v(0) = v(s_1) = 0$, $h(\tau)$ is Hölder continuous, and prove the following theorem.

THEOREM 1. $\lambda = -1$ *is not an eigenvalue of* (3.1).

Proof. If $\lambda = -1$ were an eigenvalue and $\sigma[t(s)]$ a corresponding eigenfunction, then

$$\sigma[t(\tau)] = -\int_0^{s_1} K(\tau, s)\sigma[t(s)]\, ds, \quad \sigma[t, (0)] = 0, \quad \sigma[t(s_1)] = 0.$$

Consider the harmonic function on Ω,

$$\varphi(z) = \frac{1}{\pi i}\int_{\gamma_1} dt \sigma(t)\left\{\frac{1}{t - z} - \frac{1}{t - \hat{z}}\right\}.$$

Then

$$\varphi(z) = 0 \quad \text{for } z \text{ on } \gamma_0$$

since

$$\hat{z} = z \quad \text{on } \gamma_0 \quad \text{and} \quad \sigma(z_j) = 0, \quad j = 0, 1,$$

and by assumption,

$$\operatorname{Re} \varphi(z) = 0 \quad \text{for } z \text{ on } \gamma_1.$$

Thus since $\operatorname{Re} \varphi(z)$ is harmonic in Ω, we have

$$\operatorname{Re} \varphi(z) \equiv 0 \quad \text{for } z \text{ in } \Omega \cup \gamma.$$

Let

$$\psi(z) = \frac{1}{\pi i}\int_{\gamma_1} \sigma(t)\frac{1}{t - z}\, dt;$$

then

$$\varphi(z) = \psi(z) - \psi(\hat{z})$$

and

$$\begin{aligned}
\operatorname{Re} \varphi(z) &= \tfrac{1}{2}[\psi(z)-\psi(\hat{z})+\overline{\psi(z)}-\overline{\psi(\hat{z})}] \\
&= \tfrac{1}{2}[\psi(z)-\overline{\psi(\hat{z})}]+\tfrac{1}{2}[\overline{\psi(z)}-\psi(\hat{z})] \\
&= \tfrac{1}{2}\chi(z)+\tfrac{1}{2}\overline{\chi(z)} \\
&= \operatorname{Re} \chi(z),
\end{aligned}$$

where

$$\chi(z)=\psi(z)-\overline{\psi(\hat{z})}$$

is an analytic function for z on $\Omega \cup \gamma_0 \cup \hat{\Omega} \cup \{\text{boundary of } \Gamma-\gamma_1-z_0-z_1\}$.

But since

$$\operatorname{Re} \varphi(z)=\operatorname{Re} \chi(z)=0 \quad \text{on } \Omega \cup \gamma,$$

we have $\chi(z)=ib$, $b=$ real constant for z in $\Omega \cup \gamma$. Since, however, $\chi(z)$ is analytic on $\Omega \cup \gamma_0 \cup \hat{\Omega} \cup \{\text{boundary of } \Gamma-\gamma_1-z_0-z_1\}$ we have $\chi(z)=ib$ there also. Thus on the smooth Jordan curve $\tilde{\Gamma}$ which is the same as Γ except in neighborhoods of z_0 and z_1 and which contains no points of γ_1, we have

$$\chi(z)=[\psi(z)-\overline{\psi(z)}]=ib,$$

i.e.,

$$\operatorname{Im} \psi(z)=\frac{1}{2i}[\psi(z)-\overline{\psi(z)}]=\frac{1}{2}b \text{ on } \tilde{\Gamma}.$$

But $\psi(z)$ is analytic off of γ_1 and thus $\operatorname{Im} \psi(z)$ is harmonic outside of Γ and takes on the value $\frac{1}{2}b$ on $\tilde{\Gamma}$. Moreover, $\psi(z)\to 0$ as $z\to\infty$, thus

$$\operatorname{Im} \psi(z)=0 \quad \text{outside } \Gamma,$$

and therefore

$$\psi(z)=\text{const.}=0 \quad \text{outside } \Gamma.$$

But $\psi(z)$ is analytic off γ_1 and thus

$$\psi(z)=0 \quad \text{off of } \gamma_1.$$

Since $\sigma(t)$ is continuous, we see by the jump discontinuity for the potential that upon approaching t_1 on γ_1 through z inside Ω and z' outside Ω that for nontangential limits

$$\lim_{z,z'\to t_1}[\psi(z)-\psi(z')]=2\sigma(t_1), \qquad |t_1-z|=|t_1-z'|.$$

From this it follows that

$$\sigma_1(t_1)=0 \quad \text{for } t_1 \quad \text{on } \gamma_1.$$

Thus $\gamma=-1$ is not an eigenvalue.

COROLLARY. *The integral equation* (3.1) *has a unique solution for every Hölder continuous* $h(\tau)$.

Proof. Since $\lambda = -1$ is not an eigenvalue of (3.1), (3.1) with $h(\tau) \equiv 0$ has only the trivial solution. By the Fredholm theorems then, the homogeneous adjoint equation has only the trivial solution. Thus for the original nonhomogeneous equation there exists a solution and it is unique since the range of the original equation is orthogonal to the nullspace of the adjoint equation.

4. Rate of convergence when γ_1 is convex. To enhance the usefulness of the integral equation as a practical method for obtaining numerical solutions by iteration, we shall give a geometrical estimate for the smallest absolute eigenvalue in the case γ_1 is convex. As the resolvent is analytic in λ for $|\lambda| < \lambda_0$, where λ_0 is the smallest absolute eigenvalue, λ_0 will give an estimate for the rate of convergence of the successive approximations.

With these ideas in mind we prove the following theorem.

THEOREM 2. *Let γ_1 be convex and such that if γ_0 and γ_1 have points z_1 and z_2 in common and z_1z_2 is the chord joining z_1 to z_2, then γ_0 is in one of the half-planes of the line through z_1z_2 and γ_1 is in the other. Let α_0 be the angle through which the tangent to γ_1 turns in going from z_1 to z_2. Let $\gamma_1 \subset$ convex set with boundary $\supset \gamma_1$,*

$$\delta(t_1) = \textit{angle through which } \hat{t}_1 t \textit{ turns as}$$

$$t \textit{ goes from } z_1 \textit{ to } z_2 \textit{ along } \gamma_1,$$

$$\delta_0 = \max_{t_1 \in \gamma_1} \delta(t_1);$$

then if λ is an eigenvalue of (3.1) *we have*

$$|\lambda| \geqq \lambda_0,$$

where

$$\lambda_0 \geqq \frac{\pi}{\delta_0 + \alpha_0} \qquad \textit{if } \delta_0 + \alpha_0 > 0$$

and

$$\lambda_0 = \infty \quad \textit{if } \delta_0 + \alpha_0 = 0.$$

Note. $\lambda_0 > 1$ if $\delta_0 + \alpha_0 < \pi$.

Proof. Since in (2.6),

$$\alpha = \text{angle between } t(s) - t(\tau) \text{ and the exterior normal to } \gamma_1 \text{ at } t(s),$$

$$\beta = \text{angle between } t(s) - \hat{t}(\tau) \text{ and the exterior normal to } \gamma_1 \text{ at } t(s),$$

see Fig. 2,

$$-(\pi/2) \leqq \alpha \leqq (\pi/2)$$

since γ_1 is convex and $t(\tau)$ is always on the same side of the support line to γ_1

though $t(s)$,

$$-(\pi/2) \leqq \beta \leqq (\pi/2)$$

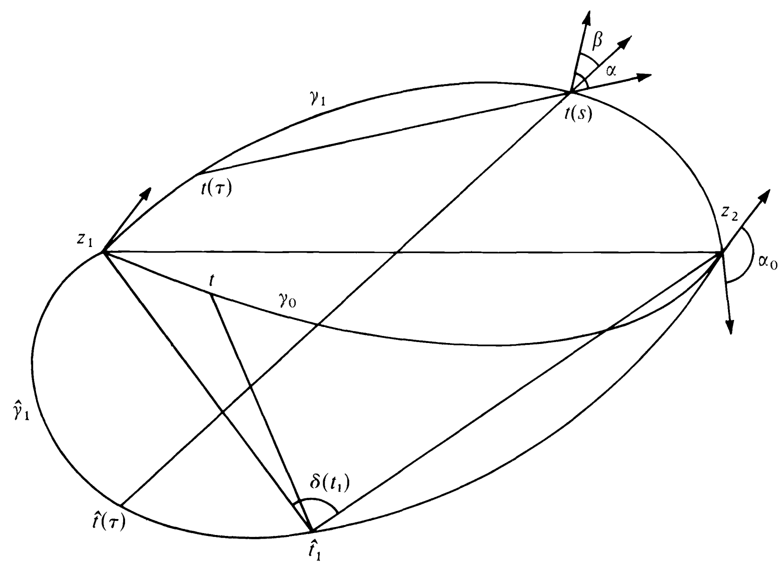

FIG. 2

since $\hat{\gamma}_1$ lies in a convex set that has as part of its boundary γ_1 and thus $\hat{t}(\tau)$ is always on the same side of the support line to γ_1 through $t(s)$, and thus in (2.6),

$$K(\tau, s) = \frac{1}{\pi}\left[\frac{\cos\alpha}{r_{s\tau}} - \frac{\cos\beta}{\rho_{s\tau}}\right] = \frac{1}{\pi}\left[\frac{d\omega}{ds} - \frac{d\hat{\omega}}{ds}\right],$$

where

$$d\omega/ds \geqq 0, \qquad d\hat{\omega}/ds \geqq 0.$$

If λ is an eigenvalue and $\nu(t)$ is a corresponding eigenfunction, then for $M = \max_{t\in\gamma_1} \nu(t)$ we have

$$\begin{aligned} |\nu(t)| &= |\lambda| \left| \int_0^{s_1} K(\tau, s)\nu(s)\, ds \right| \\ &= |\lambda| \frac{1}{\pi} \left| \int_0^{s_1} \nu(s)\left(\frac{d\omega}{ds} - \frac{d\hat{\omega}}{ds}\right) ds \right| \\ &\leqq |\lambda| \frac{1}{\pi} M \left[\int_0^{s_1} \frac{d\omega}{ds}\, ds + \int_0^{s_1} \frac{d\hat{\omega}}{ds}\, ds \right] \\ &\leqq \frac{|\lambda| M}{\pi} [\alpha_0 + \delta_0]; \end{aligned}$$

thus

$$|\lambda| \geqq \frac{\pi}{\alpha_0+\delta_0} \quad \text{if } \alpha_0+\delta_0>0,$$

and the proof is complete.

COROLLARY.

$$\delta_0 = \max_{0\leqq s\leqq s_1} \arc\cos\left[\frac{|t(0)-\hat{t}(s)|^2+|t(s_1)-\hat{t}(s)|^2-|t(0)-t(s_1)|^2}{2|t(0)-\hat{t}(s)||t(s_1)-t(s)|}\right],$$

where $z_1 = t(0)$, $z_2 = t(s_1)$, $\gamma_1 : t(s)$, $0 \leqq s \leqq s_1$.

Proof. Construct a triangle with base z_1z_2 and apex at $\hat{t}(s)$; then the corollary is simply a statement of the law of cosines.

Remark. If γ_1 is a straight line, $\alpha_0 = 0$, and it is clear that δ_0 is always $<\pi$, then for all eigenvalues λ we have

$$|\lambda| > 1.$$

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MATCHED ASYMPTOTIC EXPANSION SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER*

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Abstract. The initial value problem in a Banach space

$$\varepsilon \frac{du}{dt} = B(t, \varepsilon)u + F(t, u, \varepsilon),$$

$u(0, \varepsilon) = \zeta(\varepsilon)$, is studied. Here, $B(t, \varepsilon)$ is a linear, and $F(t, u, \varepsilon)$ a nonlinear, operator in u, and ε is a small parameter. The solution is found for the case where the null space of B is one-dimensional by a method of matched asymptotic expansions. The form of the solution is shown to depend on the nature of the projection on the null space of B of the coefficient of ε in the expansion of $F(t, u, \varepsilon)$ about $\varepsilon = 0$.

1. Introduction. An ordinary differential equation with a small parameter multiplying the highest derivative can often be solved by the method of matched asymptotic expansions. In this paper, we apply this method to solve initial value problems in a Banach space E of the form

$$\begin{aligned} \varepsilon \frac{du}{dt} &= B(t, \varepsilon)u + F(t, u, \varepsilon), \\ u(0, \varepsilon) &= \zeta(\varepsilon), \end{aligned} \tag{1}$$

where $B(t, \varepsilon)$ is a linear operator and $F(t, u, \varepsilon)$, a nonlinear operator in u.

The case where $B(t, \varepsilon)$ is stable[1] for each (t, ε) has been studied by Hoppensteadt in [4] and [7]. We shall discuss the case where B has a one-dimensional null space E^1, but where its restriction in E to a complementary subspace of E^1 is stable. This case was treated by Hoppensteadt and Gordon [10] for a problem where B is independent of (t, ε); and the small parameter is not a factor of the derivative term. In that case, a matched asymptotic expansions method is used to express the solution as the sum of a smooth solution of the differential equation, known as the outer function, and a function in a fast time variable, known as the initial layer function, that gives the solution for t near zero. Formal expansions of these functions in terms of the small parameter are substituted into the problem and the coefficients are derived to give uniformly valid approximations to the solution.

In this paper, we show that the method solution of (1) varies significantly with the nature of $F(t, u, \varepsilon)$. In many cases, the method of [10] can be directly applied. We consider the case where the projection on E^1 of the coefficient of ε in the expansion of $F(t, u, \varepsilon)$ about $\varepsilon = 0$ is nonzero. As a result, the coefficients in the expansion of the outer solution are determined by successive algebraic equations.

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[1] A linear operator is called stable if its spectrum lies strictly in the negative half-plane.

Consequently, two initial layer functions are needed to satisfy the initial condition in E^1 and in its complement in E. A crucial part of our work is showing that both initial layers satisfy the "matching condition". The coefficients in the expansions of the outer and initial layer functions are derived in detail.

A previous study of this problem, where $\zeta(\varepsilon)=0$, was made by Trenogin [15]. The determination in [15] of the asymptotic scale for the expansion is generally incorrect, as shown in [10], and we use the methods of the latter paper. Theoretically, our method can be extended to cases where zero is a multiple eigenvalue of B, but the system of equations that arises for the coefficients of the outer solution expansion may be impossible to solve. The applicability of initial value problems such as (1) to the solution of partial differential equations is demonstrated in [2], where the Benard problem is solved by the method of matched asymptotic expansions.

Conditions on (1) for the use of the matched asymptotic series method are given in § 2. The form of the solution is derived, and Theorem 1, which states the main result, as well as Theorem 2, which extends the result to the semi-infinite interval, are given in § 3. The coefficients in the expansions of the solution are derived in § 4, and the theorems are proved in § 5.

2. Restrictions on the problem. The following restrictions apply to the operators $B(t,\varepsilon)$ for $0\leqq t\leqq T$ and small $\varepsilon>0$:

I. The operators $B(t,\varepsilon)$ are closed, possibly unbounded, linear operators in E, with a common domain of definition $D(B)$ which is dense in E and independent of (t,ε).

II. $B(t,\varepsilon)$ and its adjoint $B^+(t,\varepsilon)$ have one-dimensional null spaces, spanned by ϕ and ϕ^+, respectively. We take the normalization $(\phi^+,\phi)=1$, where the notation (ϕ^+,ϕ) denotes the action of $\phi^+\in E^*$, the dual of E, on ϕ. Here, ϕ and ϕ^+ are independent of (t,ε).

We define operators P and Q in E by $Pu=(\phi^+,u)(\phi^+,u)\phi$ and $Qu=(I-P)u$. Then any element $u\in E$ may be written uniquely as $u=c\phi+w$, where $c\phi=Pu$ and $w=Qu$.

III. Let $\tilde{B}(t,\varepsilon)=QB(t,\varepsilon)Q$. Then $\tilde{B}(t,\varepsilon)$ is a closed, invertible operator acting from $QD(B)$ into QE. We assume that the resolvent of $\tilde{B}(t,\varepsilon)$, $R(\tilde{B}(t,\varepsilon),\lambda)$ exists and is a bounded operator for $\operatorname{Re}\lambda\geqq O$. Also, for some $c_o>0$, which is independent of (t,ε), and $\operatorname{Re}\lambda\geqq 0$,

$$\|R(\tilde{B}(t,\varepsilon),\lambda)\|\leqq c_0(1+\lambda)^{-1}.$$

The work of Yosida [17] shows that conditions I and III are sufficient to ensure that for each (t,ε), $\tilde{B}(t,\varepsilon)$ is the infinitesimal generator of a strongly continuous semigroup of operators, $e^{\tau\tilde{B}(t,\varepsilon)}$ for $\tau\geqq 0$. Furthermore, Hoppensteadt [4] showed that there are positive constants c_1, ν independent of (t,ε), such that

$$\|e^{\tau\tilde{B}(t,\varepsilon)}\|\leqq c_1e^{-\nu\tau}\quad\text{for }\tau\geqq 0. \tag{2}$$

We assume the following condition on the operator F:

IV. F is analytic in a sense that is subordinate to a fractional power of $\tilde{B}(t,\varepsilon)$.

In particular, for some α, $0 \leqq \alpha < 1$, and any $u \in E$,

$$F(t, Pu + \tilde{B}^{-\alpha}(t, \varepsilon)Qu, \varepsilon) = \sum_{i,j=0}^{\infty} F_{ij}(t, c, Qu)\varepsilon^j,$$

where F_{ij} is an i-linear operator in c and Qu. We assume that $F_{00} = F_{10} = 0$. Consequently, for $u \in D(\tilde{B}^{\alpha}(t, \varepsilon))$, we have that for small $\varepsilon > 0$, $F(t, u, \varepsilon) = (\|u\| + \|\tilde{B}^{\alpha}(t, \varepsilon)u\|)$ uniformly in ε as $(\|u\| + \tilde{B}^{\alpha}(t, \varepsilon)u\|) \to 0$. The properties of fractional powers are discussed in detail by Sobolevskii [14].

Finally, we assume that the problem is well posed. Sufficient conditions to ensure that the following assumption holds are given by Sobolevskii [14]:

V. There exists a function $U(t, s, \varepsilon)$, defined and strongly continuous for $0 \leqq s \leqq t \leqq T$ and $\varepsilon > 0$, uniformly differentiable in t for $t > s$, and satisfying

$$\frac{\partial}{\partial t} U(t, s, \varepsilon) + \frac{1}{\varepsilon} B(t, \varepsilon) U(t, s, \varepsilon) = 0.$$

Furthermore, (1) is equivalent to the integral equation

$$u(t, \varepsilon) = U(t, 0, \varepsilon)\zeta(\varepsilon) + \frac{1}{\varepsilon} \int_0^t U(t, s, \varepsilon) F(s, u(s, \varepsilon), \varepsilon)\, ds,$$

which has a smooth, unique solution on some interval $0 \leqq t \leqq T_0$ for each $\varepsilon > 0$.

3. The matched asymptotic series method. We now proceed to determine the scale for the asymptotic expansion of the solution in terms of ε. To do this, we rewrite problem (1) as

$$\varepsilon \frac{dc}{dt} = \hat{P}F(t, c\phi + w, \varepsilon),$$

(3)

$$\varepsilon \frac{dw}{dt} = \tilde{B}w + QF(t, c\phi + w, \varepsilon),$$

with the initial condition

(4) $$c(0, \varepsilon) = \hat{P}\zeta(\varepsilon), \qquad w(0, \varepsilon) = Q\zeta(\varepsilon),$$

where $\hat{P}$ is defined by $\hat{P}u = (\phi^+, u)$ for $u \in E$,

We let $v = \tilde{B}w$ in the steady state problem of (3), which gives the system of equations

(5) $$0 = \hat{P}F(t, c\phi + \tilde{B}^{-1}v, \varepsilon),$$

(6) $$0 = v + QF(t, c\phi + \tilde{B}^{-1}v, \varepsilon).$$

Since $QF(t, 0, 0) = 0$, and we have assumed the smoothness of $QF(t, c\phi + B^{-1}v, \varepsilon)$, we may apply the implicit function theorem to (6) to show the existence of a unique solution $v(c, \varepsilon)$ which has the expansion

$$v(c, \varepsilon) = \sum_{i,j=0}^{\infty} V_{ij} c^i \varepsilon^j,$$

where $V_{00}=0$. We substitute this expansion and the one for F into (6) and solve for the coefficients in the expansion of $v(c, \varepsilon)$. In particular, we find

$$V_{10}=0, \quad V_{01}=-QF_{01}, \quad V_{20}=-QF_{20}(\phi, \phi),$$
$$V_{11}=-QF_{11}(\phi)-QF_{20}(\phi, V_{01})-QF_{20}(V_{01}, \phi),$$
$$V_{02}=-QF_{02}-QF_{20}(V_{01}, V_{01})-QF_{11}(V_{01}), \cdots.$$

Next, we substitute the expansion of $v(c, \varepsilon)$ into (5), and by re-expanding, we get

$$\begin{aligned} 0 &= \hat{P}F_{01}\varepsilon+\hat{P}F_{20}(\phi, \phi)c^2+\hat{P}F_{11}(\phi)c\varepsilon+(\hat{P}F_{20}(V_{01}, V_{01})+\hat{P}F_{11}(V_{01}) \\ &\qquad +\hat{P}F_{02})\varepsilon^2+\cdots \\ &= \sum_{i,j=0} G_{ij}c^i\varepsilon^j, \end{aligned} \tag{7}$$

where $G_{00}=0$, $G_{01}=\hat{P}F_{01}$, $G_{20}=\hat{P}F_{20}(\phi, \phi)$, $\cdots$. We shall restrict our attention to a particular case. We assume:

VIa. $G_{i0}=0$ for $i<k$, $G_{k0}\neq 0$,

VIb. $G_{01}\neq 0$.

We use the Newton polygon method [10] to determine the asymptotic scale for an expansion of a solution $c=c(\varepsilon)$ of (7) which satisfies $c(0)=0$. Thus we find that the correct scale is $\varepsilon^{1/k}$.

In many applications, the condition VIa is satisfied, but $G_{01}=0$, and $G_{11}\neq 0$. Such cases can be treated by the method of matched asymptotic expansions described in [10], as demonstrated for the Benard problem in [2]. On the other hand, the analysis of the present case proceeds as follows. We let

$$c=\varepsilon^{1/k}x, \qquad w=\tilde{B}^{-1}v(\varepsilon^{1/k}x, \varepsilon)+\varepsilon y.$$

The problem (3), (4) becomes

$$\begin{aligned} \varepsilon^{1/k}\frac{dx}{dt} &= f(t, x, y, \varepsilon), \\ \varepsilon\frac{dy}{dt} &= \tilde{B}y+g(t, x, y, \varepsilon), \end{aligned} \tag{8}$$

$$\begin{aligned} x(0, \varepsilon) &= \xi(\varepsilon), \\ y(0, \varepsilon) &= \eta(\varepsilon), \end{aligned} \tag{9}$$

where

$$\begin{aligned} f(t, x, y, \varepsilon) &= \varepsilon^{-1}[\hat{P}F(t, \varepsilon^{1/k}x\phi+\tilde{B}^{-1}v(\varepsilon^{1/k}x, \varepsilon)+\varepsilon y, \varepsilon)] \\ &= G_{k0}x_0^k+G_{01}+O(\varepsilon^{1/k}), \\ g(t, x, y, \varepsilon) &= \varepsilon^{-1}[QF(t, \varepsilon^{1/k}x\phi+\tilde{B}^{-1}v(\varepsilon^{1/k}x, \varepsilon)+\varepsilon y, \varepsilon) \\ &\qquad -QF(t, \varepsilon^{1/k}x\phi+\tilde{B}^{-1}v(\varepsilon^{1/k}x, \varepsilon), \varepsilon)] \\ &\qquad -\tilde{B}^{-1}v_x(\varepsilon^{1/k}x, \varepsilon)\hat{P}F(t, \varepsilon^{1/k}x\phi+\tilde{B}^{-1}v(\varepsilon^{1/k}x, \varepsilon)+\varepsilon y, \varepsilon), \end{aligned}$$

$$\xi(\varepsilon)=\varepsilon^{-1/k}\hat{P}\zeta(\varepsilon),$$
$$\eta(\varepsilon)=\varepsilon^{-1}[Q\zeta(\varepsilon)-\tilde{B}^{-1}v(c(0, \varepsilon), \varepsilon)].$$

Here, $f(t, x, y, \varepsilon) = O(1)$, and $g(t, x, y, \varepsilon) = O(\varepsilon^{1/k})$, as $\varepsilon \to 0$. From the expressions for $\xi(\varepsilon)$ and $\eta(\varepsilon)$, we require that:

VII. $P\zeta(\varepsilon) = O(\varepsilon^{1/k})$, $Q\zeta(\varepsilon) = O(\varepsilon)$.

Problem (8) is a two-parameter system of equations in $\varepsilon^{1/k}$ and ε. Consequently, two initial layers, which are functions of $\theta = t/\varepsilon^{1/k}$ and $\tau = t/\varepsilon$, respectively, will be needed to satisfy the initial conditions. The form of the solution is given by the following theorem.

THEOREM 1. *Let hypotheses* I–VII *hold. Also, we assume the following*:

VIII$_O$. *If k is odd, let $G_{k0} < 0$.*

VIII$_E$. *If k is even, let $G_{01}G_{k0} < 0$, and if $\xi(0) \neq 0$, let* $\operatorname{sgn} \xi(0) = -\operatorname{sgn} G_{k0}$. *Then for each small $\varepsilon > 0$, the problem* (3), (4) *has a unique solution $c(t, \varepsilon)$, $w(t, \varepsilon)$ for $0 \leqq t \leqq T$. Moreover, the solution is of the form*

$$c(t, \varepsilon) = \varepsilon^{1/k}[x^*(t, \varepsilon) + \bar{x}(\theta, \varepsilon) + X(\tau, \varepsilon)],$$

$$w(t, \varepsilon) = \varepsilon[y^*(t, \varepsilon) + \bar{y}(\theta, \varepsilon) + Y(\tau, \varepsilon)] + \tilde{B}^{-1}v(c, \varepsilon),$$

where $\theta = t/\varepsilon^{1/k}$ and $\tau = t/\varepsilon$. Here, $x^*(t, \varepsilon)$, $y^*(t, \varepsilon)$ is a solution of the problem (8) *that is smooth at $\varepsilon = 0$; $\bar{x}(\theta, \varepsilon)$, $\bar{y}(\theta, \varepsilon)$ is a solution of the problem*

$$\frac{d\bar{x}}{d\theta} = f(\varepsilon^{1/k}\theta, x^* + \bar{x}, y^* + \bar{y}, \varepsilon) - f(\varepsilon^{1/k}\theta, x^*, y^*, \varepsilon),$$

$$\varepsilon^{k-1/k}\frac{d\bar{y}}{d\theta} = \tilde{B}y + g(\varepsilon^{1/k}\theta, x^* + \bar{x}, y^* + \bar{y}, \varepsilon) - g(\varepsilon^{1/k}\theta, x^*, y^*, \varepsilon),$$

which satisfies the initial condition $\bar{x}(0, \varepsilon) = \xi(\varepsilon) - x^*(0, \varepsilon) - X(0, \varepsilon)$ and the "matching conditions" $X(\infty, \varepsilon) = |Y(\infty, \varepsilon)|_\alpha = 0$, where the norm $|\cdot|_\alpha$ will be the problem

$$\frac{dX}{d\tau} = \varepsilon^{k-1/k}[f(\varepsilon\tau, x^* + \bar{x} + X, y^* + \bar{y} + Y, \varepsilon) - f(\varepsilon\tau, x^* + \bar{x}, y^* + \bar{y}, \varepsilon)],$$

$$\frac{dY}{d\tau} = \tilde{B}y + g(\varepsilon\tau, x^* + \bar{x} + X, y^* + \bar{y} + Y, \varepsilon) - g(\varepsilon\tau, x^* + \bar{x}, y^* + \bar{y}, \varepsilon),$$

which satisfies the initial condition $Y(0, \varepsilon) = \eta(\varepsilon) - y^*(0, \varepsilon) - \bar{y}(0, \varepsilon)$ and the "matching conditions" $X(\infty, \varepsilon) = |Y(\infty, \varepsilon)|_\alpha = 0$, where the norm $|\cdot|_\alpha$ will be defined in § 5.

These functions have Taylor expansions

$$x^*(t, \varepsilon) = \sum_{i=0}^{N} x_i^*(t)\varepsilon^{i/k} + O(\varepsilon^{N+1/k}),$$

$$y^*(t, \varepsilon) = \sum_{i=0}^{N} y_i^*(t)\varepsilon^{i/k} + O(\varepsilon^{N+1/k}),$$

and similarly for x, $\bar{y}$, X, Y. The coefficients of these expansions are determined successively. The error estimates hold uniformly in t for $0 \leqq t \leqq T$.

The following theorem extends the results of Theorem 1 to problem (1) on $[0, \infty)$.

THEOREM 2. *Let the hypotheses of Theorem* 1 *hold for $T = \infty$. Then the results of Theorem* 1 *hold for $T = \infty$.*

4. Determination of the coefficients. We now proceed to determine the coefficients in the expansion of the outer and initial layer functions described in Theorem 1.

The lowest order terms in the expansion of the outer solution satisfy the equations

$$G_{k0}x_0^{*k}+G_{01}=0,$$
$$\tilde{B}y_0^*=0.$$

Thus

$$x_0^*=\left(-\frac{G_{01}}{G_{k0}}\right)^{1/k},$$
$$y_o^*=0.$$

By hypothesis $\mathrm{VIII_E}$, $G_{01}G_{k0}<0$. Thus there is a bifurcation of two formal real outer solutions when k is even, but one formal real outer solution when k is odd.

Higher order coefficients are determined successively by the equations

$$kG_{k0}x_0^{*k-1}x_i^*+f_i^*(t, x_0^*, \cdots, x_{i-1}^*, y_0^*, \cdots, y_{i-k-1}^*)=0,$$
$$\tilde{B}y_i^*+g_i^*(t, x_0^*, \cdots, x_{i-1}^*, y_0^*, \cdots, y_{i-1}^*)=0,$$

where f_i^* and g_i^* are known functions.

The equations for $\bar{x}_0$ and $\bar{y}_0$ are

$$\begin{aligned}\frac{d\bar{x}_0}{d\theta}&=G_{k0}(x_0^*(0)+\bar{x}_0)^k-G_{k0}x_0^{*k}(0)\\ &=kG_{k0}x_0^*(0)^{k-1}\bar{x}_0+G_{k0}\sum_{i=2}^{k}\binom{k}{i}x_0^*(0)^{k-i}\bar{x}_0^i,\\ \tilde{B}y_0&=0.\end{aligned}\tag{10}$$

The initial condition is

$$\bar{x}_0(0)=\xi(0)-x_0^*(0).\tag{11}$$

To ensure that $kG_{k0}x_0^*(0)^{k-1}<0$, we require (hypothesis $\mathrm{VIII_O}$) that $G_{k0}<0$ if k is odd, and $\operatorname{sgn} x_0^*(0)=-\operatorname{sgn} G_{k0}$ if k is even. The latter condition serves to select the stable solution for $x_0^*(t)$ when k is even. We need the following lemma to show that $\bar{x}_0(\theta)$ satisfies the matching condition.

LEMMA 1. *Let k be a positive integer, $k\geqq 2$. Also, let*

$$p(\lambda)=k+\sum_{i=2}^{k}\binom{k}{i}\lambda^{i-1}.$$

Then there is a positive number α such that $p(\lambda)\geqq\alpha$ for all λ if k is odd, and for $\lambda\geqq -1$ if k is even.

We shall use Lemma 1 to prove the following result.

LEMMA 2. *Let hypotheses $\mathrm{VIII_O}$ and $\mathrm{VIII_E}$ of the theorem hold, and let $\bar{x}_0(\theta)$ be a solution of* (10), (11). *Then for some positive number β, $\bar{x}_0(\theta)=O(e^{-\beta\theta})$, as $\theta\to\infty$.*

These lemmas are proved in § 5.

Higher order coefficients satisfy the successive equations

$$\frac{d\bar{x}_i}{d\theta}=kG_{k0}(x_0^*(0)+\bar{x}_0)^{k-1}\bar{x}_i+\bar{f}_i(\bar{x}_0, \cdots, \bar{x}_{i-1}, \bar{y}_0, \cdots, \bar{y}_{i-k-1}),$$
$$\tilde{B}\bar{y}_i+\bar{g}_i(\bar{x}_0, \cdots, \bar{x}_{i-1}, \bar{y}_0, \cdots, \bar{y}_{i-k-1})=0,$$

with the initial condition

$$\bar{x}_i(0)=\frac{1}{\varepsilon^{i/k}}\xi_i(\varepsilon)|_{\varepsilon=0}-x_i^*(0)-X_i(0),$$

where $\bar{f}_i$ is a known polynomial which decays exponentially as $\theta\to\infty$, and $\bar{g}_i$ is a known operator. It follows from Lemma 2 that there exist positive integers γ, θ_1 such that $G_{k0}[x_0^*(0)+\bar{x}_0]^{k-1}<-\gamma$ for $\theta>\theta_1$. Therefo· ˎ $\bar{x}_i(\theta)$ and $\bar{y}_i(\theta)$ decay exponentially as $\theta\to\infty$.

The lowest order terms in the expansion of the second boundary layer, X_0 and Y_0, satisfy the equations

$$\frac{dX_0}{d\tau}=0,\qquad \frac{dY_0}{d\tau}=\tilde{B}Y_0.$$

The initial condition is

$$Y_0(0)=\eta(0).$$

Thus

$$X_0(\tau)=0,\qquad Y_0(\tau)=e^{\tau B(\varepsilon\tau,\varepsilon)}\eta(0).$$

We use hypothesis IV to derive the estimate $Y_0(\tau)=O(e^{-\nu\tau})$, for some positive ν.

Higher order terms are derived from the successive equations

$$\frac{dX_i}{d\tau}=\hat{f}_i(X_0,\cdots,X_{i-k},Y_0,\cdots,Y_{i-2k}),$$

$$\frac{dY_i}{d\tau}=\tilde{B}Y_i+\hat{g}_i(X_0,\cdots,X_{i-1},Y_0,\cdots,Y_{i-1}),$$

with the initial condition

$$Y_i(0)=\frac{1}{\varepsilon^{i/k}}\eta(\varepsilon)|_{\varepsilon=0}-y_i^*(0)-\bar{y}_i(0),$$

where $\hat{f}_i$ is a known polynomial which decays exponentially as $\tau\to\infty$, and $\hat{g}_i$ is a known operator. To satisfy the matching condition, we take $X_i(\tau)=-\int_\tau^\infty\hat{f}_i(X_0(s),\cdots,X_{i-k}(s),Y_0(s),\cdots,Y_{i-2k}(s))\,ds$. Since $\hat{f}_i$ and $\hat{g}_i$ may contain unbounded operators, a stronger norm is needed for the derivation of decay estimates. For $v\in D(\tilde{B}^\alpha)$, we define $|v|_\alpha=\|v\|+\|\tilde{B}^\alpha v\|$. By hypothesis III, the conditions of the following lemma, which is proved in [10], are satisfied. The proof given in [8] may now be applied to show that X and Y satisfy the matching condition.

LEMMA 3. *Let hypothesis* III *hold, and suppose that, for some* $0\leqq\alpha<1$, $G(v,\varepsilon)=O(|v|_\alpha)$ *as* $|v|_\alpha\to0$ *uniformly in* ε *near zero, where* G *is some operator in* E. *If* $v(t,\varepsilon)$ *is a continuously differentiable function of* t *which satisfies* $dv/dt=\tilde{B}v+G(v,\varepsilon)$ *for* $t\leqq0$ *with* $v(0,\varepsilon)\in D(\tilde{B}^\alpha)$, *then there are positive constants* C, δ *which are independent of* ε, *such that* $|v|_\alpha\leqq Ce^{-\delta t}$ *for* $t\leqq0$, *provided* $|v(0)|_\alpha$ *is sufficiently small.*

The construction in [9] is extended to the present case by using the norm $|\cdot|_\alpha$. Thus the results of Lemma 3 are shown to apply for all initial data which lie in the domain of attraction of the steady state $v(t)=0$. By hypothesis VII and Lemma 2, we have that the initial data in our problem satisfy that requirement.

5. Proof of the theorems. In this section, we prove the theorems and the lemmas given in the previous section.

Proof of Lemma 1. Since p is a polynomial, it suffices to show that $p(\lambda)>0$ for the given ranges of λ. This is obvious for $\lambda \geqq 0$ and follows, for $\lambda<0$, from

$$p(\lambda)=-\frac{k}{\lambda}\int_{\lambda}^{0}(1+\mu)^{k-1}\,d\mu.$$

Proof of Lemma 2. We rewrite problem (10) in the form

$$\frac{d\bar{x}_0}{d\theta}=G_{k0}x_0^*(0)^{k-1}p\left[\frac{\bar{x}_0(\theta)}{x_0^*(0)}\right]\bar{x}_0(\theta). \tag{12}$$

If k is odd, then by Lemma 1,

$$p\left[\frac{\bar{x}_0(\theta)}{x_0^*(0)}\right]>\alpha$$

for some $\alpha>0$ and $0\leqq\theta<\infty$. Also, by hypothesis $\mathrm{VIII}_{\mathrm{O}}$, $G_{k0}<0$. Therefore, for some $\beta>0$,

$$G_{k0}x_0^*(0)^{k-1}p\left[\frac{x_0(\theta)}{x_0^*(0)}\right]<-\beta \quad \text{for } 0\leqq\theta<\infty.$$

It follows that $\bar{x}_1(\theta)=O(e^{-\beta\theta})$ as $\theta\to\infty$. If k is even, and $G_{k0}>0$, then by hypothesis $\mathrm{VIII}_{\mathrm{E}}$, we have $\xi(0)\leqq 0$. Also, we chose $x_0^*(t)$ so that $x_0^*(0)<0$. From the initial condition (11), we see that

$$\frac{\bar{x}_0(0)}{x_0^*(0)}\geqq -1.$$

By Lemma 1, we have

$$p\left[\frac{\bar{x}_0(0)}{x_0^*(0)}\right]\geqq\alpha \quad \text{for some } \alpha>0.$$

Therefore

$$G_{k0}x_0^*(0)^{k-1}p\left[\frac{\bar{x}_0(\theta)}{x_0^*(0)}\right]<0 \quad \text{at } \theta=0.$$

Since these are continuous functions, this inequality holds in a neighborhood of $\theta=0$. Equation (12) implies that $|\bar{x}_0(\theta)|$ decreases in this neighborhood, so

$$\frac{\bar{x}_0(\theta)}{x_0^*(0)}\geqq -1$$

in the closure of the neighborhood. A continuation of this argument shows that

$$\frac{\bar{x}_0(\theta)}{x_0^*(0)}\geqq 1 \quad \text{for } 0\leqq\theta<\infty.$$

Using Lemma 1, we have that

$$G_{k0}x_o^*(o)^{k-1}p\left[\frac{\bar{x}_0(\theta)}{x_0^*(0)}\right] < -\beta \quad \text{for some } \beta > 0 \quad \text{and } 0 \leqq \theta < \infty.$$

Therefore,

$$\bar{x}_0(\theta) = O(e^{-\beta\theta}) \quad \text{as } \theta \to \infty.$$

In the case where k is even and $G_{k0} < 0$, the same reasoning is used to prove the lemma.

Proof of Theorem 1. To prove the theorem, it remains to be seen that the expansions of the outer and initial layer solutions are asymptotically correct. To do this, we develop these functions in Taylor expansions to order N/k. Then, we derive a system of equations for the remainders divided by $\varepsilon^{N+1/k}$. The systems are of the same form as those for the original functions. As described in detail in [1], we can set up successive approximations to the solutions which are of order $O(\varepsilon^{1/k})$. The method shown in [6] can be used to show that the successive approximations converge uniformly to solutions which satisfy the order relation uniformly in t for $0 \leqq t \leqq T$. This concludes the proof of Theorem 1.

Proof of Theorem 2. To extend the results of Theorem 1 to the case $T = \infty$, it is necessary to demonstrate the stability of the outer solution. It has been seen that the outer solution is determined by algebraic equations, whose solutions depend on G_{01} and G_{k0}. Therefore the hypotheses of the theorem, together with the results of Theorem 1, suffice to prove the stability of the outer solution.

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SELF-ADJOINT DIFFERENTIAL EXPRESSIONS IN TWO VARIABLES*

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Abstract. Self-adjoint partial differential expressions in two real variables are defined and studied. The most general self-adjoint partial differential expression in two real variables is obtained.

In [3] H. L. Krall studied the linear differential operator

$$L(y)=Ay^{(2n)}+\sum_{i=0}^{n-1}\binom{2n}{2i+1}c_{2n-2i-1}A^{(2n-2i-1)}y^{(2i+1)},$$

where $c_{2r-1}=r^{-1}(2^{2r}-1)B_{2r}$, B_2, B_4, $\cdots$, are the Bernoulli numbers $Z^{(k)}=d^kZ/dx^k$, and A is a function of x alone. He proved that L is self-adjoint, and that the most general operator satisfying $L(y)=M(y)$ (M the adjoint of L) is of the form

$$L(y)=\sum_{s=0}^{n}\sum_{k=0}^{2s}(-1)^{k+1}\binom{2s}{k}\frac{2^{2s-k+1}-1}{2s-k+1}2B_{2s-k+1}A_s^{(2s-k)}y^{(k)}.$$

Using the definition of self-adjoint given in [1], no self-adjoint differential operators of odd order exist. In [2], A. M. Krall extended the definition of self-adjoint by defining the nth order differential operator L to be self-adjoint if $L(y)=(-1)^nM(y)$. With this definition, self-adjoint operators of odd orders now exist. In particular, it was shown in [2] that the linear differential operator

$$L(y)=Ay^{(2n+1)}+\sum_{j=0}^{n}\binom{2n+1}{2j}c_{2n-2j+1}A^{(2n-2j+1)}y^{(2j)},$$

where $c_{2r-1}=r^{-1}(2^{2r}-1)B_{2r}$, and B_2, B_4, $\cdots$ are the Bernoulli numbers, is self-adjoint. Furthermore, the most general differential operator satisfying $L(y)=-M(y)$ is

$$L(y)=\sum_{s=0}^{n}\sum_{k=0}^{2s}(-1)^{k}\binom{2s+1}{k}\frac{2^{2s-k+2}-1}{2s-k+2}2B_{2s-k+2}A_s^{(2s-k+1)}y^{(k)}.$$

In [1], L. Carlitz explained the surprising presence of the Bernoulli numbers in the above formulas.

The purpose of this paper is to extend these ideas to linear partial differential operators in two variables, x and y. To simplify notation, we write D_x for $\partial/\partial x$ and D_y for $\partial/\partial y$. The adjoint of the partial differential expression

$$L(u)=\sum_{k=0}^{n}\sum_{i=0}^{k}A_{k,i}D_x^{(k-i)}D_y^{(i)}(u)$$

is the partial differential expression

$$M(u)=\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^kD_x^{(k-i)}D_y^{(i)}(A_{k,i}u),$$

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where $A_{k,i}$ are differentiable functions of x and y of class C^n. L is self-adjoint if and only if $L(u)=(-1)^n M(u)$, where n is the order of L. The following properties of partial differential expressions are easily established:

(i) There are no odd (even) ordered partial differential expressions satisfying $L(u)=M(u)$ ($L(u)=-M(u)$, respectively).

(ii) The sum (or difference) of self-adjoint partial differential expressions all of even order (or all of odd order) is self-adjoint.

(iii) For a given set $\{A_{2n,i}(x,y)\}_{i=0}^{2n}$ of functions (differentiable of class C^{2n}) there is at most one self-adjoint expression of the form

$$L(u)=\sum_{i=0}^{2n} A_{2n,i}D_x^{(2n-i)}D_y^{(i)}(u)+\sum_{k=1}^{n}\sum_{i=0}^{2k-1} A_{2k-1,i}D_x^{(2k-i-1)}D_y^{(i)}(u). \tag{1}$$

A similar statement holds for odd ordered partial differential expressions.

THEOREM 1. *The even ordered partial differential expression*

$$L_{2n}(u)=\sum_{i=0}^{2n}\binom{2n}{i}A_iD_x^{(2n-i)}D_y^{(i)}(u)+\Bigg[\sum_{i=0}^{n-1}\sum_{j=0}^{2i+1}\sum_{k=0}^{2n-2i-1}\binom{2n}{2i+1}\binom{2i+1}{j}\cdot\binom{2n-2i-1}{k}c_{2n-2i-1}(D_x^{(2n-2i-k-1)}D_y^{(k)}A_{k+j})(D_x^{(2i-j+1)}D_y^{(j)}(u))\Bigg],$$

where $c_{2r-1}=r^{-1}(2^{2r}-1)B_{2r}$ and the B_{2r}'s are Bernoulli numbers, is self-adjoint. (The A_i's are functions of x and y and are differential of class C^{2n}.)

Proof. The adjoint of $L_{2n}(u)$ is

$$M_{2n}(u)=\sum_{i=0}^{2n}\binom{2n}{i}[D_x^{(2n-i)}D_y^{(i)}(A_iu)]-\sum_{i=0}^{n-1}\sum_{j=0}^{2i+1}\sum_{k=0}^{2n-2i-1}\binom{2n}{2i+1}\cdot\binom{2i+1}{j}\binom{2n+2i-1}{k}c_{2n-2i-1}\{D_x^{(2i-j+1)}D_y^{(j)}[(D_x^{(2n-2i-k-1)}D_y^{(k)}A_{k+j})u]\}.$$

Upon differentiating, we have

$$M_{2n}(u)=\sum_{i=0}^{2n}\sum_{r=0}^{2n-i}\sum_{q=0}^{i}\binom{2n}{i}\binom{2n-i}{r}\binom{i}{q}(D_x^{(2n-i-r)}D_y^{(r)}A_i)(D_x^{(r)}D_y^{(q)}u)$$
$$-\sum_{i=0}^{n-1}\sum_{j=0}^{2i+1}\sum_{k=0}^{2n-2i-1}\sum_{r=0}^{2i+j+1}\sum_{q=0}^{j}\binom{2n}{2i+1}\binom{2i+1}{j}\binom{2n-2i-1}{k}$$
$$\cdot\binom{2i+1-j}{r}\binom{j}{q}c_{2n-2i-1}(D_x^{(2n-j-k-r)}D_y^{(k+j-q)}A_{k+j})(D_x^{(r)}D_y^{(q)}u).$$

In order to simplify the limits of summation, note that these sums run over all values for which the binomial coefficients are defined. Hence we will make no restriction if we define $\binom{N}{N+k}=0$ and $\binom{N}{-k}=0$ for $k>0$ and change the limits of all the sums from 0 to ∞. Using formulas involving the binomial coefficients, and setting $s=n-i$ and $w=j+k$ in the second term, we have

$$M_{2n}(u)=\sum_i\sum_r\sum_q\binom{2n}{r}\binom{2n-r}{q}\binom{2n-r-q}{i-q}(D_x^{(2n-i-r)}D_y^{(i-q)}A_i)(D_x^{(r)}D_y^{(q)}u)$$
$$-\sum_s\sum_j\sum_w\sum_r\sum_q\binom{2n}{r}\binom{2n-r}{q}\binom{2n-r-q}{w-q}\binom{w-q}{w-j}\binom{2n-r-w}{2s-1-w+j}$$
$$\cdot\, c_{2s-1}(D_x^{(2n-w-r)}D_y^{(w-q)}A_w)(D_x^{(r)}D_y^{(q)}u).$$

(Here all sums are from 0 to ∞.)

Consider the second term of $M_{2n}(u)$. The sum over j alone is

$$\sum_{j=0}^{\infty}\binom{w-q}{w-j}\binom{2n-w-r}{2s-1-w+j}=\sum_{j=q}^{w}\binom{w-q}{w-j}\binom{2n-w-r}{2s-1-w+j}=\binom{2n-r-q}{2s-1}.$$

Combining the two equations in Theorem 1 of [3], we have

$$\sum_{i=0}^{[(r+1)/2]}\binom{r}{2i-1}c_{2i-1}=\sum_{i=0}^{\infty}\binom{r}{2i-1}c_{2i-1}=\begin{cases}0 & \text{if } r=0,\\ 1 & \text{if } r=2,4,\cdots,\\ 1-c & \text{if } r=1,3,\cdots.\end{cases}$$

Hence the sum over s alone becomes

$$\sum_{s=0}^{\infty}\binom{2n-r-q}{2s-1}c_{2s-1}=P(2n-r-q),$$

where

$$P(2n-r-q)=\begin{cases}0 & \text{if } 2n-r-q=0,\\ 1 & \text{if } 2n-r-q=2,4,\cdots,\\ 1-c_{2n-r-q} & \text{if } 2n-r-q=1,3,\cdots.\end{cases}$$

The second term of $M_{2n}(u)$ now becomes

$$\sum_r\sum_q\sum_w\binom{2n}{r}\binom{2n-r}{q}\binom{2n-r-q}{w-q}(D_x^{(2n-w-r)}D_y^{(w-q)}A_w)(D_x^{(r)}D_y^{(q)}u)P(2n-r-q).$$

Setting $i=w$ in the first term of $M_{2n}(u)$, combining the two terms, replacing q by s using $q=s-r$ and returning to finite sums, we have

$$M_{2n}(u)=\sum_{r=0}^{2n}\sum_{s=r}^{2n}\sum_{w=s-r}^{2n-r}\binom{2n}{r}\binom{2n-r}{s-r}\binom{2n-s}{w-s+r}(D_x^{(2n-w-r)}D_y^{(w-s+r)}A_w)$$
$$\cdot\,(D_x^{(r)}D_y^{(s-r)}u)(1-p(2n-s)).$$

We now use the formula $\sum_{r=0}^{2n}\sum_{s=r}^{2n}=\sum_{s=0}^{2n}\sum_{r=0}^{s}$ and note that

$$1-P(2n-s)=\begin{cases}1 & \text{if } 2n-s=0,\\ 0 & \text{if } 2n-s=2,4,\cdots,\\ c_{2n-s} & \text{if } 2n-s=1,3,\cdots,\end{cases}$$

so that the only nonzero terms of $M_{2n}(u)$ are those terms in which s is odd ($s<2n$) or $s=2n$. By letting $s=2i+1$ and separating out the $s=2n$ term, we have

$$M_{2n}(u)=\sum_{r=0}^{2n}\binom{2n}{r}A_{2n-r}(D_x^{(r)}D_y^{(2n-r)}u)+\sum_{i=0}^{n-1}\sum_{r=0}^{2i+1}\sum_{w=2i+1-r}^{2n-r}\binom{2n}{r}$$
$$\cdot\binom{2n-r}{2n-2i-1}\binom{2n-2i+1}{w+r-2i-1}(D_x^{(2n-w-r)}D_y^{(w+r-2i-1)}A_w)$$
$$\cdot(D_x^{(r)}D_y^{(2i+1-r)}u)c_{2r-2i-1}.$$

$M_{2n}(u)$ now becomes $L_{n2}(u)$ if we let $i=2n-r$ in the first term, and replace r by $2i+1-j$ and set $w=j+k$ in the second term. Hence $L_{2n}(u)$ is self-adjoint.

This theorem, along with property (iii) above, allows us to state that the most general partial differentiable expression in two variables of order $2n$ satisfying $L(u)=M(n)$ is

$$\sum_{r=0}^{n}\sum_{i=0}^{2r}A_{2r,i}(D_x^{(2r-i)}D_y^{(i)}u)=\sum_{r=0}^{n}\sum_{i=0}^{r-1}\sum_{j=0}^{2i+1}\sum_{k=0}^{2r-2i-1}\binom{2r}{2i+1}\binom{2i+1}{j}\binom{2r-2i-1}{k}$$
$$\cdot\frac{2^{2r-2i}-1}{r-i}B_{2r-2i}(D_x^{(2r-2i-1-k)}D_y^{(k)}A_{2r,k+j})(D_x^{(2i+1-j)}D_y^{(j)}u),$$

where the B_{2r-2i} are the Bernoulli numbers, and the $A_{2r,i}$ are functions of x and y which are differentiable of class C^{2r}.

The techniques used to prove Theorem 1 can be employed to prove the following theorem.

THEOREM 2. *The odd ordered partial differential expression*

$$L_{2n+1}(u)=\sum_{i=0}^{2n+1}\binom{2n+1}{i}A_i(D_x^{(2n+1-i)}D_y^{(i)}u)+\sum_{i=0}^{n}\sum_{j=0}^{2i}\sum_{k=0}^{2n-2i+1}\binom{2n+1}{2i}$$
$$\cdot\binom{2i}{j}\binom{2n-2i+1}{k}C_{2n-2i+1}(D_x^{(2n-2i+1-k)}D_y^{(k)}A_{k+j})(D_x^{(2i-j)}D_y^{(j)}u),$$

where $C_{2r-1}=r^{-1}(2^{2r}-1)B_{2r}$ and the B_{2r}'s are Bernoulli numbers, is self-adjoint.

It follows that the most general partial differential expression satisfying $L(u)=-M(u)$ is

$$\sum_{r=0}^{n}\sum_{i=0}^{2r+1}\binom{2n+1}{i}A_{2r+1,i}(D_x^{(2r+1-i)}D_y^{(i)}u)+\sum_{r=0}^{n}\sum_{i=0}^{r}\sum_{j=0}^{2i}\sum_{k=0}^{2n-2i+1}\binom{2r+1}{2i}$$
$$\cdot\binom{2i}{j}\binom{2r-2i+1}{k}\frac{(2^{2r-2i}-1)B_{2r-2i}}{2r-2i}(D_x^{(2r-2i+1-k)}D_y^{(k)}A_{k+j})(D_x^{(2i-j)}D_y^{(j)}u).$$

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GRÜNBAUM'S INEQUALITY FOR BESSEL FUNCTIONS AND ITS EXTENSIONS*

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Abstract. Recently, F. A. Grünbaum found a new kind of inequality for the Bessel functions, namely $1+\mathscr{J}_\nu(a)\geqq\mathscr{J}_\nu(b)+\mathscr{J}_\nu(c)$ $(a^2=b^2+c^2,\ \nu\geqq 0,\ \mathscr{J}_\nu(x)=\Gamma(\nu+1)(2/x)^\nu J_\nu(x))$. Later, another proof was given by R. Askey. In his paper Grünbaum suggested the desirability of finding a proof which, when $\nu=n/2-1$, $n=2,3,\cdots$, made use of the property of these functions of being spherical functions for the corresponding symmetric spaces. Such a proof is given in the present note, and it is found that the method provides an extension of Grünbaum's inequality as well as other inequalities of a similar nature.

In [2], Grünbaum used an inequality which he had proved for Legendre polynomials [1] to obtain the following inequality for Bessel functions.

THEOREM 1. *If $a^2=b^2+c^2$, then*

$$1+J_0(a)\geqq J_0(b)+J_0(c).$$

If we write $\mathscr{J}_\nu(x)=\Gamma(\nu+1)(2/x)^\nu J_\nu(x)$, $\nu\geqq 0$, then by Sonine's first integral we have

$$\mathscr{J}_\nu(x)=2\nu\int_0^{\pi/2}\mathscr{J}_0(x\sin\theta)\sin\theta\cos^{2\nu-1}\theta\,d\theta,\qquad \nu>0,$$

from which we obtain the following corollary of Theorem 1.

THEOREM 2. *If $a^2=b^2+c^2$, then*

$$1+\mathscr{J}_\nu(a)\geqq\mathscr{J}_\nu(b)+\mathscr{J}_\nu(c),\qquad \nu\geqq 0.$$

As we stated previously, Theorem 1 was proved by Grünbaum, who also proved Theorem 2 when $\nu=n/2-1$, $n=2,3,4,\cdots$. Subsequently, Askey [3] gave a direct proof of Theorem 1 and pointed out how Theorem 2 could be deduced from it by using Sonine's result. Grünbaum states that it would be desirable to prove the inequalities corresponding to $\nu=n/2-1$, $n=2,3,4,\cdots$, by a method which made use of the property of these functions as spherical functions for the corresponding symmetric spaces. Neither his nor Askey's proofs do this, and it is the object of the present note to present such a method of proof. Indeed, the present method yields rather more, giving two-sided inequalities, and may also be used to provide other inequalities of a similar nature.

The proofs all depend on the following integral formula proved in [4]. If $S(1)$ denotes both the $(n-1)$-dimensional manifold $\|x\|=1$ in Euclidean space E^n and its volume, then we have the following lemma.

LEMMA. *If $\sum_{k=1}^n\lambda_k^2=\lambda^2$ and $\nu=n/2-1$, then*

$$\frac{1}{S(1)}\int_{S(1)}\left\{\prod_{k=1}^n\cos\lambda_kx_k\right\}\sigma_1=\mathscr{J}_\nu(\lambda),\qquad n\geqq 2. \tag{1}$$

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Here σ_1 denotes the volume element in the manifold $S(1)$. First, we shall prove the following extension of Theorem 2.

THEOREM 3. *If* $a^2=b^2+c^2$, *then*

$$1+\mathcal{J}_\nu(a)\geqq|\mathcal{J}_\nu(b)+\mathcal{J}_\nu(c)|, \qquad \nu\geqq 0.$$

Proof of Theorem 3. Taking $\lambda_1=b$ and $\lambda_k=0$, $2\leqq k\leqq n$, in (1), we get

$$\frac{1}{S(1)}\int_{S(1)}(\cos bx_1)\sigma_1=\mathcal{J}_\nu(b). \tag{2}$$

Taking $\lambda_1=b$, $\lambda_2=c$ and $\lambda_k=0$, $3\leqq k\leqq n$, and writing a for λ in (1), we get

$$\frac{1}{S(1)}\int_{S(1)}(\cos bx_1\cos cx_2)\sigma_1=\mathcal{J}_\nu(a), \qquad a^2=b^2+c^2. \tag{3}$$

Now if $\varepsilon_1=\pm 1$ and $\varepsilon_2=\pm 1$, we have

$$(1+\varepsilon_1\cos bx_1)(1+\varepsilon_2\cos cx_2)\geqq 0.$$

Multiplying out, integrating over $S(1)$ and dividing by $S(1)$, we obtain, from (2) and (3), and from (2) with b replaced by c, that

$$1+\varepsilon_1\mathcal{J}_\nu(b)+\varepsilon_2\mathcal{J}_\nu(c)+\varepsilon_1\varepsilon_2\mathcal{J}_\nu(a)\geqq 0, \qquad a^2=b^2+c^2.$$

Take $\varepsilon_1=\varepsilon_2=-1$, and then take $\varepsilon_1=\varepsilon_2=+1$, and we get

$$-1-\mathcal{J}_\nu(a)\leqq\mathcal{J}_\nu(b)+\mathcal{J}_\nu(c)\leqq 1+\mathcal{J}_\nu(a), \qquad a^2=b^2+c^2,$$

which is the desired result in the case $\nu=n/2-1$, $n=2,3,\cdots$ The extension to all $\nu\geqq 0$ can now be made from the special case $\nu=0$ by Sonine's integral, as before. This completes the proof of Theorem 3.

If, in the above proof, we had instead taken $\varepsilon_1=1$, $\varepsilon_2=-1$, and then taken $\varepsilon_1=-1$, $\varepsilon_2=+1$ and proceeded in the same way, we would have obtained the following result.

THEOREM 4. *If* $a^2=b^2+c^2$, *then*

$$1-\mathcal{J}_\nu(a)\geqq|\mathcal{J}_\nu(b)-\mathcal{J}_\nu(c)|, \qquad \nu\geqq 0.$$

Clearly, other, similar results can be obtained by changing the number of factors of the type $(1+\varepsilon_1\cos bx_1)$ used in the above analysis. The simplest result uses a single factor and proceeds from the inequality

$$1+\varepsilon\cos bx\geqq 0, \qquad \varepsilon=\pm 1,$$

yielding the trivial result

$$|\mathcal{J}_\nu(b)|\leqq 1.$$

However, if we take, say, three factors, take $n\geqq 3$ in the Lemma and start from

$$(1+\varepsilon_1\cos bx_1)(1+\varepsilon_2\cos cx_2)(1+\varepsilon_3\cos dx_3)\geqq 0, \qquad \varepsilon_k=\pm 1,$$

we obtain altogether four inequalities, one of which, for example, is the following.

THEOREM 5. *If* $a^2=b^2+c^2+d^2$, *then*

$$\begin{aligned}&|\mathcal{J}_\nu(b)+\mathcal{J}_\nu(c)+\mathcal{J}_\nu(d)+\mathcal{J}_\nu(a)|\\&\quad\leqq 1+\mathcal{J}_\nu((c^2+d^2)^{1/2})+\mathcal{J}_\nu((d^2+b^2)^{1/2})+\mathcal{J}_\nu((b^2+c^2)^{1/2}), \qquad \nu\geqq\tfrac{1}{2}.\end{aligned}$$

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ON THE EIGENVALUES OF CERTAIN INTEGRAL OPERATORS*

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Abstract. Let $V(\underline{x})$ be a real function in $L^\infty(\underline{R}_n)$ with bounded support, and let $0 < \alpha < n$. In the early 1950's, M. Kac obtained an asymptotic formula for the eigenvalues of the compact nonnegative operators

$$Sf\cdot(\underline{x}) = \int_{\underline{R}_n} V(\underline{x})|\underline{x} - \underline{y}|^{-\alpha} V(\underline{y}) f(\underline{y})\, d\underline{y}, \qquad f \in L^2[\underline{R}_n],$$

using probabilistic methods. Subsequently, more general results were obtained by various mathematicians. In the present paper a connection is established between such "Kac limit theorems" and "asymptotic distribution theorems" for finite section Toeplitz operators, a relation which we use to obtain new "Kac limit theorems".

Part I. Kac limit theorems for orthogonal polynomials.

1. Introduction. This section is intended to make clear how the ideas of this paper are related to each other. Let $\underline{T}$ be the real numbers modulo 2π and let $\sigma(d\theta) = (1/2\pi)\, d\theta$ be Haar measure on $\underline{T}$ normalized so that $\underline{T}$ has mass 1. Let $V(\theta)$ be a nonnegative function in $L^\infty[\underline{T}]$, and $L(k)$ a nonnegative function on $\underline{Z}$, the integers, vanishing at ∞. We define the operator S on $L^2[\underline{T}]$ by

$$S = M(V)F^*E(L)FM(V), \tag{1.1}$$

where

$$M(V)f\cdot(\theta) = V(\theta)f(\theta), \qquad f(\theta) \in L^2(\underline{T}),$$
$$E(L)g\cdot(k) = L(k)g(k), \qquad g(k) \in L^2(\underline{Z}),$$

and where

$$Ff\cdot(k) = \int_{\underline{T}} f(\theta)\, e^{-ik\theta}\sigma(d\theta),$$
$$F^*g\cdot(\theta) = \sum_{\underline{Z}} g(k)\, e^{ik\theta}.$$

Clearly, S is a completely continuous nonnegative operator on $L^2(\underline{T})$. If $L(k)$ is "nice enough", then it can be shown that "essentially"

$$N^+[\delta, S] \sim |\{\theta, k : V(\theta)^2 L(k) > \delta\}|_{\underline{T}\times\underline{Z}} \quad \text{as } S \to 0+, \tag{1.2}$$

where $N^+[\delta, S]$ is the number of eigenvalues of S greater than δ and $|\{\cdot\}|_{\underline{T}\times\underline{Z}}$ is the measure of the set $\{\cdot\}$ in the product measure space $\underline{T} \times \underline{Z}$ where the measure on $\underline{T}$ is $\sigma(d\theta)$ and the measure on $\underline{Z}$ is the counting measure.

Formulas like (1.2) were first investigated by Kac [6], [7] using probabilistic methods. They were subsequently studied by Rosenblatt [10], [11], Widom [15], [16], and the author [4]. We will be interested in the structure of such formulas which we call "Kac limit theorems". The demonstrations in [4] are based upon the

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following idea. For each $\varepsilon > 0$ let $\pi(\varepsilon)$ be a finite subset of $\underline{Z}$. Let Ω be a measurable subset of $\underline{T}$ and let

$$M(\Omega)f \cdot (\theta) = \chi_\Omega(\theta)f(\theta), \qquad f \in L^2[\underline{T}],$$
$$E(\varepsilon)g \cdot (k) = \chi_{\pi(\varepsilon)}(k)g(k), \qquad g \in L^2[\underline{Z}].$$

We define the family of operators

$$P_\Omega(\varepsilon) = M(\Omega)F^*E(\varepsilon)FM(\Omega), \qquad \varepsilon > 0, \tag{1.3}$$

on $L^2[\underline{T}]$. $P_\Omega(\varepsilon)$ satisfies $0 \leqq P_\Omega(\varepsilon) \leqq I$ and has rank $\pi(\varepsilon)^\#$. (Throughout, $\{\cdot\}^\#$ counts the number of elements of the set $\{\cdot\}$). Let us now assume that if $Q(\varepsilon, m)$ is the number of representations of m in the form $k - j$ where $k, j \in \pi(\varepsilon)$, then

$$\lim_{\varepsilon \to 0+} Q(\varepsilon, m)/\pi(\varepsilon)^\# = 1, \quad \text{all } m \in \underline{Z}. \tag{1.4}$$

Under this assumption, we can show that for any $\delta > 0$,

$$\begin{aligned} N^+[1-\delta, P_\Omega(\varepsilon)] &\sim \sigma(\Omega)\pi(\varepsilon)^\#, \\ N^+[\delta, P_\Omega(\varepsilon)] &\sim \sigma(\Omega)\pi(\varepsilon)^\#, \end{aligned} \qquad \varepsilon \to 0+. \tag{1.5}$$

If $L(k)$ is sufficiently "nice", then we can choose $\pi(\varepsilon)$ in such a way that S can be estimated by suitable linear combinations of $P_\Omega(\varepsilon)$'s, and (1.5) (together with similar related results) can be used to obtain (1.2). Such arguments are very efficient in that they apply in all compact symmetric spaces.

Let us now turn our attention to the Toeplitz operator

$$T_L(\varepsilon) = E(\varepsilon)FM(L)F^*E(\varepsilon).$$

Here $M(L)g \cdot (\theta) = L(\theta)g(\theta)$ and, because $F^*E(\varepsilon)$ maps $L^2[\underline{Z}]$ into $C(\underline{T})$, L may be any real function in $L^1[\underline{T}]$. A generalization due to H. Krieger [8] of a famous result of Szegö asserts that if (1.4) holds, if

$$\sigma\{\theta : L(\theta) = a\} = \sigma\{\theta : L(\theta) = b\} = 0,$$

and if $T_L^{(r)}(\varepsilon)$ is the restriction to $E(\varepsilon)L^2[\underline{Z}^+]$ of $T_L(\varepsilon)$, then as $\varepsilon \to 0+$,

$$N[(a,b); T_L^{(r)}(\varepsilon)]/\pi(\varepsilon)^\# \sim \sigma\{\theta \in \underline{T} : a < L(\theta) < b\}. \tag{1.6}$$

Here $N[(a,b); T_L^{(r)}(\varepsilon)]$ is the number of eigenvalues of $T_L^{(r)}(\varepsilon)$ lying in the interval (a, b). In his thesis [9], D. Liang has recently given a new proof of (1.6) along lines different from Krieger's, but related to the ideas above. Note that in both Krieger's and Liang's work ε can belong to a directed set which need not be $\underline{R}$. We briefly sketch Liang's proof. Let

$$T_\Omega(\varepsilon) = E(\varepsilon)FM(\Omega)F^*E(\varepsilon). \tag{1.7}$$

Liang shows that $0 \leqq T_\Omega(\varepsilon) \leqq I$, and that

$$\begin{aligned} N^+[1-\delta, T_\Omega(\varepsilon)] &\sim \sigma(\Omega)\pi(\varepsilon)^\#, \\ N^+[\delta, T_\Omega(\varepsilon)] &\sim \sigma(\Omega)\pi(\varepsilon)^\#, \end{aligned} \qquad \text{as } \varepsilon \to 0+ \tag{1.8}$$

for each $\delta > 0$, provided that (1.4) holds. Using (1.8) (together with other related results), (1.6) is proved by approximating $T_L(\varepsilon)$ by suitable linear combinations of

$T_\Omega(\varepsilon)$'s. Liang's argument works for all Abelian groups with compactly generated dual, as does Krieger's. In addition, it works for all compact symmetric spaces. Moreover, Liang shows that (1.4) (or its analogues in the other cases considered) is necessary as well as sufficient.

At this point it begins to seem plausible that $P_\varepsilon(\Omega)$ and $T_\varepsilon(\Omega)$ may be unitarily equivalent. We will show that this is true. However, this fact does not in itself advance the theory of either Kac or Szegö limit theorems on groups. This is because (1.5) and (1.8) are equally easy to prove, and because the ways in which they are made to yield (1.6) and (1.2), respectively, are rather different. However, there exist a substantial number of "Szegö limit theorems" in which the functions $\{e^{in\theta}\}_{n\in Z}$ are replaced by one or another orthonormal set. See Grenander and Szegö [2]. Specializing these results to the appropriate analogues of the operators $T_\Omega(\varepsilon)$ and using the unitary equivalence, which persists between the analogues of the $T_\Omega(\varepsilon)$'s and the $P_\Omega(\varepsilon)$'s, we are in a position to use, virtually without change, the arguments of [4] to obtain Kac limit theorems quite unconnected with groups.

2. General orthogonal polynomials. Let $W(d\theta)$ be a finite measure with infinite support on the Borel subsets of $\underline{I} = [0, \pi]$. We suppose thoughout that if

$$W(d\theta) = W_a(\theta)\, d\theta + W_s(d\theta)$$

is the decomposition of $W(d\theta)$ into its absolutely continuous and singular parts with respect to Lebesgue measure, then

$$\int_{\underline{I}} \log W_a(\theta)\, d\theta < -\infty. \tag{2.1}$$

Let $L^2[W]$ be the Hilbert space of those complex Borel measurable functions $f(\theta)$ on $\underline{I}$ for which $\|f\|_W$ is finite, where

$$\|f\|_W = \left\{ \int_{\underline{I}} |f(\theta)|^2 W(d\theta) \right\}^{1/2}.$$

Let $\underline{Z}^+$ be the nonnegative integers and let $\{p(k, \theta)\}_{k\in \underline{Z}^+}$ be the orthonormal polynomials obtained by applying the Gram–Schmidt process to $(\cos\theta)^k$ $k \in \underline{Z}^+$ in $L^2[W]$, where the $p(k, \theta)$ are normalized by the condition that the coefficient of $\cos^k\theta$ in $p(k, \theta)$ is positive. Let $L^2[\underline{Z}^+]$ be the Hilbert space of these complex functions $g(k)$ on $\underline{Z}^+$ for which $\|g\|_{\underline{Z}^+}$ is finite, where

$$\|g\|_{\underline{Z}^+} = \left\{ \sum_{k\in \underline{Z}^+} |g(k)|^2 \right\}^{1/2}.$$

If we set

$$Ff \cdot (k) = \int_{\underline{I}} f(\theta) p(k, \theta) W(d\theta), \qquad f \in L^2[W],$$

and

$$F^*g \cdot (\theta) = \sum_{k\in \underline{Z}^+} g(k) p(k, \theta), \qquad g \in L^2[\underline{Z}^+],$$

then F is a unitary mapping of $L^2[W]$ into $L^2[\underline{Z}^+]$ and F^* is its inverse.

Let $V(\theta)$ be a nonnegative bounded Borel measurable function on $\underline{I}$ and $L(k)$ a positive function on $\underline{Z}^+$ such that $L(k) \to 0$ as $k \to \infty$. Then

$$M(V)f \cdot (\theta) = V(\theta)f(\theta), \qquad f \in L^2[W],$$

is a bounded nonnegative linear transformation on $L^2[W]$, and

$$E(L)g \cdot (k) = L(k)g(k), \qquad g \in L^2[\underline{Z}^+],$$

is a compact nonnegative linear transformation on $L^2[\underline{Z}^+]$. It follows that

$$S = M(V)F^*E(L)FM(V)$$

is a nonnegative compact linear transformation on $L^2[W]$. Let $\{\lambda_S(k)\}_1^\infty, \lambda_S(1) \geqq \lambda_S(2) \geqq \cdots$ be the eigenvalues of S. Our goal is to obtain an asymptotic formula for $\lambda_S(k)$ as $k \to \infty$.

In this section it will be more convenient to replace the parameter $\varepsilon > 0$ by $n = 0, 1, 2, \cdots$ (in which case $n \to \infty$ corresponds to $\varepsilon \to 0+$). For Ω a Borel set in $\underline{I}$, we define $M(\Omega)f \cdot (\theta) = \chi_\Omega(\theta)f \cdot (\theta)$ for $f(\theta) \in L^2[W]$, and we define $E(n)$ by $E(n)g(k) = \chi_{[0,n]}(k)g(k)$ for $g(k) \in L^2[\underline{Z}^+]$. As we remarked in § 1, the arguments necessary to obtain the asymptotic formula just alluded to fall into two parts. Because the second part is well understood and is developed in [4] in exactly the form we need, it is sufficient to carry out the arguments which constitute the first part. In order to make clear what it is we need, we state these results considerably in advance of their demonstration. We set

$$P_\Omega(n) = M(\Omega)F^*E(n)FM(\Omega)$$

and

$$P_{\Omega_1,\Omega_2}(n) = M(\Omega_1)F^*E(n)FM(\Omega_2) + M(\Omega_2)F^*E(n)FM(\Omega_1),$$

where Ω, Ω_1 and Ω_2 are Borel measurable sets in $\underline{I}$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let $\mu(d\theta) = (1/\pi)\, d\theta$ for $\theta \in \underline{I}$. The results we must prove are the following.

THEOREM 2a. *For each δ, $0 < \delta < 1$, we have*

$$\left.\begin{aligned} &N^+[1 - \delta, P_\Omega(n)]/(n + 1) \to \mu(\Omega) \\ &N^+[\delta, P_\Omega(n)]/(n + 1) \to \mu(\Omega) \end{aligned}\right\} \quad \text{as } n \to \infty.$$

THEOREM 2b. *For each δ, $0 < \delta$,*

$$N[\delta, P_{\Omega_1,\Omega_2}(n)]/(n + 1) \to 0 \quad \text{as } n \to \infty.$$

Here $N[\delta, P_{\Omega_1,\Omega_2}(n)]$ is the number of eigenvalues of the self-adjoint operator $P_{\Omega_1,\Omega_2}(n)$ greater than δ in absolute value.

The ease with which we can prove these results is due to the fact that we can bring to bear the following special case of a famous result of Szegö.

THEOREM 2c. *Let* (2.1) *be satisfied. If*

$$T_\Omega(n) = E(n)F^*M(\Omega)FE(n)$$

and if $a < b$ and $a, b \neq 0$ or 1, then

$$\lim_{n \to \infty} N[(a, b); T_\Omega^{(r)}(n)]/(n + 1) = \mu\{\theta : \chi_\Omega(\theta) \in (a, b)\}.$$

Here $N[(a,b);\ T_\Omega^{(r)}(n)]$ is the number of eigenvalues of $T_\Omega^{(r)}(n)$ contained in (a,b), where $T_\Omega^{(r)}(n)$ is the restriction of $T_\Omega(n)$ to its range, $\mathscr{R}[T_\Omega(n)]$. Let us set $\underline{E}(n) = F^*E(n)F$; then $F^*T_\Omega(n)F = \underline{E}(n)M(\Omega)\underline{E}(n)$ and $P_\Omega(n) = M(\Omega)\underline{E}(n)M(\Omega)$. It is simple to verify that $\mathscr{R}[F^*T_\Omega(n)F]$ is the $(n+1)$-dimensional subspace of $L^2[W]$ spanned by the functions $\{p(k,\theta)\}_{k=0}^n$ and that $\mathscr{R}[P_\Omega(n)]$ is the $(n+1)$-dimensional subspace of $L^2[W]$ spanned by the functions $\{\chi_\Omega(\theta)p(k,\theta)\}_{k=0}^n$. Simple computations show that

$$[\chi_{\hat{\Omega}}(k,j)]_{j,k=0}^n, \quad \text{where } \chi_{\hat{\Omega}}(k,j) = \int_{\underline{I}} p(k,\theta)p(j,\theta)\chi_\Omega(\theta)W(d\theta),$$

is the matricial form of $F^*T_\Omega(n)F$ restricted to $\mathscr{R}[F^*T_\Omega(n)F]$ relative to the basis $\{p(k,\theta)\}_{k=0}^n$, and at the same time the matricial form of $P_\Omega(n)$ restricted to $\mathscr{R}[P_\Omega(n)]$ relative to the basis $\{\chi_\Omega(\theta)p(k,\theta)\}_{k=0}^n$. As a consequence of this observation, we have the following result.

LEMMA 2d. *$T_\Omega(n)$ restricted to $\mathscr{R}[T_\Omega(n)]$ and $P_\Omega(n)$ restricted to $\mathscr{R}[P_\Omega(n)]$ are similar and therefore have the same eigenvalues (with the same multiplicities).*

That Theorem 2a is true now follows from Theorem 2c and Lemma 2d.

LEMMA 2e. *We have if $\Omega_1 \cap \Omega_2 = \varnothing, \Omega = \Omega_1 \cup \Omega_2$,*

$$\operatorname{tr}[P_\Omega(n)^2] = \operatorname{tr}[P_{\Omega_1}(n)^2] + \operatorname{tr}[P_{\Omega_2}(n)^2] + \operatorname{tr}[P_{\Omega_1,\Omega_2}(n)^2]. \tag{2.2}$$

Proof. In order to shorten our notation, let us set $F^*E(n)F(n) = \underline{E}(n)$. Then, since $M(\Omega)^2 = M(\Omega)$,

$$\begin{aligned} P_\Omega(n)^2 &= [M(\Omega)\underline{E}(n)M(\Omega)]\,[M(\Omega)\underline{E}(n)M(\Omega)], \\ &= M(\Omega)\underline{E}(n)M(\Omega)\underline{E}(n)M(\Omega). \end{aligned}$$

Using $\operatorname{tr} AB = \operatorname{tr} BA$, we see that because $\underline{E}(m)$ has finite rank

$$\begin{aligned} \operatorname{tr}[P_\Omega(n)^2] &= \operatorname{tr}[M(\Omega)M(\Omega)\underline{E}(n)M(\Omega)\underline{E}(n)], \\ &= \operatorname{tr}[M(\Omega)\underline{E}(n)M(\Omega)\underline{E}(n)], \\ &= \operatorname{tr}[\{M(\Omega_1) + M(\Omega_2)\}\underline{E}(n)\{M(\Omega_1) + M(\Omega_2)\}\underline{E}(n)], \qquad (2.3) \\ &= \operatorname{tr}[M(\Omega_1)\underline{E}(n)M(\Omega_1)\underline{E}(n)] + \operatorname{tr}[M(\Omega_2)\underline{E}(n)M(\Omega_2)\underline{E}(n)] \\ &\quad + \operatorname{tr}[M(\Omega_1)\underline{E}(n)M(\Omega_2)\underline{E}(n)] + \operatorname{tr}[M(\Omega_2)\underline{E}(n)M(\Omega_1)\underline{E}(n)]. \end{aligned}$$

By the argument given above, the first two terms in (2.3) are equal to

$$\operatorname{tr}[P_{\Omega_1}(n)^2] \quad \text{and} \quad \operatorname{tr}[P_{\Omega_2}(n)^2].$$

On the other hand, since $M(\Omega_1)M(\Omega_2) = M(\Omega_2)M(\Omega_1) = 0$, we see that

$$\begin{aligned} P_{\Omega_1,\Omega_2}(n)^2 &= [M(\Omega_1)\underline{E}(n)M(\Omega_2) + M(\Omega_2)\underline{E}(n)M(\Omega_1)]^2 \\ &= M(\Omega_1)\underline{E}(n)M(\Omega_2)\underline{E}(n)M(\Omega_1) + M(\Omega_2)\underline{E}(n)M(\Omega_1)\underline{E}(n)M(\Omega_2), \end{aligned}$$

from which it follows as above that

$$\operatorname{tr}[P_{\Omega_1,\Omega_2}(n)^2] = \operatorname{tr}[M(\Omega_1)\underline{E}(n)M(\Omega_1)\underline{E}(n)] + \operatorname{tr}[M(\Omega_2)\underline{E}(n)M(\Omega_2)\underline{E}(n)].$$

Combining these results, Lemma 2e follows.

We can now prove Theorem 2b. If we divide (2.2) through by $(n + 1)$ and let $n \to \infty$, then using Theorem 2a, we find that

$$\lim_{n\to\infty} \operatorname{tr}[P_{\Omega_1,\Omega_2}(n)^2]/(n+1) = \mu(\Omega) - \mu(\Omega_1) - \mu(\Omega_2) = 0,$$

and since

$$N[\delta, P_{\Omega_1,\Omega_2}(n)] \leqq \delta^{-2} \operatorname{tr}[P_{\Omega_1,\Omega_2}(n)^2],$$

it follows that

$$\lim_{n\to\infty} N[\delta, P_{\Omega_1,\Omega_2}(n)]/(n+1) = 0.$$

Theorem 2b is thus a corollary of Theorem 2a. In [4], the analogue of Theorem 2b is proved directly by an argument parallel to the argument used there to prove Theorem 2a. However, such an argument is unavailable here.

3. A Kac limit theorem for orthogonal polynomials. In the present section as in § 2, we set $\pi(n) = [0, 1, \cdots, n]$. Since the $\pi(n)$'s are not a very "rich" family of subsets of $\underline{Z}^+$, our sufficient conditions for a Kac limit theorem are correspondingly restrictive.

DEFINITION 3a. The real function $L(k) \geqq 0$, $k \in \underline{Z}^+$, is said to be *sufficiently regular* if $L(k)$ can be represented in the form

$$L(k) = L_1(k) + L_2(k),$$

where the following five conditions are satisfied:

(i) $L_1(0) \geqq L_2(1) \geqq L_3(2) \geqq \cdots$;

(ii) $L_1(k) \to 0$ as $k \to +\infty$.

For $\varepsilon > 0$, let $n(\varepsilon)$ be the largest integer k in $\underline{Z}^+$ for which $L_1(k) > \varepsilon$. (If ε is too large, $n(\varepsilon)$ may be undefined.)

(iii) for each fixed $a > 0$, $n(a\varepsilon) = O[n(\varepsilon)]$ as $\varepsilon \to 0+$;

(iv) $n(\varepsilon_1) = o[n(\varepsilon)]$ as $\varepsilon_1, \varepsilon \to 0+$ if $\varepsilon = o(\varepsilon_1)$;

(v) $L_2(k) \in \operatorname{Re} L^\infty[\underline{Z}^+]$ and for every $\delta > 0$, $E(\delta)^\#$ is finite where $E(\delta) = \{k : |L_2(k)| \geqq \delta L_1(k)\}$.

Let us define

$$\Psi(\varepsilon) = |\{(\theta, k) : |V(\theta)|^2 L_1^\frown(k) > \varepsilon\}|_{(I,d\mu)\times \underline{Z}^+}.$$

THEOREM 3b. *For $V(\theta) \in L^\infty[W]$ and $L(k)$ "sufficiently regular" in the sense of Definition* 3a, *we have for each δ, $0 < \delta < 1$,*

$$\left.\begin{aligned} N^+[\varepsilon, S] &\gtrsim \Psi\left[\frac{\varepsilon}{1-\delta}\right] \\ N^+[\varepsilon, S] &\lesssim \Psi\left[\frac{\varepsilon}{1+\delta}\right] \end{aligned}\right\} \quad \text{as } \varepsilon \to 0+,$$

where $S = M(V)F^*E(L)FM(V)$.

Proof. As we mentioned earlier, the demonstration of this result falls into two parts, the first of which we have just carried out. The second part, a systematic exposition of which is given in [4] shows that Theorems 2a and 2b imply that for any fixed $\delta > 0$,

$$\left.\begin{aligned} N^+[\varepsilon, S_1] &\lesssim \Psi\left[\frac{\varepsilon}{1+\delta}\right] \\ N^+[\varepsilon, S_1] &\gtrsim \Psi\left[\frac{\varepsilon}{1-\delta}\right] \end{aligned}\right\} \text{ as } \varepsilon \to 0+,$$

where the kernel of S_1 is $M(V)F^*E(L_1)FM(V)$.

Let us define $S_2(\delta)$ to be the integral operator on $L^2[W]$ of rank $E(\delta)^{\#}$ whose kernel is

$$\left[\sum_{k \in E(\delta)} p_k(\theta)p_k(\varphi)\right]\|\hat{L_2}\|_\infty.$$

It is apparent that

$$S \leqq (1+\delta)S_1 + S_2(\delta),$$
$$S \geqq (1-\delta)S_1 - S_2(\delta).$$

By the minimax principle (see [1, pp. 132–134]),

$$N^+[\varepsilon, S] \leqq N^+[\varepsilon, (1+\delta)S_1] + N^+[0, S_2(\delta)],$$
$$N^+[\varepsilon, S] \leqq N^+\left[\frac{\varepsilon}{1+\delta}, S_1\right] + E(\delta)^{\#}.$$

It follows from this that

$$N^+[\varepsilon, S] \lesssim \Psi\left[\frac{\varepsilon}{(1+\delta)^2}\right] \quad \text{as } \varepsilon \to \infty.$$

A similar argument shows that

$$N^+[\varepsilon, S] \gtrsim \Psi\left[\frac{\varepsilon}{(1-\delta)^2}\right] \quad \text{as } \varepsilon \to \infty.$$

Since $\delta > 0$ is arbitrary, our proof is complete.

4. A Kac limit theorem for Jacobi polynomials. In this section, we establish a Kac limit theorem associated with the Jacobi polynomials of index α, β where $\alpha > -1$, $\beta > -1$, of the same level of generality as the Krieger generalization of the Szegö limit theorem for T quoted in § 1. Let $p_{\alpha,\beta}(k, \theta)$ be the polynomials of § 2 corresponding to the weight function

$$W_{\alpha,\beta}(d\theta) = 2^{\alpha+\beta}\left(\sin\frac{\theta}{2}\right)^{2\alpha+1}\left(\cos\frac{\theta}{2}\right)^{2\beta+1} d\theta. \tag{4.1}$$

It is easily checked that

$$p_{\alpha,\beta}(k, \theta) = \{h^{(\alpha,\beta)}\}^{-1/2} P_k^{(\alpha,\beta)}(\cos\theta),$$

where the $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials as usually given (see [12]).

Ultimately, we will focus on the $p_{\alpha,\beta}(k, \theta)$'s, but at the beginning, it is just as easy to discuss general $p(k, \theta)$'s. Moreover, it is instructive to note at precisely what point we use properties which are not shared by all sets of orthogonal polynomials.

It is well known that the $p(k, \theta)$'s satisfy a recursion relation

$$\cos\theta\, p(n,\theta) = A(n)p(n-1,\theta) + B(n)p(n,\theta) + C(n)p(n-1,\theta), \tag{4.2}$$

where $A(0) = 0$ and $A(n+1) = C(n) > 0$, $n = 0, 1, 2, \cdots$. It is shown in [12, Chap. 12] that if (2.1) holds, then

$$\lim_{n\to\infty} A(n) = \lim_{n\to\infty} C(n) = \tfrac{1}{2}, \qquad \lim_{n\to\infty} B(n) = 0. \tag{4.3}$$

Note further that

$$\cos\theta\cos k\theta = \tfrac{1}{2}\cos[(k+1)\theta] + \tfrac{1}{2}\cos[(k+1)\theta], \qquad k \in \underline{Z}^+. \tag{4.4}$$

For each $\varepsilon > 0$, let $\pi(\varepsilon)$ be a finite subset of $\underline{Z}^+$ such that if $Q(\varepsilon, m)$ is defined as in § 1,

$$\lim_{\varepsilon\to 0+} Q(\varepsilon, m)/\pi(\varepsilon)^{\#} = 1 \tag{4.5}$$

for all $m \in \underline{Z}^+$ (which implies that it holds for all $m \in \underline{Z}$). Let $L(\theta) \in \operatorname{Re} L^1[\underline{I}, \mu]$. If we extend $L(\theta)$ to $[-\pi, \pi]$ so that it is even and write Krieger's theorem in matricial form, we obtain the following.

LEMMA 4a. *Let $L(\theta) \in \operatorname{Re} L^1[\underline{I}, \mu]$, and let (4.5) hold. If $\mu\{\theta : L(\theta) = a\} = \mu\{\theta : L(\theta) = b\} = 0$, then, in an evident notation,*

$$\lim_{\varepsilon\to 0+} N[(a,b) : \mathbf{T}(L,\varepsilon)]/\pi(\varepsilon)^{\#} = \mu\{\theta : a < L(\theta) < b\},$$

where

$$\mathbf{T}(L,\varepsilon) = \left[\int_{\underline{I}} L(\theta)\cos[(k-j)\theta]\mu(d\theta)\right]_{j,k\in\pi(\varepsilon)}.$$

LEMMA 4b. *Let $q(x)$ be a real polynomial of degree r, and let $\{p(n,\theta)\}_{n\in\underline{Z}^+}$ be the orthonormal polynomials on $\underline{I}$ associated with the weight function $W(d\theta)$ satisfying (2.1). Then*

$$\lim_{j,k\to\infty} \int_{\underline{I}} p(j,\theta)p(k,\theta)q(\cos\theta)W(d\theta) = \int_{\underline{I}} q(\cos\theta)\cos[(k-j)\theta]\mu(d\theta).$$

Moreover, if $|j-k| > r$, then

$$\int_{\underline{I}} p(j,\theta)p(k,\theta)q(\cos\theta)W(d\theta) = \int_{\underline{I}} q(\cos\theta)\cos[(k-j)\theta]\mu(d\theta) = 0.$$

Proof. Our assertions are immediate consequences of (4.2), (4.3) and (4.4).

THEOREM 4c. *Under the above assumptions, if*

$$T_W(q,\varepsilon) = \left[\int_{\underline{I}} p(j,\theta)p(k,\theta)q(\cos\theta)W(d\theta)\right]_{j,k\in\pi(\varepsilon)}$$

and if $\mu\{\theta : q(\cos\theta) = a\} = \mu\{\theta : q(\cos\theta) = b\} = 0$, then, in an evident notation,

$$\lim_{\varepsilon\to 0+} N[(a,b); T_W(q,\varepsilon)]/\pi(\varepsilon)^{\#} = \mu\{\theta : a < q(\cos\theta) < b\}.$$

Proof. It follows from Lemma 4b that

$$|||T_W(q,\varepsilon) - T(q,\varepsilon)|||_2^2 = o(\pi(\varepsilon)^{\#}) \quad \text{as } \varepsilon \to 0+,$$

where $|||\cdot|||_2$ is the Hilbert–Schmidt norm of the $\pi(\varepsilon) \times \pi(\varepsilon)$ matrix in question. In conjunction with Lemma 4a, this yields, via the Weyl–Courant inequalities (see [1, pp. 132–134]) the desired result.

Theorem 4c is more general than Theorem 2c in that it permits a richer family of sets $\pi(\varepsilon)$, but it is more restrictive in that it has been shown to be valid only for polynomials in $\cos\theta$. To deal with this deficiency, we note that if $L(\theta) \in L^1[I, W]$ and $L(\theta) \geqq 0$, then $T_L(\varepsilon) \geqq 0$, and that

$$\operatorname{tr}[T_W(L,\varepsilon)] = \sum_{k\in\pi(\varepsilon)} \int_I p(k,\theta)^2 L(\theta) W(d\theta).$$

Let us *suppose* that there exists a finite measure $W^*(d\theta)$ on the Borel sets of I such that

$$|p(k,\theta)|^2 W(d\theta) \leqq W^*(d\theta) \qquad k \in \underline{Z}^+. \tag{4.6}$$

Since $p(0,\theta)$ is a positive constant, $L^1[W^*] \subset L^1[W]$. If (4.6) holds, it is apparent that

$$\operatorname{tr}[T_W(L,\varepsilon)] \leqq \pi(\varepsilon)^{\#} \int_I L(\theta) W^*(d\theta). \tag{4.7}$$

THEOREM 4d. *Let* (2.1), (4.5) *and* (4.6) *hold. If* $L(\theta) \in \operatorname{Re} L^1[W^*]$ *and if*

$$\mu\{\theta : L(\theta) = a\} = \mu\{L(\theta) = b\} = 0, \tag{4.8}$$

then

$$\lim_{\varepsilon\to 0+} N[(a,b): T_L^{(r)}(\varepsilon)]/\pi(\varepsilon)^{\#} = \mu\{\theta : a < L(\theta) < b\}. \tag{4.9}$$

Here $T_L^{(r)}(\varepsilon)$ *is the restriction of* $T_L(\varepsilon)$ *to* $E(\varepsilon)\underline{Z}^+$.

Proof. Using (4.7), the set of functions L in $\operatorname{Re} L^1[W^*]$ for which Theorem 4d is true can be shown to be closed under monotone limits. Since this set includes all real polynomials in $\cos\theta$, it includes $\operatorname{Re} L^1[W^*]$. See [5, § 7] for a similar argument carried out in detail.

THEOREM 4e. *Set* $W_{\alpha,\beta}(\theta)$ *is defined as in* § 2. *Then if* $-1 < \alpha, \beta$, (4.6) *holds with*

$$W^*_{\alpha,\beta}(d\theta) = c(\alpha,\beta)\left(\sin\frac{\theta}{2}\right)^{\min(0,2\alpha+1)}\left(\cos\frac{\theta}{2}\right)^{\min(0,2\beta+1)} d\theta, \tag{4.10}$$

where $c(\alpha,\beta)$ *is a constant depending only on* α *and* β.

Proof. Equation (4.10) follows from Theorem 3.32.2 of Szegö [12].

THEOREM 4f. *Let* $-1 < \alpha$, $-1 < \beta$, *and let* (4.5) *and* (4.6) *hold* (*with* $W^*_{\alpha,\beta}$ *given by* 4.10). *Then if* (4.8) *is satisfied*, (4.9) *is true.*

Proof. This is a corollary of our previous results.

Note that Theorem 4f is *not* a generalization of Szegö's theorem. Theorem 2c, since, although the sets $\pi(\varepsilon)$ are more general, $L(\theta)$ is more restricted! However, if Ω is any Borel set in I, $\chi_\Omega(\theta) \in \operatorname{Re} L^1[W^*_{\alpha,\beta}]$, and we can thus apply the arguments of § 2 and § 3 to obtain the following.

THEOREM 4g. *Let $L_1(k)$, $k \in \underline{Z}^+$ be a real positive function such that $L_1(k) \to 0$ as $k \to +\infty$, and for each $\varepsilon > 0$ let*

$$\pi(\varepsilon) = \{k \in \underline{Z}^+ : L_1(k) > \varepsilon\}.$$

We assume that:

(a) *for each $a > 0$, $\pi(a\varepsilon)^\# = O[\pi(\varepsilon)^\#]$ as $\varepsilon \to \infty$;*

(b) *$\pi(\varepsilon_1)^\# = o[\pi(\varepsilon)^\#]$ if $\varepsilon_1, \varepsilon \to \infty$, $\varepsilon = o(\varepsilon_1)$;*

(c) *$\lim_{\varepsilon \to 0+} [\pi(\varepsilon) \cap \pi(\varepsilon) - m]^\# / \pi(\varepsilon)^\# = 1$ all $m \in \underline{Z}^+$.*

Let

$$L(k) = L_1(k) + L_2(k),$$

where we further assume that:

(d) *for each $\delta > 0$, $E(\delta)^\#$ is finite, where*

$$E(\delta) = \{k \in \underline{Z}^+ : |L_2(k)| \geqq \delta L_1(k)\}.$$

If $V(x)$ is a positive bounded Borel measurable function on $\underline{I}$, then for each δ, $0 < \delta < 1$,

$$\left.\begin{aligned} N^+[\varepsilon, S] &\gtrsim \Psi\left(\frac{\varepsilon}{1-\delta}\right) \\ N^+[\varepsilon, S] &\lesssim \Psi\left(\frac{\varepsilon}{1+\delta}\right) \end{aligned}\right\} \quad \text{as } \varepsilon \to 0+,$$

where (in an obvious notation)

$$S = M(V)F^*_{\alpha,\beta}E(L)F_{\alpha,\beta}M(V),$$

and where Ψ is defined as in § 3.

Part II. The Kac limit theorem for $\underline{R}_n$.

5. Basic estimates. We depart here from our principal theme in order to reprove by our methods a variant of Widom's "Kac limit theorem for $\underline{R}_n$". This variant has an interesting application and we will need the ideas introduced here in Part III. Let $\underline{R}_n$ denote n-dimensional Euclidean space. We use letters such as $\underline{x} = (x_1, \cdots, x_n)$, $\underline{y} = (y_1, \cdots, y_n)$, $\underline{t} = (t_1, \cdots, t_n)$, to denote points of $\underline{R}_n$, and set

$$\underline{x} \cdot \underline{y} = x_1 y_1 + \cdots + x_n y_n, \qquad |\underline{x}| = [\underline{x} \cdot \underline{x}]^{1/2},$$
$$a\underline{x} = (ax_1, \cdots, ax_n), \qquad \underline{x} + \underline{y} = (x_1 + y_1, \cdots, x_n + y_n).$$

We shall in the present section apply the methods of [4] to $\underline{R}_n$ which is a symmetric space but is *not* compact. The first part of the development proceeds quite as before. The final steps, however, involve significant changes. In what follows, we will carry out in detail only those arguments which are substantially different from those given in [4].

We recall a few formulas from Fourier analysis on $\underline{R}_n$ in order to fix our notations. Let $\sigma(d\underline{x}) = (2\pi)^{-n/2}\, d\underline{x}$

$$Ff \cdot (\underline{t}) = (2\pi)^{-n} \int_{\underline{R}_n} e^{-i\underline{t}\cdot\underline{x}} f(\underline{x})\sigma(d\underline{x})$$

and if $f \in L^2[\underline{R}_n]$, then

$$\|Ff\|_2 = \|f\|_2,$$

and

$$f = F^*[Ff],$$

where

$$F^*f \cdot (x) = \int_{\underline{R}_n} e^{it \cdot \underline{x}} f(\underline{t})\, \sigma(d\underline{t}).$$

We set

$$f * g \cdot (\underline{x}) = (2\pi)^{-n} \int_{\underline{R}_n} f(\underline{x} - \underline{y}) g(\underline{y})\, \sigma(d\underline{y}).$$

We proceed to make a list of various assumptions we will require. (Later it will be seen to be very convenient to have these collected in one place).

Let $V(\underline{x})$ be a measurable function on $\underline{R}_n$ satisfying:

(5.1)
(i) $V(\underline{x}) \geqq 0, \quad \underline{x} \in \underline{R}_n$,
(ii) $V(\underline{x}) \in L^\infty[\underline{R}_n]$,
(iii) $V(\underline{x})$ has bounded support.

Let $L(\underline{t})$ be a real measurable function on $\underline{R}_n$ satisfying the conditions:

(5.2)
(i) $L(\underline{t}) \geqq 0, \quad \underline{t} \in \underline{R}_n$,
(ii) $L(\underline{t}) \in L^\infty[\underline{R}_n]$,
(iii) $L(\underline{t}) \to 0$ as $\underline{t} \to \infty$.

For C a measurable set in $\underline{R}_n$, $C^\#$ is the σ-measure of C. Define

$$\pi(\varepsilon) = \{\underline{t} \in \underline{R}_n : L(\underline{t}) > \varepsilon\}.$$

We assume that $\pi(\varepsilon)$ satisfies the conditions:

(5.3)
(i) $\lim_{\varepsilon \to 0+} [\pi(\varepsilon) \cap \pi(\varepsilon) - \underline{t}]^\# / \pi(\varepsilon)^\# = 1$ in measure on each set of finite measure in $\underline{R}_n$;
(ii) $\pi(\varepsilon_1)^\# = o[\pi(\varepsilon)^\#]$ as $\varepsilon_1, \varepsilon \to 0+$ if $\varepsilon = o(\varepsilon_1)$;
(iii) for any constant $a > 0$, there exists a constant A depending upon a such that

$$|\log \pi(\varepsilon_1)^\# - \log \pi(\varepsilon_2)^\#| \leqq A$$

if ε_1 and ε_2 are sufficiently small and if

$$|\log \varepsilon_1 - \log \varepsilon_2| \leqq a.$$

We remark that this implies that there exist constants η_1 and $\eta_2 > 0$ such that

$$\pi(\varepsilon)^{\#} = O(\varepsilon^{-\eta_1}), \qquad 1/\pi(\varepsilon)^{\#} = O(\varepsilon^{\eta_2}) \quad \text{as } \varepsilon \to 0+.$$

There exists a constant $E > 0$ such that

$$\text{ess sup}\,\{|\underline{t}| : \underline{t} \in \pi(\varepsilon)\}/[\pi(\varepsilon)^{\#}]^{1/n} \leqq E \tag{5.4}$$

for all sufficiently small $\varepsilon > 0$.

For V satisfying (ii) of (5.1), let

$$M(V)f \cdot (\underline{x}) = V(\underline{x})f(\underline{x}), \qquad f \in L^2[\underline{R}_n],$$

and for L satisfying (i), (ii) and (iii) of (5.2), let

$$E(L)g \cdot (\underline{t}) = L(\underline{t})g(\underline{t}), \qquad g \in L^2[\underline{R}_n].$$

Our goal is to study the eigenvalues of the operator

$$S = M(V)F^*E(L)FM(V).$$

The following familiar result is proved in slightly greater generality than is necessary.

THEOREM 5a. *If $V(\underline{x}) \in L^2(\underline{R}_n) \cap L^\infty(\underline{R}_n)$ and if $L(\underline{t}) \in L^\infty(\underline{R}_n)$, $L(\underline{t}) \to 0$ as $\underline{t} \to \infty$, then S is compact.*

Proof. Given $r > 0$, we define

$$L_1(r, \underline{t}) = \begin{cases} L(\underline{t}) & \text{if } |\underline{t}| < r, \\ 0 & \text{if } |\underline{t}| \geqq r, \end{cases}$$

and

$$L_2(r, \underline{t}) = \begin{cases} 0 & \text{if } |\underline{t}| < r, \\ L(\underline{t}) & \text{if } |\underline{t}| \geqq r. \end{cases}$$

For $f \in L^2[\underline{R}_n]$,

$$Sf \cdot (\underline{x}) = S_1(r)f \cdot (\underline{x}) + S_2(r)f \cdot (\underline{x}),$$

where, letting $L_1 = L_1(r, \underline{t})$, etc., we have

$$S_1(r)f(\underline{x}) = V(\underline{x}) \cdot [L_1(Vf)\hat{}\,]\check{}\cdot(\underline{x}),$$

$$S_2(r)f(\underline{x}) = V(\underline{x}) \cdot [L_2(Vf)\hat{}\,]\check{}\cdot(\underline{x}).$$

Here we have written $\hat{}$ for F and $\check{}$ for F^*. It is easy to see that

$$|S_1(r)f \cdot (\underline{x})| \leqq |V(\underline{x})|\, \|L_1(r)\|_1 \|V\|_2 \|f\|_2,$$

$$\frac{\partial}{\partial x_i}[L_1(r)(Vf)\hat{}\,]\check{}(\underline{x}) \leqq r\|L_1(r)\|_1 \|V\|_2 \|f\|_2, \qquad i = 1, \cdots, n, \tag{5.5}$$

$$\|S_2(r)f\|_2 \leqq \|V\|_\infty^2 \|L_2(r)\|_\infty \|f\|_2.$$

The first two formulas in conjunction with Arzela's theorem imply that $S_1(r)$ is compact for each $r > 0$. The third formula implies that $\|S_2(r)\|$ can be made as small as we please by taking r large. Since compact operators are closed in the Banach algebra of all bounded linear operators on a Hilbert space, our theorem follows.

COROLLARY 5b. *If in addition, $V(\underline{x})$ and $L(\underline{t})$ are nonnegative, then S is nonnegative.*

For $\pi(\varepsilon)$ defined as in (5.3), let

$$E(\varepsilon)g \cdot (\underline{t}) = \chi_{\pi(\varepsilon)}(\underline{t})g(\underline{t}), \qquad g(\underline{t}) \in L[\underline{R}_n],$$

and for Ω a set of finite measure in $\underline{R}_n$, let

$$M(\Omega)f \cdot (\underline{x}) = \chi_\Omega(\underline{x})f(\underline{x}), \qquad f(\underline{x}) \in L^2[\underline{R}_n].$$

Our goal is to analyze

$$P_\Omega(\varepsilon) = M(\Omega)F^*E(\varepsilon)FM(\Omega)$$

and

$$P_{\Omega_1,\Omega_2}(\varepsilon) = M(\Omega_1)F^*E(\varepsilon)FM(\Omega_2) + M(\Omega_2)F^*E(\varepsilon)FM(\Omega_1).$$

Here Ω_1 and Ω_2 are sets of finite measure in $\underline{R}_n$.

THEOREM 5c. *Let conditions* (i), (ii) *and* (iii) *of* (5.2) *and* (i) *of* (5.3) *hold. Then $P_\Omega(\varepsilon)$ is a nonnegative compact operator. If $\lambda(\varepsilon, 1) \geqq \lambda(\varepsilon, 2) \geqq \cdots$ are the nonzero eigenvalues of $P_\Omega(\varepsilon)$ repeated according to their multiplicities, then*

$$\text{(i) } 0 < \lambda(\varepsilon, k) \leqq 1, \qquad k = 1, 2, \cdots,$$

$$\text{(ii) } \sum_k \lambda(\varepsilon, k) \leqq \Omega^\# \pi(\varepsilon)^\#,$$

$$\text{(iii) } \sum_k \lambda(\varepsilon, k)^2 = \Omega^\# \pi^\# [1 - r_1(\varepsilon)],$$

where $0 \leqq r_1(\varepsilon) \leqq 1$ and where $r_1(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

The demonstration of this result follows the pattern of [4] except that there is a (fortunately unimportant) slip there in the assertion that Mercer's theorem implies that (ii) holds with equality. The correct assertion (given above) follows from a simple corollary of Mercer's theorem (see § 8). Actually, in the context of [4], equality does obtain in (ii), but this must be shown by a direct calculation, not by an appeal to Mercer's theorem.

THEOREM 5d. *If* (i), (ii) *and* (iii) *of* (5.2) *and* (i) *of* (5.3) *hold, then for $r_1(\varepsilon) \leqq \frac{1}{2}$,*

$$N^+[1 - r_1(\varepsilon)^{1/2}, P_\Omega(\varepsilon)] \geqq \Omega^\# \pi(\varepsilon)^\# [1 - 4r_1(\varepsilon)^{1/2}],$$

$$N^+[r_1(\varepsilon)^{1/2}, P_\Omega(\varepsilon)] \leqq \Omega^\# \pi(\varepsilon)^\# [1 + 4r_1(\varepsilon)^{1/2}].$$

The reader who consults [4] will notice that the argument there is carried through under the assumption that equality holds in conclusion (ii) of Theorem 5c. However, it is easy to see that only the inequality is actually used.

Let $N[\delta, P_{\Omega_1,\Omega_2}(\varepsilon)]$ denote the number of eigenvalues of $P_{\Omega_1,\Omega_2}(\varepsilon)$ which exceed $\delta > 0$ in absolute value.

THEOREM 5e. *If* (i), (ii) *and* (iii) *of* (5.2) *and* (i) *of* (5.3) *hold, then for any $\delta > 0$,*

$$\left.\begin{array}{l} N^+[(1 - \delta), P_\Omega(\varepsilon)] \sim \Omega^\# \pi^\# \\ N^+[\delta, P_\Omega(\varepsilon)] \sim \Omega^\# \pi^\# \end{array}\right\} \quad \text{as } \varepsilon \to 0+,$$

and

$$N[\delta, P_{\Omega_1,\Omega_2}(\varepsilon)] = o(\pi(\varepsilon)^\#) \quad \text{as } \varepsilon \to 0+.$$

Proof. The proof is routine.

6. Basic estimates continued. In the case where $\underline{R}_n$ is replaced by $\underline{T}_n$, the results of Theorem 5e are sufficient to enable us to establish our principal result. However, because $\underline{R}_n$ is not compact, an additional result is needed. The following theorem can be regarded as a far-reaching although very crude analogue of the precise but special asymptotic formulas obtained by Widom in [16]. The proof is patterned after a similar argument in [15].

THEOREM 6a. *Let the conditions* (5.2), (5.3) *and* (5.4) *hold, and let* Ω *be a set of finite measure and bounded support. Then for each real c,*

$$N^+[\varepsilon^c, P_\Omega(\varepsilon)] = O[\pi(\varepsilon)^\#] \quad \text{as } \varepsilon \to 0+. \tag{6.1}$$

Proof. Because Ω is bounded, the vector difference $\Omega - \Omega$ is bounded. Elementary considerations based on homogeneity show that there is no loss of generality in assuming that the closure of $\Omega - \Omega$ lies in the interior of $\underline{T}_n$ which we identify with that cube in $\underline{R}_n$ with center at the origin, sides of length 2π and faces perpendicular to the coordinate axes. Let $\varphi(\underline{x}) \in C^\infty[\underline{R}_n]$ be 1 on $\Omega - \Omega$ and 0 outside $\underline{T}_n$. If

$$P(\varepsilon, \underline{x}) = \int_{\pi(\varepsilon)} e^{i\underline{x}\cdot t}\,\sigma(d\underline{t}),$$

then

$$P_\Omega(\varepsilon)f\cdot(\underline{x}) = \int_{\underline{R}_n} \chi_\Omega(\underline{x})P(\varepsilon, \underline{x} - \underline{y})\chi_\Omega(\underline{y})f(\underline{y})\,\sigma(d\underline{y}),$$

$$= \int_{\underline{T}_n} \chi_\Omega(\underline{x})P(\varepsilon, \underline{x} - \underline{y})\varphi(\underline{x} - \underline{y})\chi_\Omega(\underline{y})f(\underline{y})\,\sigma(d\underline{y}).$$

Let $\underline{Z}_n$ be the points in $\underline{R}_n$ with integral coordinates. For $\underline{x} \in \underline{T}_n$, we have

$$P(\varepsilon, \underline{x})\varphi(\underline{x}) = \sum_{\underline{k}\in \underline{Z}_n} e^{i\underline{k}\cdot\underline{x}}(2\pi)^{-n}\int_{\underline{T}_n} e^{-i\underline{k}\cdot\xi}\,d\xi\,\varphi(\xi)\int_{\pi(\varepsilon)} e^{i\underline{t}\cdot\xi}\sigma(d\underline{t})$$

$$= \sum_{\underline{Z}_n} e^{i\underline{k}\cdot\underline{x}}\int_{\pi(\varepsilon)} d\underline{t}(2\pi)^{-n}\int_{\underline{T}_n} \varphi(\xi)\,e^{i(\underline{t}-\underline{k})\cdot\xi}\,\sigma(d\xi)$$

$$= \sum_{\underline{Z}_n} e^{i\underline{k}\cdot\underline{x}}\int_{\pi(\varepsilon)} \varphi\hat{\ }(\underline{k} - \underline{t})\,\sigma(d\underline{t}).$$

Repeated integration by parts shows that for any integer $r \geqq 0$, there is a constant B_r such that

$$|\varphi\hat{\ }(\underline{x})| \leqq B_r|\underline{x}|^{-rn-n}, \qquad \underline{x} \in \underline{R}_n. \tag{6.2}$$

Define

$$Q'(\varepsilon) = \{\underline{k} : |\underline{k}| < 2E[\pi(\varepsilon)^\#]^{1/n}\},$$

$$Q''(\varepsilon) = \{\underline{k} : |\underline{k}| \geqq 2E[\pi(\varepsilon)^\#]^{1/n}\},$$

where E is the constant in (5.4), and set

$$P'(\varepsilon, \underline{x}) = \sum_{\underline{k} \in Q'(\varepsilon)} e^{i\underline{k}\cdot\underline{x}} \int_{\pi(\varepsilon)} \varphi\hat{\ }(\underline{k} - \underline{t})\, \sigma(d\underline{t}),$$

$$P''(\varepsilon, \underline{x}) = \sum_{\underline{k} \in Q''(\varepsilon)} e^{i\underline{k}\cdot\underline{x}} \int_{\pi(\varepsilon)} \varphi\hat{\ }(\underline{k} - \underline{t})\, \sigma(d\underline{t}).$$

Let $P'_\Omega(\varepsilon)$ and $P''_\Omega(\varepsilon)$ be the integral operators with kernels

$$\chi_\Omega(\underline{x}) P'(\varepsilon, \underline{x} - \underline{y}) \chi_\Omega(\underline{y})$$

and

$$\chi_\Omega(\underline{x}) P''(\varepsilon, \underline{x} - \underline{y}) \chi_\Omega(\underline{y}).$$

Let us denote by $Q'(\varepsilon)^\#$ the number of points $\underline{k}$ in $Q'(\varepsilon)$. Clearly,

$$\operatorname{rank}\{P'_\Omega(\varepsilon)\} = Q'(\varepsilon)^\# = O[\pi(\varepsilon)^\#].$$

It follows from (5.4) that if $\underline{t} \in \pi(\varepsilon)$ and if $\underline{k} \in Q''(\varepsilon)$, then $|\underline{k} - \underline{t}| \geqq (1/2)|k|$. Using this and (6.2), we see that there is a constant B'_r such that for $\underline{k} \in Q''(\varepsilon)$

$$\left| \int_{\pi(\varepsilon)} \varphi\hat{\ }(\underline{k} - \underline{t})\, \sigma(d\underline{t}) \right| \leqq B'_r |k|^{-rn-n}.$$

It follows that if $\|P''_\Omega(\varepsilon)\|$ is the operator norm of $P''_\Omega(\varepsilon)$, then

$$\|P''_\Omega(\varepsilon)\| = O([\pi(\varepsilon)^\#]^{-r}) \quad \text{as } \varepsilon \to 0+.$$

By (5.3) (iii), there is an integer $r > 0$ such that $[\pi(\varepsilon)^\#]^{-r} < \varepsilon^c$ for $\varepsilon > 0$ sufficiently small. For such ε,

$$\begin{aligned} N^+[\varepsilon^c, P_\Omega(\varepsilon)] &\leqq N^+[\varepsilon^c, P''_\Omega(\varepsilon)] + N^+[0, P'_\Omega(\varepsilon)] \\ &\leqq 0 + \operatorname{rank}\{P'_\Omega(\varepsilon)\} = O[\pi(\varepsilon)]^\#, \end{aligned}$$

which proves (6.1).

7. The Kac limit theorem for $\underline{R}_n$.

THEOREM 7a. *Let conditions* (5.1)–(5.4) *hold. Then for each* δ, $0 < \delta < 1$,

$$\text{(i)}\quad N^+[\varepsilon, S] \gtrsim \Psi_S\left[\frac{\varepsilon}{1 - \delta}\right] \quad \text{as } \varepsilon \to 0+,$$

$$\text{(ii)}\quad N^+[\varepsilon, S] \lesssim \Psi_S\left[\frac{\varepsilon}{1 + \delta}\right] \quad \text{as } \varepsilon \to 0+,$$

where

$$\Psi_S[\varepsilon] = |\{\underline{x}, \underline{t} : V(\underline{x})^2 L(\underline{t}) > \varepsilon\}|_{(\underline{R}_n, \sigma) \times (\underline{R}_n, \sigma)}.$$

Here $|\{\cdot\}|$ *is the measure of the set* $\{\cdot\}$ *in the product measure space.*

Proof. We consider in detail only the special case $V(\underline{x}) = \chi_\Omega(\underline{x})$ where Ω is a bounded measurable set in $\underline{R}_n$. That the general result can be demonstrated by similar although more complicated arguments will then be evident from [4].

Let

$$T = M(\Omega)F^*E(L)FM(\Omega).$$

For any $\varepsilon > 0$, it follows from the definition of $P^{\hat{}}(\varepsilon, \underline{t})$ that

$$L(\underline{t}) \geqq \varepsilon P^{\hat{}}(\varepsilon, \underline{t}),$$

which implies that

$$T \geqq \varepsilon P_{\Omega}(\varepsilon).$$

Thus by Theorem 5d,

$$\begin{aligned} N^+[\varepsilon(1-\delta), T] \geqq N^+[\varepsilon(1-\delta), \varepsilon P_{\Omega}(\varepsilon)] &= N^+[(1-\delta), P_{\Omega}(\varepsilon)] \\ &\sim \Omega^{\#}\pi(\varepsilon)^{\#} \equiv \Psi_T[\varepsilon] \quad \text{as } \varepsilon \to 0+. \end{aligned}$$

Replacing $\varepsilon(1-\delta)$ by ε, we have

$$N^+[\varepsilon, T] \gtrsim \Psi_T\left[\frac{\varepsilon}{1-\delta}\right] \quad \text{as } \varepsilon \to 0+.$$

Let ε_1 be a function of ε such that $\varepsilon_1 \to 0+$ as $\varepsilon \to 0+$, $\varepsilon = o(\varepsilon_1)$, and $\varepsilon_1 = o[\min\{\varepsilon r_1(\varepsilon)^{(-1/2)}, \varepsilon^{(1/2)}\}]$. If $A = \|L\|_\infty$, then

$$L(\underline{t}) \leqq \varepsilon + \varepsilon_1 P^{\hat{}}(\varepsilon, \underline{t}) + AP^{\hat{}}(\varepsilon_1, \underline{t}),$$

which implies that

$$T \leqq \varepsilon I + \varepsilon_1 P_{\Omega}(\varepsilon) + AP_{\Omega}(\varepsilon_1).$$

By the Weyl–Courant lemma,

$$N^+[\varepsilon(1+\delta), T] \leqq N^+[\varepsilon, \varepsilon I] + N^+[\tfrac{1}{2}\delta\varepsilon, \varepsilon_1 P_{\Omega}(\varepsilon)] + N^+[\tfrac{1}{2}\delta\varepsilon, AP_{\Omega}(\varepsilon_1)].$$

We have

$$N^+[\varepsilon, \varepsilon I] = 0.$$

Since $\varepsilon_1 = o(\varepsilon r_1(\varepsilon)^{(-1/2)})$, it follows from Theorem 5d that

$$N^+[\tfrac{1}{2}\delta\varepsilon, \varepsilon_1 P_{\Omega}(\varepsilon)] \sim \Omega^{\#}\pi(\varepsilon)^{\#} \equiv \Psi_T[\varepsilon] \quad \text{as } \varepsilon \to 0+.$$

By the definition of ε_1, Theorem 6a and (5.3) (ii), we have

$$N^+[\tfrac{1}{2}\delta\varepsilon, AP_{\Omega}(\varepsilon_1)] \leqq N^+[\varepsilon_1^2, P_{\Omega}(\varepsilon_1)] = O[\pi(\varepsilon_1)^{\#}] = o[\pi(\varepsilon)^{\#}]. \tag{7.1}$$

Combining these results, we have shown that

$$N^+[\varepsilon(1+\delta), T] \lesssim \Omega^{\#}\pi(\varepsilon)^{\#} \equiv \Psi_T[\varepsilon].$$

To finish, replace $(1+\delta)\varepsilon$ by ε.

In [4] where $\underline{T}_n$ replaced $\underline{R}_n$, we had

$$N^+[0, AP_{\Omega}(\varepsilon_1)] = o[\pi(\varepsilon)^{\#}] \quad \text{as } \varepsilon \to 0+, \tag{7.2}$$

simply because $P_{\Omega}(\varepsilon_1)$ had rank $\pi(\varepsilon_1)^{\#}$ and $\pi(\varepsilon_1)^{\#} = o[\pi(\varepsilon)^{\#}]$. This trivial argument fails for $\underline{R}_n$ which is why it was necessary to prove Theorem 6a. The reader should compare this demonstration with that given in [13]. Arguments used by Widom in [13, § 6] can be applied without change to yield the following. Let $S(r)$ be the sphere in $\underline{R}_n$ with center at the origin and radius r.

THEOREM 7b. *Let conditions* (5.1)–(5.4) *hold except that condition* (5.2) (ii) *is replaced by the* (*more general*) *assumption*: *there exists an* $r > 0$ *such that*

$$L\hat{\ }(\underline{t}) \in L^1[S(r)], L(\underline{t}) \in L^\infty[\underline{R}_n \backslash \sigma(r)].$$

Then for every δ, $0 < \delta < 1$, *we have*

$$\begin{aligned} N^+[\varepsilon, S] &\gtrsim \Psi_S\left[\frac{\varepsilon}{1-\delta}\right] \quad as\ \varepsilon \to 0+, \\ N^+[\varepsilon, S] &\lesssim \Psi_S\left[\frac{\varepsilon}{1+\delta}\right] \quad as\ \varepsilon \to 0+, \end{aligned} \tag{7.3}$$

where

$$\Psi_S[\varepsilon] = |\{(\underline{x}, \underline{t}) : V(\underline{x})^2 L(\underline{t}) > \varepsilon\}|_{(\underline{R}_n,\sigma)\times(\underline{R}_n,\sigma)}.$$

Widom's argument is reproduced in § 12 in a different context, so we do not sketch it here.

Widom also shows in [15, § 6] that if $L(\underline{t})$ satisfies the assumptions of Theorem 7b and if $K(\underline{t}) \in L^1(\underline{R}_n)$ satisfies

$$\lim_{\underline{t}\to\infty} K(\underline{t})/L(\underline{t}) = 1,$$

then (7.3) holds with L replaced by K. His argument used in the proof of Theorem 3b can be applied without change and, moreover, is reproduced in § 12.

We conclude this section with an example. Let

$$L(\underline{t}) = |\underline{t}|^{-\omega}\Phi(\underline{t}), \qquad 0 < \omega < n,$$

where $\Phi(\underline{t})$, defined for $\underline{t} \in \underline{R}_n \backslash \{0\}$, is a bounded nonnegative measurable function, homogeneous of degree 0, and not 0 almost everywhere. We claim that $L\hat{\ }(\underline{t})$ satisfies the assumptions of Theorem 7b. Indeed, since

$$\begin{aligned} \pi(\varepsilon) &= \{\underline{t} \in \underline{R}_n : |\underline{t}|^{-\omega}\Phi(\underline{t}) > \varepsilon\} \\ &= \{\underline{t} \in \underline{R}_n \backslash \{0\} : |\underline{t}| < \varepsilon^{-1/\omega}\Phi(\underline{t})^{1/\omega}\}, \end{aligned}$$

we have

$$\pi(\varepsilon_1) = (\varepsilon/\varepsilon_1)^{1/\omega}\pi(\varepsilon).$$

Our assertion follows immediately from this formula. Thus if $V(\underline{x})$ satisfies (5.1), the formulas (7.3) hold for the eigenvalues of the integral operator on $\underline{R}_n$ whose kernel is $V(\underline{x})L(\underline{x} - y)V(y)$, a fact which has application to the theory of very large eigenvalues of Toeplitz operators developed in an as yet unpublished work.

8. Mercer's theorem. Let $\underline{K}(\underline{x}(, \underline{y}), \underline{x}, \underline{y} \in \underline{R}_n$, satisfy

$$K(\underline{x}, \underline{y}) = \overline{K(\underline{y}, \underline{x})}, \tag{8.1}$$

and

$$\int_{\underline{R}_n}\int_{\underline{R}_n} |K(\underline{x}, \underline{y})|^2 \, d\underline{x} \, d\underline{y} < \infty. \tag{8.2}$$

Under these assumptions, the operator

$$U_K f \cdot (\underline{x}) = \int_{\underline{R}_n} K(\underline{x}, \underline{y}) f(\underline{y}) \, d\underline{y}, \qquad f \in L^2(\underline{R}_n),$$

is self-adjoint and compact. If $\{\lambda_k\}_{k=1}^{\infty}$ are the eigenvalues of U_K repeated according to their multiplicities, then (as is well known)

$$\sum_{k=1}^{\infty} \lambda_k^2 = \int_{\underline{R}_n}\int_{\underline{R}_n} |K(\underline{x}, \underline{y})|^2 \, d\underline{x} \, d\underline{y}.$$

Suppose now that in addition,

$$U_K \text{ is positive semidefinite.} \tag{8.3}$$

This implies that the λ_k's are nonnegative. It will be convenient to suppose that $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots$. Let us further assume that

$$K(\underline{x}, \underline{y}) \in C(\underline{R}_n \times \underline{R}_n), \tag{8.4}$$

$$\|K(\underline{x}, \cdot)\|_2 \in C(\underline{R}_n), \tag{8.5}$$

and

$$\int_{\underline{R}_n} K(\underline{x}, \underline{x}) \, d\underline{x} < \infty. \tag{8.6}$$

Then the arguments given in [1, pp. 138–140] can be applied to give Mercer's theorem,

$$\sum_{k=1}^{\infty} \lambda_k = \int_{\underline{R}_n} K(\underline{x}, \underline{x}) \, d\underline{x}. \tag{8.7}$$

(In [1], only the case where $\underline{R}_n$ is replaced by a finite interval in $\underline{R}_1$ is explicitly considered).

COROLLARY 8a. *Let $K(\underline{x}, \underline{y})$ satisfy* (8.1), (8.2) *and* (8.6) *and for each* $r = 1, 2, \cdots$, *let $K_r(\underline{x}, \underline{y})$ satisfy* (8.1)–(8.6). *Then if*

$$\lim_{r \to \infty} \int_{\underline{R}_n}\int_{\underline{R}_n} |K(\underline{x}, \underline{y}) - K_r(\underline{x}, \underline{y})|^2 \, d\underline{x} \, d\underline{y} = 0, \tag{8.8}$$

$$\lim_{r \to \infty} \int_{\underline{R}_n} K_r(\underline{x}, \underline{x}) \, d\underline{x} = \int_{\underline{R}_n} K(\underline{x}, \underline{x}) \, d\underline{x}, \tag{8.9}$$

it follows that $K(\underline{x}, \underline{y})$ is positive definite, and if $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots$ are its eigenvalues, then

$$\sum_{k=1}^{\infty} \lambda_k \leqq \int_{\underline{R}_n} K(\underline{x}, \underline{x}) \, d\underline{x}.$$

Proof. Using (8.8) and standard results from elementary perturbation theory [13, pp. 56–59], we have

$$\lim_{r\to\infty} \lambda_k^{(r)} = \lambda_k, \qquad k = 1, 2, \cdots,$$

from which, by what is essentially Fatou's lemma, we see that

$$\sum_1^\infty \lambda_k \leqq \lim_{r\to\infty} \sum_{k=1}^\infty \lambda_k^{(r)}.$$

Now use (8.7) for K_r and (8.9).

Part III. A Kac limit theorem for Hankel transforms.

9. Introduction. In order to state the problem we wish to solve, we first need various facts concerning Hankel transforms. A systematic account of this material is given in [3, § 1 and 2].

Fix $\nu \geqq 0$. Note, however, that for $\nu = 0$ a few of our formulas require changes. Let $\underline{R}^+ = [0, \infty)$ and set

$$\begin{aligned} C_\nu &= 2^{\nu-(1/2)}\Gamma(\nu + (1/2)), \\ w_\nu(dx) &= C_\nu^{-1} x^{2\nu}\, dx, \\ \mathbf{J}_\nu(x) &= C_\nu x^{(1/2)-\nu} J_{\nu-(1/2)}(x). \end{aligned}$$

Let

$$\begin{aligned} \| f \|_p &= \left[\int_0^\infty |f(x)|^p w_\nu(dx) \right]^{1/p}, \qquad 1 \leqq p < \infty, \\ \| f \|_\infty &= \operatorname*{ess\,sup}_{x \in \underline{R}^+} |f(x)|. \end{aligned}$$

If we define

$$\mathbf{J}_\nu f \cdot (u) = \int_0^\infty f(x) \mathbf{J}_\nu(xu) w_\nu(dx), \qquad f \in L^1[\underline{R}^+, w_\nu], \tag{9.1}$$

then (in an evident notation)

$$\| \mathbf{J}_\nu f \|_\infty \leqq \| f \|_1. \tag{9.2}$$

It is not difficult to show that

$$\| \mathbf{J}_\nu f \|_2 = \| f \|_2 \quad \text{for } f \in L^1[\underline{R}^+, w_\nu] \cap L^2[\underline{R}^+, w_\nu].$$

Since $L^1[\underline{R}^+, w_\nu] \cap L^2[\underline{R}^+, w_\nu]$ is dense in $L^2[\underline{R}^+, w_\nu]$, $\mathbf{J}_\nu$ has a unique continuous extension to all of $L^2[\underline{R}^+, w_\nu]$. Denoting this extension by $\mathbf{J}_\nu$ (which will cause no confusion), we have

$$\| \mathbf{J}_\nu f \|_2 = \| f \|_2, \qquad f \in L^2[\underline{R}^+, w_\nu]. \tag{9.3}$$

Moreover, it can be shown that

$$\mathbf{J}_\nu(\mathbf{J}_\nu f) = f, \qquad f \in L^2[\underline{R}^+, w_\nu].$$

For $V(x) \in L^\infty[\underline{R}^+, w_v] \cap L^2[\underline{R}^+, w_v]$ let

$$M(V)f(x) = V(x)f(x), \qquad f \in L^2[\underline{R}^+, w_v].$$

Let $L(t) \in \operatorname{Re} L^\infty[\underline{R}^+, w_v]$ be such that $L(t) \to 0$ as $t \to \infty$, and set

$$E(L)g(t) = L(t)g(t), \qquad g \in L^2[\underline{R}^+, w_v].$$

We define

$$Sf = M(V)\mathbf{J}_v E(L)\mathbf{J}_v M(V)f, \qquad f \in L^2[\underline{R}^+, w_v].$$

THEOREM 9a. *The operator S is compact.*

Proof. We have only to imitate the demonstration of Theorem 5a.

COROLLARY 9b. *If, in addition, $V(x)$ and $L(t)$ are nonnegative, then S is nonnegative.*

Our goal in Part III is to determine the asymptotic behavior of the eigenvalues of S under assumptions parallel to those in § 2 and § 3. Since v is fixed throughout our discussion, we may and do omit it for the most part.

10. Basic estimates, I. Let $E(t) = E[\chi_{[0,t]}]$, $M(\Omega) = M[\chi_\Omega]$, and let

$$P_\Omega(t) = M(\Omega)\underline{J}E(t)\underline{J}M(\Omega),$$

$$P_{\Omega_1,\Omega_2}(t) = M(\Omega_1)\underline{J}E(t)\underline{J}M(\Omega_2) + M(\Omega_2)\underline{J}E(t)\underline{J}M(\Omega_1),$$

where Ω, Ω_1 and Ω_2 are Borel measurable sets in $\underline{R}^+$ and $\Omega_1 \cap \Omega_2 = \varnothing$. Let $\mu(dx) = (1/\pi)\,dx$ for $x \in \underline{R}^+$. (Note that $\mu(dx)$ in Part III must not be confused with $\mu(d\theta)$ in Part I.) We suppose Ω, Ω_1 and Ω have finite μ measure.

THEOREM 10a. *We have, as $t \to \infty$,*

$$N^+[\delta, P_\Omega(t)]/t \to \mu(\Omega) \text{ or } 0 \quad \text{as } 0 < \delta < 1 \text{ or } \delta > 1,$$

$$N[\delta, P_{\Omega_1,\Omega_2}(t)/t \to 0, \qquad 0 < \delta.$$

The proof of these results depends upon the fact that if $T_\Omega(t) = E(t)\underline{J}M(\Omega)\underline{J}E(t)$, then

$$N^+[\delta, T_\Omega(t) \to \mu(\Omega) \text{ or } 0 \quad \text{as } 0 < \delta < 1 \text{ or } \delta > 1.$$

See Grenander and Szegö [2, § 8.7], together with the relations

$$\mathscr{R}[\underline{J}T_\Omega(t)\underline{J}] = \left\{ \int_0^t J_v(ux)f(u)W_v(du); f \in L^2[W_v] \right\},$$

$$\mathscr{R}[P_\Omega(t)] = \left\{ \chi_\Omega(\chi) \int_0^t J_v(ux)f(u)W_v(du); f \in L^2[W_v] \right\}.$$

THEOREM 10b. *Let $T_\Omega(t) = E(t)\underline{J}M(\Omega)\underline{J}E(t)$. Then*

$$\lim_{t\to\infty} N^+[a; T_\Omega(t)]/t = \begin{cases} \mu(\Omega), & 0 < a < 1, \\ 0, & a > 1. \end{cases}$$

11. Basic estimates, II. In order to proceed, we need some additional information regarding Hankel transforms. Once again we refer the reader to [3] for details.

For $v \geqq 0$ fixed, let

$$(11.1) \qquad D_v(x, y, z) = \frac{2^{(3v-(5/2))}\Gamma(v + (1/2))^2}{\Gamma(v)\pi^{1/2}}(xyz)^{-2v+1}A(x, y, z)^{2v-2},$$

where $A(x, y, z)$ is the area of the triangle with sides x, y, z if there is such a triangle. Otherwise $D_v(x, y, z)$ is taken to be zero. (If $v = 0$, this formula must be modified.) We have

$$(11.2) \qquad D_v(x, y, z) \geqq 0, \qquad 0 \leqq x, y, z < \infty.$$

The relation

$$(11.3) \qquad \int_0^\infty D_v(x, y, z)\mathbf{J}_v(zu)w(dz) = \mathbf{J}_v(xu)\mathbf{J}_v(yu)$$

is valid for $0 < x, y, z < \infty$, $0 \leqq u < \infty$. See [12, p. 367 and p. 411]. The special case $u = 0$ yields

$$(11.4) \qquad \int_0^\infty D_v(x, y, z)w(dz) = 1, \qquad 0 \leqq x, y < \infty.$$

We note in particular that

$$(11.5) \qquad D_v(x, y, z) = 0$$

unless one of the three equivalent conditions

$$(11.5') \qquad \begin{aligned} |x - y| &< z < x + y, \\ |y - z| &< x < y + z, \\ |z - x| &< y < z + x, \end{aligned}$$

is satisfied.

These formulas are the basis of the convolution theory of the Hankel transform. As in § 10, we now regard v as fixed and omit it as a subscript.

Given $f(x) \in L^1[\underline{R}^+, w]$, we define

$$(11.6) \qquad f(x, y) = \int_0^\infty f(u)D(x, y, u)w(du);$$

$f(x, y)$ in the present theory is analogous to $f(x - y)$ in Fourier analysis on the line. Multiplying (11.6) by $\mathbf{J}(xz)$ and integrating with respect to x we find, using Tonelli's theorem, that

$$\begin{aligned} \int_0^\infty f(x, y)\mathbf{J}(xz)w(dx) &= \int_0^\infty \mathbf{J}(xz)w(dx)\int_0^\infty f(u)D(x, y, u)w(du) \\ &= \int_0^\infty f(u)w(du)\int_0^\infty D(x, y, u)\mathbf{J}(xz)w(dx) \\ &= \int_0^\infty f(u)\mathbf{J}(zy)\mathbf{J}(zu)w(du). \end{aligned}$$

The analogue of convolution for Hankel transforms is as follows. If $f, g \in L^1[\underline{R}^+, w]$, then

$$f * g \cdot (x) = \int_0^\infty f(x, y)g(y)w(dy) = \int_0^\infty f(y)g(x, y)w(dy).$$

It is easy to verify that, for example, if $f, g \in L^1[\underline{R}^+ w]$, then

$$\mathbf{J}[F * g] \cdot (u) = \mathbf{J}f \cdot (u) \times \mathbf{J}g \cdot (u). \tag{11.7}$$

Let $0 < \gamma_1 < \gamma_2 < \cdots$ be the necessarily real zeros of $\mathbf{J}(x)$. For future use, we note that

$$\gamma_m \sim m\pi \quad \text{as } m \to +\infty. \tag{11.8}$$

See [14, pp. 504–505]. It follows from the elementary theory of Fourier–Bessel series developed in [14, Chap. 18] that the functions

$$\{\mathbf{J}(\gamma_m x)\}_{m=1}^\infty$$

form a complete orthonormal set in $L^2[[0, 1]; w(dx)]$. Consequently, if

$$1/\omega(m) = \int_0^1 \mathbf{J}(\gamma_m x)^2 w(dx)$$

and if $f \in L^2[[0, 1]; w(dx)]$, then

$$f(x) = \sum_{m=1}^\infty f\hat{\ }(m)\mathbf{J}(\gamma_m x) \quad \text{in } L^2[[0, 1], w(dx)], \tag{11.9}$$

where

$$f\hat{\ }(m) = \omega(m) \int_0^1 f(x)\mathbf{J}(\gamma_m x)w(dx). \tag{11.9'}$$

Using the asymptotic formula for Bessel functions, one easily sees that

$$\omega(m) = O(1) \quad \text{as } m \to \infty.$$

(Needless to say, much more precise information is available.)

Our goal in this section is to establish the following result.

THEOREM 11a. *For Ω a bounded Borel set in $\underline{R}^+$, we have for each $c > 0$,*

$$N^+[t^{-c}, P_\Omega(t)] = O(t), \qquad t \to +\infty.$$

Proof. Since Ω is bounded in $\underline{R}^+$, the set

$$|\Omega \pm \Omega| = \{|x \pm y| : x, y \in \Omega\}$$

is bounded in $\underline{R}^+$. From homogeneity considerations, we see that we can assume that the closure of $|\Omega \pm \Omega|$ is contained in $[0, 1)$. Let $\varphi(x)$ be an even function in $C^\infty(-\infty, \infty)$ such that $\varphi(x) = 1$ for $x \in |\Omega \pm \Omega|$ and $\varphi(x) = 0$ for $x \in [1, \infty)$. We assert that if $p(t, x) = \int_0^\infty \chi_{[0,t]}(u)\mathbf{J}(ux)w(du)$,

$$[p(t, \cdot)\varphi(\cdot)](x, y) = p(t; x, y) \quad \text{for } x, y \in \Omega. \tag{11.10}$$

We have

$$[p(t, \cdot)\varphi(\cdot)](x, y) = \int_0^\infty p(t, z)\varphi(z)D(x, y, z)w(dz).$$

Let $x, y \in \Omega$. Since $D(x, y, z) = 0$ if $z \notin [\Omega \pm \Omega|$ and since $\varphi(z) = 1$ if $z \in |\Omega \pm \Omega|$, we see that

$$[p(t, \cdot)\varphi(\cdot)](x, y) = \int_0^\infty p(t, z)D(x, y, z)w(dz) = p(t; x, y)$$

as desired.

We next expand $p(t, x)\varphi(x)$ in a Fourier–Bessel series. Let $\chi_t(\xi)$ be the characteristic function of $[0, t]$. Then

$$p(t, x)\varphi(x) = \sum_{m=1}^{\infty} a(t, m)\mathbf{J}(\gamma_m x)$$

if

$$\begin{aligned} a(t, m)/\omega(m) &= \int_0^1 p(t, x)\varphi(x)\mathbf{J}(\gamma_m x)w(dx) \\ &= \int_0^\infty p(t, x)\varphi(x)\mathbf{J}(\gamma_m x)w(dx). \end{aligned}$$

Using

$$p(t, x) = \int_0^\infty \chi_t(u)\mathbf{J}(ux)w(du)$$

and Fubini's theorem, we obtain

$$\begin{aligned} a(t, m)/\omega(m) &= \int_0^\infty \varphi(x)\mathbf{J}(\gamma_m x)w(dx) \int_0^\infty \chi_t(u)\mathbf{J}(xu)w(du), \\ &= \int_0^\infty \chi_t(u)w(du) \int_0^\infty \varphi(x)\mathbf{J}(\gamma_m x)\mathbf{J}(ux)w(dx), \\ &= \int_0^\infty \chi_t(u)w(du) \int_0^\infty \varphi(x)w(dx) \int_0^\infty \mathbf{J}(xv)D(u, \gamma_m, v)w(dv), \\ &= \int_0^\infty \chi_t(u)w(du) \int_0^\infty D(u, \gamma_m, v)w(dv) \int_0^\infty \varphi(x)\mathbf{J}(xv)w(dx), \\ &= \int_0^\infty \chi_t(u)w(du) \int_0^\infty D(u, \gamma_m, v)\hat{\varphi}(v)w(dv). \end{aligned}$$

Thus finally,

$$a(t, m)/\omega(m) = \int_0^t w(du) \int_0^\infty \hat{\varphi}(v)D(u, \gamma_m, v)w(dv). \tag{11.11}$$

Here $\hat{\varphi}(v) = \mathbf{J}\varphi \cdot (v)$.

We recall that if

$$\Delta h \cdot (v) = h''(v) + \frac{2v}{v}h'(v),$$

then

$$\Delta\mathbf{J}(uv) = -u^2\mathbf{J}(uv).$$

In view of this,

$$(-1)^r u^{2r}\varphi\hat{}(u) = \int_0^\infty \varphi(v)[(-1)^r u^{2r}]\mathbf{J}(uv)w(dv) = \int_0^\infty \varphi(v)\Delta^r\mathbf{J}(uv)w(dv).$$

Integrating by parts $2r$ times, we obtain

$$(-1)^r u^{2r}\varphi\hat{}(u) = \int_0^\infty [\Delta^r\varphi\cdot(v)]\mathbf{J}(uv)w(dv)$$

from which it follows that for each integer $r \geqq 0$, we have

$$|\varphi\hat{}\cdot(u)| \leqq A_r u^{-2r}, \qquad u > 0. \tag{11.12}$$

It follows from this that if $\pi m \geqq 2t$, then, using (11.11),

$$|a(t,m)| \leqq A_r m^{-2r} t^{2\nu+1}. \tag{11.13}$$

We next assert that

$$\int_0^1 \sum_{m=1}^\infty a(t,m)\mathbf{J}(\gamma_m u)D(x,y,u)w(du) = \sum_{m=1}^\infty a(t,m)\mathbf{J}(\gamma_m x)\mathbf{J}(\gamma_m y)$$

if $x, y \in \Omega$. This is equivalent to showing that

$$\int_1^\infty \mathbf{J}(\gamma_m x)D(x,y,u)w(du) = 0, \qquad x, y \in \Omega,$$

for each $m = 1, 2, \cdots$, which is true because if $x, y \in \Omega$ and $u \geqq 1$, then $D(x,y,u) = 0$.

We have thus finally shown that

$$\begin{aligned}&\int_{\underline{R}^+} \chi_\Omega(x)p(t;x,y)\chi_\Omega(y)f(y)w(dy)\\ &\quad = \sum_{m=1}^\infty a(t,m)\chi_\Omega(x)\mathbf{J}(\gamma_m x)\int_{\underline{R}^+}\chi_\Omega(y)\mathbf{J}(\gamma_m y)f(y)w(dy).\end{aligned} \tag{11.14}$$

We can now complete the proof of Theorem 11a. Let

$$\begin{aligned}P'_\Omega(t;x,y) &= \sum_{\gamma_m \leqq 2t} a(m,t)\chi_\Omega(x)\mathbf{J}(\gamma_m x)\mathbf{J}(\gamma_m y)\chi_\Omega(y),\\ P''_\Omega(t;x,y) &= \sum_{\gamma_m > 2t} a(m,t)\chi_\Omega(x)\mathbf{J}(\gamma_m x)\mathbf{J}(\gamma_m y)\chi_\Omega(y),\end{aligned} \tag{11.15}$$

and let $P'_\Omega(t)$ and $P''_\Omega(t)$ be the operators on $L^2[\underline{R}^+, w]$ corresponding to these kernels. Then it follows from (11.15) that

$$\operatorname{rank} P'_\Omega(t) = O(t) \quad \text{as } t \to +\infty. \tag{11.16}$$

From (11.13) and (11.15), we see that

$$\|P''_\Omega(t,x,y)\|_\infty \leqq B_r t^{2\nu+2-2r},$$

and thus a fortiori, that if $\|P''_\Omega(t)\|$ is the operator norm of $P''_\Omega(t)$, then

$$\|P''_\Omega(t)\| = O(t^{2\nu+2-2r}).$$

Choosing r sufficiently large, we see that

$$\|P''_\Omega(t)\| = o(t^{-c}). \tag{11.17}$$

Since

$$N^+[t^{-c}, P_\Omega(t)] \leqq N^+[0, P'_\Omega(t)] + N^+[t^{-c}, P''_\Omega(t)],$$

it follows that

$$N^+[t^{-c}, P_\Omega(t)] = O(t) \quad \text{as } t \to \infty.$$

12. The limit theorem.

DEFINITION 12a. The real function $L(t)$ $t \in \underline{R}^+$ is said to be *sufficiently regular* if $L(t)$ can be represented in the form

$$L(t) = L_1(t) + L_2(t),$$

where $L_1(t)$ and $L_2(t)$ are real and where the following conditions are satisfied:

(i) $L_1(t)$ is bounded and decreasing,

(ii) $L_1(t) \to 0$ as $t \to +\infty$.

For $\varepsilon > 0$, let $t(\varepsilon)$ be the least upper bound of the values of t for which $L_1(t) > \varepsilon$. (If ε is too large, $t(\varepsilon)$ may be undefined).

(iii) for each fixed $a > 0$ $t(a\varepsilon) = O[t(\varepsilon)]$ as $\varepsilon \to 0+$,

(iv) $t(\varepsilon_1) = o[t(\varepsilon)]$ as $\varepsilon_1, \varepsilon \to 0+$ if $\varepsilon = o(\varepsilon_1)$,

(v) $L_2(t)$ is bounded and for every $\delta > 0$, $E(\delta)$ is bounded, where

$$E(\delta) = \{t \in \underline{R}^+ : |L_2(t)| \geqq \delta L_1(t)\}.$$

The information we have generated now enables us to use familiar techniques to prove the theorem below. Let us define

$$\Psi(\varepsilon) = |\{(x, t) : V(x)^2 L(t) > \varepsilon|_{(\underline{R}^+, \mu) \times \underline{Z}^+}.$$

THEOREM 12b. *Let $V(x)$ be a bounded Borel measurable function with bounded support on $\underline{R}^+$ and let $L(t)$ be "sufficiently regular" in the sense of Definition* 12a. *Then for each δ, $0 < \delta < 1$,*

$$\begin{aligned} N^+[\varepsilon, S] &\gtrsim \Psi\left[\frac{\varepsilon}{1-\delta}\right] \\ N^+[\varepsilon, S] &\lesssim \Psi\left[\frac{\varepsilon}{1+\delta}\right] \end{aligned} \qquad \text{as } \varepsilon \to 0+, \tag{12.1}$$

where

$$S = M(V)\mathbf{J}E(L)\mathbf{J}M(V).$$

THEOREM 12c. *Let $L(t)$ satisfy the conditions of Definition* 12a *except that* (i) *is replaced by*

(i′) *$L(t)$ is decreasing and $L(t) \in L^1[[0, a], \mu(dt)]$ for some $a > 0$.*

Then (12.1) *continuous to be valid.*

Proof. For any $a > 0$, let

$$L_a(u) = [L(u) - L(a)]\chi_{[0,a]}(u), \; K_a(u) = \min\{L(u), L(a)\}.$$

Then

$$S = M(V)\mathbf{J}E(L_a)\mathbf{J}M(V) + M(V)\mathbf{J}E(K_a)\mathbf{J}M(V) = S' + S''.$$

Let, as in the proof of Theorem 11a, $\varphi(x)$ be an even C^∞-function on $\underline{R}$ with compact support, which is 1 on $|\Omega \pm \Omega|$. We do not, however, assume $|\Omega \pm \Omega| \subset [0, 1)$. We set

$$L_a'(u) = \int_0^\infty \int_0^\infty D(u, \xi, \eta) L_a(\xi) \varphi^\wedge(\eta) w(d\xi) w(d\eta). \tag{12.2}$$

An argument like that used to prove (11.3) but simpler gives

$$\begin{aligned} &L_a'(u) \in L^\infty[\underline{R}^+], \\ &L_a'(u) = O(u^{-r}) \quad \text{as } u \to +\infty, \end{aligned} \tag{12.3}$$

for every $r > 0$, and that

$$\chi_\Omega(x)[L_a']^\wedge(x, y)\chi_\Omega(y) = \chi_\Omega(x)[L_a]^\wedge(x, y)\chi_\Omega(y). \tag{12.4}$$

It follows from (12.4) that

$$S' = M(V)\mathbf{J}E(L_a')\mathbf{J}M(V).$$

Using (12.3) and applying Theorem 12b to suitable decreasing majorants of $\max(0, L_a')$ and $\max(0, -L_a')$, we see that

$$N(\varepsilon, S'] = O(\varepsilon^c) \quad \text{as } \varepsilon \to 0+ \tag{12.5}$$

for every real c.

If we set

$$\Psi_a(\varepsilon) = \{(x, t) : V(x)^2 K_a(t) > \varepsilon|_{(\underline{R}^+, \mu) \times \underline{Z}^+},$$

then it follows from Theorem 12b that for every $\delta > 0$

$$\left.\begin{aligned} N^+[\varepsilon, S''] &\lesssim \Psi_a\left[\frac{\varepsilon}{1 - \delta}\right] \\ N^+[\varepsilon, S''] &\gtrsim \Psi^a\left[\frac{\varepsilon}{1 + \delta}\right] \end{aligned}\right. \quad \text{as } \varepsilon \to 0+. \tag{12.6}$$

It is easy to see that

$$\Psi_a(\varepsilon) \leqq \Psi(\varepsilon) \leqq \Psi_a(\varepsilon) + O(1). \tag{12.7}$$

Combining (12.5), (12.6) and (12.7), we obtain our desired result.

It seems virtually certain that it is possible to prove a result which stands in the same relation to Theorem 12c as Theorem 4g stands to Theorem 3b. However, I have admittedly not tried to carry out the details.

13. Errata. We use this opportunity to correct some misprints[1] in [4]. On page 321, line 3, $\varphi_v(c, c', n \times n^{-1})$ should be $\varphi_\gamma(c, c', n \times n^{-1})$. On page 321, line 4*, Theorem 2.7 should be Theorem 2.6. On page 323, the formula asserted in Theorem 2.12 should read

$$d_K(\gamma, c) \int_N \int_K \chi(\gamma, c, xkn^{-1}yn)\, dk\, dn = \chi(\gamma, c, x)\chi(\gamma, c, y).$$

[1] "line 3" means the 3rd line from the top of the page; "line 4*" means the 4th line from the bottom of the page.

On page 332, in line 3, $\gtrsim$ should be $\geqq$ and in line 5, $\lesssim$ should be $\leqq$. Line 7 is irrelevant and can be omitted. On page 337, line 2, $\Psi(x)$ should be $\Psi(\varepsilon)$. On page 339, line 6, $h_k^2(\varepsilon, \alpha, a)$ should be $h_k^2 R\hat{}(\varepsilon, \alpha, a)$.

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